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Asymptotic derivation of models for anisotropic piezoelectric beams and shallow arches

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Abstract

Asymptotic derivation of models for anisotropic piezoelectric beams and shallow arches

Due to the wide use of anisotropic solid structures, the study of anisotropic piezoelectricity becomes increasingly important. The purpose of this thesis is to extend the research of Álvarez-Dios & Viaño [1993, 1996, 2003] and Trabucho & Viaño [1996] to anisotropic elastic beams, and present asymptotic models for anisotropic piezoelectric beams.

These models are derived in a rigorous way from a three-dimensional problem for an anisotropic piezoelectric beam under an electric potential applied on one of two types of boundary: first type when the electric potentials are applied in both extremities of the beam and second type when the voltage acts on its lateral surface. In order to analyse the asymptotic behaviour of the solution of the three-dimensional piezoelectricity problem for a beam which has diameter of the cross section much smaller than the length, the asymptotic method was implemented by considering the diameter as a small parameter.

We start by briefly introducing the three-dimensional equations that describe the linear piezoelectricity theory: a coupled system of mechanical and electrical equilibrium boundary value partial differential equations; weak and strong formulations. After, we expand the solution as an expansion with respect to the small parameter and we obtain a sequence of problems which allow us characterize some terms of the development. As a result of this analysis, we establish the following models for piezoelectric anisotropic beams:

- The one-dimensional beam theory from the three-dimensional problem for an anisotropic piezoelectric beam of class 2 under an applied electric potential on both ends is derived in Chapter 3. The characterization of the second-order displacements is an essential step to achieve this model and to demonstrate the strong convergence.

- An approach for anisotropic piezoelectric beam of class 2 in response to an applied electric potential acting on its lateral surface is derived in Chapter 4. The weak convergence result is also discussed in this chapter for anisotropic piezoelectric beam of the subclass *6mm*, where it is concluded that the displacement vector field and the electric potential weakly converge towards the leading terms of the *displacement - electric potential* expansions.

- A zeroth-order model for a transversely isotropic - *6mm* symmetry class - piezoelectric shallow arch submitted to an electric potential at the both ends was determined in Chapter 5 by the asymptotic expansion *displacement* - *stress* - *electric potential* - *electric displacement*.

Resumo

Determinação assimptótica de modelos para vigas piezoeléctricas anisotrópicas

O estudo do fenómeno piezoeléctrico aumentou consideravelmente devido à crescente utilização dos materiais piezoeléctricos anisotrópicos em aplicações de engenharia. A presente tese tem como principais objectivos, a determinação e justificação assimptótica de modelos de vigas piezoeléctricas anisotrópicas, para, desta forma, generalizar a teoria de vigas elásticas propostas por Álvarez-Dios & Viaño [1993, 1996, 2003] e Trabucho & Viaño [1996].

Os modelos desenvolvidos são baseados na teoria da piezoelectricidade linearizada para vigas feitas dum material piezoeléctrico anisotrópico, sujeitas a um potencial eléctrico que actua em dois tipos de fronteira. No tipo I, o potencial eléctrico é aplicado nas extremidades da viga e, no tipo II, a voltagem é induzida numa área lateral da viga. Tendo em vista a análise do comportamento da solução do problema piezoeléctrico, numa viga em que o comprimento é muito maior do que as dimensões da sua secção transversal e o diâmetro tende para zero, utiliza-se o método assimptótico, tomando-se o diâmetro da viga como pequeno parâmetro.

Nesta documento, começa-se por apresentar, de forma resumida, as equações tridimensionais que descrevem a piezoelectricidade linear num sólido: equações diferenciais parciais do acoplamento electromecânico; formulação forte e fraca. De seguida, expandindo a solução numa série assimptótica em função do pequeno parâmetro, obtemos uma série de problemas através dos quais se caracterizam alguns termos do desenvolvimento. Como resultado desta análise, propõe-se, nesta tese, os seguintes modelos para vigas piezoeléctricas anisotrópicas:

- Modelo unidimensional para o problema electromecânico, definido numa viga cujo

material piezoeléctrico pertence à classe 2, e em resposta a um potencial eléctrico induzido numa região do tipo I. A caracterização do segundo termo dos deslocamentos é essencial para a determinação deste modelo e para a demonstração da convergência forte. Estes cálculos encontram-se detalhados no Capítulo 3.

- No Capítulo 4, é justificado um modelo para uma viga piezoeléctrica anisotrópica de classe 2 submetida a um potencial eléctrico numa área do tipo II. Demonstra-se ainda que, se a viga é feita de uma material piezoeléctrico pertencente à classe de simetria *6mm*, então o vector dos deslocamentos e o potencial eléctrico convergem fracamente para os primeiros termos dos respectivos desenvolvimentos.

- Um modelo de primeira ordem para vigas, "debilmente" curvas, constituídas por material piezoeléctrico transversalmente isotrópico - classe *6mm* - e com potencial eléctrico aplicado nos extremos da viga, é apresentado no Capítulo 5. A formulação mista, juntamente com o desenvolvimento do tipo *deslocamento-potencial eléctrico-tensão-deslocamento eléctrico*, mostrou ser uma solução eficaz na obtenção do respectivo modelo e na demonstração do resultado de convergência forte.

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Conventions and notations

The conventions and notations follow mostly Trabucho & Viaño [1996] and Álvarez-Dios & Viaño [1996]. For the sake of completeness, the most relevant are summarized below.

Conventions

The notations follow in general these guidelines:

1. Tensors and vectors are denoted by boldface.

2. The standard Einstein summation is adopted, and Latin indices range over the values 1,2,3 and Greek indices over the values 1,2.

3. The symbol "'h" designates a parameter that is > 0 and approaches zero.

4. Superscripts h will be dropped when equal to one, so that

$$\Omega = \Omega^1, \quad \omega = \omega^1, \quad \boldsymbol{x} = \boldsymbol{x}^1 = (x_1, x_2, x_3), \quad \Gamma_{eD} = \Gamma^1_{eD}, \quad ...$$

5. Differential operators $\partial/\partial x_i^h$, $\partial/\partial x_i$ are denoted by ∂_i^h, ∂_i.

6. For functions z depending on the variable x_3 we use the notations z', z'', ... for its derivatives, and in some cases the alternative $\partial_3 z$, $\partial_{33} z$,...

Notations

We collect here the definitions of some of the most frequently used symbols; others are defined where they first appear in the text.

Piezoelectricity

$\boldsymbol{e}$: the strain tensor in which the components are represented by (e_{ij}) .

$\boldsymbol{E}$: the electric vector field in which the components are represented by (E_i) .

$\boldsymbol{\sigma}$: the stress tensor in which the components are represented by (σ_{ij}).

$\boldsymbol{D}$: the electric displacement in which the components are represented by (D_i).

U : the internal energy density.

H : the electric enthalpy function.

$\boldsymbol{C}$: the elastic tensor of the material of components $C_{ijkl} = \frac{\partial \sigma_{ij}}{\partial e_{kl}}$.

$\boldsymbol{P}$: the piezoelectric tensor of components $P_{kij} = \frac{\partial \sigma_{ij}}{\partial E_k}$.

$\boldsymbol{\varepsilon}$: the dielectric tensor of the material of components $\varepsilon_{ij} = \frac{\partial D_i}{\partial E_j}$.

F : the elastic enthalpy function.

$\bar{\boldsymbol{C}}$: the elastic tensor of components $\bar{C}_{ijkl}$.

$\bar{\boldsymbol{P}}$: the piezoelectric tensor of the material of components $\bar{P}_{kij}$.

$\bar{\boldsymbol{\varepsilon}}$: the dielectric tensor of components $\bar{\varepsilon}_{ij}$.

λ, μ : Lamé's constants.

Y, ν : Young's modulus and Poisson's ratio.

Ω : the piezoelectric body.

$\boldsymbol{\chi}$: deformation map.

$\boldsymbol{id}$: the identity transformation.

$\boldsymbol{x}$: a generic point in the set Ω.

$\boldsymbol{u}(\boldsymbol{x})$: the displacement vector at the point $\boldsymbol{x}$ in the body Ω.

$\boldsymbol{n}$: the unit vector outward normal to the surface.

ϵ_0 : the symmetric positive permittivity tensor.

μ_0 : the dielectric impermeability.

Common notations

x, $\boldsymbol{x}$: the scalar $x \in \mathbb{R}$, and vector $\boldsymbol{x} \in \mathbb{R}^d$.

u, $\boldsymbol{u}$: the scalar valued function u, vector valued function $\boldsymbol{u}$.

Ω, Γ: the bounded domain (open and connected subset of $\mathbb{R}^d$) with sufficiently smooth boundary $\Gamma = \partial\Omega$.

$\overline{\Omega} = \Omega \cup \partial\Omega$: the closure of Ω.

∂_i : $\partial_i = \partial/\partial x_i$, $\boldsymbol{x} = (x_i) \in \Omega$.

$C(\Omega)$, $C(\overline{\Omega})$: the functions continuous in Ω and $\overline{\Omega}$, respectively.

$L^s(\Omega)$:	the Lebesgue space of s-integrable functions u with the norm $\|u\|_{L^s(\Omega)} = \left(\int_\Omega |u|^s d\boldsymbol{x}\right)^{1/s}, 1 \le s < \infty.$
$L^2(\Omega)$:	the Lebesgue space of scalar square-integrable functions on Ω.
$\|\cdot\|_\Omega,\ \|\cdot\|_{0,\Omega}$:	$\|u\|_{L^2(\Omega)} = (u,u)^{\frac{1}{2}}_{L^2(\Omega)}$.
$(u,v)_{L^2(\Omega)},\ (u,v)_0$:	$(u,v)_{L^2(\Omega)} = (u,v)_0 = \int_\Omega u(x)v(x)dx.$
$H^m(\Omega)$:	the Sobolev space $H^m(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega),\ |\alpha| = \alpha_1 + \cdots + \alpha_n \le m\}.$ which is a Banach space for the norm $\|u\|_{H^m(\Omega)} = \|u\|_{m,\Omega} = \left(\sum_{|\alpha|\le m} \int_\Omega |\partial^\alpha u(x)|^2\, dx\right)^{1/2}.$
$H^1_0(\Omega),\ H^{1/2}(\Gamma_N)$:	Sobolev spaces and subsets.
$\mathcal{D}(\Omega)$:	the linear space of infinitely differentiable functions, with compact support in Ω.
$\mathcal{D}'(\Omega)$:	the dual space of $\mathcal{D}(\Omega)$.
div :	the divergence operator, div $\boldsymbol{u}(\boldsymbol{x}) = \sum_{i=1}^d \frac{\partial u_i(\boldsymbol{x})}{\partial x_i}$ for a vector valued $\boldsymbol{u} = (u_1(\boldsymbol{x}), \cdots, u_d(\boldsymbol{x}))^T$ for $\boldsymbol{x} \in \mathbb{R}^d$.
$\boldsymbol{\nabla}$:	the gradient operator, $\boldsymbol{\nabla} u(\boldsymbol{x}) = \left(\frac{\partial u(\boldsymbol{x})}{\partial x_1}, \cdots, \frac{\partial u(\boldsymbol{x})}{\partial x_d}\right)$ for $\boldsymbol{x} \in \mathbb{R}^d$.
curl, curl $(N=2)$:	the rotational operator for distributions ϕ of $\mathcal{D}'(\Omega)$ and $\boldsymbol{u}$ of $\mathcal{D}'(\Omega)^2$, $\mathbf{curl}\ \phi = \left(\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1}\right)$, curl $\boldsymbol{u} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$.
curl $(N=3)$:	the rotational operator of a distribution $\boldsymbol{u}$ of $\mathcal{D}'(\Omega)^3$, $\mathbf{curl}\ \boldsymbol{u} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$.
Δ:	the Laplace operator, $\Delta u(\boldsymbol{x}) = \sum_{i=1}^d \frac{\partial^2 u(\boldsymbol{x})}{\partial^2 x_i}$ for $\boldsymbol{x} \in \mathbb{R}^d$.
$v_n \to v$:	the converge in norm (strong convergence).
$v_n \rightharpoonup v$:	the weak convergence.
$\Gamma_{dD}\ (\Gamma_{eD})$:	the portion of the surface with mechanical (electrical) Dirichlet boundary conditions.
$\Gamma_{dN}\ (\Gamma_{eN})$:	the portion of the surface with mechanical (electrical) Neumann boundary conditions.
$V_0(\Omega)$:	the spaces of admissible displacements.
$V_{w,0}(\Omega)$:	the space of admissible displacements which satisfy the weak boundary conditions.

$\Psi_0(\Omega)$: the spaces of admissible electric potential

$\Psi_2(\Omega)$: a closed and convex subset of $H^1(\Omega)$.

$\Psi(\Omega)$: the spaces of admissible electric potential.

$X_0(\Omega)$: the space $X_0(\Omega) = V_0 \times \Psi_0,\ (X_{0,w}(\Omega) = V_{w,0} \times \Psi_0)$.

$X_1(\Omega)$: the space $X_1(\Omega) = [L^2(\Omega)]_s^9 \times L^2(\Omega)$.

$X_2(\Omega)$: the space $X_2(\Omega) = V_0 \times \Psi_2\ (X_{2,w}(\Omega) = V_{w,0} \times \Psi_0)$.

$d\boldsymbol{x}$: the volume element in Ω.

$\boldsymbol{p} = (p_i)$: the surface forces density at the extremities of the beam.

$\boldsymbol{f} = (f_i)$: the body force density.

$\boldsymbol{g} = (g_i)$: the surface traction density.

Piezoelectric beam

ω : the domain in $\mathbb{R}^2$ (open, bounded, connetected subset with the Lipschitz - continuous boundary $\gamma = \partial\omega$).

$d\gamma$: the length element along γ.

γ_0 : the measurable subset of γ with length $\gamma_0 > 0$.

L : the length of the beam.

$\Omega = \omega \times (0, L)$.

$\Omega^h = \omega^h \times (0, L)$.

$\Gamma_0^h = \omega^h \times \{0\}$: the left extremity of the beam.

$\Gamma_L^h = \omega^h \times \{L\}$: the right extremity of the beam.

φ_0^h : the electric boundary.

Γ_{dD}^h : the portion of the surface where a beam is clamped.

Γ_{eD}^h : the portion of the surface where is applied an electric potential φ_0^h.

$\Gamma_{dN}\,(\Gamma_{eN})$: the portion of the surface that is subjected to the action of external applied forces (potential).

$\boldsymbol{u}^h = (u_i^h)$: the displacement vector at a point $\boldsymbol{x}^h$ in the body Ω^h. An arbitrary admissible displacement vector is denoted $\boldsymbol{v}^h$.

φ^h : the electric potential vector at a point $\boldsymbol{x}^h$ in the body Ω^h. An arbitrary admissible electric field is denoted ψ^h.

$\hat{\varphi}^h$: an extension of φ_0^h in $H^1(\Omega^h)$.

$\bar{\varphi} = \varphi - \hat{\varphi}$.

Π^h :	the bijection from $\overline{\Omega}$ onto $\overline{\Omega}^h$, defined by $\Pi^h(\boldsymbol{x}) = \boldsymbol{x}^h$.
$d\Gamma$:	the area element along $\partial\Omega$.
$d\Gamma^h$:	the area element along $\partial\Omega^h$.
$\boldsymbol{p}^h = (p_i^h)$:	the surface force density at the extremities of the beam.
$\boldsymbol{f}^h = (f_i^h)$:	the body force density.
$\boldsymbol{g}^h = (g_i^h)$:	the surface traction density.
$\boldsymbol{n}^h = (n_i^h)$:	the unit outer vector along the boundary $\partial\Omega^h$.
$\boldsymbol{e}^h = (e_{ij}^h)$:	the strain tensor.
$\boldsymbol{E}^h = (E_i^h)$:	the electric vector field, typically given by $E_i^h = -\partial_i^h \varphi^h$.
$\boldsymbol{\sigma}^h = (\sigma_{ij}^h)$:	the stress tensor, typically given by $\sigma_{ij}^h = C_{ijkl}^h e_{kl}^h - P_{kij}^h E_k^h$.
$\boldsymbol{D}^h = (D_i^h)$:	the electric displacement, typically given by $D_k^h = P_{kij}^h e_{ij}^h + \varepsilon_{kl}^h E_l^h$.
$\boldsymbol{u}(h) = (u_i^h)$:	the scaled displacement field.
$u_i(h) : \overline{\Omega} \to \mathbb{R}$:	the component of scaled displacements.
$\varphi(h) : \overline{\Omega} \to \mathbb{R}$:	the scaled electric potential.
$\boldsymbol{\sigma}(h) = (\sigma_{ij})h))$:	the scaled stress tensor.
$\boldsymbol{D}(h) = (D_i(h))$:	the scaled electric displacement.
$A(\omega)$:	the area of ω.
I_α :	$I_\alpha = \displaystyle\int_\omega x_\alpha^2 \, d\omega$.
$X_{\alpha\beta},\ Y_\alpha,\ Z,\ L$:	the constants of geometry.
$\overline{X}_{\alpha\beta},\ \overline{Y}_\alpha,\ \overline{Z}$:	the constants of geometry.
J, Y :	the new elastic constant defined in (4.75).
J :	the torsion constant $J > 0$.

Piezoelectric shallow arch

ω :	the domain in $\mathbb{R}^2$ (open, bounded, connetected subset with Lipschitz continuous boundary.
$(\boldsymbol{t}^{*,h}, \boldsymbol{n}^{*,h}, \boldsymbol{b}^{*,h})$:	the Frenet trihedron.
$\boldsymbol{\Theta}^h$:	mapping that defines the reference configuration of the shallow arch.

$b^h_{ij}(\boldsymbol{x}^h)$:	the components of the Jacobian matrix, $((\boldsymbol{\nabla}^h\boldsymbol{\Theta}^h(\boldsymbol{x}^h))^{-1})_{ij}$.
o^h :	the determinant of the Jacobian, $\det \boldsymbol{\nabla}^h\boldsymbol{\Theta}^h(\boldsymbol{x}^h)$.
$\breve{n}^h_i$:	unit outer normal vector along the boundary $\breve{\partial}\breve{\Omega}^h$ of $\breve{\Omega}^h$.
$\{\breve{\omega}^h\}^-$:	the cross section of the shallow arch.
$\left\{\breve{\Omega}^h\right\}^-$:	the reference configuration of the shallow arch, where $\breve{\Omega}^h = \boldsymbol{\Theta}^h(\Omega^h)$.
$\breve{\boldsymbol{x}}^h = (\breve{x}^h_i)$:	a generic point in the set $\left\{\breve{\Omega}^h\right\}^-$.
$\breve{\boldsymbol{u}}^h = (\breve{u}^h_i)$:	the displacement vector at a point $\breve{\boldsymbol{x}}^h$ in the body $\breve{\Omega}^h$. An arbitrary admissible displacement vector is denoted $\breve{\boldsymbol{v}}^h$.
$\breve{\varphi}^h$:	the electric potential vector at a point $\breve{\boldsymbol{x}}^h$ in the body $\breve{\Omega}^h$.
$\boldsymbol{\Theta}^h(\gamma \times (0, L))$:	the lateral surface of the set $\left\{\breve{\Omega}^h\right\}^-$.
$\breve{\Gamma}^h_0 = \boldsymbol{\Theta}^h(\Gamma_0)$:	the left end of beam.
$\breve{\Gamma}^h_L = \boldsymbol{\Theta}^h(\Gamma_L)$:	the right end of beam.
$\breve{\Gamma}^h_D = \boldsymbol{\Theta}^h(\Gamma_D)$:	the portion of the boundary where the shallow arch is clamped.
$\breve{\Gamma}^h_{eD} = \boldsymbol{\Theta}^h(\Gamma_D)$:	the portion of the boundary where the shallow arch is subjected an electric voltage $\breve{\varphi}^h_0$.
$\breve{\Gamma}^h_{dN}$.	the portion of the boundary where the shallow arch is clamped.
$\breve{\Gamma}^h_{eN} = \breve{\Gamma}^h_0 \cup \breve{\Gamma}^h_L$.	the portion of the boundary where the shallow arch has a Neumann condition
$\breve{\partial}^h_j \breve{v}^h_i(\breve{\boldsymbol{x}}^h)$:	the partial derivative.
$\breve{\boldsymbol{e}}^h = (\breve{e}^h_{ij})$:	the strain tensor.
$\breve{\boldsymbol{E}}^h = (\breve{E}^h_k)$:	the electric field vector.
$\breve{\boldsymbol{p}}^h = (\breve{p}^h_i)$:	the body force density.
$\breve{\boldsymbol{f}}^h = (\breve{f}^h_i)$:	the body force density.
$\breve{\boldsymbol{g}}^h = (\breve{g}^h_i)$:	the surface traction density.
$\breve{\boldsymbol{e}}^h = (e^h_i)$:	the strain tensor.
$\breve{\boldsymbol{E}}^h = (\breve{E}^h_i)$:	the electric vector field.
$\breve{\boldsymbol{\sigma}}^h = (\breve{\sigma}^h_{ij})$:	the stress tensor, typically given by $\breve{\sigma}^h_{ij} = \breve{C}^h_{ijkl}\breve{e}^h_{kl} - \breve{P}^h_{kij}\breve{E}^h_k$.
$\breve{\boldsymbol{D}}^h = (\breve{D}^h_i)$:	the electric displacement, typically given by $\breve{D}^h_k = \breve{P}^h_{kij}\breve{e}^h_{ij} + \varepsilon^h_{kl}\breve{E}^h_l$.
$\breve{\boldsymbol{n}}^h = (\breve{n}^h_i)$:	the outward unit vector normal to the boundary.
(t^*, n^*, b^*) :	the Frenet trihedron.
$\kappa^h(x_3)$, $\tau^h(x_3)$:	the centreline's curvature and torsion.
$\boldsymbol{e}^\phi = (e^\phi_{ij})$:	the strain tensor.

$\boldsymbol{E}^{\phi} = (E_k^{\phi})$: the electric field vector.

Chapter 1

Introduction

1.1 Background

Piezoelectric Solid (PS) has been defined in the literature as a solid that *produce an electric field when deformed and, conversely, undergo deformation when subjected to an electric field* [see e.g. Hwu *et al.*, 2004]. The electromechanical coupling is the key physical property of the piezoelectric materials and was investigated by Crawley & Luis [1987] via experiments and analytic models. These piezoelectric materials are widely used as sensors and actuators in many important applications, such as structural, aerospace, robotic, various medical purposes, etc. With the increasing successful technological applications of piezoelectric actuators and sensors, piezoelectric solids have attracted considerable attention from scientific researchers, in order to understand the basic phenomena responsible for their particular properties.

In mathematical terms the electromechanical coupling is described through a set of partial differential equations of second order, in which the displacement components and electric potential are taken as the essential unknowns (see e.g. Tiersten [1969], Jackson [1975], Nye [1985], Ikeda [1990]). Due to the complexity of the coupling effect between mechanical and electrical fields it is very difficult, if not impossible, to find an exact solution to the piezoelectric problem. Many simplifying assumptions, such as simplification of geometry and restrictions on the piezoelectric material's electromechanical behavior, have been made to obtain approximative solutions. Several approximations have been developed by Robbins & Reddy [1991], Crawley & Luis [1987], Bisegna & Maceri [1996b], Saravanos & Heyliger [1999], Vidoli & Batra [2000] and Vidoli *et al.* [2000] for modelling of piezoelectric structures.

One way to generate models for piezoelectric problems in *thin piezoelectric solids*, in which one or more dimensionals are small, compared to the others, is to reduce the original problem to a new one, in a lower dimensional space, where the small dimensions disappear.

An approach is deduced by the study of the asymptotic behavior of the three-dimensional problem depending on a small parameter, as this one goes to zero.

Based on this method, we intend, with this work, to derive and to mathematically justify lower-dimensional equations for linearly piezoelectric beams and shallow arches, as the diameter approaches zero.

We present next an overview that tries to cover the main techniques and results of which we are aware and are most closely related to the aim of the present thesis.

The application of the asymptotic expansion to justify plate theories was initiated in the pioneering work of Maugin & Attou [1990], where it was used to derive a two-dimensional theory for piezoelectric plates belonging to the class 6 *mm* of piezoelectric crystals. This method was also successfully applied by Bisegna & Maceri [1996a], Rahmoune *et al.* [1998], Sene [2001], Miara [2001], Licht & Weller [2003], Weller & Licht [2002], Raoult & Sene [2003], Figueiredo & Leal [2005] and Weller & Licht [2007] in deriving two-dimensional models for thin piezoelectric plates. Extensions of this method can be found in Collard & Miara [2003] for thin piezoelectric shells.

The classical mechanical Kirchhoff-Love theory for piezoelectric solids was established by Maugin & Attou [1990] and successively found in many other works mentioned above. Rahmoune *et al.* [1998] introduced the idea of dependence between the electric assumptions and the electric boundary conditions. An uncoupled electromechanical problem for homogeneous and orthorhombic piezoelectric plates was determined, for which only the mechanical problem should be solved - the electric potentials can be entirely deduced from the mechanical displacement.

The work of Sene [2001] justifies mathematically the theory of Destuynder *et al.* [1992] (which is fully described in all admissible piezoelectric crystals) for a piezoelectric plate. In this work, Sene [2001], states that the electric potential affects the mechanical equations only through the difference of potential between the horizontal faces. This paper also justifies the *a priori* assumptions that the electric potential is a second polynomial order with respect to the thickness variable, assumption originally proposed by Bernadou & Haenel [2003]. The paper by Raoult & Sene [2003] generalizes the previous model by considering a magnetic effect accompanying the dynamic behavior. In the work by Licht & Weller [2003] is shown that, according to the type of boundary conditions, the asymptotic analysis of thin linearly piezoelectric plate as the thickness approaches zero leads to two distinct models, linked to sensor or actuator behavior. Further, they proposed in Weller & Licht [2007] four different models of linearly electromagnetic-elasticity thin plate according to the type of electromagnetic boundary conditions. The lower-dimensional equations for a nonhomogeneous anisotropic plates has been proposed by Figueiredo & Leal [2005], which extends the previous work of Sene [2001] for homogeneous and isotropic materials.

Latter papers by Collard & Miara [2003] and Sabu [2002, 2003] published two-dimensional models for boundary value problems considering piezoelectric shells. A two-dimensional nonlinear shell model was proposed by Collard & Miara [2003], where they derived two-dimensional membrane equations and flexural models written on the middle surface of the shell, and verified that the coupling between the electric limit displacement field and the limit electric potential inherent to piezoelectricity appears only in the membrane model. The two-dimensional theories for the vibrations of thin piezoelectric flexural and shallow shells were proposed by Sabu [2002, 2003], by considering the limit, as the thickness goes to zero, the eigenvalue problem for piezoelectric respective flexural and shallow shells.

Pioneering work for modelling of thin linearly isotropic piezoelectric beams was performed by Viaño *et al.* [2005a,b], where a purely mechanical Bernoulli-Navier beam theory emerges. Recent models of the phenomena piezoelectric beams were described by Figueiredo & Leal [2006], Viriyasrisuwattana *et al.* [2007] and Weller & Licht [2008]. A new mathematical model for a linearly nonhomogeneous anisotropic thin beam was proposed by Figueiredo & Leal [2005], which is a system of coupled equations, with generalized Bernoulli-Navier equations and reduced Maxwell-Gauss equations but it was not possible to prove the uniqueness of the solution for the limit equations. A one-dimensional asymptotic models for a linearly piezoelectric slender beams can be found in Weller & Licht [2008].

In this work, we take advantage of previous experience, varying the applied boundary electric conditions as suggested by Weller & Licht [2002] and using the asymptotic method mentioned above, and we find various models for linearly piezoelectric beams and shallow arches as the diameter of its cross section approaches zero. We complete the asymptotic analysis, proving weak and strong convergence results and characterizing completely the limit models, closing several open questions form above mentioned papers.

1.2 Main results of this thesis

We present next, in a simple setting, the principal aspects of this thesis.

Consider the three-dimensional piezoelectric beam (as illustrated in Figure 1.1)

$$\Omega^h = \omega^h \times (0, L)$$

where h is a small positive number representing the diameter of the cross section ω^h, a bounded domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ^h. The boundary of the domain Ω^h, $\Gamma^h = \partial\Omega^h$, is composed of $\Gamma^h = \Gamma^h_{dD} \cup \Gamma^h_{dN}$, $\Gamma^h_{dD} \cap \Gamma^h_{dN} = \varnothing$ for the mechanical boundary conditions and $\Gamma^h = \Gamma^h_{eD} \cup \Gamma^h_{eN}$, $\Gamma^h_{eD} \cap \Gamma^h_{eN} = \varnothing$ for the electrical boundary conditions. We assume that the beam is clamped along the boundary Γ^h_{dD}, and subjected to an electric potential φ^h_0 on Γ^h_{eD}. The beam is also subjected to applied body forces of density $(f^h_i) : \Omega^h \to \mathbb{R}^3$ acting inside Ω^h, and on surface forces (g^h_i) on Γ^h_{dN}. We denote a typical point in Ω^h by $\boldsymbol{x}^h = (x^h_\alpha, x_3)$.

We denote by $\boldsymbol{C}^h = (C^h_{ijkl})$, $\boldsymbol{P}^h = (P^h_{ikl})$ and $\boldsymbol{\varepsilon}^h = \varepsilon^h_{ij}$, respectively, the elastic tensor field, the piezoelectric tensor field, and the dielectric tensor field that characterize the material.

In the framework of small deformations and linear piezoelectricity, the three-dimensional static equations for the piezoelectric solid Ω^h are the following: *Find a displacement vector field* $\boldsymbol{u}^h : \Omega^h \to \mathbb{R}^3$ *and an electric potential* $\varphi^h : \Omega^h \to \mathbb{R}$ *such that* (see Chapter 2 with $h = 1$)

$$\sigma^h_{ij}\left(\boldsymbol{u}^h, \varphi^h\right) = C^h_{ijkl} e^h_{kl}(\boldsymbol{u}^h) - P^h_{kij} E^h_k(\varphi^h) \quad \text{in} \quad \Omega^h,$$

$$D^h_k\left(\boldsymbol{u}^h, \varphi^h\right) = P^h_{kij} e^h_{ij}(\boldsymbol{u}^h) + \varepsilon^h_{kj} E_j(\varphi^h) \quad \text{in} \quad \Omega^h,$$

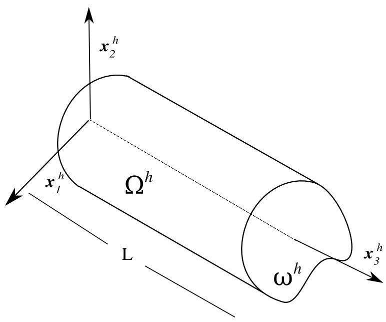
and

$$\partial^h_j \sigma^h_{ij}\left(\boldsymbol{u}^h, \varphi^h\right) = f^h_i, \quad \text{in} \quad \Omega^h, \qquad \partial^h_i D^h_i\left(\boldsymbol{u}^h, \varphi^h\right) = 0, \quad \text{in} \quad \Omega^h,$$

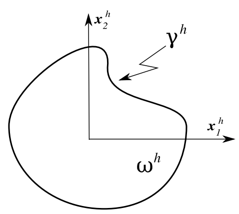
$$\sigma^h_{ij}\left(\boldsymbol{u}^h, \varphi^h\right)\ n^h_j = g^h_i \quad \text{in} \quad \Gamma^h_{dN}, \qquad D^h_k\left(\boldsymbol{u}^h, \varphi^h\right) n^h_k = 0 \quad \text{in} \quad \Gamma^h_{eN},$$

$$u^h_i = 0, \quad \text{on} \quad \Gamma^h_{dD}, \qquad \varphi^h = \varphi^h_0 \quad \text{on} \quad \Gamma^h_{eD},$$

where $e^h_{ij}(\boldsymbol{u}^h) = 1/2(\partial^h_i u^h_j + \partial^h_j u^h_i)$ denote the components of the linearized strain tensor, $E^h_i(\varphi^h)$ is the components of the static electric vector field defined by $E^h_i(\varphi^h) = -\partial_i \varphi^h$, and (n^h_i) is the unit outer normal vector along Γ^h. This problem can be put in a variational form, which consists in finding $(\boldsymbol{u}^h, \varphi^h) \in V^h_0(\Omega^h) \times H^1(\Omega^h)$ such that $\varphi^h = \varphi^h_0$ on Γ^h_{eD}



(a) General view.



(b) Cross-section.

Figure 1.1: Notations for a piezoelectric beam.

and (Section 2.3.2.1)

$$\int_{\Omega^h} \left[C^h_{ijkl} e^h_{kl}(\boldsymbol{u}^h) - P^h_{mij}\, E^h_m\left(\varphi^h\right)\right] e^h_{ij}(\boldsymbol{v}^h) d\boldsymbol{x}^h$$
$$+ \int_{\Omega^h} \left[P^h_{mij}\, e^h_{ij}(\boldsymbol{u}^h) + \varepsilon^h_{mi}\, E^h_i\left(\varphi^h\right)\right] E^h_m\left(\psi^h\right) d\boldsymbol{x}^h = \int_{\Omega^h} f^h_i v^h_i d\boldsymbol{x}^h + \int_{\Gamma^h_{dN}} g^h_i v^h_i d\Gamma^h,$$

for all

$$\boldsymbol{v}^h \in V^h_0 := V^h_0(\Omega^h) = \left\{\boldsymbol{v}^h \in \left[H^1(\Omega^h)\right]^3 : \boldsymbol{v}^h = 0 \ \text{ on } \ \Gamma^h_{dD}\right\},$$
$$\psi^h \in \Psi^h_0 := \Psi^h_0(\Omega^h) = \left\{\psi^h \in H^1(\Omega^h) : \psi^h = 0 \ \text{ on } \ \Gamma^h_{eD}\right\}.$$

In Section 2.3.2.1 we show that if there is an extension $\hat{\varphi}^h$ of φ^h_0 in $H^1(\Omega^h)$, then the solution $(\boldsymbol{u}^h, \varphi^h)$ is derived from $\varphi^h = \bar{\varphi}^h + \hat{\varphi}^h$ with $(\boldsymbol{u}^h, \bar{\varphi}^h) \in V^h_0(\Omega^h) \times \Psi^h_0(\Omega^h)$ satisfying:

$$\int_{\Omega^h} \left[C^h_{ijkl} e^h_{kl}(\boldsymbol{u}^h) - P^h_{mij}\, E^h_m\left(\bar{\varphi}^h\right)\right] e^h_{ij}(\boldsymbol{v}^h) d\boldsymbol{x}^h$$
$$+ \int_{\Omega^h} \left[P^h_{mij}\, e^h_{ij}(\boldsymbol{u}^h) + \varepsilon^h_{mi}\, E^h_i\left(\bar{\varphi}^h\right)\right] E^h_m\left(\psi^h\right) d\boldsymbol{x}^h$$
$$= \int_{\Omega^h} \left[f^h_i v^h_i + P^h_{kij} E_k(\hat{\varphi}^h) e^h_{ij}(\boldsymbol{v}^h) - \varepsilon^h_{kl} E_l(\hat{\varphi}^h) E^h_k(\psi^h)\right] d\boldsymbol{x}^h + \int_{\Gamma^h_{dN}} g^h_i v^h_i d\Gamma^h.$$

The existence and uniqueness of solution $(\boldsymbol{u}^h, \bar{\varphi}^h)$ and $(\boldsymbol{u}^h, \varphi^h)$ follows from the classical Korn's and Poincaré's inequalities, by assuming some regularity to the data (see Proposition 1 and Corollary 1).

In Chapter 3, we consider that the beam is made of a piezoelectric crystal material of class 2 which the components of the elastic, piezoelectric and dielectric material satisfy (cf. (3.4))

$$C^h_{iijj} \neq 0, \quad C^h_{klkl} \neq 0, \qquad \text{for} \quad k \neq l,$$
$$P^h_{3\alpha\beta} \neq 0, \quad P^h_{\alpha 3\beta} \neq 0, \quad P^h_{333} \neq 0, \quad \varepsilon^h_{\alpha\beta} \neq 0, \quad \varepsilon^h_{33} \neq 0.$$

We also assume that the beam is *weakly clamped* (clamped in mean) along at one end Γ^h_0, i.e. the displacement field $\boldsymbol{u}^h$ being such that

$$\int_{\Gamma^h_0} u^h_i \, d\omega^h = 0, \quad \int_{\Gamma^h_0} (x^h_j u^h_i - x^h_i u^h_j)\, d\omega^h = 0,$$

which we denote by $\langle \boldsymbol{u} \rangle = \mathbf{0}$ on Γ^h_0, the other end being controlled by surface forces

$\boldsymbol{p}^h = (p_i^h)$. Furthermore, the *electric potentials* $\varphi_0^{h,0}$ and $\varphi_0^{h,L}$ (constants) are induced on both ends $\Gamma_0^h := \omega^h \times \{0\}$ and $\Gamma_L^h := \omega^h \times \{L\}$, respectively. So, the boundary sets should be defined by $\Gamma_{dD}^h = \Gamma_0^h$ and $\Gamma_{dN}^h = \Gamma_N^h \cup \Gamma_L^h$, where $\Gamma_N^h = \gamma^h \times (0, L)$ and $\Gamma_{eD}^h = \Gamma_0^h \cup \Gamma_L^h$, and the variational equations become: *find* $(\boldsymbol{u}^h, \varphi^h) \in V_{w,0}^h \times \Psi_2^h$ *such that*

$$\begin{aligned}
&\int_{\Omega^h} \left[C_{ijkl}^h e_{kl}^h(\boldsymbol{u}^h) - P_{mij}^h\, E_m^h\left(\varphi^h\right)\right] e_{ij}^h(\boldsymbol{v}^h) d\boldsymbol{x}^h \\
&\qquad + \int_{\Omega^h} \left[P_{mij}^h\, e_{ij}^h(\boldsymbol{u}^h) + \varepsilon_{mi}^h\, E_i^h\left(\varphi^h\right)\right] E_m^h\left(\psi^h\right) d\boldsymbol{x}^h \\
&= \int_{\Omega^h} f_i^h v_i^h d\boldsymbol{x}^h + \int_{\Gamma_N^h} g_i^h v_i^h d\Gamma^h + \int_{\Gamma_L^h} p_i^h v_i^h d\Gamma^h, \quad \forall (\boldsymbol{v}^h, \psi^h) \in V_{w,0}^h \times \Psi_0^h,
\end{aligned}$$

where $V_{w,0}^h$ *and* Ψ_2^h *are defined by (see Section 3.3)*

$$V_{w,0}^h := V_{w,0}^h(\Omega^h) = \left\{\boldsymbol{v} \in \left[H^1(\Omega)\right]^3 : \langle \boldsymbol{v} \rangle = \boldsymbol{0} \ \text{ on } \ \Gamma_{dD}\right\}, \tag{1.1}$$

$$\Psi_2^h := \Psi_2^h(\Omega^h) = \left\{\psi^h \in H^1(\Omega^h) : \psi^h - \hat{\varphi}^h \in \Psi_0^h\right\}. \tag{1.2}$$

The existence and unicity of solution is also obtained by classical results of elliptic variational equations.

In Section 3.2 we define an equivalent problem, but now posed over set $\bar{\Omega} = \bar{\omega} \times (0, L)$, which is independent of h. We denote by $\boldsymbol{x} = (x_1, x_2, x_3)$ a generic point in $\bar{\Omega}$, and with each point $\boldsymbol{x} \in \bar{\Omega}$, we associate the point $\boldsymbol{x}^h \in \bar{\Omega}^h$ through the bijection

$$\Pi^h : \boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega} \to \boldsymbol{x}^h = (hx_1, hx_2, x_3) \in \bar{\Omega}^h. \tag{1.3}$$

Then, the unknowns are scaled, by letting

$$\begin{aligned}
&u_\alpha(h)(\boldsymbol{x}) = h\, u_\alpha^h(\boldsymbol{x}^h), \quad u_3(h)(\boldsymbol{x}) = u_3^h(\boldsymbol{x}^h), \quad \bar{\varphi}(h)(\boldsymbol{x}) = \bar{\varphi}^h(\boldsymbol{x}^h), \\
&\sigma_{\alpha\beta}^h(\boldsymbol{x}^h) = h^2 \sigma_{\alpha\beta}(h)(\boldsymbol{x}), \quad \sigma_{3\alpha}(h)(\boldsymbol{x}) = h^{-1}\sigma_{3\alpha}^h(\boldsymbol{x}^h), \quad \sigma_{33}(h)(\boldsymbol{x}) = \sigma_{33}^h(\boldsymbol{x}^h), \\
&D_\alpha(h)(\boldsymbol{x}) = h^{-1} D_\alpha^h(\boldsymbol{x}^h), \quad D_3(h)(\boldsymbol{x}) = D_3^h(\boldsymbol{x}^h),
\end{aligned}$$

and it is assumed that there exists functions $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_N)$, $p_i \in L^2(\Gamma_L)$ and $\varphi_0 \in H^{1/2}(\Gamma_{eD})$ independent of h, such that:

$$\begin{aligned}
&f_\alpha^h(\boldsymbol{x}^h) = h f_\alpha(\boldsymbol{x}), \quad f_3^h(\boldsymbol{x}^h) = f_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h, \\
&g_\alpha^h(\boldsymbol{x}^h) = h^2 g_\alpha(\boldsymbol{x}), \quad g_3^h(\boldsymbol{x}^h) = h g_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma_N^h, \\
&p_\alpha^h(\boldsymbol{x}^h) = h p_\alpha(\boldsymbol{x}), \quad p_3^h(\boldsymbol{x}^h) = p_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma_L^h
\end{aligned}$$

$$C^h_{ijkl}(\boldsymbol{x}^h) = C_{ijkl}(\boldsymbol{x}),\ \ P^h_{kij}(\boldsymbol{x}^h) = P_{kij}(\boldsymbol{x}),\ \ \varepsilon^h_{ij}(\boldsymbol{x}^h) = \varepsilon_{ij}(\boldsymbol{x}),\ \text{ for all } \boldsymbol{x}^h \in \Omega^h,$$

$$\hat{\varphi}^h(x^h_3) = \hat{\varphi}(x_3), \quad \text{for all } x^h_3 \in [0, L],$$

where $\hat{\varphi}(x_3) = \frac{1}{L}(L - x_3)\varphi^0_0 + \frac{1}{L}x_3\varphi^L_0 \in H^1(\Omega)$ is an extension of $\varphi_0 \in H^{1/2}(\Gamma_{eD})$ and define $\varphi(h) = \bar{\varphi}(h) + \hat{\varphi}$.

It is found in this fashion that the scaled unknown $(\boldsymbol{u}(h), \varphi(h))$ satisfies a variational problem of the form (Proposition 2 in Section 3.2)

$$\begin{cases} \text{Find } (\boldsymbol{u}(h), \varphi(h)) \in V_{0,w} \times \Psi_2 \text{ such that} \\ h^{-4}a_{-4}((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) + h^{-2}a_{-2}((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) + a_0((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) = l(\boldsymbol{v}, \psi), \\ \forall(\boldsymbol{v}, \psi) \in V_{0,w} \times \Psi_0, \end{cases}$$

where the bilinear forms a_{-4}, a_{-2} and a_0 and the linear forms l are defined in Chapter 3 and are independent of h.

The specific form of this variational problem suggests that we use the method of formal asymptotic expansion, i.e., we let

$$\boldsymbol{u}(h) = \boldsymbol{u}^0 + h^2\boldsymbol{u}^2 + \text{h.o.t.}, \quad \boldsymbol{u}^i \in V_{0,w},$$

$$\varphi(h) = \varphi^0 + h^2\varphi^2 + \text{h.o.t}, \quad \varphi^0 - \hat{\varphi},\, \varphi^p \in \Psi_0, \quad \text{with } p \geq 1,$$

in the variational equations, and then we equate to zero the factors of h^q, $q \geq -4$. We need to characterize the terms $\boldsymbol{u}^0$, $\boldsymbol{u}^2$ and φ^0 (see Theorem 4 in Section 3.4) to yield the complete determination of the leading terms. We then establish in Section 3.3 the main results of the chapter by showing that the scaled unknown $(\boldsymbol{u}(h), \varphi(h))$ strongly converges in $[H^1(\Omega)]^3 \times H^1(\Omega)$ to the leading term $(\boldsymbol{u}^0, \varphi^0)$ (see Theorem 7) and it is obtained by solving a one-dimensional problem:

(i) The vector field $\boldsymbol{u}^0$ is a Bernoulli-Navier displacement field, i.e.: the functions u^0_α depend only on x_3 and displacement u^0_3 takes the form $u^0_3(x_1, x_2, x_3) = \xi_3(x_3) - x_\beta\xi'_\beta(x_3)$, with $\xi_3 \in H^1(0, L)$ satisfying $\xi_3(0) = 0$, and $\xi_\beta \in H^2(0, L)$ satisfying $\xi_\beta(0) = \xi'_\beta(0) = 0$.

(ii) The scalar function $\varphi^0(x_1, x_2, x_3) = z_3(x_3)$ with $z_3 \in H^1(0, L)$, satisfying $z_3(0) = \varphi^0_0$ and $z_3(L) = \varphi^0_L$.

(iii) The vector field (ξ_i, z_3) solves a one-dimensional coupled boundary value problem which represents the one-dimensional piezoelectric model in the fixed domain:

$$\begin{cases} -C^*_{33}\, A(\omega)\, \xi''_3 - P^*_3 A(\omega)\, z''_3 = F_3 \quad \text{in } (0,L), \\ \varepsilon^*_3 A(\omega)\, z''_3 - P^*_3 A(\omega)\, \xi''_3 = 0 \qquad \text{in } (0,L), \\ \xi_3\,(0) = 0, \quad z_3(0) = \varphi^0_0, \quad z_3(L) = \varphi^L_0, \\ C^*_{33}\, A(\omega)\, \xi'_3(L) + P^*_3 A(\omega)\, z'_3(L) = F^L_3 \end{cases}$$

and (no sum on β)

$$\begin{cases} C^*_{33}\, I_\beta\, \xi^{(4)}_\beta = F_\beta + M'_\beta \quad \text{in } (0,L), \\ \xi_\beta\,(0) = 0, \quad \xi'_\beta(0) = 0, \\ C^*_{33}\, I_\beta\, \xi''_\beta(L) = -M^L_\alpha, \qquad -C^*_{33}\, I_\beta\, \xi'''_\beta(L) = F^L_\beta - M_\beta(L), \end{cases}$$

where the constants, defined in Theorem 5, are given by

$$C^*_{33} = C_{3333} - C_{33\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}C_{\rho\rho 33} + \tilde{C}_{\alpha\beta 12}C_{1233}] = A^c_{33}\bar{\varepsilon}_{33},$$

$$P^*_3 = P_{333} - C_{33\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}P_{3\rho\rho} + \tilde{C}_{\alpha\beta 12}P_{312}] = A^c_{33}\bar{P}_{333},$$

$$\varepsilon^*_3 = \varepsilon_{33} + P_{3\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}C_{\rho\rho 33} + \tilde{C}_{\alpha\beta 12}C_{1233}] = A^c_{33}\bar{C}_{3333},$$

and the loading dependent terms are defined by

$$F_i = \int_\omega f_i\, d\omega + \int_{\gamma_N} g_i\, d\gamma, \quad M_\alpha = \int_\omega x_\alpha f_3\, d\omega + \int_{\gamma_N} x_\alpha g_3\, d\gamma,$$

$$M_3 = \int_\omega (x_2 f_1 - x_1 f_2)\; d\omega + \int_{\gamma_N} (x_2 g_1 - x_1 g_2)\; d\gamma,$$

$$F^L_i = \int_\omega p_i\, d\omega,$$

$$M^L_\alpha = \int_\omega x_\alpha p_3\, d\omega, \qquad M^L_3 = \int_\omega (x_2 p_1 - x_1 p_2)\; d\omega.$$

Naturally, these equation have to be “de-scaled”, so as to be expressed in terms of “physical” unknowns and data. These equations are presented in Section 3.5.

In Chapter 4, we suppose that the beam is made of a piezoelectric crystal material that satisfy:

$$C^h_{3\rho 33} = C^h_{3\theta\alpha\beta} = 0, \quad P^h_{\theta\rho\sigma} = P^h_{33\alpha} = P^h_{\beta 33} = 0, \quad \varepsilon^h_{3\theta} = 0,$$

and that the beam is weakly clamped at the extremities $\Gamma^h_0 \cup \Gamma^h_L$, i.e. $\langle \boldsymbol{v} \rangle = \boldsymbol{0}$ on $\Gamma^h_0 \cup \Gamma^h_L$. As shown in Figure 1.2, we assume that the beam has electric potential φ^h_0 on $\Gamma^h_{eD} = \gamma^h_{eD} \times (0, L)$ with $\gamma^h_{eD} \subset \gamma^h$ and $measure(\gamma^h_{eD}) > 0$. Due to the force loading and electric potential, the pair $(\boldsymbol{u}^h, \varphi^h)$ is derived from $\varphi^h = \bar{\varphi}^h + \hat{\varphi}^h$ with $(\boldsymbol{u}^h, \bar{\varphi}^h)$ satisfying the problem (2.62) defined in Section 2.3.2.1 for Ω^h.

Like the variational problem in Section 3.2, the problem (2.62) is then transformed into an analogous one, but now posed over the fixed domain $\bar{\Omega} := \bar{\omega} \times (0, L)$. As in Section 3.2, we apply the bijection

$$\Pi^h : \boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega} \to \boldsymbol{x}^h = (hx_1, hx_2, x_3) \in \bar{\Omega}^h, \tag{1.4}$$

and we take a new scaling to electrical part. In other words, we use the appropriate scalings

$$\bar{\varphi}(h)(\boldsymbol{x}) = h^{-1}\bar{\varphi}^h(\boldsymbol{x}^h), \qquad D_\alpha(h)(\boldsymbol{x}) = D^h_\alpha(\boldsymbol{x}^h), \qquad D_3(h)(\boldsymbol{x}) = hD^h_3(\boldsymbol{x}^h),$$

and adequate assumptions on the data:

$$\begin{cases} \hat{\varphi}^h(\boldsymbol{x}^h) = h\hat{\varphi}(h)(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h, \\ \hat{\varphi}(h) \in H^1(\Omega) \text{ a trace lifting of } \varphi_0 \text{ and define } \varphi(h) = \bar{\varphi}(h) + \hat{\varphi}(h). \end{cases}$$

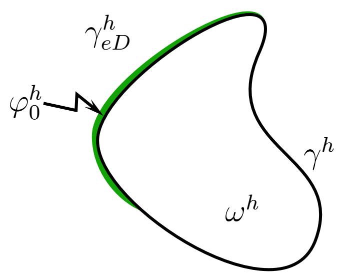


Figure 1.2: Electrical boundary in each cross-section.

In this way, the scaled principle virtual work reads: *Find* $(\boldsymbol{u}(h), \bar{\varphi}(h)) \in \boldsymbol{X}_0$ *such that*

$$\int_\Omega \sigma_{ij}(h) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_k(h) E_k(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\, d\Gamma, \quad \forall\, (\boldsymbol{v}, \psi) \in \boldsymbol{X}_0,$$

where $\boldsymbol{\sigma}(h) = \boldsymbol{\sigma}(h)(\boldsymbol{u}(h), \varphi(h))$ and $\boldsymbol{D}(h) = \boldsymbol{D}(h)(\boldsymbol{u}(h), \varphi(h))$ are given by

$$\begin{aligned}
&\sigma_{\alpha\beta}(h) = h^{-4} C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}(h)) + h^{-2} C_{\alpha\beta 33} e_{33}(\boldsymbol{u}(h)) - h^{-1} P_{3\alpha\beta} E_3(\varphi(h)),\\
&\sigma_{3\alpha}(h) = 2h^{-2} C_{3\alpha 3\beta} e_{3\beta}(\boldsymbol{u}(h)) - h^{-1} P_{\beta 3\alpha} E_\beta(\varphi(h)),\\
&\sigma_{33}(h) = h^{-2} C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}(h)) + C_{3333} e_{33}(\boldsymbol{u}(h)) - h P_{333} E_3(\varphi(h)),\\
&D_\alpha(h) = 2h^{-1} P_{\alpha 3\beta} e_{3\beta}(\boldsymbol{u}(h)) + \varepsilon_{\alpha\beta} E_\beta(\varphi(h)),\\
&D_3(h) = h^{-1} P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}(h)) + h P_{333} e_{33}(\boldsymbol{u}(h)) + h^2 \varepsilon_{33} E_3(\varphi(h)),
\end{aligned}$$

and $\boldsymbol{X}_0 = V_0 \times \Psi_0$ or $\boldsymbol{X}_0 = V_{0,w} \times \Psi_0$. Inserting the developments mentioned above into the previous problem results in a set of variational equations that must be satisfied for all $h > 0$ and consequently the terms at the successive powers of h must be zero. This procedure allows to show that the negative terms can be cancelled, and consequently to get the characterization of the terms $\boldsymbol{u}^0$ (Theorem 10), $\boldsymbol{u}^1$ (Theorem 11) and $\boldsymbol{u}^2$ (Theorem 14) of the development of $\boldsymbol{u}(h)$. Summarizing, we show that:

(i) The first term in the asymptotic expansion of the scaled displacement field $\boldsymbol{u}^0 = (u_i^0) \in H^1(\Omega)$ is a Bernoulli-Navier displacement field as defined above, where the transverse components ξ_α and the stretch component ξ_3 of the zeroth order displacement field are, respectively, the unique solutions of the following variational problems (no sum on α):

$$\begin{cases} \xi_\alpha \in H_0^2(0, L), \\ \displaystyle\int_0^L Y I_\alpha \xi_\alpha'' \chi_\alpha''\, dx_3 = \int_0^L F_\alpha \chi_\alpha\, dx_3 - \int_0^L M_\alpha \chi_\alpha'\, dx_3, \quad \text{for all } \chi_\alpha \in H_0^2(0, L), \end{cases}$$

$$\begin{cases} \xi_3 \in H_0^1(0, L), \\ \displaystyle\int_0^L Y A(\omega) \xi_3' \chi_3'\, dx_3 = \int_0^L F_3 \chi_3\, dx_3, \qquad \text{for all } \chi_3 \in H_0^1(0, L), \end{cases}$$

where $Y = \frac{\det \boldsymbol{C}}{\det \boldsymbol{M} \det \boldsymbol{N}}$, and $\boldsymbol{C}$, $\boldsymbol{M}$ and $\boldsymbol{N}$ are defined in Sect.4.3.0.2.

(ii) The leading term in asymptotic expansion of the scaled electric potential is determined by the pair $(r, \bar{\varphi}^0)$ where

$$(\underline{r}, \underline{\bar{\varphi}}^0)(x_1, x_2) = \frac{1}{L} \int_0^L (r, \bar{\varphi}^0)(x_1, x_2, s) ds$$

is the unique solution of the following 2D variational problem:

$$
\begin{aligned}
&(\underline{r}, \underline{\bar{\varphi}}^0) \in T(\omega) = Q(\omega) \times S(\omega) \text{ such that} \\
&\int_\omega C_{3\alpha 3\beta} \partial_\beta \underline{r} \partial_\alpha \rho \, d\omega + \int_\omega \varepsilon_{\alpha\beta} \partial_\alpha \underline{\bar{\varphi}}^0 \partial_\beta \psi \, d\omega + \int_\omega P_{\beta 3\alpha} (\partial_\alpha \underline{r} \partial_\beta \psi - \partial_\beta \underline{\bar{\varphi}}^0 \partial_\alpha \rho) \, d\omega \\
&\quad = \int_\omega P_{\beta 3\alpha} \partial_\beta \underline{\varphi}_0 \partial_\alpha \rho \, d\omega - \int_\omega \varepsilon_{\alpha\beta} \partial_\alpha \underline{\varphi}_0 \partial_\beta \psi \, d\omega, \quad \text{for all } (\rho, \psi) \in T.
\end{aligned}
$$

The spaces $Q(\omega)$, $S(\omega)$, the function w and the torsion constant $J > 0$ are defined in Section 4.3.0.3. In particular, we have that, for a beam of class *6mm* material, the electric potential satisfies the following Laplace's equation,

$$
\begin{aligned}
&\bar{\varphi}^0 \in S(\omega) \text{ such that a.e. } x_3 \in [0, L], \\
&\int_\omega \left(P_{14} \frac{P_{14}}{C_{44}} + \varepsilon_{11} \right) \partial_\beta \bar{\varphi}^0 \partial_\beta \psi d\omega = - \int_\omega \left(P_{14} \frac{P_{14}}{C_{44}} + \varepsilon_{11} \right) \partial_\beta \varphi_0 \partial_\beta \psi d\omega, \\
&\text{for all } \psi \in S(\omega).
\end{aligned}
$$

(iii) The families $(\boldsymbol{\kappa}(h))_{h>0}$ and $(\boldsymbol{\vartheta}(h))_{h>0}$, defined in Section 4.2 by (4.16) and (4.17), weakly converge to $\boldsymbol{\kappa}$ and $\boldsymbol{\vartheta}$. By Korn's inequality and Poincaré-Friedrichs's inequality, we then establish that the family $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ converge weakly in the spaces $[H^1(\Omega)]^3$ and $L^2(\Omega)$, respectively, to $\boldsymbol{u}$ and φ as $h \to 0$. Thus, the limits κ_{33} and $\boldsymbol{\vartheta}$ are characterized with respect to $\boldsymbol{u}$ and φ, respectively.

(iv) In Section 4.4.2 we prove that, for a beam of class 2 material, the sequence $(\boldsymbol{u}(h))_{h>0}$ converges weakly to the first terms the asymptotic expansion of the scaled displacement field, $\boldsymbol{u}^0$, while that for the homogeneous transversely isotropic beam model, we show that the sequence $(\boldsymbol{u}(h), \varphi(h))_{h>0}$ converges weakly towards the unique solution $(\boldsymbol{u}^0, \varphi^0)$ of the variational equations derived in Section 4.3.

For the homogeneous transversely isotropic beam model, we prove the uniqueness of weak solution which improve some results given by Figueiredo & Leal [2006]. However, as in the work of Figueiredo & Leal [2006], the strong convergence does not emerge in our study.

Let us now briefly outline the content of Chapter 5. In Section 5.2, we consider a three-dimensional linearly piezoelectric shallow arch occupying in its reference configuration the set $\{\breve{\Omega}^h\}^- = \boldsymbol{\Theta}^h(\bar{\Omega}^h)$, where $\breve{\Omega}^h := \boldsymbol{\Theta}^h(\Omega^h)$, $\Omega^h = \omega^h \times (0, L)$, and the mapping $\boldsymbol{\Theta}^h : \breve{\bar{\Omega}}^h \to \mathbb{R}^3$ is given by

$$
\boldsymbol{\Theta}^h \left(\boldsymbol{x}^h \right) = \boldsymbol{\phi}^h(x_3) + x_1^h \boldsymbol{n}^*(x_3) + x_2^h \boldsymbol{b}^*(x_3) \in \{\breve{\Omega}^h\}^-,
$$

where $\phi_\alpha^h(x_3)$ are given functions verifying $\phi_\alpha^h \in C^3[0,L]$, and $\boldsymbol{n}^*$ and $\boldsymbol{b}^*$ are the normal and the binormal vectors of the Frenet trihedron $(\boldsymbol{t}^*, \boldsymbol{n}^*, \boldsymbol{b}^*)$ associated to the curve C^h defined by ϕ_α^h (see Section 5.1). The shallow arch is clamped on the portion $\breve{\Gamma}_{dD}^h = \breve{\Gamma}_0^h$ and submitted to a mechanical volume force of density $\breve{\boldsymbol{f}}^h$ in $\breve{\Omega}^h$, to a mechanical surface force $\breve{\boldsymbol{g}}^h$ in lateral surface $\breve{\Gamma}_N^h$ and to a surface force $\breve{\boldsymbol{p}}^h$ on the end $\breve{\Gamma}_L^h$. A prescribed electrical potential $\breve{\varphi}_0^h(\breve{x}_3)$ on $\breve{\Gamma}_{eD}^h$ is imposed. Then, the body undergoes a mechanical displacement field $\breve{\boldsymbol{u}}^h = (\breve{u}_i^h) : \bar{\breve{\Omega}}^h \to \mathbb{R}^3$ and an electrical potential $\breve{\varphi}^h : \bar{\breve{\Omega}}^h \to \mathbb{R}$ satisfying the following mixed variational equations:

$$\int_{\breve{\Omega}^h} \left(\breve{C}_{ijkl}^h \breve{\sigma}_{kl}^h + \breve{P}_{kij}^h \breve{D}_k^h\right) \breve{\tau}_{ij}^h \, d\breve{\boldsymbol{x}}^h + \int_{\breve{\Omega}^h} \left(-\breve{P}_{kij}^h \breve{\sigma}_{ij}^h + \breve{\varepsilon}_{kl}^h \, \breve{D}_l^h\right) \breve{d}_k^h \, d\breve{\boldsymbol{x}}^h = \int_{\breve{\Omega}^h} \breve{\tau}_{ij}^h \, \breve{e}_{ij}^h(\breve{\boldsymbol{u}}^h) \, d\breve{\boldsymbol{x}}^h$$
$$+ \int_{\breve{\Omega}^h} \breve{d}_k^h \, \breve{E}_k^h(\breve{\varphi}^h) \, d\breve{\boldsymbol{x}}^h,$$

for all $(\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h) \in \breve{\boldsymbol{X}}_1^h$

$$\int_{\breve{\Omega}^h} \breve{\sigma}_{ij}^h \, \breve{e}_{ij}^h(\breve{\boldsymbol{v}}^h) \, d\breve{\boldsymbol{x}}^h + \int_{\breve{\Omega}^h} \breve{D}_k^h \, \breve{E}_k^h(\breve{\psi}^h) \, d\breve{\boldsymbol{x}}^h = \int_{\breve{\Omega}^h} \breve{f}_i^h \, \breve{v}_i^h \, d\breve{\boldsymbol{x}}^h + \int_{\breve{\Gamma}_N^h} \breve{g}_i^h \, \breve{v}_i^h \, d\breve{\Gamma}^h$$
$$+ \int_{\breve{\Gamma}_L^h} \breve{p}_i^h \, \breve{v}_i^h \, d\breve{\Gamma}^h,$$

for all $(\breve{\boldsymbol{v}}^h, \breve{\psi}^h) \in \breve{\boldsymbol{X}}_{0,w}^h$

where $\breve{C}_{ijkl}^h$, $\breve{P}_{kij}^h$ and $\breve{\varepsilon}_{ij}^h$ are the elastic, piezoelectric and dielectric tensor field that characterize the material by the following behavior law

$$\breve{\boldsymbol{e}}^h(\breve{\boldsymbol{u}}^h) = \breve{C}_{ijkl}^h \, \breve{\sigma}_{kl}^h + \breve{P}_{kij}^h \, \breve{D}_k^h$$
$$\breve{\boldsymbol{E}}^h(\breve{\varphi}^h) = -\breve{P}_{ikl}^h \, \breve{\sigma}_{kl}^h + \breve{\varepsilon}_{ij} \, \breve{D}_j^h,$$

obtained from the *elastic enthalpy function*, defined in Section 2.1.3.

Let $\Omega = \omega \times (0, L)$. We define the scaled displacement field $\boldsymbol{u}(h)$, the scaled electric potential $\varphi(h)$, the scaled stress tensor $\boldsymbol{\sigma}(h)$ and the scaled electric displacement $\boldsymbol{D}(h)$ by, for all $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h\left(\Pi^h(\boldsymbol{x})\right) \in \{\breve{\Omega}^h\}^-$,

$$u_\alpha(h)(\boldsymbol{x}) = h \, u_\alpha^h(\breve{\boldsymbol{x}}^h), \quad u_3(h)(\boldsymbol{x}) = \breve{u}_3^h(\breve{\boldsymbol{x}}^h), \quad \bar{\varphi}(h)(\boldsymbol{x}) = \bar{\varphi}^h(\boldsymbol{x}^h),$$

$$\sigma_{\alpha\beta}(h)(\boldsymbol{x}) = h^{-2} \breve{\sigma}_{\alpha\beta}^h(\breve{\boldsymbol{x}}^h), \quad \sigma_3(h)(\boldsymbol{x}) = h^{-1} \breve{\sigma}_{3\alpha}^h(\breve{\boldsymbol{x}}^h), \quad \sigma_{33}(h)(\boldsymbol{x}) = \breve{\sigma}_{33}^h(\breve{\boldsymbol{x}}^h),$$

$$D_\alpha(h)(\boldsymbol{x}) = h^{-1} \breve{D}_\alpha^h(\breve{\boldsymbol{x}}^h), \quad D_3(h)(\boldsymbol{x}) = \breve{D}_3^h(\breve{\boldsymbol{x}}^h).$$

We also assume that the data is such that

$$\breve{f}^h_\alpha(\breve{\boldsymbol{x}}^h) = h\, f_\alpha(\boldsymbol{x}), \quad \breve{f}^h_3(\breve{\boldsymbol{x}}^h) = f_3(\boldsymbol{x}), \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}\left(\Pi^h(\boldsymbol{x})\right) \in \{\breve{\Omega}^h\}^-,$$

$$\breve{g}^h_\alpha(\breve{\boldsymbol{x}}^h) = h^2\, g_\alpha(\boldsymbol{x}), \quad \breve{g}^h_3(\breve{\boldsymbol{x}}^h) = h\, g_3(\boldsymbol{x}), \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}\left(\Pi^h(\boldsymbol{x})\right) \in \breve{\Gamma}^h_N,$$

$$\breve{p}^h_\alpha(\breve{\boldsymbol{x}}^h) = h^2\, p_\alpha(\boldsymbol{x}), \quad \breve{p}^h_3(\breve{\boldsymbol{x}}^h) = h\, p_3(\boldsymbol{x}), \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}\left(\Pi^h(\boldsymbol{x})\right) \in \breve{\Gamma}^h_L,$$

and

$$\begin{aligned} \breve{\hat{\varphi}}^h(\breve{x}^h_3) &= \hat{\varphi}(x_3), \\ \phi^h_\alpha(\breve{x}^h_3) &= h\phi_\alpha(x_3) \text{ for all } x^h_3 \in [0, L]. \end{aligned}$$

As we referred in the last chapter, the constants of the material satisfy the following conditions:

$$\begin{cases} \breve{C}^h_{ijkl}(\boldsymbol{x}^h) = C_{ijkl}(\boldsymbol{x}), \\ \breve{P}^h_{kij}(\boldsymbol{x}^h) = P_{kij}(\boldsymbol{x}), \qquad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}\left(\Pi^h(\boldsymbol{x})\right) \in \{\breve{\Omega}^h\}^-. \\ \breve{\varepsilon}^h_{ij}(\boldsymbol{x}^h) = \varepsilon_{ij}(\boldsymbol{x}). \end{cases}$$

As a consequence of these scalings and assumptions, the scaled unknowns satisfy the mixed scaled problem, described in Section 5.3,

$$\begin{cases} \text{Find } \left(\left(\boldsymbol{\sigma}(h), \boldsymbol{D}(h)\right), \left(\boldsymbol{u}(h), \varphi(h)\right)\right) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w} \quad \text{such that} \\ a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ \quad + h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))\right) = 0, \quad \forall (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \\ b_H\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \qquad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}, \end{cases}$$

with $\boldsymbol{X}_{0,w} = V_{0,w} \times \Psi_0$, $\boldsymbol{X}_1 = [L^2(\Omega)]^9_s \times L^2(\Omega)$ and $\boldsymbol{X}_{2,w} = V_{0,w} \times \Psi_2$, where

$$V_{0,w} = V_{0,w}(\Omega) = \left\{ \boldsymbol{v} \in \left[H^1(\Omega)\right]^3 : \langle \boldsymbol{v} \rangle = \boldsymbol{0} \text{ on } \Gamma_{dD} \right\},$$

$$\Psi_0 = \Psi_0(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{eD} \right\},$$

$$\Psi_2 = \Psi_2(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi - \hat{\varphi} \in \Psi_0 \right\},$$

and $a_{H,i}\left(\cdot, \cdot\right) : \boldsymbol{X}_1 \times \boldsymbol{X}_1 \to \mathbb{R}$ and $b_H\left(\cdot, \cdot\right) : \boldsymbol{X}_1 \times \boldsymbol{X}_{0,w} \to \mathbb{R}$ defined in Section 5.3.

Assuming that the scaled unknowns can be expanded as

$$(\boldsymbol{\sigma}(h), \boldsymbol{D}(h)) = h^{-4}(\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}) + h^{-2}(\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}) + (\boldsymbol{\sigma}^{0}, \boldsymbol{D}^{0}) + \dots$$

and

$$(\boldsymbol{u}(h), \varphi(h)) = (\boldsymbol{u}^0 + h^2\boldsymbol{u}^2 + \dots, \varphi^0 + h^2\varphi^2 + \dots),\ \left(\boldsymbol{u}^0, \varphi^0 - \hat{\varphi}\right),\ \left(\boldsymbol{u}^{2p}, \varphi^{2p}\right) \in \boldsymbol{X}_{0,w},$$

we find that the leading term satisfy, $\left((\boldsymbol{\sigma}^0, \boldsymbol{D}^0), ((\boldsymbol{u}^0, \varphi^0)\right) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w}$

$$\begin{cases} a_{H,0}\left((\boldsymbol{\sigma}^0, \boldsymbol{D}^0), (\boldsymbol{\tau}, \boldsymbol{d})\right) = -a_{H,2}\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ \qquad -a_{H,4}\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{\tau}, \boldsymbol{d})\right) - b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}^0, \varphi^0)\right) \\ \qquad -a_H^{\#}(h, \phi)\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{d})\right), \qquad \forall (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \\ b_H\left((\boldsymbol{\sigma}^0, \boldsymbol{D}^0), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \qquad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}. \end{cases}$$

In Section 5.5.2.1 we show that the cancelation of the factors of h^q, $-4 \leq q \leq -1$, implies that the formal expansion of the tensor and electric displacement does not contain any negative powers of h, and consequently we derive an expression for the components u_α^2 of the displacement $\boldsymbol{u}^2$, taking into account that $\boldsymbol{u}^0$ and φ^0 belong respectively to the spaces V_{BN}^ϕ and $\hat{\varphi} + \Psi_3^\phi$ defined by

$$\begin{aligned} V_{BN}^\phi = \ & \{\boldsymbol{v} : \Omega \to \mathbb{R}^3 : v_\alpha(x_1, x_2, x_3) = \zeta_\alpha(x_3),\ \zeta_\alpha \in V_0^2(0, L), \\ & \quad v_3(x_1, x_2, x_3) = \zeta_3(x_3) - \chi_\alpha^b(x_1, x_2, x_3)\zeta_\alpha'(x_3),\ \zeta_3 \in V_0^1(0, L)\right\}, \\ \hat{\varphi} + \Psi_3^\phi = \ & \hat{\varphi} + \left\{\psi \in H^1(\Omega) : \psi(x_1, x_2, x_3) = z(x_3),\ z \in H_0^1(0, L)\right\}, \end{aligned}$$

and

$$b_\alpha = \phi_\alpha''/\left((\phi_1'')^2 + (\phi_2'')^2\right)^{1/2}, \qquad b_1^2 + b_2^2 = 1,$$

$$\begin{pmatrix} \chi_1^b(x_1, x_2, x_3) \\ \chi_2^b(x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} b_1(x_3) & -b_2(x_3) \\ b_2(x_3) & b_1(x_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

The last mixed variational equations become

$$a_{H,0}\left((\boldsymbol{\sigma}^0, \boldsymbol{D}^0), (\boldsymbol{\tau}, \boldsymbol{D})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}^0, \varphi^0)\right) = 0, \quad \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1,$$

$$b_H\left((\boldsymbol{\sigma}^0, \boldsymbol{D}^0), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \quad \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_0 = V_{BN}^\phi \times \Psi_3^\phi,$$

and then lead to show that the leading terms can be fully identified solving a one-dimensional problem. In Section 5.4 is proved that there is a constant C such that

$$|\psi|^{\phi}_{\Psi_3^{\phi}} := \left|E_3^{\phi}(\psi)\right|_{0,\Omega} \geq C\,\|\psi\|_{1,\Omega}\,, \quad \forall \psi \in \Psi_3^{\phi},$$

and therefore we also obtain the existence and uniqueness of the solution of the previous limit problem in $V_{BN}^{\phi} \times \Psi_3^{\phi}$. Our main result (Theorem 20) then consists in showing that the family $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ strongly converge in the spaces $[H^1(\Omega)]^3$ and $H^1(\Omega)$, respectively, as $h \to 0$, and that $(\boldsymbol{u}^0, \varphi^0) = \lim_{h\to 0}(\boldsymbol{u}(h), \varphi(h))$ can be obtained from the solution of a coupled one-dimensional problem; more concretely, we prove that:

(i) The element $(\xi_\alpha, \xi_3, q_3) \in [V_0^2(0,L)]^2 \times V_0^1(0,L) \times (\hat{\varphi} + H^1(0,L))$ solves the following variational equations:

$$\left\{\begin{array}{l}
\text{Find } \xi_\beta \in V_0^2(0,L) \text{ such that} \\
I_{\alpha\beta}\displaystyle\int_0^L C_{33}^*\,\xi_\alpha''\zeta_\beta''dx_3 + A(\omega)\int_0^L \left\{C_{33}^*\left(\xi_3' + \phi_\beta'\xi_\beta'\right) + P_3^* q_3'\right\}\phi_\beta'\zeta_\beta' dx_3 \\
= \displaystyle\int_0^L \left(F_\alpha + M_\alpha'\right)\zeta_\alpha dx_3 - M_\alpha(L)\zeta_\alpha'(L) + F_\alpha^L\zeta_\alpha(L) + M_\alpha^L\zeta_\alpha'(L), \qquad \forall \zeta_\beta \in V_0^2(0,L),
\end{array}\right.$$

$$\left\{\begin{array}{l}
\text{Find } (\xi_3, q_3) \in V_0^1(0,L) \times (\hat{\varphi} + H^1(0,L)) \\
A(\omega)\displaystyle\int_0^L \left\{C_{33}^*\left(\xi_3' + \phi_\beta'\xi_\beta'\right) + P_3^* q_3'\right\}\zeta_3' dx_3 \\
-A(\omega)\displaystyle\int_0^L \left\{P_3^*\left(\xi_3' + \phi_\beta'\xi_\beta'\right) - \varepsilon_3^* q_3'\right\} z_3' dx_3 \\
= \displaystyle\int_0^L F_3\zeta_3 dx_3 + F_3^L\zeta_3(L), \qquad \forall (\zeta_3, z_3) \in V_0^1(0,L) \times H_0^1(0,L).
\end{array}\right.$$

where the admissible spaces $V_0^1(0,L)$, $V_0^2(0,L)$ and $\hat{\varphi} + H^1(0,L)$ defined by

$$V_0^1(0,L) = \left\{\eta \in H^1(0,L) : \eta(0) = 0\right\},$$

$$V_0^2(0,L) = \left\{\eta \in H^2(0,L) : \eta(0) = \eta'(0) = 0\right\},$$

$$\hat{\varphi} + H^1(0,L) := \left\{z \in H^1(0,L) : z - \hat{\varphi} \in H_0^1(0,L)\right\},$$

and the reduced constants read

$$C_{33}^* = \bar{\varepsilon}_{33}A_{33}^d, \quad P_3^* = \bar{P}_{333}A_{33}^d, \quad \varepsilon_3^* = \bar{C}_{3333}A_{33}^d, \quad A_{33}^d = \frac{1}{\bar{C}_{3333}\bar{\varepsilon}_{33} + \bar{P}_{333}\bar{P}_{333}}.$$

Chapter 2

Piezoelectricity theory

For a better description of the piezoelectrical phenomena we discuss in this chapter some relevant theoretical aspects. In recent years a vast literature has flourished describing the piezoelectric effect [see e.g. Ikeda, 1990; Nye, 1985; Taylor *et al.*, 1985; Royer & Dieulesaint, 2000].

Section 2.1.1 begins with a short description of the piezoelectric phenomena and a review of the constitutive equations that describe the piezoelectric's response, based on thermodynamical principles. Simplifications of the coupling matrices due to crystal symmetries are also presented in this section. The field equations governing linear piezoelectric solids are given in Section 2.2. Section 2.3.2 gives the primal and mixed variational formulation for electromechanical boundary value problem as well as the existence, uniqueness and regularity results for the weak solutions.

2.1 Material laws

2.1.1 Piezoelectric material

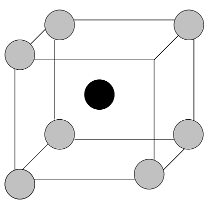
A piezoelectric material has the ability to produce an electric field when subjected to mechanical stress [see e.g. Rovenski *et al.*, 2007]. This phenomenon is known as the "direct" effect (the word comes from the Greek *piezein* which means "to press ") and was first discovered in 1880 by Jacques and Pierre Curie. The "converse" (or indirect) piezoelectric effect, by which certain materials deform when subjected to an electric field, was mathematically deduced from fundamental thermodynamical principles in 1881 by Lippmann and confirmed experimentally by the Curie brothers.

2.1.2 Crystal classes

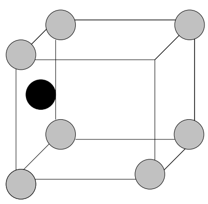
The two piezoelectric effects are specific to the piezoelectric's crystals' structure that lacks a center of symmetry [Boor & Tabaka, 2008]. Of the total 32 crystal classes, 21 are noncentrosymmetric and with the exception of one class, all of these are piezoelectric. The 32 classes are divided into seven groups: triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal and cubic. There is a relationship between these groups and the elastic nature of the material: triclinic represents anisotropic material, orthorhombic represents orthotropic material and cubic are usually isotropic materials.

Using Maugin and Hermann's notation, the 20 piezo-classes are: triclinic class 1; monoclinic classes 2 and m; orthorhombic classes 222 and $mm2$; tetragonal classes 4, -4, 422, $4mm$, and $-42m$; trigonal classes 3, 32, and $3m$; hexagonal classes 6, -6, 622, $6mm$, and $-62m$; and cubic classes 23 and $-43m$.

To understand the mechanism that causes a material to possess piezoelectric properties it is necessary to consider its behavior at the molecular level. The crystal structure of a material or the arrangement of atoms in a crystal can be composed of a lattice of atoms, which can be deformed by an applied force or change in electric field. Figure 2.1 illustrates a cube with one negatively charged atom at each corner and a single positively charged atom in the center. If the positive atom is exactly at the center of the cube (figure 2.1(a)), then the center of positive and negative charge remains fixed when a force is applied because of the center, and therefore the material is not piezoelectric. In acentric crystals (figure 2.1(b)), the center of the positive and negative charge are displaced by an



(a) Structure is non piezoelectric.



(b) Structure is piezoelectric.

Figure 2.1: Configuration of the structure.

applied force, and therefore, the crystal may possess *polarization* or net electric charge.

More detail about piezoelectric material equations can be found in Nye [1985], Ikeda [1990] and Royer & Dieulesaint [2000].

2.1.3 Thermodynamic description of piezoelectricity

For piezoelectric materials the constitutive equations, which relate the *strain tensor* $\boldsymbol{e}$ and the *electric field vector* $\boldsymbol{E}$ to the *stress tensor* $\boldsymbol{\sigma}$ and the *electric displacement* $\boldsymbol{D}$, are derived from thermodynamic potentials and they are described in detail in several texts [see e.g. Maugin, 1988; Tiersten, 1969; Mechkour, 2004; Royer & Dieulesaint, 2000]. Following these works we can observe that the constitutive equations can assume different forms depending on the state field. We take the strain tensor components e_{ij} and the electric field components E_k as independent variables; that is to say, the state of the crystal, and in particular the stress tensor components σ_{ij} and the electric displacement components D_k, are determined when the quantities e_{ij} and E_k are given. Accordingly, we may write

$$\sigma_{ij} = \frac{\partial \sigma_{ij}}{\partial e_{kl}} e_{kl} + \frac{\partial \sigma_{ij}}{\partial E_k} E_k, \qquad D_k = \frac{\partial D_k}{\partial e_{ij}} e_{ij} + \frac{\partial D_k}{\partial E_l} E_l, \tag{2.1}$$

which are known by *piezoelectric constitutive relationship* and were first given by Voigt.

One way is to define a function

$$H(\boldsymbol{e}, \boldsymbol{E}) = U - \boldsymbol{E}\boldsymbol{D}, \tag{2.2}$$

where U represents the internal energy density. From the first law of thermodynamics for piezoelectric continuum we have

$$dU = \sigma_{ij}\, de_{ij} + E_k\, dD_k. \tag{2.3}$$

To obtain U with respect to the variables e_{ij} and E_k, we build the first derivative of (2.2) and apply (2.3)

$$\frac{\partial H}{\partial t} = \frac{\partial U}{\partial t} - \frac{\partial E_i}{\partial t} D_i - E_i \frac{\partial D_i}{\partial t} = \sigma_{ij} \frac{\partial e_{ij}}{\partial t} - D_i \frac{\partial E_i}{\partial t}. \tag{2.4}$$

Hence H defined in (2.2) is also a function of (e_{ij}, E_k) and we may write

$$\frac{\partial H}{\partial t} = \frac{\partial H}{\partial e_{ij}} \frac{\partial e_{ij}}{\partial t} + \frac{\partial H}{\partial E_i} \frac{\partial E_i}{\partial t}, \tag{2.5}$$

and, by comparing coefficients in equations (2.4) and (2.5), we deduce that

$$\sigma_{ij} = \frac{\partial H}{\partial e_{ij}}, \quad D_i = -\frac{\partial H}{\partial E_i}. \tag{2.6}$$

Hence, by further differentiations,

$$\frac{\partial^2 H}{\partial E_k \partial e_{ij}} = \frac{\partial \sigma_{ij}}{\partial E_k} = -\frac{\partial D_k}{\partial e_{ij}} = P_{kij}. \tag{2.7}$$

In a similar way,

$$\frac{\partial^2 H}{\partial e_{ij} \partial e_{kl}} = \frac{\partial \sigma_{ij}}{\partial e_{kl}} = C_{ijkl} = C_{klij} = \frac{\partial \sigma_{kl}}{\partial e_{ij}} = \frac{\partial^2 H}{\partial e_{kl} \partial e_{ij}}, \tag{2.8}$$

$$\frac{\partial^2 H}{\partial E_i \partial E_j} = \frac{\partial D_i}{\partial E_j} = \varepsilon_{ij} = \varepsilon_{ji} = \frac{\partial D_j}{\partial E_i} = \frac{\partial^2 H}{\partial E_j \partial E_i}. \tag{2.9}$$

From (2.7)-(2.9), equations (2.1) may now be written as follows

$$\sigma_{ij} = C_{ijkl} e_{kl} - P_{kij} E_k, \qquad D_k = P_{kij} e_{ij} + \varepsilon_{kj} E_j, \\ 1 \leq i, j, k \leq 3, \tag{2.10}$$

where $\mathbf{C} = (C_{ijkl})$ is the forth-order elasticity tensor, $\mathbf{P} = (P_{kij})$ is the third-order piezoelectric tensor and $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ is the second-order dielectric tensor.

When the matrices in equations (2.10) are denoted by single letters we finally have the following compact expression:

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{e} - \boldsymbol{P}\boldsymbol{E}, \qquad \boldsymbol{D} = \boldsymbol{P}\boldsymbol{e} + \boldsymbol{\varepsilon}\boldsymbol{E}.$$

Now, substituting (2.10) into equation (2.6), and integrating both equations of (2.6) with respect to mechanical and electrical strains, respectively, yields,

$$H(\boldsymbol{e}) = \frac{1}{2} C_{ijkl} e_{kl} e_{ij} - P_{kij} E_k e_{ij} \tag{2.11}$$

and

$$H(\boldsymbol{E}) = -P_{kij} E_k e_{ij} - \frac{1}{2} \varepsilon_{ij} E_i E_j. \tag{2.12}$$

Combining both conservative fields, *the electric enthalpy* reads

$$H(\boldsymbol{e}, \boldsymbol{E}) = \frac{1}{2} C_{ijkl} e_{kl} e_{ij} - P_{kij} E_k e_{ij} - \frac{1}{2} \varepsilon_{ij} E_i E_j. \tag{2.13}$$

Other symmetries are implied by the symmetry of the strain tensor, such as

$$(\mathbf{H}^c_{21}) \qquad C_{ijrs} = C_{jirs} = C_{ijsr}, \ P_{kij} = P_{kji}. \tag{2.14}$$

For stable materials, both $\mathbf{C}$ and $\boldsymbol{\varepsilon}$ are positive definite, i.e., there exists a constant $c_1 > 0$

$$(\mathbf{H}^c_{22}) \qquad C_{ijkl}\tau_{ij}\tau_{kl} \geq c_1 \sum_{i,j=1}^{3} (\tau_{ij})^2, \qquad \varepsilon_{ij} d_i d_j \geq c_1 \sum_{i=1}^{3} (d_i)^2, \tag{2.15}$$

for all $\boldsymbol{d} = (d_i) \in \mathbb{R}^3$ and $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{R}^9, \tau_{ij} = \tau_{ji}$.

2.1.3.1 Matrix notation

The array A_{ij} written out as

$$\boldsymbol{A} = \left(\begin{array}{cccccc|ccc} C_{1111} & C_{1211} & C_{1122} & C_{1113} & C_{1123} & C_{1133} & -P_{111} & -P_{211} & -P_{311} \\ C_{1112} & C_{1212} & C_{1222} & C_{1213} & C_{1223} & C_{1233} & -P_{112} & -P_{212} & -P_{312} \\ C_{1122} & C_{1222} & C_{2222} & C_{2213} & C_{2223} & C_{2233} & -P_{122} & -P_{222} & -P_{322} \\ C_{1113} & C_{1213} & C_{2213} & C_{3131} & C_{3132} & C_{3133} & -P_{113} & -P_{213} & -P_{313} \\ C_{1123} & C_{1223} & C_{2223} & C_{3132} & C_{3232} & C_{3233} & -P_{123} & -P_{223} & -P_{323} \\ C_{1133} & C_{1233} & C_{2233} & C_{3133} & C_{3233} & C_{3333} & -P_{133} & -P_{233} & -P_{333} \\ \hline P_{111} & P_{112} & P_{122} & P_{131} & P_{132} & P_{133} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ P_{211} & P_{212} & P_{222} & P_{231} & P_{232} & P_{233} & \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ P_{311} & P_{312} & P_{322} & P_{331} & P_{332} & P_{333} & \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{array} \right)$$

is a matrix of coefficients on the right-side of equations (2.10). Consequently, the behaviour law (2.10) reads:

$$\begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{D} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{E} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \boldsymbol{C} & -\boldsymbol{P}^T \\ \boldsymbol{P} & \boldsymbol{\varepsilon} \end{pmatrix}. \tag{2.16}$$

The simplifications introduced by (2.8)-(2.9) and (2.14) mean that the corresponding matrix $\boldsymbol{A}$ has at most 45 independent coefficients as follows:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ & \varepsilon_{22} & \varepsilon_{23} \\ sym. & & \varepsilon_{33} \end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix} P_{111} & P_{112} & P_{122} & P_{113} & P_{123} & P_{133} \\ P_{211} & P_{212} & P_{222} & P_{213} & P_{223} & P_{233} \\ P_{311} & P_{312} & P_{322} & P_{313} & P_{323} & P_{333} \end{bmatrix}, \tag{2.17}$$

$$\boldsymbol{C} = \begin{bmatrix} C_{1111} & C_{1112} & C_{1122} & C_{1113} & C_{1123} & C_{1133} \\ & C_{1212} & C_{1222} & C_{1213} & C_{1223} & C_{1233} \\ & & C_{2222} & C_{2213} & C_{2222} & C_{2233} \\ & & & C_{1313} & C_{1323} & C_{1333} \\ & sym. & & & C_{2323} & C_{2333} \\ & & & & & C_{3333} \end{bmatrix}. \tag{2.18}$$

The material is referred to as triclinic material.

Combining the properties (2.8)-(2.9) with conditions ($\mathbf{H}_{21}^c$) and ($\mathbf{H}_{22}^c$) we can expressed the strain tensor and the electric vector field by the following way

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{E} \end{pmatrix} = \begin{pmatrix} \bar{\boldsymbol{C}} & \bar{\boldsymbol{P}}^T \\ -\bar{\boldsymbol{P}} & \bar{\varepsilon} \end{pmatrix} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{D} \end{pmatrix}, \tag{2.19}$$

where

$$\bar{\varepsilon} = (\boldsymbol{\varepsilon} + \boldsymbol{P}\boldsymbol{C}^{-1}\boldsymbol{P}^T)^{-1}, \quad \bar{\boldsymbol{P}} = \bar{\varepsilon}\boldsymbol{P}\boldsymbol{C}^{-1} \text{ and } \bar{\boldsymbol{C}} = \boldsymbol{C}^{-1}(\boldsymbol{I} - \boldsymbol{P}^T\bar{\varepsilon}\bar{\boldsymbol{P}}C^{-1}). \tag{2.20}$$

The blocks of the constitutive matrix $\boldsymbol{A}^{-1}$ that characterize the constitutive equation (2.19) has similar expressions to the matrix $\boldsymbol{A}$, i.e., the blocks $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{P}}$, $\bar{\varepsilon}$ are of the form (2.17)-(2.18).

The effect of increasing crystal symmetry and the choice of reference axes allows us to reduce the number of independent components needed to specify completely the properties of the crystal.

In this research work we will consider, at most, the monoclinic crystal structural - class 2. This means that it can be anything up to crystal symmetry class 2, including the most popular piezoceramics and piezopolymers - the Lead Zirconate Titanate (PZT) and the Polyvinylidene Fluoride (PVDF).

In next sections, we give a brief summary of the properties of the material that crystalize in the monoclinic crystal system (class 2 and class m) and in the class *6mm* of the hexagonal crystal system.

2.1.3.2 Monoclinic crystal system

As already mentioned, there are 3 crystallographic classes in the monoclinic group: $2/m$, m and 2. The class 2 and the class m for which the corresponding elasticity tensor has at most thirteen non-null components [see e.g. Royer & Dieulesaint, 2000]

$$C_{1111},\ C_{1112},\ C_{1122},\ C_{1133},\ C_{1212},\ C_{1222},\ C_{1233},\ C_{2222},\ C_{2213},\ C_{1313},\ C_{1323},\ C_{2323},\ C_{3333},$$

and four non-null components of the permittivity tensor

$$\varepsilon_{11},\, \varepsilon_{12},\, \varepsilon_{22},\, \varepsilon_{33},$$

whereas the piezoelectric stress tensor has eight non-null components if the material belongs to class 2:

$$P_{311},\, P_{312},\;\, P_{322},\;\, P_{333},\;\, P_{113},\, P_{123},\, P_{213},\, ,\, P_{223},$$

and ten non-null components if the material belongs to class m:

$$P_{111},\, P_{112},\;\, P_{122},\;\, P_{211},\;\, P_{212},\, P_{222},\, P_{133},\, P_{233},\, P_{313},\;\, P_{323}.$$

2.1.3.3 Transversely isotropic crystal - 6mm symmetric class

A transversely isotropic structures of hexagonal symmetry - *6mm* symmetric class - is characterized by ten non-zero independent matrix elements consisting of 5 independent elastic constants, 3 independent piezoelectric constants and 2 independent dielectric constants. For these materials, the reduced matrix form of the above constitutive relationships can now be written as:

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & 0 & C_{13} & 0 & 0 & C_{16} \\ & \frac{C_{11}-C_{13}}{2} & 0 & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & C_{16} \\ & & & C_{44} & 0 & 0 \\ & sym. & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix},$$

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 0 & P_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & P_{14} & 0 \\ P_{31} & 0 & P_{31} & 0 & 0 & P_{36} \end{bmatrix},\; \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ & \varepsilon_{11} & 0 \\ sym. & & \varepsilon_{33} \end{bmatrix},$$

where the non-zero components are given by the following relations

$$C_{11} = C_{1111} = C_{2222}, \quad C_{13} = C_{1122}, \quad C_{16} = C_{1133} = C_{2233}, \tag{2.21}$$

$$C_{44} = C_{1313} = C_{2323}, \quad C_{1212} = \frac{C_{11} - C_{13}}{2}, \quad C_{66} = C_{3333}, \tag{2.22}$$

$$P_{14} = P_{113} = P_{223}, \quad P_{31} = P_{311} = P_{322}, \quad P_{36} = P_{333}. \tag{2.23}$$

2.2 Field quantities and equations

The mechanical behavior of any material is governed by certain physical laws, which relate stress, strain, electric fields and electric displacement together. In any points, including points on the boundary, it must satisfy three basic equations which are Cauchy's equations of motion, kinematics equations and constitutive equations (defined before).

2.2.1 Strains and electric fields

We consider a piezoelectric body continuum occupying a region Ω in a "stress-free" configuration (in the absence of any electric field or mechanical load), henceforth called the *reference configuration* of the body. The goal of this section is to describe the deformation in response to given forces.

Let $\boldsymbol{\chi} = \boldsymbol{id} + \boldsymbol{u} : \boldsymbol{x} \in \Omega \longmapsto (\boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})) \in \mathbb{R}^3$ be a standard C^1-deformation, with $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ denoting the mechanical displacement at $\boldsymbol{x} \in \Omega$. Let us denote the electric potential at point $\boldsymbol{x} \in \Omega$ as $\varphi(\boldsymbol{x})$. We let $\boldsymbol{x} = (x_i)$ denote a generic point in the set Ω. The associated generalized deformations are the *linearized strain* $\mathbf{e}$ and the *electric field* $\mathbf{E}$, which are expressed as a function of $\boldsymbol{u}$ and φ through the following equations of kinematical compatibility:

$$\boldsymbol{e}(\boldsymbol{u}) := (e_{ij}(\boldsymbol{u})) = \left(\frac{1}{2}(\partial_i u_j + \partial_j u_i)\right), \tag{2.24}$$

$$\boldsymbol{E}(\varphi) := (-\partial_i \varphi), \qquad \text{(static)}, \tag{2.25}$$

where $\partial_i = \partial / \partial x_i$, $\boldsymbol{x} = (x_i) \in \Omega$. The expression (2.25) will be deduced in Section 2.2.3.

2.2.2 Mechanical balance laws

One the basic principles of mechanics is the balance of momentum. In the presence of a body force $\boldsymbol{f}$, this law takes the form (static case)

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f}, \tag{2.26}$$

or, in componentwise,

$$-\partial_i \sigma_{ij} = f_j. \tag{2.27}$$

2.2.3 Maxwell's equations

In this section, we deduce the main Maxwell's equations for the electric field variables in the absence of magnetic fields, free currents and electric charges.

The Maxwell's equations of electromagnetism are written as

$$\mathbf{curl}\ \boldsymbol{H} = \partial_t \boldsymbol{D} + \boldsymbol{J}, \tag{2.28}$$

$$\mathbf{curl}\ \boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{2.29}$$

$$\operatorname{div} \boldsymbol{B} = \mathbf{0}, \tag{2.30}$$

$$-\operatorname{div} \boldsymbol{D} = \mathbf{0}, \tag{2.31}$$

where $\boldsymbol{E}$ denotes the electric field as before, $\boldsymbol{D}$ the electric displacement, $\boldsymbol{H}$ the magnetic field, $\boldsymbol{B}$ the magnetic induction and $\boldsymbol{J}$ the current density. The magnetic filed and the magnetic induction are related by

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}),$$

where $\boldsymbol{M}$ is the magnetic field (will be neglected) and μ_0 is the dielectric impermeability. The electric field and the electric displacement are related by

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P},$$

where $\mathbf{P}$ is the electric polarization and ϵ_0 is the (symmetric positive definite) permittivity tensor. From equation (2.30) we deduce the existence of $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \mathbf{curl}\boldsymbol{A}.$$

The previous equation together with (2.29) imply that $\boldsymbol{E} + \frac{\partial \mathbf{A}}{\partial t}$ admits a rotational null, derives of one scalar electrical potential φ, and therefore

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{2.32}$$

where $\boldsymbol{\nabla} = (\frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}, \frac{\partial}{\partial_z})$. As the magnetic part is neglected, the following conditions are considered for magnetic part

$$\boldsymbol{A} = 0, \quad \mu_0 = 0,$$

and the electric field can be treated as quasi-static [see Gantner, 2005]. Therefore, the

Maxwell equation (2.29) reads

$$\mathbf{curl}\, \boldsymbol{E} = 0,$$

and the electric field can be represented as the gradient of an electric scalar potential according to

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi, \tag{2.33}$$

in which *the electric field* is represented by the negative gradient of *the electric potential* φ. We also assume that the body is a *perfect dielectric*, i.e., non charge electric in (2.31). Thus, the only relevant Maxwell equation is

$$\operatorname{div} \boldsymbol{D} = \mathbf{0}. \tag{2.34}$$

Combining all the above, we have the so-called field equations for linear piezoelectric problems:

$$\begin{aligned} \boldsymbol{e} &= \tfrac{1}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{u} + (\boldsymbol{\nabla}\cdot\boldsymbol{u})^T), & \boldsymbol{E} &= -\boldsymbol{\nabla}\varphi, \\ \boldsymbol{\sigma} &= \mathbf{C}\boldsymbol{e} - \boldsymbol{P}\boldsymbol{E}, & \boldsymbol{D} &= \boldsymbol{P}\boldsymbol{e} + \boldsymbol{\varepsilon}\boldsymbol{E}, \\ -\operatorname{div}\boldsymbol{\sigma} &= \boldsymbol{f}, & \operatorname{div}\boldsymbol{D} &= \mathbf{0}. \end{aligned} \tag{2.35}$$

2.2.4 Boundary conditions

System (2.35) is not a well-posted problem unless we provide it with appropriate boundary conditions.

Let Ω be a region occupied by a piezoelectric body. The set Ω is assumed to be an open bounded subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary. On the surface $\partial\Omega$ of the solid, mechanical and electrical boundary conditions are applied. We consider that the surface $\Gamma = \partial\Omega$ is composed by

$$\Gamma = \Gamma_{dN} \cup \Gamma_{dD}, \qquad \Gamma_{dN} \cap \Gamma_{dD} = \emptyset,$$

for the mechanical boundary conditions and

$$\Gamma = \Gamma_{eN} \cup \Gamma_{eD}, \qquad \Gamma_{eN} \cap \Gamma_{eD} = \emptyset,$$

for the electric boundary conditions. An illustration of this decomposition is depicted in Figures 2.2 and 2.3, where $\mathbf{n}$ is the unit vector, normal to a surface element $\partial\Omega$. The boundary conditions for the displacements, the surface traction, $\boldsymbol{g} = (g_i)$, the electric potential, and the electric charge, q are defined by

$$\begin{aligned} u_i = 0 \quad \text{on} \quad \Gamma_{dD} \quad \text{and} \quad \sigma_{ij} n_j = g_i \quad \text{on} \quad \Gamma_{dN}, \\ \varphi = \varphi_0 \quad \text{on} \quad \Gamma_{eD} \quad \text{and} \quad D_k n_k = q \quad \text{on} \quad \Gamma_{eN}. \end{aligned} \tag{2.36}$$

In our case, we also assume that $q = 0$. Note that the lower subscripts ${}_{eN}$ and ${}_{eD}$ in Γ_{eN} and Γ_{eD} refer to electric (e), Neumann (N) and Dirichelet (D) boundary conditions, respectively, while the lower subscripts ${}_{dN}$ and ${}_{dD}$ in Γ_{dN} and Γ_{dD} refer to displacement (Neumann and Dirichelet) boundary conditions.

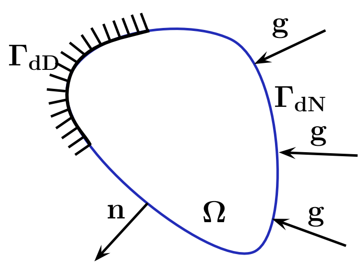


Figure 2.2: Mechanical boundary conditions

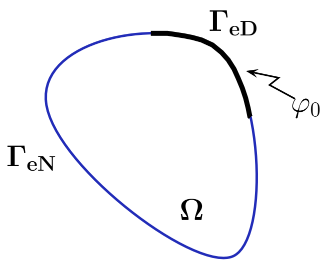


Figure 2.3: Electrical boundary conditions

2.3 Equations for a piezoelectric problem

In this section, we derive the Boundary Value Problem (BVP) for the piezoelectric problem described before and its variational formulation.

2.3.1 Coupled piezoelectric equations

Combining equations (2.26), (2.34) with previous mechanical and electrical boundary conditions (2.36), we arrive at the BVP for piezoelectricity:

BVP 1 *In a domain* $\Omega \subset \mathbb{R}^3$*, find the displacement field* $\boldsymbol{u} : \bar{\Omega} \to \mathbb{R}^3$ *and the scalar* $\varphi : \bar{\Omega} \to \mathbb{R}$ *satisfying*

$$\left\{ \begin{aligned} -\partial_j \sigma_{ij}(\boldsymbol{u}, \varphi) &= f_i \quad in \quad \Omega, \\ \sigma_{ij}(\boldsymbol{u}, \varphi) n_j &= g_i \quad on \quad \Gamma_{dN}, \\ u_i &= 0 \quad on \quad \Gamma_{dD}, \end{aligned} \right. \tag{2.37}$$

$$\begin{cases} \partial_k D_k(\boldsymbol{u}, \varphi) & = & 0 & in & \Omega, \\ D_k(\boldsymbol{u}, \varphi) n_k & = & 0 & on & \Gamma_{eN}, \\ \varphi & = & \varphi_0 & on & \Gamma_{eD}. \end{cases} \tag{2.38}$$

Hypotheses 1 *Throughout this work, we are going to make the following standard assumptions on the data:*

$$C_{ijkl} \in L^{\infty}\left(\bar{\Omega}\right), \quad P_{kij} \in L^{\infty}\left(\bar{\Omega}\right), \quad \varepsilon_{ij} \in L^{\infty}\left(\bar{\Omega}\right), \tag{2.39}$$

$$\boldsymbol{f} \in [L^2(\Omega)]^3, \quad \boldsymbol{g} \in [L^2(\Gamma_{dN})]^3, \quad \varphi_0 \in H^{1/2}(\Gamma_{eD}). \tag{2.40}$$

2.3.2 Variational formulation: existence and uniqueness of a solution

There are two different ways to formulate the variational formulation of the BVP 1, defined in previous section: the primal variational principle and the mixed variational principle. We begin by introducing the appropriate spaces and then derive a variational (weak) formulation of our system of partial differential equations.

2.3.2.1 Primal formulation

Let $H^1(\Omega)$ be the usual Sobolev space of functions, whose generalized derivatives of order at most 1 are squared integrable, that is, they belong to $L^2(\Omega)$. Let the essential spaces for the piezoelectric problem be given by

$$V_0 := V_0(\Omega) = \left\{ \boldsymbol{v} \in \left[H^1(\Omega)\right]^3 : \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{dD} \right\}, \tag{2.41}$$

$$\Psi_0 := \Psi_0(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{eD} \right\}. \tag{2.42}$$

We also introduce the non-empty closed convex subset Ψ_2 of $H^1(\Omega)$

$$\Psi_2 := \Psi_2(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi = \varphi_0 \text{ on } \Gamma_{eD} \right\}. \tag{2.43}$$

The spaces $V_0(\Omega)$ and $\Psi_0(\Omega)$ are equipped with norms

$$||\boldsymbol{v}||_{V_0} = |e(\boldsymbol{v})|_{0,\Omega} = \left(\sum_{i,j=1}^{3} |e_{ij}(\boldsymbol{v})|_{0,\Omega}^2 \right)^{1/2}, \qquad \forall \boldsymbol{v} \in V_0(\Omega), \tag{2.44}$$

$$||\psi||_{\Psi_0} = ||\psi||_{H^1(\Omega)}, \qquad \forall \psi \in \Psi_0(\Omega). \tag{2.45}$$

Let $\boldsymbol{v} \in V_0$ be the test vector function. Take the scalar product of the first equation of the system (2.37) with the test vector function $\boldsymbol{v} \in V_0$, and integrate it over Ω, one have

$$\int_\Omega -\frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{u}, \varphi) v_i \, d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x}. \tag{2.46}$$

Apply the generalized divergence theorem to the left hand side of (2.46), and get

$$-\int_{\Gamma_{dN}} v_i \sigma_{ij} n_j d\Gamma - \int_{\Gamma_{dD}} v_i \sigma_{ij} n_j d\Gamma + \int_\Omega \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x}. \tag{2.47}$$

Since $\boldsymbol{v} \in V_0$, which implies $\boldsymbol{v} = \boldsymbol{0}$ on Γ_{dD}, the second term on the right hand side is zero. Then using the Newmann mechanical boundary condition, the previous equation becomes

$$-\int_{\Gamma_{dN}} v_i g_i d\Gamma + \int_\Omega \sigma_{ij} \frac{\partial v_i}{\partial x_j} d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x}. \tag{2.48}$$

Next, using the fact that $\sigma_{ij} = \sigma_{ji}$, we obtain that

$$\sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = \sigma_{ij} e_{ij}(\boldsymbol{v}), \tag{2.49}$$

and consequently, we get

$$\int_\Omega \sigma_{ij} e_{ij}(\boldsymbol{v}) d\boldsymbol{x} = \int_{\Gamma_{dN}} g_i v_i d\Gamma + \int_\Omega f_i v_i d\boldsymbol{x}. \tag{2.50}$$

Applying constitutive laws (2.10), we achieve

$$\int_\Omega \left[C_{ijkl} e_{kl}(\boldsymbol{u}) - P_{mij} E_m(\varphi) \right] e_{ij}(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i d\Gamma. \tag{2.51}$$

Analogously, let $\psi \in \Psi_0$. Multiplying both sides of the first equation of (2.38) and integrating it over a domain Ω it leads to

$$\int_\Omega \operatorname{div} \boldsymbol{D}(\mathbf{u}, \varphi) \psi d\boldsymbol{x} = 0.$$

Combining the Green's formula with Neumann's electric boundary conditions (2.38).

$$\int_\Omega \operatorname{div} \boldsymbol{D}(\mathbf{u},\varphi)\psi d\boldsymbol{x} = \int_{\Gamma_{eN}} (D_i n_i)\psi d\Gamma - \int_\Omega D_i \partial_i \psi d\boldsymbol{x} = -\int_\Omega D_i \partial_i \psi d\boldsymbol{x},$$

and substituting the electric displacement by expression (2.10), we obtain

$$\int_\Omega \left[P_{ikl}\, e_{kl}(\boldsymbol{u}) + \varepsilon_{ik} E_k(\varphi)\right] E_i\,(\psi)\, d\boldsymbol{x} = 0. \tag{2.52}$$

Adding equations (2.51) and (2.52), we obtain the variational formulations to the problem (2.37)-(2.38) [see Haenel, 2000]:

$$\begin{cases} \text{Find } \ (\boldsymbol{u},\varphi) \in V_0 \times \Psi_2 \text{ such that} \\ a((\boldsymbol{u},\varphi)\,, (\boldsymbol{v},\psi)) = l\,(\boldsymbol{v},\psi)\,, \quad \forall (\boldsymbol{v},\psi) \in V_0 \times \Psi_0, \end{cases} \tag{2.53}$$

where

$$\begin{aligned} a((\boldsymbol{u},\varphi)\,, (\boldsymbol{v},\psi)) = & \int_\Omega \left[C_{ijkl} e_{kl}(\boldsymbol{u}) - P_{mij}\, E_m\,(\varphi)\right] e_{ij}(\boldsymbol{v}) d\boldsymbol{x} \\ & + \int_\Omega \left[P_{mij}\, e_{ij}(\boldsymbol{u}) + \varepsilon_{mi}\, E_i\,(\varphi)\right] E_m\,(\psi)\, d\boldsymbol{x}, \qquad (2.54) \\ l\,(\boldsymbol{v},\psi) = & \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i\, d\Gamma. \end{aligned}$$

Now we give an alternative variational formulation which derives from the non-homogeneous condition (2.36).

From hypothesis $\varphi_0 \in H^{1/2}(\Gamma_{eD})$, there exists an extension $\hat{\varphi}$ of φ_0 in $H^1(\Omega)$, i.e. a function $\hat{\varphi} \in H^1(\Omega)$ such that $\hat{\varphi} = \varphi_0$ on Γ_{eD}.

We define

$$\bar{\varphi} = \varphi - \hat{\varphi} \in \Psi_0, \tag{2.55}$$

and substituting (2.55) into laws (2.10), we obtain

$$\sigma_{ij}(\boldsymbol{u},\varphi) = \sigma_{ij}(\boldsymbol{u},\bar{\varphi}) - P_{kij} E_k(\hat{\varphi}), \qquad D_k(\boldsymbol{u},\varphi) = D_k(\boldsymbol{u},\bar{\varphi}) + \varepsilon_{kl} E_l(\hat{\varphi}), \tag{2.56}$$

and consequently equations (2.37) and (2.38) become

$$-\partial_j \sigma_{ij}(\boldsymbol{u},\bar{\varphi}) = f_i - \partial_j\left(P_{kij} E_k(\hat{\varphi})\right), \qquad \partial_k D_k(\boldsymbol{u},\bar{\varphi}) \quad = \partial_k\left(\varepsilon_{kl} E_l(\hat{\varphi})\right), \tag{2.57}$$

with the following boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_{dD}, \qquad \bar{\varphi} = 0 \quad \text{on} \quad \Gamma_{eD}, \tag{2.58}$$

and

$$\sigma_{ij}(\boldsymbol{u}, \bar{\varphi})\; n_j = g_i + P_{kij} E_k(\hat{\varphi}) n_j \quad \text{on} \quad \Gamma_{dN}, \tag{2.59}$$

$$D_k(\boldsymbol{u}, \bar{\varphi})\; n_k = -\varepsilon_{kl} E_l(\hat{\varphi}) n_k \quad \text{on} \quad \Gamma_{eN}. \tag{2.60}$$

We first consider the equilibrium equations (2.57). Multiplication of both sides of the equations (2.57) by respectively test functions $\boldsymbol{v} \in V_0$ and $\psi \in \Psi_0$ and integration over a domain Ω lead to

$$\begin{aligned} &\int_\Omega [-\partial_j \sigma_{ij}(\boldsymbol{u}, \bar{\varphi})\; v_i + \partial_k D_k(\boldsymbol{u}, \bar{\varphi})\; \psi]\, d\boldsymbol{x} \\ &= \int_\Omega [f_i\; v_i - \partial_j \left(P_{kij} E_k(\hat{\varphi})\right) v_i + \partial_k \left(\varepsilon_{kl} E_l(\hat{\varphi})\right) \psi]\, d\boldsymbol{x}. \end{aligned}$$

Integration by parts of the first term of the above equations results in

$$\begin{aligned} &\int_\Omega \sigma_{ij}(\boldsymbol{u}, \bar{\varphi})\, e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} - \int_\Omega D_k(\boldsymbol{u}, \bar{\varphi})\, E_k(\psi)\, d\boldsymbol{x} \\ &\qquad - \int_{\Gamma_{dN}} \sigma_{ij}(\boldsymbol{u}, \bar{\varphi}) n_j v_i\, d\Gamma + \int_{\Gamma_{eN}} D_k(\boldsymbol{u}, \bar{\varphi}) n_k \psi\, d\Gamma \\ &= \int_\Omega [f_i v_i + P_{kij} E_k(\hat{\varphi}) e_{ij}(\boldsymbol{v}) - \varepsilon_{kl} E_l(\hat{\varphi}) E_k(\psi)]\, d\boldsymbol{x} \\ &\qquad - \int_{\Gamma_{dN}} P_{kij} E_k(\hat{\varphi}) v_i n_j d\Gamma + \int_{\Gamma_{eN}} \varepsilon_{kl} E_l(\hat{\varphi}) \psi n_k\, d\Gamma \end{aligned} . \tag{2.61}$$

Substituting the stress tensor and the electric displacement by expressions (2.10) and using (2.59)-(2.60) we deduce that [see e.g. Haenel, 2000; Mechkour, 2004] the solution $(\boldsymbol{u}, \varphi)$ of (2.53)-(2.54) is derived from $\varphi = \bar{\varphi} + \hat{\varphi}$ with $(\boldsymbol{u}, \bar{\varphi})$ satisfying:

$$\begin{cases} \text{Find } (\boldsymbol{u}, \bar{\varphi}) \in V_0 \times \Psi_0 \text{ such that} \\ a((\boldsymbol{u}, \bar{\varphi}), (\boldsymbol{u}, \psi)) = l_2(\boldsymbol{v}, \psi) \qquad \forall (\boldsymbol{v}, \psi) \in V_0 \times \Psi_0 \end{cases} \tag{2.62}$$

where the bilinear form $a(\cdot, \cdot)$ was defined in (2.54) and the linear form $l_2(\cdot)$ reads

$$l_2(\boldsymbol{v}, \psi) \;\; = \;\; \int_\Omega [f_i v_i + P_{kij} E_k(\hat{\varphi}) e_{ij}(\boldsymbol{v}) - \varepsilon_{kl} E_l(\hat{\varphi}) E_k(\psi)]\, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i d\Gamma. \tag{2.63}$$

Remark 1 *From (2.43) and (2.55) we have*

$$\Psi_2 = \hat{\varphi} + \Psi_0 = \left\{ \psi \in H^1(\Omega) : \psi - \hat{\varphi} \in \Psi_0 \right\}.$$

An equivalent formulation problem of (2.53)-(2.54) is the following variational inequality (see Viriyasrisuwattana *et al.* [2007]):
Find $(\boldsymbol{u}, \varphi) \in V_0 \times \Psi_2$ *such that*

$$a((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi - \varphi)) \geq l(\boldsymbol{v}, \psi - \varphi), \qquad \forall (\boldsymbol{v}, \psi) \in V_0 \times \Psi_2. \tag{2.64}$$

We note that the problem (2.64) is equivalent to the problem (2.53)-(2.54).

2.3.2.2 Existence and uniqueness of solution

Let us first introduce some results to the linear and bilinear forms (see Haenel [2000] and Mechkour [2004]), which allow us to guarantee the existence and uniqueness of solution to the problem (2.53)-(2.54) by applying the Lax-Milgram's Lemma (see Viriyasrisuwattana *et al.* [2007], Weller & Licht [2008]).

Lemma 1 *Assume that* $\boldsymbol{f} = (f_i) \in [L^2(\Omega)]^3$ *and* $\boldsymbol{g} = (g_i) \in [L^2(\Gamma_{dN})]^3$*. Then the linear form* $l_2 : V_0 \times \Psi_0 \to \mathbb{R}$ *defined in (2.63) is continuous.*

Lemma 2 *Assume that meas* $(\Gamma_{dD}) > 0$*, meas* $(\Gamma_{eD}) > 0$*. Then the bilinear form* $a(\cdot, \cdot)$ *is continuous and* $V_0 \times \Psi_0$ *- elliptic.*

Proposition 1 *Assume that meas* $(\Gamma_{dD}) > 0$*, meas* $(\Gamma_{eD}) > 0$*,* $\boldsymbol{f} = (f_i) \in [L^2(\Omega)]^3$*,* $\boldsymbol{g} = (g_i) \in [L^2(\Gamma_{dN})]^3$*. Then (2.62)-(2.63) has a unique solution* $(\boldsymbol{u}, \bar{\varphi}) \in V_0 \times \Psi_0$*.*

Corollary 1 *Assume the hypotheses of Proposition 1 and also* $\varphi_0 \in H^{1/2}(\Gamma_{eD})$*. Then the variational problem (2.53)-(2.54) has one and only one solution* $(\boldsymbol{u}, \varphi) \in V_0 \times \Psi_2$*. Moreover,* $\varphi = \bar{\varphi} + \hat{\varphi}$*, where* $(\boldsymbol{u}, \bar{\varphi})$ *is the only solution of problem (2.62)-(2.63) and* $\hat{\varphi} \in H^1(\Omega)$ *is an extension of* φ_0*.*

2.3.2.3 Mixed formulation

In this section, we establish a mixed variational formulation of problem (2.37)-(2.38) using the spaces $\boldsymbol{X}_1$, $\boldsymbol{X}_0$ and $\boldsymbol{X}_2$:

$$\boldsymbol{X}_1 := \boldsymbol{X}_1(\Omega) = [L^2(\Omega)]^9_s \times L^2(\Omega), \qquad \|(\cdot,\cdot)\|_{\boldsymbol{X}_1} = \left[|\cdot|^2_{L^2(\Omega)} + |\cdot|^2_{L^2(\Omega)}\right]^{1/2},$$

$$\boldsymbol{X}_0 := \boldsymbol{X}_0(\Omega) = V_0 \times \Psi_0, \qquad \|(\cdot,\cdot)\|_{\boldsymbol{X}_0} = \left[||\cdot||^2_{V_0} + ||\cdot||^2_{H^1(\Omega)}\right]^{1/2},$$

$$\boldsymbol{X}_2 := \boldsymbol{X}_2(\Omega) = V_0 \times \Psi_2, \qquad \|(\cdot,\cdot)\|_{\boldsymbol{X}_2} = \left[||\cdot||^2_{V_0} + ||\cdot||^2_{H^1(\Omega)}\right]^{1/2}.$$

To deduce the mixed variational formulation of BVP 1, we write the constitutive law for piezoelectric material in the inverse formulation as follows (cf. (2.19)-(2.20))

$$e_{ij} = \bar{C}_{ijkl}\sigma_{kl} + \bar{P}_{kij}D_k, \qquad E_i = -\bar{P}_{ikl}\sigma_{kl} + \bar{\varepsilon}_{ik}D_k. \tag{2.65}$$

From properties of coerciveness and symmetry of tensors C_{ijkl} and ε_{ij}, and symmetry of P_{kij}, the following properties are derived:

($\mathbf{H}_1^m$) $\bar{C}_{ijkl} = \bar{C}_{klij} = \bar{C}_{jikl}, \quad \bar{\varepsilon}_{ij} = \bar{\varepsilon}_{ji}, \quad \bar{P}_{mij} = \bar{P}_{mji}.$

($\mathbf{H}_2^m$) There exists $c > 0$ such that, for any $\boldsymbol{d} = (d_i) \in \mathbb{R}^3$, $\boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{R}^{3\times 3}$,

$$\bar{C}_{ijkl}\tau_{ij}\tau_{kl} \geq c\sum_{i,j=1}^{3}(\tau_{ij})^2, \quad \bar{\varepsilon}_{ij}d_id_j \geq c\sum_{i=1}^{3}(d_i)^2. \tag{2.66}$$

The constitutive equation (2.65) is now considered. The weak form of this equation, obtained by multiplying both sides of the equation (2.65) by a test function $(\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1$ and integrating by parts over Ω, reads as follows

$$\int_\Omega \left(\bar{C}_{ijkl}\sigma_{kl} + \bar{P}_{kij}D_k\right)\tau_{ij}\, d\boldsymbol{x} + \int_\Omega \left(-\bar{P}_{kij}\sigma_{ij} + \bar{\varepsilon}_{kl}\, D_l\right) d_k d\boldsymbol{x}$$
$$- \int_\Omega e_{ij}(\boldsymbol{u})\tau_{ij}\, d\boldsymbol{x} - \int_\Omega E_k(\varphi)d_k\, d\boldsymbol{x} = 0.$$

Combining the previous equation with the equation (2.61) we obtain

$$\begin{cases} \text{Find } ((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{u}, \varphi)) \in \boldsymbol{X}_1 \times \boldsymbol{X}_2, \\ a_H\left((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{\tau}, \boldsymbol{d})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}, \varphi)\right) = 0, \quad \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \\ b_H\left((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \quad \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_0, \end{cases}$$

where

$$a_H\left((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{\tau}, \boldsymbol{d})\right) = \int_\Omega \left(\bar{C}_{ijkl}\sigma_{kl} + \bar{P}_{kij}D_k\right)\tau_{ij}\, d\boldsymbol{x} + \int_\Omega \left(-\bar{P}_{kij}\sigma_{ij} + \bar{\varepsilon}_{kl}D_l\right) d_k\, d\boldsymbol{x} \tag{2.67}$$

$$b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}, \varphi)\right) = -\int_\Omega e_{ij}(\boldsymbol{u})\tau_{ij}\, d\boldsymbol{x} - \int_\Omega E_k(\varphi)d_k\, d\boldsymbol{x} \tag{2.68}$$

$$l_H(\boldsymbol{v}, \psi) = -\int_\Omega f_iv_i d\boldsymbol{x} - \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma, \tag{2.69}$$

which is the *Hellinger-Reissner variational principle* for problem (2.37)-(2.38).

Using the relation $\varphi = \bar{\varphi} + \hat{\varphi}$ established in Section 2.3.2.1, the mixed problem (2.67)-(2.69) is: *Find* $((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{u}, \bar{\varphi})) \in \boldsymbol{X}_1 \times \boldsymbol{X}_0$, *such that*

$$\begin{cases} \text{Find } (\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{u}, \bar{\varphi})) \in \boldsymbol{X}_1 \times \boldsymbol{X}_0 \text{ such that} \\ a_H\left((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{\tau}, \boldsymbol{d})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}, \bar{\varphi})\right) = l_{1,H}(\boldsymbol{\tau}, \boldsymbol{d}), \quad \forall (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1 \\ b_H\left((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{v}, \psi)\right) = l_{2,H}(\boldsymbol{v}, \psi), \quad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{X}_0 \end{cases} \tag{2.70}$$

where

$$\begin{aligned} l_{1,H}(\boldsymbol{\tau}, \boldsymbol{d}) &= \int_\Omega E_k(\hat{\varphi}) d_k \, d\boldsymbol{x}, \\ l_{2,H}(\boldsymbol{v}, \psi) &= l_H(\boldsymbol{v}, \psi) = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i \, v_i \, d\Gamma. \end{aligned} \tag{2.71}$$

2.3.2.4 Existence and uniqueness of solution

To show the existence and uniqueness of solution for the above mixed formulation, it is enough to show that:

Theorem 1 *An unique solution* $((\boldsymbol{\sigma}, \boldsymbol{D}), (\boldsymbol{u}, \bar{\varphi})) \in \boldsymbol{X}_1 \times \boldsymbol{X}_0$ *to the problem defined by equations (2.70) and (2.71) exists, provided that*

1. $\boldsymbol{K}_0$ *- ellipticity of* a_H. *That is, there exists a constant* $\beta_1 > 0$ *such that*

$$|a_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{\tau}, \boldsymbol{d})\right)| \geq \beta_1 ||(\boldsymbol{\tau}, \boldsymbol{d})||^2_{\mathbf{X}_1} \qquad \forall (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{K}_0, \tag{2.72}$$

where

$$\boldsymbol{K}_0 = \left\{ (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1 : b_H((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{v}, \psi)) = 0 \quad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{X}_0 \right\}.$$

2. *(Babuška-Brezzi condition) Given* $(\boldsymbol{v}, \psi) \in \boldsymbol{X}_0$ *there exists a constant* β_2 *such that*

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1} \frac{-b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{v}, \psi)\right)}{||(\boldsymbol{\tau}, \boldsymbol{d})||_{\boldsymbol{X}_1}} \geq \beta_2 \|(\boldsymbol{v}, \psi)\|_{\boldsymbol{X}_0}. \tag{2.73}$$

Proof. Letting $(\boldsymbol{\tau}, \boldsymbol{d}) = (\boldsymbol{e}(\boldsymbol{v}), \boldsymbol{E}(\psi)) \in \boldsymbol{X}_1$ in the bilinear form $b_H(\cdot, \cdot)$, by virtue of Poincaré's and Korn's inequalities, we prove the existence of $\beta_2 > 0$ satisfying the above inf-sup condition (2.73). The elliptic property (2.72) is obvious since the ellipticity of $\bar{\boldsymbol{C}}$

and $\bar{\boldsymbol{\varepsilon}}$

$$\begin{aligned} a_H\left((\boldsymbol{\tau},\boldsymbol{d}),(\boldsymbol{\tau},\boldsymbol{d})\right) &= \int_{\dot{\Omega}} \left(\bar{C}_{ijkl}\,\tau_{kl}\,\tau_{ij} + \bar{\varepsilon}_{kl}\,d_l\,d_k\right)\,d\boldsymbol{x} \\ &\geq c\left(\|\boldsymbol{\tau}\|^2_{L^2(\Omega)} + \|\boldsymbol{d}\|^2_{L^2(\Omega)}\right), \quad \forall(\boldsymbol{\tau},\boldsymbol{d}) \in \boldsymbol{X}_0. \end{aligned}$$

We are now in a position to apply Babuška-Brezzi's Theorem (see e.g. Girault & Raviart [1986]): there exists a unique pair of functions $((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{u},\bar{\varphi})) \in \boldsymbol{X}_1 \times \boldsymbol{X}_0$ satisfying (2.70)-(2.71). ■

Chapter 3

One-dimensional piezoelectric model for a cantilever beam with electric potential applied on both ends.

In this chapter we initiate the asymptotic study of the piezoelectric problem for a linear beam that belongs to crystal symmetry class 2 in response to an electric potential acting on both ends. Through this analysis we derive a one-dimensional model for piezoelectric beams as the cross sectional goes to zero.

The analysis in this chapter uses asymptotic methods employed, in this field, by Maugin & Attou [1990]. Following Trabucho & Viaño [1996], we organize this chapter as follows. In Section 3.1, we recall the three-dimensional piezoelectricity problem and the equilibrium equations are written in the variational form (principle of virtual work). In Section 3.2, a set of scalings is used to maintain constant the beam diameter, and assign appropriate orders to the components of the displacement, the stress, the electric potential and the electric displacement. The scaled variational formulation of the three-dimensional problem posed over a fixed domain is also defined in this section. The weak convergence of the solution to this problem and the "limit" variational problem are studied in Section 3.3, as the small parameter tends to zero. In Section 3.4, we introduce the scaled principle of virtual work, and applying the *displacement-electric potential* approach, we prove that the scaled stress and electric displacement developments do not contain any negative power of h. In Section 3.4.0.2, we obtain the limit model whose leading term of the development is unknown, as expected. In Theorem 7, established in Section 3.4.1, the strong convergence results follows. Finally in Section 3.5, we find the boundary value problem to the limit, which consist in two partial differential equation of fourth order and two coupled partial differential equation of second order, posed over the one-dimensional set $(0, L)$.

3.1 The mechanical problem

Here, we are going to study the BVP 1, introduced in Section 2.3, for a family of a linearly piezoelectric beams.

3.1.1 Reference configuration, loading and boundary conditions

In its reference configuration, the beam occupies the domain

$$\bar{\Omega}^h = \omega^h \times [0, L],$$

having L as length and $\omega^h \subset \mathbb{R}^2$ as its cross-section. We assume that $\omega^h = h\omega$, where $\omega \subset \mathbb{R}^2$ is a bounded, open set of $\mathbb{R}^2$, of area $A = A(\omega)$ and boundary $\gamma = \partial\omega$ Lipschitz continuous. Then, the area of cross section ω^h is $A^h = h^2 A$ and the diameter of order h is assumed very small when compared with L.

An arbitrary point of Ω^h will be denoted by $\boldsymbol{x}^h = (x_1^h, x_2^h, x_3^h)$ and the unit outer normal vector to the boundary $\Gamma^h = \partial\Omega^h$ by $\mathbf{n}^h = (n_i^h)$. The coordinate system $Ox_1^h x_2^h x_3^h$ will be assumed a principal system of inertia associated to ω^h, which means that

$$\int_{\omega^h} x_\alpha^h \, d\omega^h = \int_{\omega^h} x_1^h x_2^h \, d\omega^h = 0. \tag{3.1}$$

We consider two decompositions of the boundary $\Gamma^h = \partial\Omega^h$, which correspond to the mechanical and electrical boundary conditions:

$$\Gamma^h = \partial\Omega^h = \Gamma_{dD}^h \cup \Gamma_{dN}^h \quad \text{with} \quad \Gamma_{dD}^h \cap \Gamma_{dN}^h = \emptyset \quad \text{and} \quad \text{meas}(\Gamma_D^h) > 0,$$

$$\Gamma^h = \partial\Omega^h = \Gamma_{eD}^h \cup \Gamma_{eN}^h \quad \text{with} \quad \Gamma_{eD}^h \cap \Gamma_{eN}^h = \emptyset \quad \text{and} \quad \text{meas}(\Gamma_{eD}^h) > 0.$$

Further we define the boundary sets:

$$\Gamma_0^h = \bar{\omega}^h \times \{0\}, \qquad \text{the left end of the beam,}$$

$$\Gamma_L^h = \bar{\omega}^h \times \{L\}, \qquad \text{the right end of the beam,}$$

$$\Gamma_N^h = \partial\omega^h \times (0, L), \quad \text{the lateral surface.}$$

In this Chapter, we assume the following loadings to the system (as illustrated in Figure 3.1.1):

i) The beam is weakly clamped at $\Gamma_{dD}^h = \Gamma_0^h$, which means that it is clamped in mean,

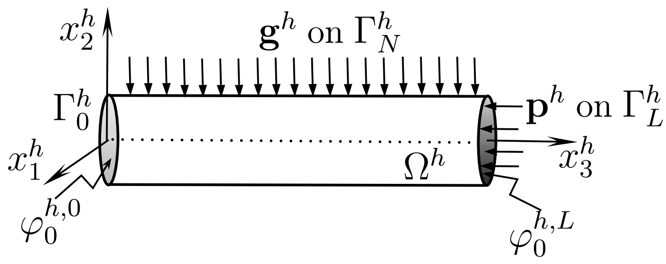


Figure 3.1: Schematic representation of the solid domain and their mechanical and electrical boundary conditions.

and reads (see e.g. Trabucho & Viaño [1996]),

$$\int_{\Gamma_0^h} u_i^h \, d\omega^h = 0, \quad \int_{\Gamma_0^h} (x_i^h u_j^h - x_j^h u_i^h) \, d\omega^h = 0; \tag{3.2}$$

We abbreviate the conditions (3.2) to

$$\langle \boldsymbol{u}^h \rangle_{\Gamma_0^h} = \mathbf{0}. \tag{3.3}$$

ii) The surface forces $\boldsymbol{g}^h$ are acting on Γ_N^h and $\boldsymbol{p}^h$ on Γ_L^h. This corresponds to put $\Gamma_{dN}^h = \Gamma_N^h \cup \Gamma_L^h$.

iii) An electric potential φ_0^h is applied on $\Gamma_{eD}^h = \Gamma_0^h \cup \Gamma_L^h$. More specifically, we denote $\varphi_0^h\big|_{\Gamma_0^h} = \varphi_0^{h,0} = \varphi_0^h(\cdot,\cdot,0)$ and $\varphi_0^h\big|_{\Gamma_L^h} = \varphi_0^{h,L} = \varphi_0^h(\cdot,\cdot,L)$.

Remark 2 *As we will see, the use of the "average clamping" condition (3.3) instead of a strong clamping condition such as $\mathbf{u}^h = \mathbf{0}$ on Γ_0^h allows to avoid the "boundary layer problem" (see Lions [1973]) that arises when a strong clamping condition is used - which turns out to be related to the fact that in general a strong clamping of the beam is physically impossible (see e.g. Trabucho & Viaño [1996] and references therein).*

We denote by C_{ijkl}^h, P_{kij}^h and ε_{kl}^h the components of elasticity tensor, piezoelectric tensor and dielectric tensor of the material, where Ω^h is made of a *monoclinic piezoelectric* material of class 2, i.e., the components C_{ijkl}^h, ε_{kl}^h and P_{kij}^h satisfy the following conditions:

$$C_{3\alpha\theta\rho}^h = C_{333\alpha}^h = \varepsilon_{3\alpha}^h = P_{\alpha\beta\rho}^h = P_{33\alpha}^h = P_{\alpha 33}^h = 0. \tag{3.4}$$

The conditions $C_{3\alpha\theta\rho}^h = C_{333\alpha}^h = 0$ reflet that the longitudinal axis Ox_3 is a principal directional of piezoelectricity (see Lekhnitskii [1981]), Royer & Dieulesaint [2000]) Therefore the stress tensor $\boldsymbol{\sigma}^h = (\sigma_{ij}^h) : \Omega^h \to \mathbb{R}_s^9$ and the electric displacement vector $\mathbf{D}^h = (D_i^h) : \Omega^h \to \mathbb{R}^3$ are related to the linear strain tensor, $e_{ij}^h(\boldsymbol{u}^h) = \frac{1}{2}(\partial_i^h u_j^h + \partial_j^h u_i^h)$, and the gradient of the electric potential, $E_i^h(\varphi^h) = -\partial_i^h \varphi^h$, through the following behavior laws:

$$\begin{cases} \sigma_{\alpha\beta}^h(\boldsymbol{u}^h,\varphi^h) = C_{\alpha\beta 33}^h e_{33}^h(\boldsymbol{u}^h) + C_{\alpha\beta\theta\rho}^h e_{\theta\rho}^h(\boldsymbol{u}^h) - P_{3\alpha\beta}^h E_3^h(\varphi^h), & \text{in } \Omega^h, \\ \sigma_{3\alpha}^h(\boldsymbol{u}^h,\varphi^h) = 2C_{3\alpha 3\beta}^h e_{3\beta}^h(\boldsymbol{u}^h) - P_{\beta 3\alpha}^h E_\beta^h(\varphi^h), & \text{in } \Omega^h, \\ \sigma_{33}^h(\boldsymbol{u}^h,\varphi^h) = C_{3333}^h e_{33}^h(\boldsymbol{u}^h) + C_{33\alpha\beta}^h e_{\alpha\beta}^h(\boldsymbol{u}^h) - P_{333}^h E_3^h(\varphi^h), & \text{in } \Omega^h, \\ D_\alpha^h(\boldsymbol{u}^h,\varphi^h) = 2P_{\alpha 3\beta}^h e_{3\beta}^h(\boldsymbol{u}^h) + \varepsilon_{\alpha\beta}^h E_\beta^h(\varphi^h), & \text{in } \Omega^h, \\ D_3^h(\boldsymbol{u}^h,\varphi^h) = P_{3\alpha\beta}^h e_{\alpha\beta}^h(\boldsymbol{u}^h) + P_{333}^h e_{33}^h(\boldsymbol{u}^h) + \varepsilon_{33}^h E_3^h(\varphi^h), & \text{in } \Omega^h. \end{cases} \tag{3.5}$$

From (3.4) and (3.5), we have

$$\begin{pmatrix} \sigma_{31}^h(\boldsymbol{u}^h,\varphi^h) \\ \sigma_{32}^h(\boldsymbol{u}^h,\varphi^h) \\ D_1^h(\boldsymbol{u}^h,\varphi^h) \\ D_2^h(\boldsymbol{u}^h,\varphi^h) \end{pmatrix} = \begin{pmatrix} 2C_{3131}^h & 2C_{3132}^h & -P_{131}^h & -P_{231}^h \\ 2C_{3231}^h & 2C_{3232}^h & -P_{132}^h & -P_{232}^h \\ 2P_{131}^h & 2P_{132}^h & \varepsilon_{11}^h & \varepsilon_{12}^h \\ 2P_{231}^h & 2P_{232}^h & \varepsilon_{21}^h & \varepsilon_{22}^h \end{pmatrix} \begin{pmatrix} e_{31}^h(\boldsymbol{u}^h) \\ e_{32}^h(\boldsymbol{u}^h) \\ E_1^h(\varphi^h) \\ E_2^h(\varphi^h) \end{pmatrix}, \tag{3.6}$$

and

$$\begin{pmatrix} \sigma_{11}^h(\boldsymbol{u}^h,\varphi^h) \\ \sigma_{12}^h(\boldsymbol{u}^h,\varphi^h) \\ \sigma_{22}^h(\boldsymbol{u}^h,\varphi^h) \\ \sigma_{33}^h(\boldsymbol{u}^h,\varphi^h) \\ D_3^h(\boldsymbol{u}^h,\varphi^h) \end{pmatrix} = \begin{pmatrix} C_{1111}^h & 2C_{1112}^h & C_{1122}^h & C_{1133}^h & -P_{311}^h \\ C_{1211}^h & 2C_{1212}^h & C_{1222}^h & C_{1233}^h & -P_{312}^h \\ C_{2211}^h & 2C_{2212}^h & C_{2222}^h & C_{2233}^h & -P_{322}^h \\ C_{3311}^h & 2C_{3312}^h & C_{3322}^h & C_{3333}^h & -P_{333}^h \\ P_{311}^h & 2P_{312}^h & P_{322}^h & P_{333}^h & \varepsilon_{33}^h \end{pmatrix} \begin{pmatrix} e_{11}^h(\boldsymbol{u}^h) \\ e_{12}^h(\boldsymbol{u}^h) \\ e_{22}^h(\boldsymbol{u}^h) \\ e_{33}^h(\boldsymbol{u}^h) \\ E_3^h(\varphi^h) \end{pmatrix}. \tag{3.7}$$

Equivalently,

$$\begin{pmatrix} e_{31}^h(\boldsymbol{u}^h) \\ e_{32}^h(\boldsymbol{u}^h) \\ E_1^h(\varphi^h) \\ E_2^h(\varphi^h) \end{pmatrix} = \begin{pmatrix} 2C_{3131}^h & 2C_{3132}^h & -P_{131}^h & -P_{231}^h \\ 2C_{3231}^h & 2C_{3232}^h & -P_{132}^h & -P_{232}^h \\ 2P_{131}^h & 2P_{132}^h & \varepsilon_{11}^h & \varepsilon_{12}^h \\ 2P_{231}^h & 2P_{232}^h & \varepsilon_{21}^h & \varepsilon_{22}^h \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{31}^h(\boldsymbol{u}^h,\varphi^h) \\ \sigma_{32}^h(\boldsymbol{u}^h,\varphi^h) \\ D_1^h(\boldsymbol{u}^h,\varphi^h) \\ D_2^h(\boldsymbol{u}^h,\varphi^h) \end{pmatrix},$$

and

$$\begin{pmatrix} e^h_{11}(\boldsymbol{u}^h) \\ e^h_{12}(\boldsymbol{u}^h) \\ e^h_{22}(\boldsymbol{u}^h) \\ e^h_{33}(\boldsymbol{u}^h) \\ E^h_3(\varphi^h) \end{pmatrix} = \begin{pmatrix} C^h_{1111} & 2C^h_{1112} & C^h_{1122} & C^h_{1133} & -P^h_{311} \\ C^h_{1211} & 2C^h_{1212} & C^h_{1222} & C^h_{1233} & -P^h_{312} \\ C^h_{2211} & 2C^h_{2212} & C^h_{2222} & C^h_{2233} & -P^h_{322} \\ C^h_{3311} & 2C^h_{3312} & C^h_{3322} & C^h_{3333} & -P^h_{333} \\ P^h_{311} & 2P^h_{312} & P^h_{322} & P^h_{333} & \varepsilon^h_{33} \end{pmatrix}^{-1} \begin{pmatrix} \sigma^h_{11}(\boldsymbol{u}^h,\varphi^h) \\ \sigma^h_{12}(\boldsymbol{u}^h,\varphi^h) \\ \sigma^h_{22}(\boldsymbol{u}^h,\varphi^h) \\ \sigma^h_{33}(\boldsymbol{u}^h,\varphi^h) \\ D^h_3(\boldsymbol{u}^h,\varphi^h) \end{pmatrix}.$$

On the other hand from (2.19)-(2.20) we have

$$\begin{aligned} e^h_{3\alpha} &= \ 2\bar{C}^h_{3\alpha3\theta}\,\sigma^h_{3\theta} + \bar{P}^h_{\beta3\alpha}\,D^h_\beta, && \text{in } \Omega^h, \\ E^h_\alpha &= \ -2\bar{P}^h_{\alpha3\beta}\,\sigma^h_{3\beta} + \bar{\varepsilon}^h_{\alpha\beta}\,D^h_\beta, && \text{in } \Omega^h, \\ e^h_{\alpha\beta} &= \bar{C}^h_{\alpha\beta33}\,\sigma^h_{33} + \bar{C}^h_{\alpha\beta\theta\rho}\,\sigma^h_{\theta\rho} + \bar{P}^h_{3\alpha\beta}\,D^h_3, && \text{in } \Omega^h, \\ e^h_{33} &= \ \bar{C}^h_{3333}\,\sigma^h_{33} + \bar{C}_{33\theta\rho}\,\sigma^h_{\theta\rho} + \bar{P}^h_{333}\,D^h_3, && \text{in } \Omega^h, \\ E^h_3 &= \ -\bar{P}^h_{3\alpha\beta}\,\sigma^h_{\alpha\beta} - \bar{P}^h_{333}\,\sigma^h_{33} + \bar{\varepsilon}^h_{33}\,D^h_3, && \text{in } \Omega^h, \end{aligned} \tag{3.8}$$

and therefore we arrive (see (2.65))

$$\bar{C}^h_{3\alpha\theta\rho} = \bar{C}^h_{333\alpha} = \bar{\varepsilon}^h_{3\alpha} = \bar{P}^h_{\alpha\beta\rho} = \bar{P}^h_{\alpha33} = \bar{P}^h_{33\alpha} = 0, \tag{3.9}$$

$$\begin{pmatrix} \bar{C}^h_{3131} & \bar{C}^h_{3132} & -\bar{P}^h_{131} & -\bar{P}^h_{231} \\ \bar{C}^h_{3231} & \bar{C}^h_{3232} & -\bar{P}^h_{132} & -\bar{P}^h_{232} \\ \bar{P}^h_{131} & \bar{P}^h_{132} & \bar{\varepsilon}^h_{11} & \bar{\varepsilon}^h_{12} \\ \bar{P}^h_{231} & \bar{P}^h_{232} & \bar{\varepsilon}^h_{21} & \bar{\varepsilon}^h_{22} \end{pmatrix} = \begin{pmatrix} 2C^h_{3131} & 2C^h_{3132} & -P^h_{131} & -P^h_{231} \\ 2C^h_{3231} & 2C^h_{3232} & -P^h_{132} & -P^h_{232} \\ 2P^h_{131} & 2P^h_{132} & \varepsilon^h_{11} & \varepsilon^h_{12} \\ 2P^h_{231} & 2P^h_{232} & \varepsilon^h_{21} & \varepsilon^h_{22} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} \bar{C}^h_{1111} & \bar{C}^h_{1112} & \bar{C}^h_{1122} & \bar{C}^h_{1133} & \bar{P}^h_{311} \\ \bar{C}^h_{1211} & \bar{C}^h_{1212} & \bar{C}^h_{1222} & \bar{C}^h_{1233} & \bar{P}^h_{312} \\ \bar{C}^h_{2211} & \bar{C}^h_{2212} & \bar{C}^h_{2222} & \bar{C}^h_{2233} & \bar{P}^h_{322} \\ \bar{C}^h_{3311} & \bar{C}^h_{3312} & \bar{C}^h_{3322} & \bar{C}^h_{3333} & \bar{P}^h_{333} \\ -\bar{P}^h_{311} & -\bar{P}^h_{312} & -\bar{P}^h_{322} & -\bar{P}^h_{333} & \bar{\varepsilon}^h_{33} \end{pmatrix} = \begin{pmatrix} C^h_{1111} & 2C^h_{1112} & C^h_{1122} & C^h_{1133} & -P^h_{311} \\ C^h_{1211} & 2C^h_{1212} & C^h_{1222} & C^h_{1233} & -P^h_{312} \\ C^h_{2211} & 2C^h_{2212} & C^h_{2222} & C^h_{2233} & -P^h_{322} \\ C^h_{3311} & 2C^h_{3312} & C^h_{3322} & C^h_{3333} & -P^h_{333} \\ P^h_{311} & 2P^h_{312} & P^h_{322} & P^h_{333} & \varepsilon^h_{33} \end{pmatrix}^{-1}.$$

To guarantee the regularity to the data we increase one condition to the Hypothesis 1 introduced in previous chapter:

Hypotheses 2 $\boldsymbol{p}^h \in [L^2(\Gamma_L^h)]^3$.

3.1.2 Variational problem: primal and mixed formulation

As a consequence of the weakly clamped condition the functional spaces of admissible displacements and electric potential take the form:

$$V_{0,w}^h = V_{0,w}^h(\Omega^h) = \left\{ \begin{array}{c} \boldsymbol{v}^h \in \left[H^1(\Omega^h)\right]^3 : \displaystyle\int_{\omega^h \times \{0\}} v_i^h \, d\omega^h = 0, \\ \displaystyle\int_{\omega^h \times \{0\}} (x_j^h v_i^h - x_i^h v_j^h) \, d\omega^h = 0 \end{array} \right\}, \tag{3.10}$$

$$\Psi_0^h = \Psi_0^h(\Omega^h) = \{\psi^h \in H^1(\Omega^h) : \psi^h = 0 \text{ on } \Gamma_{eD}^h\},$$

$$\Psi_2^h = \Psi_2^h(\Omega^h) = \{\psi^h \in H^1(\Omega^h) : \psi^h = \varphi_0^h \text{ on } \Gamma_{eD}^h\}.$$

The spaces $V_{0,w}^h(\Omega^h)$ and $\Psi_0^h(\Omega^h)$ equipped with the norms

$$||\boldsymbol{v}^h||_{V_{0,w}^h(\Omega^h)} = \left(\sum_{i,j=1}^{3} ||e_{ij}^h(\boldsymbol{v}^h)||_{0,\Omega^h} \right)^{1/2}, \quad \forall \boldsymbol{v}^h \in V_{0,w}^h(\Omega^h),$$

$$||\psi^h||_{\Psi_0^h(\Omega^h)} = ||\nabla \psi^h||_{0,\Omega^h}, \quad \forall \psi^h \in \Psi_0^h(\Omega^h),$$

becomes Hilbert spaces (Mechkour [2004]).

The mechanical problem corresponds to the following variational problem (see (2.53)-(2.54)):

$$\begin{aligned} &\text{Find } (\boldsymbol{u}^h, \varphi^h) \in V_{0,w}^h \times \Psi_2^h, \\ &a^h((\boldsymbol{u}^h, \varphi^h), (\boldsymbol{v}^h, \psi^h)) = l^h(\boldsymbol{v}^h, \psi^h), \quad \forall (\boldsymbol{v}^h, \psi^h) \in V_{0,w}^h \times \Psi_0^h, \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} a^h((\boldsymbol{u}^h, \varphi^h), (\boldsymbol{v}^h, \psi^h)) &= \int_{\Omega^h} \sigma_{ij}^h(\boldsymbol{u}^h, \varphi^h) e_{ij}^h(\boldsymbol{v}^h) \, d\boldsymbol{x}^h + \int_{\Omega^h} D_k^h(\boldsymbol{u}^h, \varphi^h) E_k^h(\psi^h) \, d\boldsymbol{x}^h \\ &= \int_{\Omega^h} [C_{ijkl}^h e_{kl}^h(\boldsymbol{u}^h) - P_{mij}^h E_m^h(\varphi^h)] e_{ij}^h(\boldsymbol{v}^h) \, d\boldsymbol{x}^h \\ &\quad + \int_{\Omega^h} [P_{mij}^h e_{ij}^h(\boldsymbol{u}^h) + \varepsilon_{mi}^h E_i^h(\varphi^h)] E_m(\psi^h) \, d\boldsymbol{x}^h, \end{aligned} \tag{3.12}$$

and

$$l^h(\boldsymbol{v}^h, \psi^h) = \int_{\Omega^h} f_i^h v_i^h \, d\boldsymbol{x}^h + \int_{\Gamma_N^h} g_i^h v_i^h \, d\Gamma^h + \int_{\Gamma_L^h} p_i^h v_i^h \, d\Gamma^h. \tag{3.13}$$

which corresponds to the principle of virtual work.

Remark 3 *In this formulation, the only difference with the strong clamped condition is* $V_{0,w}^h \times \Psi_2^h$ *that replaces* $V_0^h \times \Psi_0^h$ *(*V_0^h *defined in Chapter 2).*

Next, we prove that the problem (3.11) has an unique solution. We remark that the BVP associated with (3.11) is not simple to write. In fact, we obtain:

$$\left\{ \begin{array}{rcll} -\partial_j^h \sigma_{ij}^h \left(\boldsymbol{u}^h, \varphi^h\right) & = & f_i^h & \text{in} \quad \Omega^h, \\ \sigma_{ij}^h \left(\boldsymbol{u}^h, \varphi^h\right) n_j^h & = & g_i^h & \text{on} \quad \Gamma_N^h, \\ \langle u_i^h \rangle & = & 0 & \text{on} \quad \Gamma_{dD}^h, \\ \sigma_{i3}^h \left(\boldsymbol{u}^h, \varphi^h\right) & = & p_i^h & \text{on} \quad \Gamma_L^h, \\ \displaystyle\int_{\Gamma_0^h} \sigma_{i3}^h \left(\boldsymbol{u}^h, \varphi^h\right) v_i^h & = & 0 & \forall \boldsymbol{v}^h = (v_i^h) \in V_{0,w}^h, \end{array} \right. \tag{3.14}$$

$$\left\{ \begin{array}{rcll} \partial_k^h D_k^h \left(\boldsymbol{u}^h, \varphi^h\right) & = & 0 & \text{in} \quad \Omega^h, \\ D_k^h \left(\boldsymbol{u}^h, \varphi^h\right) n_k^h & = & 0 & \text{on} \quad \Gamma_{eN}^h, \\ \varphi^h & = & \varphi_0^h & \text{on} \quad \Gamma_{eD}^h. \end{array} \right. \tag{3.15}$$

The condition $(3.14)_5$ imposes a restriction on the form of σ_{3i}^h on Γ_0^h that cannot be expressed in a strong way.

Remark 4 *Let* $\hat{\varphi}^h \in H^1(\Omega)$ *be an extension of* $\varphi_0^h \in H^{1/2}(\Gamma_{eD}^h)$*. For simplicity we take the* x_3^h *- interpolant*

$$(\mathbf{H}_{31}^a) \qquad \hat{\varphi}^h(x_3^h) = \frac{1}{L}(L - x_3^h)\varphi_0^{h,0} + \frac{1}{L} x_3^h \varphi_0^{h,L}. \tag{3.16}$$

To achieve the limit problem we will suppose that the electric potential applied in both ends of the beam is constant and independent on x_1^h *and* x_2^h*. Therefore,*

$$\varphi_0^{h,0} \quad \textit{and} \quad \varphi_0^{h,L} \quad \textit{are constants,} \tag{3.17}$$

and therefore $\hat{\varphi}^h$ *depends only on variable* x_3^h*.*

To summarize, we have

$$\Psi_2^h = \Psi_2^h(\Omega) = \hat{\varphi}^h + \Psi_0^h, \tag{3.18}$$

and

$$\varphi^h = \hat{\varphi}^h + \bar{\varphi}^h, \quad \bar{\varphi}^h \in \Psi_0^h. \tag{3.19}$$

Moreover, $(\boldsymbol{u}^h, \bar{\varphi}^h)$ is the unique solution of the following problem:

$$\begin{aligned} &(\boldsymbol{u}^h, \bar{\varphi}^h) \in V_{0,w}^h \times \Psi_0^h, \\ &a^h((\boldsymbol{u}^h, \bar{\varphi}^h), (\boldsymbol{v}^h, \psi^h)) = l_2^h(\boldsymbol{v}^h, \psi^h), \quad \forall\, (\boldsymbol{v}^h, \psi^h) \in V_{0,w}^h \times \Psi_0^h, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} l_2^h(\boldsymbol{v}^h, \psi^h) &= l^h(\boldsymbol{v}^h, \psi^h) - a^h((0, \hat{\varphi}^h), (\boldsymbol{v}^h, \psi^h)) \\ &= \int_{\Omega^h} f_i^h v_i^h \, d\boldsymbol{x}^h + \int_{\Gamma_N^h} g_i^h v_i^h \, d\Gamma^h + \int_{\Gamma_L^h} p_i^h v_i^h \, d\Gamma^h \\ &\quad - \int_{\Omega^h} \varepsilon_{33}^h E_3^h(\hat{\varphi}^h) E_3^h(\psi^h) \, d\boldsymbol{x}^h + \int_{\Omega^h} P_{333}^h E_3^h(\hat{\varphi}^h) e_{33}^h(\boldsymbol{v}^h) \, d\boldsymbol{x}^h \\ &\quad + \int_{\Omega^h} P_{3\alpha\beta}^h E_3^h(\hat{\varphi}^h) e_{\alpha\beta}^h(\boldsymbol{v}^h) \, d\boldsymbol{x}^h. \end{aligned} \tag{3.21}$$

3.1.2.1 Existence and uniqueness of solution

Theorem 2 *The problem (3.20) has an unique solution* $(\boldsymbol{u}^h, \bar{\varphi}^h) \in V_{0,w}^h \times \Psi_0^h$.

Proof. Firstly, we need to prove that l^h is a continuous linear form on $V_{0,w}^h \times \Psi_0^h$, which is an elementary conclusion from hypothesis:

$$(\mathbf{H}_1^d) \qquad \boldsymbol{f}^h \in [L^2(\Omega^h)]^3, \quad \boldsymbol{g}^h \in [L^2(\Gamma_N^h)]^3, \quad \boldsymbol{p}^h \in [L^2(\Gamma_L^h)]^3. \tag{3.22}$$

Next, we need to prove that a^h is a $V_{0,w}^h \times \Psi_0^h$-elliptic. This is a consequence of Korn's and Poincaré's inequalities because of Korn's inequality is still valid in $V_{0,w}$, space that contains V_0 (see Trabucho & Viaño [1996]). Then, the Lax-Milgram's Lemma can be applied to conclude. ■

3.1.2.2 Mixed formulation

Following the same steps done in Section 2.3.2.4 we obtain the mixed formulation of problem (3.20)-(3.21). Defining

$$\boldsymbol{X}_1^h = [L^2(\Omega^h)]_s^9 \times [L^2(\Omega^h)]^3, \quad \boldsymbol{X}_{0,w}^h = V_{0,w}^h \times \Psi_0^h \quad \text{and} \quad \boldsymbol{X}_{2,w}^h = V_{0,w}^h \times \Psi_2^h,$$

we have (see (2.67)):

$$\text{Find } \big((\boldsymbol{\sigma}^h, \boldsymbol{D}^h), (\boldsymbol{u}^h, \varphi^h)\big) \in \boldsymbol{X}_1^h \times \boldsymbol{X}_{2,w}^h \text{ such that}$$

$$\begin{cases} a_H^h\big((\boldsymbol{\sigma}^h, \boldsymbol{D}^h), (\boldsymbol{\tau}^h, \boldsymbol{d}^h)\big) + b_H^h\big((\boldsymbol{\tau}^h, \boldsymbol{d}^h), (\boldsymbol{u}^h, \varphi^h)\big) = 0 \\ \text{for all } \big(\boldsymbol{\tau}^h, \boldsymbol{d}^h\big) \in \boldsymbol{X}_1^h \end{cases} \tag{3.23}$$

$$\begin{cases} b_H^h\big(\big(\boldsymbol{\sigma}^h, \boldsymbol{D}^h\big), \big(\boldsymbol{v}^h, \psi^h\big)\big) = l_H^h(\boldsymbol{v}^h, \psi^h) \\ \text{for all } \big(\boldsymbol{v}^h, \psi^h\big) \in \boldsymbol{X}_{0,w}^h \end{cases} \tag{3.24}$$

where

$$a_H^h\Big((\bar{\boldsymbol{\tau}}^h, \bar{\boldsymbol{d}}^h), (\boldsymbol{\tau}^h, \boldsymbol{d}^h)\Big) = \int_{\Omega^h} \big(\bar{C}_{ijkl}^h \, \bar{\tau}_{kl}^h + \bar{P}_{kij}^h \bar{d}_k^h\big) \, \tau_{ij}^h \, d\boldsymbol{x}^h + \int_{\Omega^h} \big(-\bar{P}_{kij}^h \bar{\tau}_{ij}^h + \bar{\varepsilon}_{kl}^h \, \bar{d}_l^h\big) \, d_k^h \, d\boldsymbol{x}^h, \tag{3.25}$$

$$b_H^h\Big((\bar{\boldsymbol{\tau}}^h, \bar{\boldsymbol{d}}^h), \big(\boldsymbol{v}^h, \psi^h\big)\Big) = -\int_{\Omega^h} \bar{\tau}_{ij}^h \, e_{ij}^h(\boldsymbol{v}^h) d\boldsymbol{x}^h - \int_{\Omega^h} \bar{d}_k^h \, E_k^h(\psi^h) d\boldsymbol{x}^h, \tag{3.26}$$

$$l_H^h(\boldsymbol{v}^h, \psi^h) = -\int_{\Omega^h} f_i^h \, v_i^h \, d\boldsymbol{x}^h - \int_{\Gamma_N^h} g_i^h \, v_i^h d\Gamma^h - \int_{\Gamma_L^h} p_i^h \, v_i^h d\Gamma^h. \tag{3.27}$$

Existence and uniqueness of solution of the problem (3.23)-(3.27) is obtained as in Theorem 2 because all arguments are valid replacing $\boldsymbol{X}_{2,w}^h$ ($\boldsymbol{X}_{0,w}^h$) by $\boldsymbol{X}_2^h$ ($\boldsymbol{X}_0^h$), respectively

3.2 Change of variable to the reference beam Ω

The major geometric feature of a three-dimensional beam is the fact that the largest cross sectional dimension is very small compared to its length ($h \ll L$), causing ill-conditioning of the three-dimensional problem. We take advantage of this property to use an asymptotic expansion method (see Lions [1973]) with respect to the small parameter h as usually done in the elastic beam case (see e.g. Bermúdez & Viaño [1984], Trabucho & Viaño [1996] and references therein). We will study the dependence of the solution

$(\boldsymbol{u}^h, \varphi^h)$ with respect to h. The technique of change of variable to a fixed domain and subsequent rescalling of the displacement and electric potential will allow us to derive a variational problem equivalent to (3.11)-(3.13) or (3.20)-(3.21) where h shows up in an explicit way in the rescaled equations.

To start, we perform a change of variable to the reference domain $\Omega = \omega \times (0, L)$ through the following transformation (Figure 3.2)

$$\begin{aligned} \Pi^h : \boldsymbol{x} = (x_1, x_2, x_3) \in \bar{\Omega} \\ \rightarrow \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) = (x_1^h, x_2^h, x_3^h) = (hx_1, hx_2, x_3) \in \bar{\Omega}^h. \end{aligned} \tag{3.28}$$

All notations refereed to domain Ω are obtained from Ω^h for $h = 1$ and this index is dropped. For example:

$$\Gamma = \partial\Omega, \quad \Gamma_0 = \Gamma_0^1 = \omega \times \{0\}, \quad \Gamma_N = \gamma \times [0, L].$$

Furthermore, condition (3.1) becomes now

$$\int_\omega x_\alpha \, d\omega = \int_\omega x_1 x_2 \, d\omega = 0, \tag{3.29}$$

that is, the system "$Ox_1x_2x_3$" is a principal system of inertia for Ω. In the view of (3.2) we will represent the boundary condition

$$\int_{\Gamma_0} v_i \, d\omega = 0, \quad \int_{\Gamma_0} (x_j v_i - x_i v_j) \, d\omega = 0,$$

by $\langle \boldsymbol{v} \rangle = \mathbf{0}$ on Γ_0, that is, taking (3.29) into account,

$$\int_{\Gamma_0} v_i \, d\omega = 0, \quad \int_{\Gamma_0} \delta_\alpha v_\alpha \, d\omega = 0, \quad \int_{\Gamma_0} x_\alpha v_3 \, d\omega = 0, \tag{3.30}$$

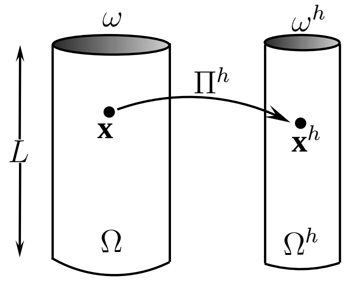


Figure 3.2: Change of variable between the set $\bar{\Omega}$ and the set $\bar{\Omega}^h$.

where $\delta_1(x_1,x_2)=x_2$, $\delta_2(x_1,x_2)=-x_1$. We now define the spaces (cf. (3.10))

$$V_{0,w} := V_{0,w}(\Omega) = \{\boldsymbol{v} \in [H^1(\Omega)]^3 : \langle \boldsymbol{v}\rangle|_{\Gamma_0} = \boldsymbol{0}\},$$

$$\Psi_0 := \Psi_0(\Omega) = \{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_0 \cup \Gamma_L\}, \tag{3.31}$$

$$\Psi_2 := \Psi_2(\Omega) = \hat{\varphi} + \Psi_0(\Omega) = \{\psi \in H^1(\Omega) : \psi - \hat{\varphi} \in \Psi_0\}, \tag{3.32}$$

$$\boldsymbol{X}_{0,w} := \boldsymbol{X}_{0,w}(\Omega) = V_{0,w}(\Omega) \times \Psi_0(\Omega),$$

$$\boldsymbol{X}_{2,w} := \boldsymbol{X}_{2,w}(\Omega) = V_{0,w}(\Omega) \times \Psi_2(\Omega),$$

endowed with the following norms equivalent to the usual Sobolev norms:

$$\|\boldsymbol{v}\|_{V_{0,w}} = |\boldsymbol{e}(\boldsymbol{v})|_{0,\Omega}\,, \quad \|\psi\|_{\Psi_0} = |\boldsymbol{\nabla}\psi|_{0,\Omega}\,, \quad \|(\boldsymbol{v},\psi)\|_{\boldsymbol{X}_{0,w}} = \left(\|\boldsymbol{v}\|^2_{V_{0,w}} + \|\psi\|^2_{\Psi_0}\right)^{1/2}$$

and

$$\boldsymbol{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v}))\,, \quad e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

Remark 5 *In what follows we shall make use of the following decomposition of* $V_{0,w}(\Omega)$ *(cf. Trabucho & Viaño [1996])*

$$V_{0,w}(\Omega) = W_1(\Omega) \times W_2(\Omega), \tag{3.33}$$

$$W_1(\Omega) = \left\{\eta \in H^1(\Omega) : \int_{\omega\times\{0\}} \eta = \int_{\omega\times\{0\}} x_\alpha \eta = 0\right\}, \tag{3.34}$$

$$W_2(\Omega) = \left\{\widehat{\rho} = (\rho_\alpha) \in \left[H^1(\Omega)\right]^2 : \int_{\omega\times\{0\}} \rho_\alpha = \int_{\omega\times\{0\}} (x_1\rho_2 - x_2\rho_1) = 0\right\}. \tag{3.35}$$

In order to obtain a problem in Ω equivalent to (3.11) we associate it to the unknowns and test displacement fields $\boldsymbol{u}^h$, $\boldsymbol{v}^h$ in $V^h_{0,w}$, the (unknown and test) scaled displacement fields $\boldsymbol{u}(h) = (u_i(h))$ and $\boldsymbol{v}(h) = (v_i(h))$ in $V_{0,w}$ defined by the following scalings valid for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$:

$$u_\alpha(h)(\boldsymbol{x}) = hu^h_\alpha(\boldsymbol{x}^h), \quad u_3(h)(\boldsymbol{x}) = u^h_3(\boldsymbol{x}^h), \tag{3.36}$$

$$v_\alpha(h)(\boldsymbol{x}) = hv^h_\alpha(\boldsymbol{x}^h), \quad v_3(h)(\boldsymbol{x}) = v^h_3(\boldsymbol{x}^h). \tag{3.37}$$

Similarly, the electric potential φ^h and the test function ψ^h in Ψ^h_0 are associated to the scaled potential $\varphi(h)$ and the scaled (test) function $\psi(h)$ in Ψ_0 using the following scaling for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$:

$$\varphi(h)(\boldsymbol{x}) = \varphi^h(\boldsymbol{x}^h), \qquad \psi(h)(\boldsymbol{x}) = \psi^h(\boldsymbol{x}^h). \tag{3.38}$$

Moreover, to the stress tensor $\boldsymbol{\sigma}^h = (\sigma^h_{ij}) : \bar{\Omega}^h \to \mathbb{R}^9_s$ and to the electric displacement $\boldsymbol{D}^h = (D^h_k) : \bar{\Omega}^h \to \mathbb{R}^3$ we associate, respectively, the scaled stress tensor field $\boldsymbol{\sigma}(h) = (\sigma_{ij}(h)) : \bar{\Omega} \to \mathbb{R}^9_s$ and the electric displacement vector $\boldsymbol{D}(h) = (D_k(h)) : \bar{\Omega} \to \mathbb{R}^3$ defined by

$$\begin{cases} \sigma^h_{\alpha\beta}(\boldsymbol{x}^h) = h^2\sigma_{\alpha\beta}(h)(\boldsymbol{x}), \quad \sigma^h_{3\alpha}(\boldsymbol{x}^h) = h\sigma_{3\alpha}(h)(\boldsymbol{x}), \quad \sigma^h_{33}(\boldsymbol{x}^h) = \sigma_{33}(h)(\boldsymbol{x}), \\ D^h_\alpha(\boldsymbol{x}^h) = hD_\alpha(h)(\boldsymbol{x}), \quad D^h_3(\boldsymbol{x}^h) = D_3(h)(\boldsymbol{x}), \end{cases} \tag{3.39}$$

valid for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$. Furthermore, we consider the following hypothesis on the magnitude of the data with respect to the diameter of the beam cross-section h:

1. There exist functions $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_N)$ and $p_i \in L^2(\Gamma_L)$, independent of h, such that:

$$\begin{cases} f^h_\alpha(\boldsymbol{x}^h) = hf_\alpha(\boldsymbol{x}), \quad f^h_3(\boldsymbol{x}^h) = f_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h, \\ g^h_\alpha(\boldsymbol{x}^h) = h^2 g_\alpha(\boldsymbol{x}), \quad g^h_3(\boldsymbol{x}^h) = hg_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma^h_N, \\ p^h_\alpha(\boldsymbol{x}^h) = hp_\alpha(\boldsymbol{x}), \quad p^h_3(\boldsymbol{x}^h) = p_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma^h_L. \end{cases} \tag{3.40}$$

2. There exists a function $\hat{\varphi} \in H^1(0, L)$, independent of h, such that:

$$\hat{\varphi}(x_3) = \hat{\varphi}^h(x^h_3), \quad \text{for all } x^h_3 = \Pi^h(x_3),\ x_3 \in [0, L]. \tag{3.41}$$

   As mentioned before, $\hat{\varphi}$ is the trace lifting of φ_0. If $\hat{\varphi}^h$ is given by condition $(\mathbf{H}^a_{31})$, then

$$\hat{\varphi}(x_3) = \frac{1}{L}(L - x_3)\varphi^0_0 + \frac{1}{L}x_3\varphi^L_0. \tag{3.42}$$

3. The piezoelectric constants are such that

$$C^h_{ijkl}(\boldsymbol{x}^h) = C_{ijkl}(\boldsymbol{x}), \quad P^h_{kij}(\boldsymbol{x}^h) = P_{kij}(\boldsymbol{x}), \quad \varepsilon^h_{ij}(\boldsymbol{x}^h) = \varepsilon_{ij}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{3.43}$$

   where C_{ijkl}, P_{kij} and ε_{ij} are independent of the size of the cross section and satisfy Hypothesis 1.

Combining (3.28) with the notations (3.36)-(3.38) we have for all $\boldsymbol{v}^h \in V^h_{0,w}$, $\psi^h \in \Psi^h_0$ and

$\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$:

$$\begin{cases} e_{\alpha\beta}(\boldsymbol{v}(h))(\boldsymbol{x}) = h^2 e^h_{\alpha\beta}(\boldsymbol{v}^h)(\boldsymbol{x}^h), \quad e_{3\beta}(\boldsymbol{v}(h))(\boldsymbol{x}) = h e^h_{3\beta}(\boldsymbol{v}^h)(\boldsymbol{x}^h), \\ e_{33}(\boldsymbol{v}(h))(\boldsymbol{x}) = e^h_{33}(\boldsymbol{v}^h)(\boldsymbol{x}^h), \end{cases} \tag{3.44}$$

$$\begin{cases} E_\alpha(\psi(h))(\boldsymbol{x}) &= -\partial_\alpha(\psi(h))(\boldsymbol{x}) = -h\partial^h_\alpha \psi^h(\boldsymbol{x}^h) = hE^h_\alpha(\psi^h)(\boldsymbol{x}^h), \\ E_3(\psi(h))(\boldsymbol{x}) &= -\partial_3(\psi(h))(\boldsymbol{x}) = -\partial^h_3 \psi^h(\boldsymbol{x}^h) = E^h_3(\psi^h)(\boldsymbol{x}^h). \end{cases} \tag{3.45}$$

Using the scalings defined previously for the displacement vector and for the electric potential together with the above assumptions, we can reformulate the variational problem (3.11)-(3.13) into another variational problem posed in the domain Ω independent of h. We have the following result:

Proposition 2 *The scaled pair* $(\boldsymbol{u}(h), \varphi(h))$ *is the unique solution of the following problem*

$$\begin{cases} (\boldsymbol{u}(h), \varphi(h)) \in \boldsymbol{X}_{2,w} = V_{0,w} \times \Psi_2, \\ h^{-4} a_{-4}((\boldsymbol{u}(h), \varphi(h)), (\boldsymbol{v}, \psi)) + h^{-2} a_{-2}((\boldsymbol{u}(h), \varphi(h)), (\boldsymbol{v}, \psi)) \\ \quad + a_0((\boldsymbol{u}(h), \varphi(h)), (\boldsymbol{v}, \psi)) = l(\boldsymbol{v}, \psi), \quad \textit{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w} = V_{0,w} \times \Psi_0, \end{cases} \tag{3.46}$$

where the bilinear forms $a_{-4}(\cdot,\cdot)$, $a_{-2}(\cdot,\cdot)$ *and* $a_0(\cdot,\cdot)$ *are defined by*

$$a_{-4}((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) = \int_\Omega C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x}, \tag{3.47}$$

$$\begin{aligned} a_{-2}((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) = &\int_\Omega C_{\alpha\beta 33} \left(e_{33}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v}) + e_{\alpha\beta}(\boldsymbol{u}) e_{33}(\boldsymbol{v})\right) d\boldsymbol{x} \\ &+ 4\int_\Omega C_{\alpha 33\theta} e_{3\theta}(\boldsymbol{u}) e_{3\alpha}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \varepsilon_{\theta\alpha} E_\alpha(\varphi) E_\theta(\psi)\, d\boldsymbol{x} \\ &- \int_\Omega P_{3\alpha\beta}[E_3(\varphi) e_{\alpha\beta}(\boldsymbol{v}) - e_{\alpha\beta}(\boldsymbol{u}) E_3(\psi)]\, d\boldsymbol{x} \\ &- 2\int_\Omega P_{\theta 3\alpha}[E_\theta(\varphi) e_{3\alpha}(\boldsymbol{v}) - e_{3\alpha}(\boldsymbol{u}) E_\theta(\psi)]\, d\boldsymbol{x}, \end{aligned} \tag{3.48}$$

$$\begin{aligned} a_0((\boldsymbol{u}, \varphi), (\boldsymbol{v}, \psi)) = &\int_\Omega C_{3333} e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \varepsilon_{33} E_3(\varphi) E_3(\psi) d\boldsymbol{x} \\ &- \int_\Omega P_{333}[E_3(\varphi) e_{33}(\boldsymbol{v}) - e_{33}(\boldsymbol{u}) E_3(\psi)]\, d\boldsymbol{x}, \end{aligned} \tag{3.49}$$

and the linear form reads

$$l(\boldsymbol{v},\psi) = \int_\Omega f_i v_i \, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i \, d\Gamma + \int_{\Gamma_L} p_i v_i \, d\Gamma. \tag{3.50}$$

The existence and uniqueness of solution of the problem (3.46)-(3.50) is due to Lax-Milgram's Lemma since both a and l are $\boldsymbol{X}_{0,w}$-continuous and a is $\boldsymbol{X}_{0,w}$-elliptic as the reader can easily check having in mind properties (3.22), conditions ($\mathbf{H}_{21}^c$)-($\mathbf{H}_{22}^c$) and (2.39)-(2.40) for $\boldsymbol{C}$, $\boldsymbol{P}$, $\boldsymbol{\varepsilon}$, and that $0 < h \leq 1$.

Remark 6 *We note that the restrictions (3.4) about the material coefficients C_{ijkl}, P_{kij} and ε_{ij} allow us to avoid the odd powers for h in (3.46). This is a fundamental fact because it eliminates some coupled effects in the beam and is the basis to order to complete the following asymptotic analysis. This analysis would be far more complicated if (3.4) was not satisfied.*

Next we write problem (3.46)-(3.50) in an equivalent form that exhibits the following scaled principle of virtual work and scaled constitutive law.

Proposition 3 *Problem (3.46)-(3.50) is formally equivalent to*

$$\begin{aligned}
&(\boldsymbol{u}(h),\varphi(h)) \in \boldsymbol{X}_{2,w} = V_{0,w} \times \Psi_2,\\
&\int_\Omega \sigma_{ij}(h) e_{ij}(\boldsymbol{v}) \, d\boldsymbol{x} + \int_\Omega D_k(h) E_k(\psi) \, d\boldsymbol{x} = \int_\Omega f_i v_i \, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i \, d\Gamma \\
&\qquad + \int_{\Gamma_L} p_i v_i \, d\Gamma, \qquad \forall \; (\boldsymbol{v},\psi) \in \boldsymbol{X}_{0,w} = V_{0,w} \times \Psi_0,
\end{aligned} \tag{3.51}$$

where the scaled stress tensor $\boldsymbol{\sigma}(h) = (\sigma_{ij}(h))$ and the scaled electrical displacement $\boldsymbol{D}(h) = (D_k(h))$ are related to $(\boldsymbol{u}(h),\varphi(h))$ by the following scaled constitutive piezo-electric law (compatible with (3.39), (3.44) and (3.45))

$$\begin{aligned}
&\sigma_{\alpha\beta}(h) := h^{-4} C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}(h)) + h^{-2} C_{\alpha\beta 33} e_{33}(\boldsymbol{u}(h)) - h^{-2} P_{3\alpha\beta} E_3(\varphi(h)),\\
&\sigma_{3\alpha}(h) := 2h^{-2} C_{\alpha 33\theta} e_{3\theta}(\boldsymbol{u}(h)) - h^{-2} P_{\theta\alpha 3} E_\theta(\varphi(h)),\\
&\sigma_{33}(h) := h^{-2} C_{33\theta\rho} e_{\theta\rho}(\boldsymbol{u}(h)) + C_{3333} e_{33}(\boldsymbol{u}(h)) - P_{333} E_3(\varphi(h)),\\
&D_\theta(h) := 2h^{-2} P_{\theta 3\alpha} e_{3\alpha}(\boldsymbol{u}(h)) + h^{-2} \varepsilon_{\theta\alpha} E_\alpha(\varphi(h)),\\
&D_3(h) := h^{-2} P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}(h)) + P_{333} e_{33}(\boldsymbol{u}(h)) + \varepsilon_{33} E_3(\varphi(h)).
\end{aligned} \tag{3.52}$$

3.2.0.3 The scaled mixed formulation

The following mixed formulation of scaled problem is immediately obtained from (3.38)-(3.39), (3.44) and (3.51).

The scaled unknown $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ satisfies the following variational problem: *Find* $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w}$ *such that*

$$\begin{cases} a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ +h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))\right) = 0, \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \mathbf{X}_1, \end{cases} \tag{3.53}$$

$$\begin{cases} b_H\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \\ \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}, \end{cases} \tag{3.54}$$

where $b_H(\cdot,\cdot) : \boldsymbol{X}_1 \times \boldsymbol{X}_{0,w} \to \mathbb{R}$ and $a_{H,i}(\cdot,\cdot) : \boldsymbol{X}_1 \times \boldsymbol{X}_1 \to \mathbb{R}$ are the following bilinear form

$$b_H((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{v}, \psi)) = -\int_\Omega \tau_{ij}\, e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} - \int_\Omega d_k\, E_k(\psi) d\boldsymbol{x}, \tag{3.55}$$

$$a_{H,4}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) = \int_\Omega \bar{C}_{\alpha\beta\theta\rho}\, \bar{\tau}_{\theta\rho} \tau_{\alpha\beta}\, d\boldsymbol{x}, \tag{3.56}$$

$$\begin{aligned} a_{H,2}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) &= \int_\Omega \left(\bar{C}_{\alpha\beta33}\, \bar{\tau}_{33} + \bar{P}_{3\alpha\beta} \bar{d}_3\right) \tau_{\alpha\beta}\, d\boldsymbol{x} + \int_\Omega \bar{C}_{33\theta\rho}\, \bar{\tau}_{\theta\rho} \tau_{33}\, d\boldsymbol{x} \\ &+ 2\int_\Omega \left(2\bar{C}_{3\alpha3\theta}\, \bar{\tau}_{3\theta} + \bar{P}_{\theta3\alpha} \bar{d}_\theta\right) \tau_{3\alpha}\, d\boldsymbol{x} - \int_\Omega \bar{P}_{3\alpha\beta}\, \bar{\tau}_{\alpha\beta}\, d_3\, d\boldsymbol{x} \\ &+ \int_\Omega \left(-2\bar{P}_{\theta3\alpha}\, \bar{\tau}_{3\alpha} + \bar{\varepsilon}_{\theta\alpha}\, \bar{d}_\alpha\right) d_\theta\, d\boldsymbol{x} \end{aligned} \tag{3.57}$$

$$\begin{aligned} a_{H,0}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) &= \int_\Omega \left(\bar{C}_{3333}\, \bar{\tau}_{33} + \bar{P}_{333} \bar{d}_3\right) \tau_{33}\, d\boldsymbol{x} \\ &+ \int_\Omega \left(-\bar{P}_{333} \bar{\tau}_{33} + \bar{\varepsilon}_{33}\, \bar{d}_3\right) d_3\, d\boldsymbol{x}, \end{aligned} \tag{3.58}$$

and the linear form $l_H(\cdot) : \mathbf{X}_{0,w} \to \mathbb{R}$ read

$$l_H(\boldsymbol{v}, \psi) = -\int_\Omega f_i\, v_i\, d\boldsymbol{x} - \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma - \int_{\Gamma_L} p_i\, v_i\, d\Gamma. \tag{3.59}$$

The pair $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ can also characterized as the unique solution of this problem thanks to the Babuška-Brezzi condition.

3.3 Convergence of the scaled unknowns as $h \to 0$.

3.3.1 Weak convergence

In this section we prove that the family $(\boldsymbol{u}(h), \varphi(h))_{h>0}$ weakly converge to $(\boldsymbol{u}, \varphi)$ in $[H^1(\Omega)]^3 \times H^1(\Omega)$, as $h \to 0$, and identify the "limit" variational problem solved by $(\boldsymbol{u}, \varphi)$. We follow the method introduced by Trabucho & Viaño [1996].
Then the following weak convergence are guaranteed.

Proposition 4 *There exists $C > 0$, independent of h, such that for all $0 < h \leq 1$ the solution $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ of problem (3.53)-(3.59) verifies*

$$|\sigma_{33}(h)|_{0,\Omega} \leq C, \qquad h\,|\sigma_{\alpha 3}(h)|_{0,\Omega} \leq C, \qquad h^2\,|\sigma_{\alpha\beta}(h)|_{0,\Omega} \leq C, \tag{3.60}$$

$$h\,|D_\alpha(h)|_{0,\Omega} \leq C, \qquad |D_3(h)|_{0,\Omega} \leq C, \tag{3.61}$$

$$\|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq C, \qquad \|(\boldsymbol{u}(h), \varphi(h))\|_{\boldsymbol{X}_{2,w}} \leq C. \tag{3.62}$$

Proof. Let $\boldsymbol{S}(h) \in [L^2(\Omega)]^9_s$ and $\boldsymbol{T}(h) \in [L^2(\Omega)]^3$ be the following elements:

$$S_{33}(h) = \sigma_{33}(h), \qquad S_{3\alpha}(h) = h\sigma_{3\alpha}(h), \qquad S_{\alpha\beta}(h) = h^2\sigma_{\alpha\beta}(h), \tag{3.63}$$

$$T_\beta(h) = hD_\alpha(h), \qquad T_3(h) = D_3(h). \tag{3.64}$$

We have

$$\begin{aligned}
&a_T\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) := a_{H,0}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right)\\
&\qquad + a_{H,2}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) + a_{H,4}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right)\\
&= a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\sigma}(h), \boldsymbol{d}(h))\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\sigma}(h), \boldsymbol{D}(h))\right)\\
&\qquad + h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\sigma}(h), \boldsymbol{D}(h))\right).
\end{aligned}$$

Hence letting $(\boldsymbol{\tau}, \boldsymbol{d}) = (\boldsymbol{\sigma}(h), \boldsymbol{D}(h))$ in equation (3.53) we obtain

$$\begin{aligned}
a_T((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))) &= \int_\Omega [\bar{C}_{3333} S_{33}(h) S_{33}(h) + \bar{\varepsilon}_{33} T_3(h) T_3(h)] d\boldsymbol{x}\\
&\quad + 2\int_\Omega [\bar{C}_{33\theta\rho} S_{\theta\rho}(h) S_{33}(h) + 4\bar{C}_{3\alpha 3\theta} S_{3\theta}(h) S_{3\alpha}(h)] d\boldsymbol{x}\\
&\quad + \int_\Omega \bar{\varepsilon}_{\alpha\theta} T_\theta(h) T_\alpha(h) d\boldsymbol{x} + \int_\Omega \bar{C}_{\alpha\beta\theta\rho} S_{\theta\rho}(h) S_{\alpha\beta}(h) d\boldsymbol{x}\\
&= -b_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))
\end{aligned}$$

and from $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \bar{\varphi}(h)) \in \mathbf{X}_{0,w}$ in equation (3.54) with $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \bar{\varphi}(h)) \in \mathbf{X}_{0,w}$ (note that $\bar{\varphi}(h) = \varphi(h) - \hat{\varphi}$) one has

$$a_T((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))) = \int_\Omega f_i u_i(h) d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i(h) d\Gamma$$
$$+ \int_{\Gamma_L} p_i u_i(h) d\Gamma + \int_\Omega D_3(h) E_3(\hat{\varphi}) d\boldsymbol{x}. \tag{3.65}$$

Using properties (2.66) for $\bar{C}_{ijkl}$, $\bar{\varepsilon}_{ij}$, we have

$$a_T((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))) \geq C\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{0,\Omega}^2 = C\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{\boldsymbol{X}_1}^2, \tag{3.66}$$

on the one hand. On the other hand, from (3.65), we deduce that

$$\begin{aligned} &a_T((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))) \\ &\leq C\left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{0,\Gamma_N} + \|\boldsymbol{p}\|_{0,\Gamma_L}\right)\|\boldsymbol{u}(h)\|_{1,\Omega} + C\|\hat{\varphi}\|_{1,(0,L)}\|D_3(h)\|_{0,\Omega} \\ &\leq C\left\|\boldsymbol{u}(h)\right\|_{1,\Omega} + C\|T_3(h)\|_{0,\Omega} \leq C\left\|(\boldsymbol{u}(h), \bar{\varphi}(h))\right\|_{\boldsymbol{X}_{0,w}} + C\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{\boldsymbol{X}_1}. \end{aligned} \tag{3.67}$$

Combining (3.66) with (3.67), and using the inequality $2ab \leq \frac{a^2}{m} + mb^2$, $m > 0$, we deduce the existence of the constants C such that

$$\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{\boldsymbol{X}_1}^2 \leq C\left\|(\boldsymbol{u}(h), \bar{\varphi}(h))\right\|_{\boldsymbol{X}_{0,w}} + C. \tag{3.68}$$

On the other hand, taking $(\boldsymbol{\tau}, \boldsymbol{d}) = \big(\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)\big) \in \boldsymbol{X}_1$ in equation (3.53) we have

$$\begin{aligned} &- b_H\left((\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))\right) = -b_H\left((\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)), (\boldsymbol{u}(h), \varphi(h) - \hat{\varphi}(h))\right) \\ &= a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h))\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h))\right) \\ &\quad + h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h))\right) + b_H\left((\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)), (\boldsymbol{0}, \hat{\varphi}(h))\right), \end{aligned} \tag{3.69}$$

and taking into account the fact that $0 < h \leq 1$ we guarantee the existence of a positive constant C satisfying

$$\begin{aligned} &|-b_H((\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h)))| \\ &\leq C\left\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\right\|_{\boldsymbol{X}_1}\|(\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h))\|_{\boldsymbol{X}_1} + C\|(\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h))\|_{\boldsymbol{X}_1} \\ &\leq \left[C\left\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\right\|_{\boldsymbol{X}_1} + C\right]\|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}}. \end{aligned} \tag{3.70}$$

Expliciting in (3.70) the bilinear form $b(\cdot, \cdot)$ and using the inequalities of Korn and

Poincaré we have

$$
\begin{aligned}
-b_H((\boldsymbol{e}(h), \bar{\boldsymbol{E}}(h)), (\boldsymbol{u}(h),\bar{\varphi}(h))) &= \int_\Omega e_{ij}(\boldsymbol{u}(h))e_{ij}(\boldsymbol{u}(h))\, d\boldsymbol{x} + \int_\Omega E_k(\bar{\varphi}(h))E_k(\bar{\varphi}(h))\, d\boldsymbol{x} \\
&= \|\boldsymbol{e}(\boldsymbol{u}(h))\|_{0,\Omega}^2 + \|\nabla\bar{\varphi}(h))\|_{0,\Omega}^2 \\
&\geq C\|\boldsymbol{u}(h)\|_{1,\Omega} + C\|\bar{\varphi}(h))\|_{1,\Omega} \geq C\|(\boldsymbol{u}(h),\bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}}. \qquad (3.71)
\end{aligned}
$$

From (3.70) and (3.71) we deduce

$$
\|(\boldsymbol{u}(h),\bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq C\,\|(\boldsymbol{S}(h),\boldsymbol{T}(h))\|_{\boldsymbol{X}_1} + C. \qquad (3.72)
$$

Combining (3.71) and (3.72) we obtain that

$$
\|(\boldsymbol{S}(h),\boldsymbol{T}(h))\|_{\boldsymbol{X}_1} \leq C, \qquad \|(\boldsymbol{u}(h),\bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq C. \qquad (3.73)
$$

Finally, we have:

$$
\begin{aligned}
\|(\boldsymbol{u}(h),\varphi(h))\|_{\boldsymbol{X}_{2,w}} &= \|(\boldsymbol{u}(h),\bar{\varphi}(h)) + (\boldsymbol{0},\hat{\varphi}(h))\|_{\boldsymbol{X}_{2,w}} \\
&\leq C\|(\boldsymbol{u}(h),\bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} + C\|\hat{\varphi}\|_{H^1(0,L)}. \qquad (3.74)
\end{aligned}
$$

■

Corollary 2 *There exists a subsequence, still parameterized by h, and there exist $\boldsymbol{u} \in V_{0,w}$, $\boldsymbol{\Sigma} \in [L^2(\Omega)]_s^9$, $\varphi \in L^2(\Omega)$ and $\boldsymbol{\mathfrak{D}} \in [L^2(\Omega)]^3$, such that the following weak convergence hold when h tends to zero:*

$$
\sigma_{33}(h) \rightharpoonup \Sigma_{33}, \qquad h\sigma_{\alpha 3}(h) \rightharpoonup \Sigma_{\alpha 3}, \qquad h^2\sigma_{\alpha\beta}(h) \rightharpoonup \Sigma_{\alpha\beta}, \qquad \text{in } L^2(\Omega) \qquad (3.75)
$$

$$
hD_\alpha(h) \rightharpoonup \mathfrak{D}_\alpha, \qquad D_3(h) \rightharpoonup \mathfrak{D}_3, \qquad \text{in } L^2(\Omega) \qquad (3.76)
$$

$$
\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}, \qquad \text{in } V_{0,w}(\Omega), \qquad (3.77)
$$

$$
\varphi(h) \rightharpoonup \varphi, \qquad \text{in } \Psi_2(\Omega), \qquad (3.78)
$$

$$
\bar{\varphi}(h) \rightharpoonup \bar{\varphi} = \varphi - \hat{\varphi}, \qquad \text{in } \Psi_0(\Omega). \qquad (3.79)
$$

Moreover, the limits $\boldsymbol{\Sigma}$, $\boldsymbol{\mathfrak{D}}$, $\boldsymbol{u}$ and φ satisfy the following properties:

$$
e_{33}(\boldsymbol{u}) = \bar{C}_{33\theta\rho}\Sigma_{\theta\rho} + \bar{C}_{3333}\Sigma_{33} + \bar{P}_{333}\mathfrak{D}_3, \qquad (3.80)
$$

$$
e_{3\alpha}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) = 0, \qquad (3.81)
$$

$$E_3(\varphi) = -\bar{P}_{3\theta\rho}\Sigma_{\theta\rho} - \bar{P}_{333}\Sigma_{33} + \bar{\varepsilon}_{33}\mathfrak{D}_3, \tag{3.82}$$

$$E_\alpha(\varphi) = 0, \tag{3.83}$$

$$\bar{C}_{\alpha\beta\theta\rho}\Sigma_{\theta\rho} + \bar{C}_{\alpha\beta33}\Sigma_{33} + \bar{P}_{3\alpha\beta}\mathfrak{D}_3 = 0, \tag{3.84}$$

$$2\bar{C}_{3\alpha3\theta}\Sigma_{3\theta} + \bar{P}_{\theta3\alpha}\mathfrak{D}_\theta = 0, \tag{3.85}$$

$$\bar{\varepsilon}_{\theta\alpha}\mathfrak{D}_\theta - 2\bar{P}_{\theta3\alpha}\Sigma_{3\alpha} = 0. \tag{3.86}$$

$$\int_\omega \Sigma_{\alpha\beta}e_{\alpha\beta}(\boldsymbol{v})d\omega = 0, \;\; \forall\, \boldsymbol{v} = (v_1, v_2, 0) \in V_{0,w}(\Omega), \tag{3.87}$$

$$\int_\omega \Sigma_{\alpha\beta}d\omega = \int_\omega x_\gamma \Sigma_{\alpha\beta}d\omega = 0, \tag{3.88}$$

$$\int_\omega \mathfrak{D}_3 E_3(\psi)d\omega = 0, \quad \forall\, \psi \in H_0^1(0, L), \tag{3.89}$$

Proof. We immediately see from the previous theorem, the existence of a subsequence, still indexed by h, and there exists an element denoted by $(\boldsymbol{\Sigma}, \boldsymbol{\mathfrak{D}}, \boldsymbol{u}, \varphi)$ such that convergences (3.75)-(3.79) hold. Passing equation (3.53) to the limit as h goes to zero, we obtain the relations (3.80)-(3.83) in $L^2(\Omega)$. To derive (3.84) we take $\tau_{33} = d_3 = 0$ in the first equation of the mixed problem (3.53), multiplying by h^{-2} and passing to the limit taking into account that $e_{3\alpha}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) = E_\alpha(\varphi) = 0$. Now, we choose $\tau_{33} = \tau_{\alpha\beta} = d_3 = 0$ in (3.53). Multiplying by h^{-1} and passing to the limit we deduce (3.85) and (3.86). To prove (3.87) we multiply equation (3.51) by h^2 and pass to the limit as h goes to zero. Choosing appropriate test functions $v_i \in V_{0,w}$ in (3.87) we obtain immediately (3.88) (see Trabucho & Viaño [1996]). The properties (3.89) are obtained from (3.51) choosing $\boldsymbol{v} = \boldsymbol{0}$ and passing to the limit for appropriate Ψ_0. ■

From (3.81) we have that the limit displacement $\boldsymbol{u}$ is a Bernoulli-Navier displacement, i.e.:

$$V_{BN}(\Omega) = \{\boldsymbol{u} \in V_{0,w} : e_{\alpha\beta}(\boldsymbol{u}) = e_{3\alpha}(\boldsymbol{u}) = 0\}. \tag{3.90}$$

In Trabucho & Viaño [1996] this space was characterized by the following equivalent definition:

$$\begin{aligned} V_{BN} := V_{BN}(\Omega) = \big\{ &\boldsymbol{v} = (v_i) : v_\alpha(x_1, x_2, x_3) = \chi_\alpha(x_3) \in V_0^2(0, L), \\ &v_3(x_1, x_2, x_3) = \chi_3(x_3) - x_\beta\chi'_\beta(x_3),\; \chi_3 \in V_0^1(0, L)\big\}, \end{aligned} \tag{3.91}$$

where

$$V_0^1(0,L) = \left\{\eta \in H^1(0,L) : \eta(0) = 0\right\}, \tag{3.92}$$

$$V_0^2(0,L) = \left\{\eta \in H^2(0,L) : \eta(0) = \eta'(0) = 0\right\}. \tag{3.93}$$

Consequently, we have

$$\begin{cases} u_\alpha(x_1,x_2,x_3) = \xi_\alpha(x_3), & \xi_\alpha \in V_0^2(0,L), \\ u_3(x_1,x_2,x_3) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3), & \xi_3 \in V_0^1(0,L). \end{cases} \tag{3.94}$$

Now, we define Ψ^3 to be the space of the electric potential satisfying the condition (3.83), i.e.,

$$\Psi^3 := \Psi^3(\Omega) = \left\{\psi \in H^1(\Omega) : E_\alpha(\psi) = 0\right\}. \tag{3.95}$$

From (3.83) we deduce that φ only depends on variable x_3, that is,

$$\varphi(x_1,x_2,x_3) = z_3(x_3), \quad z_3 \in H^1(0,L). \tag{3.96}$$

By other hand, $\varphi = \hat{\varphi}$ on $\Gamma_0 \cup \Gamma_L = \Gamma_{eD}$, which give us (see (3.42))

$$z_3(0) = \varphi(x_1,x_2,0) = \hat{\varphi}(0) = \varphi_0^{1,0}, \ (x_1,x_2) \in \omega, \tag{3.97}$$

$$z_3(L) = \varphi(x_1,x_2,L) = \hat{\varphi}(L) = \varphi_0^{1,L}, \ (x_1,x_2) \in \omega. \tag{3.98}$$

with

$$\varphi_0^{1,0} \quad \text{and} \quad \varphi_0^{1,L} \quad \text{constants.} \tag{3.99}$$

In this case, φ has the following from

$$\varphi(x_1,x_2,x_3) = z_3(x_3), \quad z_3 \in H^1(0,L), \quad z_3(0) = \varphi_0^{1,0}, \quad z_3(L) = \varphi_0^{1,L}. \tag{3.100}$$

Obviously, it is possible to give an equivalent definition of Ψ^3 as follows

$$\begin{aligned} \Psi^3 = \hat{\varphi} + \Psi_0^3(\Omega) &= \hat{\varphi} + \left\{\psi \in H^1(\Omega) : \psi(x_1,x_2,x_3) = z_3(x_3), \ z_3 \in H_0^1(0,L)\right\}, \\ &= \left\{\psi(x_1,x_2,x_3) = z_3(x_3), \ z_3 \in H^1(0,L) : \psi - \hat{\varphi} \in \Psi_0^3\right\}. \end{aligned} \tag{3.101}$$

Consequently, we have

$$\varphi \in \hat{\varphi} + \Psi_0^3(\Omega) \tag{3.102}$$

$$\bar{\varphi}(x_1, x_2, x_3) = \varphi(x_1, x_2, x_3) - \hat{\varphi}(x_3) = z_3(x_3) - \frac{1}{L}(L - x_3)\varphi_0^{1,0} - \frac{1}{L}x_3\varphi_0^{1,L}, \tag{3.103}$$

$$E_3(\varphi) = -\partial_3\varphi = -z_3', \tag{3.104}$$

$$E_3(\bar{\varphi}) = -\partial_3\bar{\varphi} = -z_3' + \frac{1}{L}(\varphi_0^{1,L} - \varphi_0^{1,0}). \tag{3.105}$$

Theorem 3 *Let us that the beam is made of a class 2 piezoelectric material whose coefficients* $A_{33}^c A_{\alpha\beta}^c$ *and* $\frac{1}{\bar{\varepsilon}_{33}}\left(\bar{P}_{3\alpha\beta} - A_{33}^c \bar{P}_{333} A_{\alpha\beta}^c\right)$ *do not depend on* x_α. *The limit* $(\boldsymbol{u}, \varphi)$ *is the solution of the following problem:*

$$\begin{aligned}
&\int_\Omega A_{33}^c \left[\bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) - \bar{P}_{333} E_3(\varphi)\right] e_{33}(\boldsymbol{v}) d\boldsymbol{x} \\
&+ \int_\Omega A_{33}^c \left[\bar{P}_{333} e_{33}(\boldsymbol{u}) + \bar{C}_{3333} E_3(\varphi)\right] E_3(\psi) d\boldsymbol{x} \\
&= \int_\Omega f_i\, v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i\, v_i\, d\Gamma + \int_{\Gamma_L} p_i\, v_i\, d\Gamma, \qquad \forall (\boldsymbol{v}, \psi) \in V_{BN} \times \Psi_0^3
\end{aligned} \tag{3.106}$$

where

$$A_{33}^c = \frac{1}{\bar{\varepsilon}_{33}\bar{C}_{3333} + \bar{P}_{333}\bar{P}_{333}}.$$

Proof. Taking now $\boldsymbol{v} \in V_{BN}(\Omega)$ and $\psi \in \Psi_0^3(\Omega)$ in second equation of the mixed problem (3.53) we obtain

$$\int_\Omega \sigma_{33}(h)\, e_{33}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega D_3(h)\, E_3(\psi) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma + \int_{\Gamma_L} p_i v_i d\Gamma,$$

and passing to the limit when $h \to 0$, we get

$$\int_\Omega \Sigma_{33} e_{33}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3\, E_3(\psi) d\boldsymbol{x} = \int_\Omega f_i\, v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i\, v_i\, d\Gamma d + \int_{\Gamma_L} p_i\, v_i\, d\Gamma, \tag{3.107}$$

for all $(\boldsymbol{v}, \psi) \in V_{BN} \times \Psi_0^3$. Using (3.80) and (3.82), one has

$$\Sigma_{33} = A_{33}^c \left[\bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) - \bar{P}_{333} E_3(\varphi) - A_{\alpha\beta}^c \Sigma_{\alpha\beta}\right], \tag{3.108}$$

$$\mathcal{D}_3 = A_{33}^c \bar{P}_{333} e_{33}(\boldsymbol{u}) + A_{33}^c \bar{C}_{3333} E_3(\varphi) - \frac{1}{\bar{\varepsilon}_{33}}\left[\bar{P}_{3\theta\rho} - A_{33}^c \bar{P}_{333} A_{\alpha\beta}^c\right] \Sigma_{\alpha\beta}, \tag{3.109}$$

$$A_{33}^c = \frac{1}{\bar{\varepsilon}_{33}\bar{C}_{3333} + \bar{P}_{333}\bar{P}_{333}}, \qquad A_{\alpha\beta}^c = \bar{\varepsilon}_{33}\bar{C}_{33\alpha\beta} + \bar{P}_{333}\bar{P}_{3\alpha\beta}. \tag{3.110}$$

Putting these expressions into (3.107) we obtain

$$\begin{aligned}
&\int_\Omega A_{33}^c(\bar{\varepsilon}_{33}e_{33}(\boldsymbol{u}) - \bar{P}_{333}E_3(\varphi))e_{33}(\boldsymbol{v})d\boldsymbol{x} \\
&- \int_0^L \left(\int_\omega A_{33}^c A_{\alpha\beta}^c \Sigma_{\alpha\beta} d\omega\right) \zeta_3' dx_3 + \int_0^L \left(\int_\omega A_{33}^c A_{\alpha\beta}^c\, x_\gamma \Sigma_{\alpha\beta} d\omega\right) \zeta_\gamma'' dx_3 \\
&\qquad + \int_\Omega A_{33}^c(\bar{P}_{333}e_{33}(\boldsymbol{u}) + \bar{C}_{3333}E_3(\varphi))E_3(\psi) \\
&\qquad + \int_0^L \left[\int_\omega \frac{1}{\bar{\varepsilon}_{33}} \left(\bar{P}_{3\alpha\beta} - A_{33}^c \bar{P}_{333} A_{\alpha\beta}^c\right) \Sigma_{\alpha\beta} d\omega\right] q_3' dx_3 \\
&= \int_\Omega f_i\, v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i\, v_i\, d\Gamma + \int_{\Gamma_L} p_i\, v_i\, d\Gamma,
\end{aligned}$$

Then, if we use conditions (3.88) and the fact that the coefficients are independent of x_α, we get (3.106). ■

Corollary 3 *Any function $(\boldsymbol{u}, \varphi)$ solves (3.106) if and only if it has the form*

$$u_\alpha(x_1, x_2, x_3) = \xi_\alpha(x_3), \qquad u_3(x_1, x_2, x_3) = \xi_3(x_3) - x_\beta \xi_\beta'(x_3), \tag{3.111}$$

$$\varphi(x_1, x_2, x_3) = z_3(x_3), \tag{3.112}$$

with $\xi_\alpha \in V_0^2(0, L)$, $\xi_3 \in V_0^1(0, L)$ and $z_3 \in H^1(0, L)$, $z_3(0) = \varphi_0^{1,0}$, $z_3(L) = \varphi_0^{1,L}$, and ξ_i and z_3 solving the following variational problem (no sum on α):

$$\left\{\begin{aligned}
&\int_0^L I_\alpha A_{33}^c \bar{\varepsilon}_{33} \xi_\alpha'' \zeta_\alpha''\, dx_3 = \int_0^L (\int_\omega f_\alpha\, d\omega + \int_{\gamma_N} g_\alpha\, d\gamma)\zeta_\alpha dx_3 \\
&\qquad - \int_0^L (\int_\omega x_\alpha f_3\, d\omega + \int_{\gamma_N} x_\alpha g_3\, d\gamma)\zeta_\alpha' dx_3 + (\int_\omega p_\alpha\, d\omega)\zeta_\alpha(L) \\
&\qquad -(\int_\omega x_\alpha p_3\, d\omega)\zeta_\alpha'(L), \qquad \textit{for all } \zeta_\alpha \in V_0^2(0, L)
\end{aligned}\right. \tag{3.113}$$

$$\left\{\begin{aligned}
&\int_0^L A(\omega) A_{33}^c [(\bar{\varepsilon}_{33}\xi_3' + \bar{P}_{333} z_3')\zeta_3' - (\bar{P}_{333}\xi_3' - \bar{C}_{3333} z_3') q_3'] dx_3 \\
&= \int_0^L (\int_\omega f_3 d\omega + \int_{\gamma_N} g_3 d\gamma)\zeta_3 dx_3 + (\int_\omega p_3 d\omega)\zeta_3(L), \\
&\forall \zeta_3 \in V_0^1(0, L),\ q_3 \in H_0^1(0, L).
\end{aligned}\right. \tag{3.114}$$

We remark that the limit model, expressed by equations (3.113) and (3.114), is now written over $(0, L)$ as was considered initially. We note that the flexion model (3.113) is independent of the electric potential, and therefore can be compared with the classical bending model for an elastic beam. By other hand, the axial displacement is coupled with electrical potential as can be see in variational problem (3.114).

Our study could stop at this stage, however we intend to prove, in the next section, the strong convergence and, for that, we need to obtain additional information about limits mentioned in Corollary 2, using the asymptotic expansions method.

3.4 The method of formal asymptotic expansions: the displacement - electric potential approach

In order to complete this study, and be able to establish the strong convergence of sequence $(\boldsymbol{u}(h), \varphi(h))_{h>0}$, we next assume that the solution of the problem (3.46)-(3.50) can be expressed in the form

$$(\boldsymbol{u}(h), \varphi(h)) = (\boldsymbol{u}^0, \varphi^0) + h^2\,(\boldsymbol{u}^2, \varphi^2) + \text{h.o.t.}, \tag{3.115}$$

where

$$\left(\boldsymbol{u}^0, \varphi^0 - \hat{\varphi}\right) \in V_{0,w} \times \Psi_0,\ (\boldsymbol{u}^p, \varphi^p) \in V_{0,w} \times \Psi_0,\ p \geq 1, \tag{3.116}$$

and the successive coefficients of the powers of h are independent of h. We note that only the leading term is required to satisfy the electric boundary condition found in definition of the space $V_{0,w} \times \Psi_0$.

The assumption (3.115) induces the following expansion (see (3.52))

$$(\boldsymbol{\sigma}(h), \boldsymbol{D}(h)) := h^{-4}(\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-2}) + h^{-2}(\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-2}) + (\boldsymbol{\sigma}^0, \boldsymbol{D}^0) + \cdots, \tag{3.117}$$

where the tensor fields $(\boldsymbol{\sigma}^q, \boldsymbol{D}^q)$, $q \geq -4$, are independent of h.

Applying the *displacements - electric potential* approach to the scaled principle of virtual work (3.51)-(3.52), we try to characterize the terms $\boldsymbol{u}^0$, $\boldsymbol{u}^2$ and φ^0 in the formal expansions in order to compute $(\boldsymbol{u}^0, \varphi^0)$.

3.4.0.1 Cancelation of the factors of h^q, $-4 \leq q \leq 0$, in the scaled three-dimensional problem.

In this section we show that the factors of h^q, $-4 \leq q \leq 0$ disappear, and therefore the formal expansions (3.117) do not contain any negative terms. For the sake of clarity, the proof is divided in five parts.

From (3.115)-(3.117) and (3.52) we have

$$\sigma_{\alpha\beta}(h) = h^{-4}\sigma^{-4}_{\alpha\beta} + h^{-2}\sigma^{-2}_{\alpha\beta} + \sigma^{0}_{\alpha\beta} + \text{h.o.t..},$$

$$\sigma_{3\alpha}(h) = h^{-2}\sigma^{-2}_{3\alpha} + \sigma^{0}_{3\alpha} + \text{h.o.t.},$$

$$\sigma_{33}(h) = h^{-2}\sigma^{-2}_{33} + \sigma^{0}_{3\alpha} + \text{h.o.t.},$$

$$D_{\alpha}(h) = h^{-2}D^{-2}_{\alpha} + D^{0}_{\alpha} + \text{h.o.t.},$$

$$D_{3}(h) = h^{-2}D^{-2}_{3} + D^{0}_{3} + \text{h.o.t.},$$

with

$$\sigma^{-4}_{\alpha\beta} = C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^0), \tag{3.118}$$

$$\sigma^{-2}_{\alpha\beta} = C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^0) - P_{3\alpha\beta}E_3(\varphi^0), \tag{3.119}$$

$$\sigma^{0}_{\alpha\beta} = C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^4) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^2) - P_{3\alpha\beta}E_3(\varphi^2), \tag{3.120}$$

$$\sigma^{2p}_{\alpha\beta} = C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^{2p+4}) + C_{\alpha\beta33}e_{33}(\boldsymbol{u}^{2p+2}) - P_{3\alpha\beta}E_3(\varphi^{2p+2}), \quad p \geq 1, \tag{3.121}$$

$$\sigma^{-2}_{3\alpha} = 2C_{3\alpha3\beta}e_{3\beta}(\boldsymbol{u}^0) - P_{\theta3\alpha}E_\theta(\varphi^0), \tag{3.122}$$

$$\sigma^{0}_{3\alpha} = 2C_{\alpha33\theta}e_{3\theta}(\boldsymbol{u}^2) - P_{\theta3\alpha}E_\theta(\varphi^2), \tag{3.123}$$

$$\sigma^{2p}_{3\alpha} = 2C_{\alpha33\theta}e_{3\theta}(\boldsymbol{u}^{2p+2}) - P_{\theta3\alpha}E_\theta(\varphi^{2p+2}),\ p \geq 1, \tag{3.124}$$

$$\sigma^{-2}_{33} = C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^0), \tag{3.125}$$

$$\sigma^{0}_{33} = C_{33\theta\rho}e_{\theta\rho}(\boldsymbol{u}^2) + C_{3333}e_{33}(\boldsymbol{u}^0) - P_{333}E_3(\varphi^0), \tag{3.126}$$

$$\sigma^{2p}_{33} = C_{33\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^{2p+2}) + C_{3333}e_{33}(\boldsymbol{u}^{2p}) - P_{333}E_3(\varphi^{2p}),\ p \geq 1, \tag{3.127}$$

$$D^{-2}_{3} = P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^0), \tag{3.128}$$

$$D^{0}_{3} = P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^2) + P_{333}e_{33}(\boldsymbol{u}^0) + \varepsilon_{33}E_3(\varphi^0), \tag{3.129}$$

$$D^{2p}_{3} = P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^{p+2}) + P_{333}e_{33}(\boldsymbol{u}^{2p}) + \varepsilon_{33}E_3(\varphi^{2p}), \quad p \geq 1, \tag{3.130}$$

$$D^{-2}_{\alpha} = 2P_{\theta3\alpha}e_{3\alpha}(\boldsymbol{u}^0) + \varepsilon_{\alpha\beta}E_\beta(\varphi^0), \tag{3.131}$$

$$D^{0}_{\alpha} = 2P_{\theta3\alpha}e_{3\alpha}(\boldsymbol{u}^2) + \varepsilon_{\theta\alpha}E_\alpha(\varphi^2) \tag{3.132}$$

$$D^{2p}_{\alpha} = 2P_{\theta3\alpha}e_{3\alpha}(\boldsymbol{u}^{2p}) + \varepsilon_{\theta\alpha}E_\alpha(\varphi^{2p}), \quad p \geq 1, \tag{3.133}$$

From (3.51) we have

$$\begin{aligned} h^{-4}\int_\Omega \sigma^{-4}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\,d\boldsymbol{x} + h^{-2}\left[\int_\Omega \sigma^{-2}_{ij} e_{ij}(\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega D^{-2}_k E_k(\psi)\,d\boldsymbol{x}\right] \\ + \left[\int_\Omega \sigma^{0}_{ij} e_{ij}(\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega D^{0}_k E_k(\psi)\,d\boldsymbol{x}\right] + \text{h.o.t.} \\ = \int_\Omega f_i v_i\,d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\,d\Gamma + \int_{\Gamma_L} p_i v_i\,d\Gamma, \qquad \forall\ (\boldsymbol{v},\psi)\in V_{0,w}\times\Psi_0. \end{aligned} \tag{3.134}$$

It follows that

$$\int_\Omega \sigma^{-4}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\,d\boldsymbol{x} = \int_\Omega C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^0) e_{\alpha\beta}(\boldsymbol{v})\,d\boldsymbol{x}, \quad \text{for all } \boldsymbol{v}\in V_{0,w}. \tag{3.135}$$

Taking $\boldsymbol{v} = \boldsymbol{u}^0$, we deduce from the coerciveness of $\boldsymbol{C}$, that

$$e_{\theta\rho}(\boldsymbol{u}^0) = 0, \tag{3.136}$$

and therefore

$$\sigma^{-4}_{\alpha\beta} = \sigma^{-2}_{33} = D^{-2}_3 = 0. \tag{3.137}$$

Consequently, equation (3.134) becomes, from (3.136) and (3.137),

$$2\int_\Omega \sigma^{-2}_{3\alpha} e_{3\alpha}(\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega \sigma^{-2}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\,d\boldsymbol{x} + \int_\Omega D^{-2}_\theta E_\theta(\psi)\,d\boldsymbol{x} = 0, \tag{3.138}$$

$$\text{for all } (\boldsymbol{v},\psi)\in V_{0,w}\times\Psi_0. \tag{3.139}$$

Putting $\boldsymbol{v} = \boldsymbol{u}^0 \in V_{0,w}$ and $\psi = \varphi^0 - \hat{\varphi} \in \Psi_0$ in (3.139) and combining the resulting equation with the constitutive laws (3.122) and (3.131), we deduce

$$\begin{aligned} 2\int_\Omega \left(2C_{3\alpha3\beta} e_{3\beta}(\boldsymbol{u}^0) - P_{\theta\alpha3} E_\theta(\varphi^0)\right) e_{3\alpha}(\boldsymbol{u}^0)\,d\boldsymbol{x} \\ + \int_\Omega \left(2P_{\alpha3\beta} e_{3\beta}(\boldsymbol{u}^0) + \varepsilon_{\alpha\beta} E_\beta(\varphi^0)\right) E_\theta(\varphi^0 - \hat{\varphi})\,d\boldsymbol{x} = 0. \end{aligned} \tag{3.140}$$

Since

$$\hat{\varphi}(x_3) = \frac{1}{L}(L - x_3)\varphi_0^{1,0} + \frac{1}{L} x_3 \varphi_0^{1,L},$$

with $\varphi_0^{1,0}$ and $\varphi_0^{1,L}$ constants, then the previous equation becomes

$$4\int_\Omega C_{3\alpha3\beta} e_{3\beta}(\boldsymbol{u}^0) e_{3\beta}(\boldsymbol{u}^0)\,d\boldsymbol{x} + \int_\Omega \varepsilon_{\alpha\beta} E_\beta(\varphi^0) E_\alpha(\varphi^0)\,d\boldsymbol{x} = 0. \tag{3.141}$$

From the coerciveness of $\boldsymbol{C}$ and $\boldsymbol{\varepsilon}$ we obtain

$$e_{3\beta}(\boldsymbol{u}^0) = 0, \tag{3.142}$$
$$\partial_\alpha \varphi^0 = 0. \tag{3.143}$$

Consequently, we deduce

$$\sigma_{3\alpha}^{-2} = D_\alpha^{-2} = 0, \tag{3.144}$$

and equation (3.139) becomes:

$$\int_\Omega \sigma_{\alpha\beta}^{-2} e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in V_{0,w}, \tag{3.145}$$

with

$$\sigma_{\alpha\beta}^{-2} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^0) - P_{3\alpha\beta} E_3(\varphi^0). \tag{3.146}$$

Condition (3.116) will allow us to prove that it is not possible to obtain a solution $\boldsymbol{u}^2$ of (3.145) in the space V_0 except for some particular cases. This fact is at the origin of a boundary layer phenomenon as seen in Trabucho & Viaño [1996], Irago [1999] and Irago & Viaño [2002].

Next, we prove that there exists $\tilde{\boldsymbol{u}} \in V_{0,w}$ such that

$$C_{\alpha\beta\theta\rho} e_{\theta\rho}(\tilde{\boldsymbol{u}}) + C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^0) - P_{3\alpha\beta} E_3(\varphi^0) = 0. \tag{3.147}$$

Then, from (3.119) we obtain

$$\sigma_{\alpha\beta}^{-2} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\tilde{\boldsymbol{u}} - \boldsymbol{u}^0), \tag{3.148}$$

and therefore, from (3.145),

$$\int_\Omega C_{\alpha\beta\theta\rho} \left(e_{\theta\rho}(\boldsymbol{u}^2 - \widetilde{\boldsymbol{u}}) \right) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0. \tag{3.149}$$

Taking $\boldsymbol{u} = \boldsymbol{u}^2 - \widetilde{\boldsymbol{u}}$ we conclude that $e_{\theta\rho}(\boldsymbol{u}^2 - \widetilde{\boldsymbol{u}}) = 0$, and finally, from (3.148):

$$\sigma_{\alpha\beta}^{-2} = 0. \tag{3.150}$$

We remark that from (3.119) and (3.150) we deduce that $\boldsymbol{u}^2$ is also solution of equation (3.147). In other words, $\sigma_{\alpha\beta}^{-2} = 0$ if and only if the following problem has at least one solution in $V_{0,w}$:

$$C_{\alpha\beta\theta\rho}e_{\theta\rho}(\tilde{\boldsymbol{u}}) = -C_{\alpha\beta33}e_{33}(\boldsymbol{u}^0) + P_{3\alpha\beta}E_3(\varphi^0), \tag{3.151}$$

that is,

$$\begin{pmatrix} C_{1111} & 2C_{1112} & C_{1122} \\ C_{1211} & 2C_{1212} & C_{1222} \\ C_{2211} & 2C_{2212} & C_{2222} \end{pmatrix} \begin{pmatrix} e_{11}(\boldsymbol{u}^2) \\ e_{12}(\boldsymbol{u}^2) \\ e_{22}(\boldsymbol{u}^2) \end{pmatrix} = \begin{pmatrix} -C_{1133}e_{33}(\boldsymbol{u}^0) + P_{311}E_3(\varphi^0) \\ -C_{1233}e_{33}(\boldsymbol{u}^0) + P_{312}E_3(\varphi^0) \\ -C_{2233}e_{33}(\boldsymbol{u}^0) + P_{322}E_3(\varphi^0) \end{pmatrix}. \tag{3.152}$$

By a simple inversion of the matrix we have

$$\begin{pmatrix} e_{11}(\boldsymbol{u}^2) \\ e_{12}(\boldsymbol{u}^2) \\ e_{22}(\boldsymbol{u}^2) \end{pmatrix} = \begin{pmatrix} \tilde{C}_{1111} & \tilde{C}_{1112} & \tilde{C}_{1122} \\ \tilde{C}_{1211} & \tilde{C}_{1212} & \tilde{C}_{1222} \\ \tilde{C}_{2211} & \tilde{C}_{2212} & \tilde{C}_{2222} \end{pmatrix} \begin{pmatrix} -C_{1133}e_{33}(\boldsymbol{u}^0) + P_{311}E_3(\varphi^0) \\ -C_{1233}e_{33}(\boldsymbol{u}^0) + P_{312}E_3(\varphi^0) \\ -C_{2233}e_{33}(\boldsymbol{u}^0) + P_{322}E_3(\varphi^0) \end{pmatrix}, \tag{3.153}$$

where

$$\begin{pmatrix} \tilde{C}_{1111} & \tilde{C}_{1112} & \tilde{C}_{1122} \\ \tilde{C}_{1211} & \tilde{C}_{1212} & \tilde{C}_{1222} \\ \tilde{C}_{2211} & \tilde{C}_{2212} & \tilde{C}_{2222} \end{pmatrix} = \begin{pmatrix} C_{1111} & 2C_{1112} & C_{1122} \\ C_{1211} & 2C_{1212} & C_{1222} \\ C_{2211} & 2C_{2212} & C_{2222} \end{pmatrix}^{-1}. \tag{3.154}$$

In this way, we have established that

$$\begin{aligned} e_{\alpha\beta}(\boldsymbol{u}^2) = \tilde{C}_{\alpha\beta\rho\rho}\left[P_{3\rho\rho}E_3(\varphi^0) - C_{\rho\rho33}e_{33}(\boldsymbol{u}^0)\right] \\ + \tilde{C}_{\alpha\beta12}\left[P_{312}E_3(\varphi^0) - C_{1233}e_{33}(\boldsymbol{u}^0)\right]. \end{aligned} \tag{3.155}$$

From (3.136) and (3.142) we have that $\boldsymbol{u}^0$ is a Bernoulli-Navier displacement, i.e., $\boldsymbol{u}^0 \in V_{BN}(\Omega)$ (see (3.90) and (3.91)). Then, there exist functions ξ_α and ξ_3 depending only on variable x_3 such that:

$$\begin{cases} u_\alpha(x_1, x_2, x_3) = \xi_\alpha(x_3), & \xi_\alpha \in V_0^2(0, L), \\ u_3(x_1, x_2, x_3) = \xi_3(x_3) - x_\alpha\xi'_\alpha(x_3), & \xi_3 \in V_0^1(0, L). \end{cases} \tag{3.156}$$

By consequence:

$$e_{33}(\boldsymbol{u}^0) = \xi'_3 - x_\alpha\xi''_\alpha. \tag{3.157}$$

On the other hand, from (3.143) we deduce that φ^0 depends only on variable x_3, that is,

$$\varphi^0(x_1, x_2, x_3) = z_3(x_3), \quad z_3 \in H^1(0, L). \tag{3.158}$$

Since $\varphi^0 = \hat{\varphi}$ on $\Gamma_{eD} = \Gamma_0 \cup \Gamma_L$, then (cf. (3.99))

$$z_3(0) = \varphi(x_1, x_2, 0) = \hat{\varphi}(0) = \varphi_0^{1,0}, \quad z_3(L) = \varphi(x_1, x_2, L) = \hat{\varphi}(x_1, x_2, L) = \varphi_0^{1,L}. \tag{3.159}$$

Consequently, φ^0 has the following form:

$$\varphi^0(x_1, x_2, x_3) = z_3(x_3), \quad z_3 \in H^1(0, L), \quad z_3(0) = \varphi_0^{1,0}, \; z_3(L) = \varphi_0^{1,L}, \tag{3.160}$$

and

$$\bar{\varphi}^0(x_1, x_2, x_3) = \varphi^0(x_1, x_2, x_3) - \hat{\varphi}(x_3) = z_3(x_3) - \frac{1}{L}(L - x_3)\varphi_0^{1,0} - \frac{1}{L}x_3\varphi_0^{1,L}, \tag{3.161}$$

$$E_3(\varphi) = -\partial_3\varphi = -z_3', \tag{3.162}$$

$$E_3(\bar{\varphi}) = -\partial_3\bar{\varphi} = -z_3' + \frac{1}{L}(\varphi_0^{1,L} - \varphi_0^{1,0}). \tag{3.163}$$

Then, expression (3.155) becomes

$$\begin{aligned} e_{\alpha\beta}(\boldsymbol{u}^2) = {} & \tilde{C}_{\alpha\beta\rho\rho}\left[-P_{3\rho\rho}z_3' - C_{\rho\rho 33}(\xi_3' - x_\beta\xi_\beta'')\right] \\ & + \tilde{C}_{\alpha\beta 12}\left[-P_{312}z_3' - C_{1233}(\xi_3' - x_\beta\xi_\beta'')\right] \end{aligned} \tag{3.164}$$

In order to characterize the transverse components of $\boldsymbol{u}^2$, we begin by introducing the following constants and functions characterizing the geometry of the cross section ω.

(a) The constants I_α are given by

$$I_\alpha = \int_\omega x_\alpha^2 \, d\omega.$$

(b) For each $(x_1, x_2) \in \omega$ we define the components of the matrices

$$\boldsymbol{\Lambda}(x_1, x_2) = (\Lambda_\alpha)(x_1, x_2), \;\; \bar{\boldsymbol{\Lambda}}(x_1, x_2) = (\bar{\Lambda}_\alpha)(x_1, x_2), \;\; \boldsymbol{z}(x_1, x_2) = (z_{\alpha\beta})(x_1, x_2)$$

as follows

$$\Lambda_1 = x_1(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233}) + x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}),$$

$$\Lambda_2 = x_1(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_2(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233}),$$

$$\bar{\Lambda}_1 = x_1(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) + x_2(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312}),$$

$$\bar{\Lambda}_2 = x_2(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) + x_1(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312}),$$

$$z_{11} = \frac{x_1^2}{2}(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233}) - \frac{x_2^2}{2}(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}),$$

$$z_{12} = 2x_1x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_1x_2(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233}),$$

$$z_{21} = 2x_1x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_1x_2(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233}),$$

$$z_{22} = \frac{x_2^2}{2}(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233}) - \frac{x_1^2}{2}(\tilde{C}_{11\rho\rho}P_{3\rho\rho} + \tilde{C}_{1112}P_{312}).$$

(c) Constants $X_{\alpha\beta}$, Y_α, Z and L are defined by

$$X_{\alpha\beta} = \int_\omega z_{\alpha\beta}\, d\omega, \qquad Y_\alpha = \int_\omega z_{\alpha\beta}\delta_\beta\, d\omega,$$
$$Z = \int_\omega \Lambda_\alpha\delta_\alpha\, d\omega, \qquad L = \int_\omega \bar{\Lambda}_\alpha\delta_\alpha\, d\omega.$$

Theorem 4 *Let $\boldsymbol{u}^0 \in V_{BN}$ and $\varphi^0 \in \hat{\varphi} + \Psi_3^0$ be given by (3.156) and (3.160), respectively, and if $\xi_\alpha \in H^3(0,L)$ and $\xi_3, z_3 \in H^2(0,L)$, then every element $u_\alpha^2 \in W_2$ is of the form*

$$\widetilde{u}_\alpha = s_\alpha + \delta_\alpha s + z_{\alpha\beta}\xi_\beta'' - \Lambda_\alpha\xi_3' - \bar{\Lambda}_\alpha z_3', \tag{3.165}$$

where s, $s_\alpha \in H^1(0,L)$ are such that

$$\begin{aligned} s_\alpha(0) &= -\frac{1}{A(\omega)}X_{\alpha\beta}\xi_\beta''(0)\,, \\ s(0) &= -\frac{1}{I_1+I_2}\left(Y_\beta\xi_\beta''(0) - Z\xi_3'(0) - Lz_3'(0)\right). \end{aligned} \tag{3.166}$$

Proof. From (3.147) and (3.155) we have

$$\partial_1 u_1^2 = -z_3'(\tilde{C}_{11\rho\rho}P_{3\rho\rho} + \tilde{C}_{1112}P_{312}) - (\xi_3' - x_\beta\xi_\beta'')(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233})$$

$$\partial_2 u_2^2 = -z_3'(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) - (\xi_3' - x_\beta\xi_\beta'')(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233}),$$

$$\partial_1 u_2^2 + \partial_2 u_1^2 = -2z_3'(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312}) - 2(\xi_3' - x_\beta\xi_\beta'')(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}).$$

A direct integration of the first two equations gives

$$\begin{aligned}\widetilde{u}_1 &= k_1(x_2,x_3) - (x_1\xi_3' - \tfrac{x_1^2}{2}\xi_1'' - x_1x_2\xi_2'')(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233})\\ &\qquad -z_3'x_1(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}),\\ \widetilde{u}_2 &= k_2(x_1,x_3) - (x_2\xi_3' - x_1x_2\xi_1'' - \tfrac{x_2^2}{2}\xi_2'')(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233})\\ &\qquad -z_3'x_2(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}).\end{aligned} \tag{3.167}$$

Substituting these expressions in the third equation, we show that there exists a function s, depending only on x_3, such that

$$\begin{aligned}\partial_2 k_1(x_2,x_3) &= s - x_2\xi_1''(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) + 2x_1\xi_2''(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233})\\ &\quad - z_3'(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312}) - \xi_3'(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}),\\ \partial_1 k_2(x_1,x_3) &= -s - x_1\xi_2''(\tilde{C}_{11\rho\rho}P_{3\rho\rho} + \tilde{C}_{1112}P_{312}) + 2x_2\xi_1''(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233})\\ &\quad - z_3'(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312}) - \xi_3'(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}).\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{u}_1 = s_1 + sx_2 &- \Big[x_1(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) + x_2(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312})\Big]\, z_3'\\ &+ \left[\frac{x_1^2}{2}(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233}) - \frac{x_2^2}{2}(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312})\right]\xi_1''\\ &+ \Big[2x_1x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_1x_2(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233})\Big]\,\xi_2''\\ &- \Big[x_1(\tilde{C}_{11\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1112}C_{1233}) + x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233})\Big]\,\xi_3',\end{aligned}$$

$$\begin{aligned}\widetilde{u}_2 = s_2 - sx_1 &- \Big[x_2(\tilde{C}_{22\rho\rho}P_{3\rho\rho} + \tilde{C}_{2212}P_{312}) + x_1(\tilde{C}_{12\rho\rho}P_{3\rho\rho} + \tilde{C}_{1212}P_{312})\Big]\, z_3'\\ &+ \Big[2x_1x_2(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_1x_2(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233})\Big]\,\xi_1''\\ &+ \left[\frac{x_2^2}{2}(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233}) - \frac{x_1^2}{2}(\tilde{C}_{11\rho\rho}P_{3\rho\rho} + \tilde{C}_{1112}P_{312})\right]\xi_2''\\ &- \Big[x_1(\tilde{C}_{12\rho\rho}C_{\rho\rho 33} + \tilde{C}_{1212}C_{1233}) + x_2(\tilde{C}_{22\rho\rho}C_{\rho\rho 33} + \tilde{C}_{2212}C_{1233})\Big]\,\xi_3'.\end{aligned}$$

Therefore, the component $\widetilde{u}_\alpha$ is given by the following expression

$$\widetilde{u}_\alpha = s_\alpha + \delta_\alpha s + z_{\alpha\beta}\xi_\beta'' - \Lambda_\alpha\xi_3' - \bar{\Lambda}_\alpha z_3'.$$

Let $\widetilde{u}_\alpha \in L^2(\Omega)$. Then $\int_\omega \widetilde{u}_\alpha \, d\omega$, $\int_\omega \delta_\beta u_\beta^2 \, d\omega$, ξ_α'', ξ_3', $z_3' \in L^2(0,L)$ and (3.29) together with (3.165) lead to s_α, $s \in L^2(0,L)$.

Since we must have $u_\alpha^2 \in W_2(\Omega)$, we deduce that

$$\int_\omega u_\alpha^2 \, d\omega \in V_0^1(0,L),$$
$$\int_\omega \left(x_2 u_1^2 - x_1 u_2^2\right) \, d\omega \in V_0^1(0,L)$$

from which we obtain, after substituting (3.165),

$$A(\omega) s_\alpha + X_{\alpha\beta} \xi_\beta'' \in V_0^1(0,L),$$
$$(I_1 + I_2)\, s + Y_\theta \xi_\theta'' - Z\xi_3' - L z_3' \in V_0^1(0,L),$$

and consequently

$$\xi_\alpha \in H^3(0,L), \qquad \xi_3, z_3 \in H^2(0,L).$$

On the other hand, using the weakly clamping condition

$$\int_\omega u_\alpha^2 \, d\omega = \int_\omega \left(x_2 u_1^2 - x_1 u_2^2\right) \, d\omega = 0,$$

we have s, $s_\alpha \in H^1(0,L)$ verify

$$s_\alpha(0) = -\frac{1}{A(\omega)} X_{\alpha\beta} \xi_\beta''(0), \qquad s(0) = -\frac{1}{I_1 + I_2}\left(Y_\beta \xi_\beta''(0) - Z\xi_3'(0) - L z_3'(0)\right).$$

■

Corollary 4 *$(u_\alpha^2) \in W_2$ are of the form (3.165) and consequently $\sigma_{\alpha\beta}^{-2} = 0$ if and only if $\boldsymbol{u}^0 \in V_{BN}$ is such that $\xi_\alpha \in H^3(0,L)$, $\xi_3, z_3 \in H^2(0,L)$.*

3.4.0.2 Identification of a one-dimensional variational problem satisfied by the leading term $(\boldsymbol{u}^0, \varphi^0)$

The analysis of Section 3.4.0.1 culminated in Corollary 4, where it was shown that $\sigma_{\alpha\beta}^{-2} = 0$, and consequently the variational problem (3.134) can be written as follows

$$\int_\Omega \sigma_{ij}^0 e_{ij}(\boldsymbol{v}) \, d\boldsymbol{x} + \int_\Omega D_k^0 E_k(\psi) \, d\boldsymbol{x} + \text{h.o.t.}$$
$$= \int_\Omega f_i v_i \, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i \, d\Gamma + \int_{\Gamma_L} p_i v_i \, d\Gamma, \qquad \forall\, (\boldsymbol{v}, \psi) \in V_{0,w} \times \Psi_0. \qquad (3.168)$$

Letting $(\boldsymbol{v}, \psi) \in V_{BN} \times \Psi_0^3$ in the variational equations of problem (3.168) shows that

$$\int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_3^0 E_3(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\, d\Gamma + \int_{\Gamma_L} p_i v_i\, d\Gamma \tag{3.169}$$

where

$$\sigma_{33}^0 = C_{33\theta\rho} e_{\theta\rho}(\boldsymbol{u}^2) + C_{3333} e_{33}(\boldsymbol{u}^0) - P_{333} E_3(\varphi^0),$$
$$D_3^0 = P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^2) + P_{333} e_{33}(\boldsymbol{u}^0) + \varepsilon_{33} E_3(\varphi^0).$$

We have established that $e_{\theta\rho}(\boldsymbol{u}^2)$, found in the expressions for σ_{33}^0 and D_3^0, is in fact known function of $\boldsymbol{u}^0$ and φ^0, and therefore the unknowns of the previous variational problem are $\boldsymbol{u}^0 \in V_{BN}$ and $\varphi^0 \in \hat{\varphi} + \Psi_0^3$.

Theorem 5 *We suppose that the applied forces are such that*

$$\begin{aligned} &f_\alpha \in L^2(\Omega), \qquad g_\alpha \in L^2(\Gamma_N) \\ &f_3 \in H^1[L^2(\omega)], \qquad g_3 \in H^1[L^2(\gamma_N)], \qquad p_i \in L^2(\Gamma_L) \equiv L^2(\omega), \end{aligned} \tag{3.170}$$

then, in the displacement-electric-potential approach, the leading term $(\boldsymbol{u}^0, \varphi^0)$ of the formal expansion of the scaled displacement and electric potential $(\boldsymbol{u}(h), \varphi(h))$ solves the following problem

$$\boldsymbol{u}^0 \in V_{BN} = \{\boldsymbol{v} : e_{\alpha\beta}(\boldsymbol{v}) = e_{3\alpha}(\boldsymbol{v}) = 0\} \tag{3.171}$$

$$\varphi^0 \in \hat{\varphi} + \Psi_0^3 = \hat{\varphi} + \left\{\psi \in H^1(\Omega) : \psi(x_1, x_2, x_3) = z(x_3),\ z_3 \in H_0^1(0, L)\right\} \tag{3.172}$$

$$\int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_3^0 E_3(\psi) d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\, d\Gamma + \int_{\Gamma_L} p_i v_i\, d\Gamma \tag{3.173}$$

$$\text{for all } \boldsymbol{v} \in V_{BN},\ \psi \in \Psi_0^3, \tag{3.174}$$

where σ_{33}^0 and D_3^0 are given by

$$\begin{aligned} \sigma_{33}^0 &= C_{33}^* e_{33}(\boldsymbol{u}^0) - P_3^* E_3(\varphi^0), \\ D_3^0 &= P_3^* e_{33}(\boldsymbol{u}^0) + \varepsilon_3^* E_3(\varphi^0), \end{aligned} \tag{3.175}$$

and the new constants read

$$C_{33}^* = C_{3333} - C_{33\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho} C_{\rho\rho 33} + \tilde{C}_{\alpha\beta 12} C_{1233}],$$

$$P_3^* = P_{333} - C_{33\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}P_{3\rho\rho} + \tilde{C}_{\alpha\beta12}P_{312}] = P_{333} + P_{3\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}P_{3\rho\rho} + \tilde{C}_{\alpha\beta12}P_{312}],$$

$$\varepsilon_3^* = \varepsilon_{33} + P_{3\alpha\beta}[\tilde{C}_{\alpha\beta\rho\rho}C_{\rho\rho33} + \tilde{C}_{\alpha\beta12}C_{1233}],$$

where C_{33}^* *and* ε_3^* *are positive and bounded.*

Any function $(\boldsymbol{u}^0, \varphi^0)$ *solves this problem if and only if it has the form*

$$u_\alpha^0(x_1, x_2, x_3) = \xi_\alpha(x_3), \qquad u_3^0(x_1, x_2, x_3) = \xi_3(x_3) - x_\beta \xi_\beta'(x_3), \tag{3.176}$$

$$\varphi^0(x_1, x_2, x_3) = z_3(x_3), \tag{3.177}$$

with $\xi_\alpha \in V_0^2(0,L)$, $\xi_3 \in V_0^1(0,L)$ *and* $z_3 \in H^1(0,L)$, $z_3(0) = \varphi_0^{1,0}$, $z_3(L) = \varphi_0^{1,L}$, *and* ξ_i *and* z_3 *solving the following variational problem (no sum on* α*):*

$$\left\{\begin{aligned} &\int_0^L I_\alpha C_{33}^* \xi_\alpha'' \zeta_\alpha''\, dx_3 = \int_0^L (\int_\omega f_\alpha\, d\omega + \int_{\gamma_N} g_\alpha\, d\gamma)\zeta_\alpha dx_3 \\ &\qquad - \int_0^L (\int_\omega x_\alpha f_3\, d\omega + \int_{\gamma_N} x_\alpha g_3\, d\gamma)\zeta_\alpha' dx_3 + (\int_\omega p_\alpha\, d\omega)\zeta_\alpha(L) \\ &\qquad -(\int_\omega x_\alpha p_3\, d\omega)\zeta_\alpha'(L), \qquad \text{for all } \zeta_\alpha \in V_0^2(0,L) \end{aligned}\right. \tag{3.178}$$

$$\left\{\begin{aligned} &\int_0^L A(\omega)[(C_{33}^*\xi_3' + P_3^* z_3')\zeta_3' - (P_3^*\xi_3' - \varepsilon_3^* z_3')q_3']dx_3 \\ &= \int_0^L (\int_\omega f_3 d\omega + \int_{\gamma_N} g_3 d\gamma)\zeta_3 dx_3 + (\int_\omega p_3 d\omega)\zeta_3(L), \\ &\forall \zeta_3 \in V_0^1(0,L)\ q_3 \in H_0^1(0,L), \end{aligned}\right. \tag{3.179}$$

where the zeroth order bending moment and axial force components are given by

$$\begin{aligned} n_3 &= \int_\omega \sigma_{33}^0\, d\omega = A(\omega)\left(C_{33}^*\xi_3' + P_3^* z_3'\right), \\ m_\alpha &= \int_\omega x_\alpha\, \sigma_{33}^0\, d\omega = -I_\alpha C_{33}^* \xi_\alpha'', \qquad (\textit{no sum on } \alpha) \\ d_3 &= \int_\omega D_3^0\, d\omega = A(\omega)\left(P_3^*\xi_3' - \varepsilon_3^* z_3'\right). \end{aligned}$$

Proof. Let functions $\boldsymbol{v} = (v_i) \in V_{BN}$ and $\psi = 0 \in \Psi_0^3$ in problem (3.173) be of the particular form

$$\boldsymbol{v} = (0, 0, \zeta_3(x_3)), \quad \zeta_3 \in V_0^1(0,L).$$

We obtain

$$\int_\Omega \sigma^0_{33}\zeta_3'\,d\boldsymbol{x} = \int_\Omega f_3\zeta_3\,d\boldsymbol{x} + \int_{\Gamma_N} g_3\zeta_3\,d\Gamma + \int_{\Gamma_L} p_3\zeta_3\,d\Gamma. \tag{3.180}$$

Next, let the function $\boldsymbol{v} = (v_i) \in V_{BN}$ and $\psi \in \Psi^3_0$ in problem (3.173) be of the particular form

$$\boldsymbol{v} = (\zeta_1(x_3), \zeta_2(x_3), -x_\beta\zeta_\beta(x_3)), \quad \zeta_\beta \in V^2_0(0,L), \tag{3.181}$$

$$\psi = q_3(x_3), \quad q_3 \in H^1(0,L), \tag{3.182}$$

which gives

$$\int_\Omega x_\beta\sigma^0_{33}\zeta_\beta''\,d\boldsymbol{x} + \int_\Omega x_\beta D^0_3 q_3''\,d\boldsymbol{x} = \int_\Omega f_\beta\zeta_\beta\,d\boldsymbol{x} + \int_{\Gamma_N} g_\beta\zeta_\beta\,d\Gamma + \int_{\Gamma_L} p_\beta\zeta_\beta\,d\Gamma \tag{3.183}$$

$$-\int_\Omega x_\beta f_\beta\zeta_\beta'\,d\boldsymbol{x} - \int_{\Gamma_N} x_\beta g_\beta\zeta_\beta'\,d\Gamma - \int_{\Gamma_L} p_\beta x_\beta\zeta_\beta'\,d\Gamma, \tag{3.184}$$

or, equivalently, we have

$$\begin{aligned}
-\int_0^L m_\alpha\zeta_\alpha''\,dx_3 = & \int_0^L (\int_\omega f_\alpha\,d\omega + \int_{\gamma_N} g_\alpha\,d\gamma)\zeta_\alpha dx_3 \\
& -\int_0^L (\int_\omega x_\alpha f_3\,d\omega + \int_{\gamma_N} x_\alpha g_3\,d\gamma)\zeta_\alpha' dx_3 \\
& +(\int_\omega p_\alpha\,d\omega)\zeta_\alpha(L) - (\int_\omega x_\alpha p_3\,d\omega)\zeta_\alpha'(L)
\end{aligned} \tag{3.185}$$

$$\int_0^L n_3\zeta_3' dx_3 - \int_0^L d_3 q_3' dx_3 = \int_0^L (\int_\omega f_3 d\omega + \int_{\gamma_N} g_3 d\gamma) dx_3 + (\int_\omega p_3 d\omega)\zeta_3(L) \tag{3.186}$$

Conversely, if functions solve the variational problem (3.178))-(3.179), one sees that $(\boldsymbol{u}^0, \varphi^0) \in V_{BN} \times \Psi^3_0$, with $u^0_\alpha = \zeta_\alpha$, $u^0_3 = \zeta_3 - x_\beta\zeta_\beta'$ and $\varphi^0 = z_3$, solves problem (3.173). ■

We note that on the other hand, using regularity results for variational problem (3.178) and (3.179) together with (3.170) guarantees the regularity to $\xi_\alpha \in H^3(0,L)$ and $\xi_3, z_3 \in H^2(0,L)$.

Theorem 6 *The limit problem (3.178)-(3.179) admits a unique solution in the space* $[V^2_0(0,L)]^2 \times V^1_0(0,L) \times (\hat{\varphi} + H^1(0,L))$. *Moreover, it is equivalent to the following differ-*

ential problems:

$$\begin{cases} -A(\omega)(C^*_{33}\,\xi''_3 + P^*_3 z''_3) = F_3, & \textit{in } (0,L), \\ A(\omega)(\varepsilon^*_3 z''_3 - P^*_3 \xi''_3) = 0, & \textit{in } (0,L), \\ \xi_3(0) = 0, \quad z_3(0) = \varphi^0_0, \quad z_3(L) = \varphi^L_0, & \\ A(\omega)\,[C^*_{33}\,\xi'_3(L) + P^*_3\, z'_3(L)] = F^L_3, & \end{cases} \tag{3.187}$$

and (no sum on β*)*

$$\begin{cases} C^*_{33}\, I_\beta\, \xi^{(4)}_\beta = F_\beta + M'_\beta, & \textit{in } (0,L), \\ \xi_\beta(0) = 0, \quad \xi'_\beta(0) = 0, & \\ C^*_{33}\, I_\beta\, \xi''_\beta(L) = -M^L_\beta, \qquad -C^*_{33}\, I_\beta\, \xi'''_\beta(L) = F^L_\beta - M_\beta(L), & \end{cases} \tag{3.188}$$

where

$$\begin{aligned} I_\beta &= \int_\omega (x_\beta)^2 \, d\omega, \quad F_i = \int_\omega f_i \, d\omega + \int_{\gamma_N} g_i \, d\gamma, \\ M_\alpha &= \int_\omega x_\alpha f_3 \, d\omega + \int_{\gamma_N} x_\alpha g_3 \, d\gamma, \\ F^L_i &= \int_\omega p_i \, d\omega, \quad M^L_\alpha = \int_\omega x_\alpha p_3 \, d\omega. \end{aligned}$$

Proof. Since $C^*_{33} > 0$, equation (3.178) has an unique solution $\xi_\beta \in V^2_0(0,L)$. In order to prove the existence and uniqueness of the solution of (3.179), we write it problem as follows

$$(\xi_3, z_3) \in V^1_0(0,L) \times (\hat{\varphi} + H^1(0,L)), \tag{3.189}$$

$$A(\omega) \int_0^L (\zeta'_3, z'_3) \begin{pmatrix} C^*_{33} & P^*_3 \\ P^*_3 & -\varepsilon^*_3 \end{pmatrix} \begin{pmatrix} \varsigma'_3 \\ q'_3 \end{pmatrix} dx_3 = \int_0^L (F_3 + F^L_3, 0) \begin{pmatrix} \varsigma_3 \\ q_3 \end{pmatrix} dx_3, \tag{3.190}$$

$$\forall (\varsigma_3, q_3) \in V^1_0(0,L) \times H^1_0(0,L), \tag{3.191}$$

The bilinear form is $V^1_0(0,L) \times H^1_0(0,L)$-elliptic, due to the positive definiteness of the matrix $\begin{pmatrix} C^*_{33} & P^*_3 \\ P^*_3 & -\varepsilon^*_3 \end{pmatrix}$. By the Lax-Milgram Lemma, there exists a unique solution $(\xi_3, z_3) \in V^1_0(0,L) \times (\hat{\varphi} + H^1(0,L))$ which satisfies the variational equations (3.187). ∎

Remark 7 *We remark that*

$$C_{33}^{*} = A_{33}^{c}\bar{\varepsilon}_{33}, \qquad P_{3}^{*} = A_{33}^{c}\bar{P}_{333}, \quad \varepsilon_{3}^{*} = A_{33}^{c}\bar{C}_{3333}.$$

To prove this we use the algebraic software tools.

Corollary 5 *The sequences $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ weakly converge to the first term of the asymptotic expansions $\boldsymbol{u}^0$ and $\bar{\varphi}^0$, respectively, i.e.,*

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}^0, \quad in \ V_{w,0}, \tag{3.192}$$

$$\varphi(h) \rightharpoonup \varphi^0, \quad in \ L^2(\Omega). \tag{3.193}$$

3.4.1 Strong convergence

One of the objectives of this section is to analyze the strong convergence of the solution $(\boldsymbol{u}(h), \varphi(h))$ of the scaled problem (3.46)-(3.50) as $h \to 0$. A brief summary of results, which will help us to prove the convergence, is showed here. The pair $((\sigma_{33}^0, D_3^0), (\boldsymbol{u}^0, \varphi^0))$ solves

$$((\sigma_{33}^0, D_3^0), (\boldsymbol{u}^0, \varphi^0)) \in [L^2(\Omega)]^2 \times (V_{BN} \times (\hat{\varphi} + \Psi_0^3))$$

$$\int_\Omega \left(\bar{C}_{3333}\,\sigma_{33}^0 + \bar{P}_{333} D_3^0\right) \tau_{33} d\boldsymbol{x} + \int_\Omega \left(-\bar{P}_{333}\sigma_{33}^0 + \bar{\varepsilon}_{33}\, D_3^0\right) d_3 d\boldsymbol{x}$$

$$= \int_\Omega \tau_{33}\, e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} + \int_\Omega d_3\, E_3(\varphi^0) d\boldsymbol{x}, \tag{3.194}$$

for all $(\tau_{33}, d_3) \in [L^2(\Omega)]^3$,

$$\int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega D_3^0 E_3(\psi) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma + \int_{\Gamma_L} p_i v_i d\Gamma, \tag{3.195}$$

for all $(\boldsymbol{v}, \psi) \in V_{BN} \times \Psi_0^3$.

Theorem 7 *Let us that the beam is made of a class 2 piezoelectric material whose coefficients $\bar{C}_{\alpha\beta 33} C_{33}^* + \bar{P}_{3\alpha\beta} P_3^*$ and $\bar{C}_{\alpha\beta 33} P_3^* - \bar{P}_{3\alpha\beta}\varepsilon_3^*$ do not depend on x_α. For $0 < h < 1$, let $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w}$ be the solution of (3.53)-(3.59), then the following strong converges hold, when $h \to 0$:*

$$\left\|(\boldsymbol{u}(h), \varphi(h)) - (\boldsymbol{u}^0, \varphi^0)\right\|_{\boldsymbol{X}_{2,w}} \to 0, \tag{3.196}$$

$$\left|\sigma_{33}(h) - \sigma_{33}^0\right|_{0,\Omega} \to 0, \qquad \left|h\sigma_{3\alpha}(h)\right|_{0,\Omega} \to 0, \qquad \left|h^2\sigma_{\alpha\beta}(h)\right|_{0,\Omega} \to 0, \tag{3.197}$$

$$\left|hD_\alpha(h)\right|_{0,\Omega} \to 0, \qquad \left|D_3(h) - D_3^0\right|_{0,\Omega} \to 0. \tag{3.198}$$

Proof. Let $\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{\sigma}}^0 \in [L^2(\Omega)]^9_s$ and $\bar{\boldsymbol{T}}(h), \bar{\boldsymbol{D}}^0 \in [L^2(\Omega)]^3$ be defined by

$$\boldsymbol{S}(h) = \boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \qquad \boldsymbol{T}(h) = \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0$$

$$\bar{\sigma}^0_{33} = \sigma^0_{33}, \quad \bar{\sigma}^0_{3\beta} = 0, \quad \bar{\sigma}^0_{\alpha\beta} = 0, \quad \bar{D}_3 = D^0_3, \quad \bar{D}_\alpha = 0,$$

and

$$\bar{\boldsymbol{S}}(h) = \left(\bar{S}_{ij}\right), \qquad \bar{\boldsymbol{T}}(h) = \left(\bar{T}_{ij}\right),$$

$$\bar{S}_{33} = S_{33}, \quad \bar{S}_{3\alpha} = hS_{3\alpha}, \quad \bar{S}_{\alpha\beta} = h^2 S_{\alpha\beta}, \quad \bar{T}_3 = T_3, \qquad \bar{T}_\alpha = hT_\alpha.$$

Let

$$\begin{aligned} a_T\left((\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h)), (\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h))\right) &:= a_{H,0}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) \\ + h^2 a_{H,2}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) &+ h^4 a_{H,4}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right), \end{aligned} \tag{3.199}$$

to be the left-side of the equation (3.53). Take $(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}) = \left(\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h)\right)$ into (3.53)-(3.59). By coercivity argument, we infer that the following inequality holds for all $h > 0$

$$\begin{aligned} a_T\left(\left(\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h)\right), \left(\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h)\right)\right) &\geq C\left\{|\sigma_{33}(h) - \sigma^0_{33}|^2_{0,\Omega} + h^2 \left|\sigma_{3\alpha}(h)\right|^2_{0,\Omega}\right. \\ \left. + h^4 \left|\sigma_{\alpha\beta}(h)\right|^2_{0,\Omega} + h^2 \left|D_\alpha(h)\right|^2_{0,\Omega} + |D_3(h) - D^0_3|^2_{0,\Omega}\right\} & \end{aligned} \tag{3.200}$$

Let us examine the behavior of the left-hand side of this inequality as $h \to 0$. First, a simple computation based on variational equation (3.53) shows that

$$\begin{aligned} \Lambda(h) &= a_T\left((\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h)), (\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h))\right) \\ &= -b_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h) + \hat{\varphi}(h))) + b_H((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \bar{\varphi}(h) + \hat{\varphi})) \\ &\quad - a_{H,0}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))) - h^2 a_{H,2}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))) \\ &\quad - h^4 a_{H,4}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))) = \Lambda_1(h) + \Lambda_2(h), \end{aligned} \tag{3.201}$$

where

$$\begin{aligned} \Lambda_1(h) = &-b_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))) - b_H((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{0}, \hat{\varphi})) \\ &+ b_H((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \bar{\varphi}(h)) + b_H((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{0}, \hat{\varphi})), \end{aligned}$$

and

$$\Lambda_2(h) = -a_{H,0}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))) - h^2 a_{H,2}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h)))$$
$$- h^4 a_{H,4}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))).$$

Using the variational equation in problem (3.54) and the definition (3.55), we have

$$\Lambda_1(h) = \int_\Omega f_i u_i(h) d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i(h) d\Gamma + \int_{\Gamma_L} p_i u_i(h) d\Gamma + \int_\Omega D_3(h) E_3(\hat{\varphi}) d\boldsymbol{x}$$
$$- \int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{u}(h)) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\bar{\varphi}(h)) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\hat{\varphi}) d\boldsymbol{x},$$

and using the weak convergences established in Section 3.3.1 as $h \to 0$, we next have

$$\Lambda_1(h) \to \Lambda_1 = \int_\Omega f_i u_i d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i d\Gamma + \int_{\Gamma_L} p_i u_i d\Gamma + \int_\Omega \mathfrak{D}_3 E_3(\hat{\varphi}) d\boldsymbol{x}$$
$$- \int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\bar{\varphi}^0) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\hat{\varphi}) d\boldsymbol{x}$$
$$= \int_\Omega \mathfrak{D}_3 E_3(\hat{\varphi}) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\hat{\varphi}) d\boldsymbol{x} \tag{3.202}$$

Applying expressions (3.56)-(3.58), the function $\Lambda_2(h)$ can be written in following expansive way

$$\Lambda_2(h) = -a_{H,0}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h))) - h^2 a_{H,2}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h)))$$
$$- h^4 a_{H,4}((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{S}(h), \boldsymbol{T}(h)))$$
$$= - \int_\Omega \left(\bar{C}_{3333}\, \sigma_{33}^0 + \bar{P}_{333} D_3^0\right) \left(\sigma_{33}\,(h) - \sigma_{33}^0\right) d\boldsymbol{x}$$
$$- \int_\Omega \left(-\bar{P}_{333} \sigma_{33}^0 + \bar{\varepsilon}_{33}\, D_3^0\right) \left(D_3\,(h) - D_3^0\right) d\boldsymbol{x}$$
$$- h^2 \int_\Omega \left(\bar{C}_{\alpha\beta 33}\, \sigma_{33}^0 + \bar{P}_{3\alpha\beta} D_3^0\right) \sigma_{\alpha\beta}(h) d\boldsymbol{x},$$

which becomes, as h goes to zero,

$$\Lambda_2 = - \int_\Omega \left(\bar{C}_{3333}\, \sigma_{33}^0 + \bar{P}_{333} D_3^0\right) \left(\Sigma_{33} - \sigma_{33}^0\right) d\boldsymbol{x}$$
$$- \int_\Omega \left(-\bar{P}_{333} \sigma_{33}^0 + \bar{\varepsilon}_{33}\, D_3^0\right) \left(\mathfrak{D}_3 - D_3^0\right) d\boldsymbol{x} - \int_\Omega \left(\bar{C}_{\alpha\beta 33}\, \sigma_{33}^0 + \bar{P}_{3\alpha\beta} D_3^0\right) \Sigma_{\alpha\beta} d\boldsymbol{x}.$$

Moreover, it is noted that

$$\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\sigma_{33}^0 + \bar{P}_{3\alpha\beta} D_3^0\right) \Sigma_{\alpha\beta} d\boldsymbol{x} = 0$$

under assumptions (3.88) and (3.175). Furthermore, we are supposing that the test functions appearing in (3.194) are of the form $\tau_{33} = \sigma_{33} - \sigma_{33}^0$ and $d_3 = D_3 - D_3^0$, then the previous limit reads

$$\begin{aligned}\Lambda_2 &= \int_\Omega (\Sigma_{33} - \sigma_{33}^0) e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} + \int_\Omega (\mathfrak{D}_3 - D_3^0) E_3(\varphi^0) d\boldsymbol{x} \\ &= \int_\Omega \Sigma_{33} e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3 E_3(\varphi^0) d\boldsymbol{x} - \int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\varphi^0) d\boldsymbol{x},\end{aligned}$$

where $\varphi^0 = \bar{\varphi}^0 + \hat{\varphi}$. Choosing now $\boldsymbol{v} = \boldsymbol{u}^0 \in V_{BN}$ and $\psi = \varphi^0 - \hat{\varphi} \in \Psi_0^3$ as test functions in (3.107) and (3.195), we deduce

$$\int_\Omega \Sigma_{33} e_{33}(\boldsymbol{u}^{\mathbf{0}}) d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3\, E_3(\varphi^0 - \hat{\varphi}) d\boldsymbol{x} = \int_\Omega f_i u_i^0 d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i^0 d\Gamma + \int_{\Gamma_L} p_i u_i^0 d\Gamma \quad (3.203)$$

and

$$\int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{u}^0) d\boldsymbol{x} + \int_\Omega D_3^0 E_3(\varphi^0 - \hat{\varphi}) d\boldsymbol{x} = \int_\Omega f_i u_i^0 d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i^0 d\Gamma + \int_{\Gamma_L} p_i u_i^0 d\Gamma, \quad (3.204)$$

respectively. Combining (3.203) with (3.204), we obtain

$$\Lambda_2 = \int_\Omega (D_3^0 - \mathfrak{D}_3) E_3(\hat{\varphi}) d\boldsymbol{x}.$$

We thus infer from these relations that the remaining terms the left-hand side of the inequality (3.200) converge to zero, i.e.,

$$\Lambda = \Lambda_1 + \Lambda_2 = \int_\Omega \mathfrak{D}_3 E_3(\hat{\varphi}) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\hat{\varphi}) d\boldsymbol{x} + \int_\Omega (D_3^0 - \mathfrak{D}_3) E_3(\hat{\varphi}) d\boldsymbol{x} = 0.$$

We next show (3.196). First, we clearly have

$$\begin{aligned}&- b_H \left(((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h) - \boldsymbol{u}^0, \varphi(h) - \varphi^0)\right) = - b_H \left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))\right) + b_H \left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}^0, \varphi^0)\right), \\ &= a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ &\quad + h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) - \int_\Omega \tau_{33}\, e_{33}(\boldsymbol{u}^0)\, d\boldsymbol{x} - \int_\Omega d_3\, E_3(\varphi^0)\, d\boldsymbol{x}, \qquad (3.205)\end{aligned}$$

we thus deduce from continuity of bilinear forms $a_{H,i}(\cdot,\cdot)$ that

$$-\, b_H\left(((\boldsymbol{\tau},\boldsymbol{d}),\left(\boldsymbol{u}(h)-\boldsymbol{u}^0,\varphi(h)-\varphi^0\right)\right) \tag{3.206}$$

$$\leq C\left\{\begin{array}{c} |D_3(h)-D_3^0|_{0,\Omega}^2+|\sigma_{33}(h)-\sigma_{33}^0|_{0,\Omega}^2 \\ +h^2\,|D_\alpha(h)|_{0,\Omega}^2+h^2\,|\sigma_{\alpha 3}(h)|_{0,\Omega}^2+h^4\,|\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \end{array}\right\}^{1/2} \|(\boldsymbol{\tau},\boldsymbol{d})\|_{\mathbf{X}_1}, \tag{3.207}$$

where C is a positive constant; secondly, putting $(\boldsymbol{\tau},\boldsymbol{d})=(e_{33}(\boldsymbol{u}(h)-\boldsymbol{u}^0),E_3(\varphi(h)-\varphi^0))$ into

$$\sup_{(\boldsymbol{\tau},\boldsymbol{d})\in\boldsymbol{X}_1}\frac{|b_H\left((\boldsymbol{\tau},\boldsymbol{d})\,,(\boldsymbol{u}(h)-\boldsymbol{u}^0,\varphi(h)-\varphi^0)\right)|}{\|(\boldsymbol{\tau},\boldsymbol{d})\|_{\mathbf{X}_1}} \tag{3.208}$$

$$\leq C\left\{\begin{array}{c} |D_3(h)-D_3^0|_{0,\Omega}^2+|\sigma_{33}(h)-\sigma_{33}^0|_{0,\Omega}^2+h^2\,|D_\alpha(h)|_{0,\Omega}^2 \\ +h^2\,|\sigma_{\alpha 3}(h)|_{0,\Omega}^2+h^4\,|\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \end{array}\right\}.$$

and applying Korn's inequality and Poincaré's inequality, we find

$$\left\|\left(\boldsymbol{u}(h)-\boldsymbol{u}^0,\varphi(h)-\varphi^0\right)\right\|_{\boldsymbol{X}_{2,w}}$$

$$\leq C\left\{\begin{array}{c} |D_3(h)-D_3^0|_{0,\Omega}^2+|\sigma_{33}(h)-\sigma_{33}^0|_{0,\Omega}^2+h^2\,|D_\alpha(h)|_{0,\Omega}^2 \\ +h^2\,|\sigma_{\alpha 3}(h)|_{0,\Omega}^2+h^4\,|\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \end{array}\right\},$$

when $h\to 0$. This way, we finished to prove that $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ converge strongly. ■

3.5 The limit model on the original domain; formulation as a boundary value problem

Since $\boldsymbol{u}^0$, φ^0, $\boldsymbol{\sigma}^0$ and $\boldsymbol{D}^0$ are approximations of $\boldsymbol{u}(h)$, $\varphi(h)$, $\boldsymbol{\sigma}(h)$ and $\boldsymbol{D}(h)$, respectively, as h tends to zero, undoing the change of variables (3.28) and the scaling in the different field, we get the fields

$$\boldsymbol{u}^{h,0}(\boldsymbol{x}^h)=\left(h^{-1}\,u_\alpha^0(\boldsymbol{x}),u_3^0(\boldsymbol{x})\right), \qquad \varphi^{h,0}(\boldsymbol{x}^h)=\varphi^0(\boldsymbol{x}),$$

$$\boldsymbol{\sigma}^{h,0}(\boldsymbol{x}^h)=\left(h^2\sigma_{\alpha\beta}^0(\boldsymbol{x}),h\sigma_{3\alpha}^0(\boldsymbol{x}),\sigma_{33}^0(\boldsymbol{x})\right), \qquad \boldsymbol{D}^{h,0}(\boldsymbol{x}^h)=\left(hD_\alpha^0(\boldsymbol{x}),D_3^0(\boldsymbol{x})\right),$$

defined in Ω^h, can be considered approximations of $\boldsymbol{u}^h$, φ^h, $\boldsymbol{\sigma}^h$ and $\boldsymbol{D}^h$, solutions of problem (2.37)-(2.38).

We can then enunciate the following result as an immediate consequence of Theorem 6.

Corollary 6 *The approximations* $(\boldsymbol{u}^{h,0}, \varphi^{h,0}, \sigma_{33}^{h,0}, D_3^{h,0})$ *are uniquely characterized as follows:*

$$u_\alpha^{h,0} = \xi_\alpha^h(x_3), \quad \xi_\alpha^h \in V_0^2(0,L), \tag{3.209}$$

$$u_3^{h,0} = \xi_\alpha^h - x_\alpha \partial_3^h \xi_\alpha^h, \quad \xi_3^h \in V_0^1(0,L), \tag{3.210}$$

$$\varphi^{h,0} = z_3^h, \quad z_3^h \in H^1(0,L), \quad z_3^h(0) = \varphi_0^0, \; z_3^h(L) = \varphi_0^L, \tag{3.211}$$

$$\sigma_{\alpha\beta}^{h,0} = \sigma_{3\alpha}^{h,0} = 0, \tag{3.212}$$

$$\sigma_{33}^{h,0} = C_{33}^* \, e_{33}(\boldsymbol{u}^{h,0}) - P_3^* E_3 \left(\varphi^{h,0}\right), \tag{3.213}$$

$$D_3^{h,0} = P_3^* e_{33}(\boldsymbol{u}^{h,0}) + \varepsilon_3^* E_3 \left(\varphi^{h,0}\right), \tag{3.214}$$

where

$$\xi_\alpha^h(x_3^h) = h^{-1}\xi_\alpha(x_3), \qquad \xi_3^h(x_3^h) = \xi_3(x_3),$$
$$z_3^h(x_3^h) = z_3(x_3),$$

and $(\boldsymbol{\xi}^h, z_3^h)$ *is the solution of the following boundary value problem (no sum on* β*):*

$$\begin{cases} C_{33}^* \, I_\beta^h \, \left(\xi_\beta^h\right)^{(4)} = F_\beta^h + \left(M_\beta^h\right)' & \textit{in } (0,L), \\ \xi_\beta^h(0) = 0, \quad \left(\xi_\beta^h\right)'(0) = 0, \\ C_{33}^* \, I_\beta^h \, \left(\xi_\beta^h\right)''(L) = -\left(M_\beta^L\right)^h, \\ -C_{33}^* \, I_\beta^h \, \left(\xi_\beta^h\right)'''(L) = \left(F_\beta^L\right)^h - M_\beta^h(L), \end{cases} \tag{3.215}$$

$$\begin{cases} -A(\omega^h) \left[C_{33}^* \left(\xi_3^h\right)'' + P_3^* \left(z_3^h\right)''\right] = F_3^h, & \textit{in } (0,L), \\ A(\omega^h) \left[\varepsilon_3^* \left(z_3^h\right)'' - P_3^* \left(\xi_3^h\right)''\right] = 0, & \textit{in } (0,L), \\ \xi_3^h(0) = 0, \quad z_3^h(0) = \varphi_0^0, \quad z_3^h(L) = \varphi_0^L, \\ A(\omega^h) \left[C_{33}^* \left(\xi_3^h\right)'(L) + P_3^* \left(z_3^h\right)'(L)\right] = \left(F_3^L\right)^h, \end{cases} \tag{3.216}$$

where $A(\omega^h)$ is the area of the actual beam cross section and

$$I_\beta = \int_{\omega^h} \left(x_\beta^h\right)^2 d\omega^h, \quad F_i^h = \int_{\omega^h} f_i^h \, d\omega^h + \int_{\gamma^h} g_i^h \, d\gamma^h,$$
$$M_\alpha^h = \int_{\omega^h} x_\alpha^h f_3^h \, d\omega^h + \int_{\gamma^h} x_\alpha^h g_3^h \, d\gamma^h,$$
$$\left(F_i^L\right)^h = \int_{\omega^h} h_i^h \, d\omega^h, \quad \left(M_\alpha^L\right)^h = \int_{\omega^h} x_\alpha h_3^h \, d\omega^h.$$

The normal force and bending moments are given, respectively, by the expressions

$$q^h = \int_{\omega^h} \sigma_{33}^{h,0} \, d\omega^h = C_{33}^* \, A^h(\omega^h) \, (\xi_3^h)' + P_3^* A(\omega^h) \, (z_3^h)',$$
$$m_\beta^h = \int_{\omega^h} x_\beta^h \, \sigma_{33}^{h,0} \, d\omega^h = -C_{33}^* I_\beta^h \, (\xi_\beta^h)'' \qquad (\textit{no sum on } \beta).$$

It is important to observe that the limit model is described by a system of three partial differential equations, posed over the one-dimensional set $(0, L)$, two of the fourth order with respect to the unknowns ξ_α^h describing bending in an elastic beam, and two of the second order with respect to the axial displacement ξ_3^h and electric potential z_3^h.

Now we shall examine the particular case of a transversely isotropic piezoelectric beam load on one of its end and verify that our asymptotic piezoelectric model is consistent with asymptotic elastic beam theory. As in the work of Trabucho & Viaño [1996], we assume that the resultant of the applied loads P is parallel to axis Ox_3, so that, we have

$$f_i^h = 0, \qquad g_i^h = 0, \qquad h_1^h = 0, \qquad h_3^h = 0,$$
$$F_2^{h,L} = P,$$

then, we obtain

$$\begin{cases} C_{33}^* I_1^h (\xi_1^h)'''' = 0 \quad \text{in} \quad (0, L), \\ \xi_1^h(0) = (\xi_1^h)'(0) = 0, \\ C_{33}^* I_1^h (\xi_1^h)''(L) = 0, \\ -C_{33}^* I_1^h (\xi_1^h)'''(L) = 0 \end{cases} \qquad \begin{cases} C_{33}^* I_2^h (\xi_2^h)'''' = 0 \quad \text{in} \quad (0, L), \\ \xi_2^h(0) = (\xi_2^h)'(0) = 0, \\ C_{33}^* I_2^h (\xi_2^h)''(L) = 0, \\ -C_{33}^* I_2^h (\xi_2^h)'''(L) = P \end{cases}$$

and

$$\begin{cases} -A^h\left[C^*_{33}(\xi^h_3)'' - P^*_3(z^h_3)''\right] = 0 \quad \text{in} \quad (0,L), \\ A^h\left[-P^*_3(\xi^h_3)'' + \varepsilon^*_3(z^h_3)''\right] = 0 \quad \text{in} \quad (0,L), \\ \xi^h_3(0) = 0, \qquad z^h_3(0) = \varphi_0^{h,0}, \qquad z^h_3(L) = \varphi_0^{h,L}, \\ A^h\left[C^*_{33}(\xi^h_3)'(L) - P^*_3(z^h_3)'(L)\right] = 0. \end{cases}$$

For the asymptotic model, one has, form Corollary 6,

$$\begin{aligned} u_1^{h,0} &= \xi_1^h = 0, \\ u_2^0 &= \xi_2^h = -\frac{P}{6I_2^hC^*_{33}}\left(x_3^h - 3L\right)(x_3^h)^2, \\ u_3^{h,0} &= \xi_3^h - x_2^h(\xi_2^h)' = \frac{P^*_3}{C^*_{33}}\left(\varphi_0^{h,0} + \frac{\varphi_0^{h,L} - \varphi_0^{h,0}}{L}x_3^h\right) + \frac{P}{I_2^hC^*_{33}}\left(\frac{x_3^h}{2} - L\right)x_3x_2^h, \\ \varphi^h &= z_3^h = \varphi_0^{h,0} + \frac{\varphi_0^{h,L} - \varphi_0^{h,0}}{L}x_3^h, \end{aligned}$$

and consequently the stress tensor component and the electric displacement give

$$\begin{aligned} \sigma_{33}^{h,0} &= \frac{P}{I_2^h}(x_3^h - L)x_2^h, \\ D_3^{h,0} &= \frac{P}{I_2^hC^*_{33}}P^*_3(x_3^h - L)x_2^h + \left(\frac{P^*_3P^*_3}{C^*_{33}} + \varepsilon^*_3\right)\frac{\varphi_0^{h,L} - \varphi_0^{h,0}}{L}. \end{aligned}$$

If we ignore the electric field, we can verify that these equations coincide with the asymptotic model established by Trabucho & Viaño [1996]. We can conclude, from this example, that the asymptotic model for a transversely isotropic beam yields the corresponding classical models of engineering literature.

Chapter 4

Asymptotic analysis of a beam with electric potential applied to lateral surface

The main goal of this chapter is to characterize the asymptotic behavior of a family of scaled displacements and electric potentials that solve the three-dimensional problem for an anisotropic linearly beam of class 2 subject to an electric potential acting along its lateral surface. We assume that the beam is now assumed to be *weakly clamped* at both ends; to avoid the boundary layer phenomenon.

In this chapter, an attempt to point out the differences between the model obtained by asymptotic expansion method and the model obtained by convergence analysis, is presented. The convergence results are adapted from the work introduced by Figueiredo & Leal [2006], for a linear nonhomogeneous anisotropic thin rod.

The outline of the chapter is as follows: in Section 4.1 we recall the 3D piezoelectric problem to be studied and formulate it in the form of the principle of virtual work. In Section 4.2 the problem is posed over a domain independent of h where the scaled unknowns can be written in the form of asymptotic developments. Section 4.3 concerns the asymptotic study of the problem; in particular, we calculate the first terms in the asymptotic expansions and derive the limit model. Here, the equations for a transversally isotropic (of type of class *6mm*) homogeneous linearly piezoelectric beam are also studied.

The weak convergence is studied in Section 4.4. We start by establishing, for a homogeneous anisotropic beam of class 2, the weak convergence of the scaled displacement field to the leading term of the asymptotic expansion. For a homogeneous anisotropic beam of class *6mm*, we also show, in Section 4.4.2.2, that the electric potential weakly converges to the leading term of the expansion.

The physical interpretation of the results is done in Section 4.5.

4.1 Mathematical analysis of the piezoelectric problem

In this section we describe the boundary conditions and the mechanical problem that will be the focus of this chapter.

4.1.1 The mechanical and mathematical problem

Let us consider the following assumptions with the notations already introduced:

($\mathbf{H}_{41}^d$) The beam is weakly clamped at the extremities $\Gamma_0^h \cup \Gamma_L^h$ that is to say $\Gamma_{dD}^h = \Gamma_0^h \cup \Gamma_L^h$ with $\text{meas}(\Gamma_{dD}^h) > 0$.

($\mathbf{H}_{42}^d$) The surface traction $\boldsymbol{g}^h = (g_i^h)$ act on $\Gamma_{dN}^h = \Gamma_N^h$.

($\mathbf{H}_{43}^d$) We assume $\Gamma_{eD}^h = \gamma_{eD}^h \times (0, L)$ with $\text{meas}(\gamma_{eD}^h) > 0$, and $\Gamma_{eN}^h = \partial^h \Omega^h \backslash \Gamma_{eD}^h$.

($\mathbf{H}_{44}^d$) The electric potential φ_0^h is applied on Γ_{eD}^h (see figure 4.1.1).

($\mathbf{H}_{45}^d$) The constants of the material C_{ijkl}^h, P_{kij}^h and ε_{ij}^h satisfy the conditions ($\mathbf{H}_{21}^c$) and ($\mathbf{H}_{22}^c$) detailed in Chapter 2 and the following conditions:

$$C_{3\rho 33}^h = C_{3\theta\alpha\beta}^h = 0, \quad P_{\theta\rho\sigma}^h = P_{33\alpha}^h = P_{\beta 33}^h = 0, \quad \varepsilon_{3\theta}^h = 0. \tag{4.1}$$

($\mathbf{H}_{46}^d$) $f_i^h = L^2(\Omega^h)$, $g_i^h \in L^2(\Gamma_{dN}^h)$, and $\varphi_0^h \in H^{1/2}(\Gamma_{eD}^h)$.

As the beam is weakly clamped at the extremities $\Gamma_0^h \cup \Gamma_L^h$, the displacement field $\boldsymbol{u}^h$ satisfies the condition

$$\int_{\Gamma_a^h} u_i^h \, d\omega^h = 0, \quad \int_{\Gamma_a^h} (x_j^h u_i^h - x_i^h u_j^h) \, d\omega^h = 0, \quad a = 0, L. \tag{4.2}$$

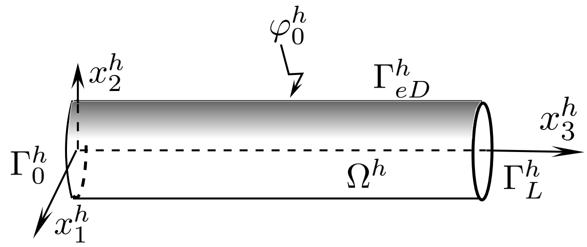


Figure 4.1: Schematic representation of the solid domain detailing the electrical boundary conditions.

In the following we will represent this boundary condition by $\langle \boldsymbol{u}^h \rangle = \mathbf{0}$ on Γ^h_{dD}.

Remark 8 *Condition ($\boldsymbol{H}^d_{45}$) means that the beam is made of a piezoelectric material having crystal symmetry corresponding to one of the following classes (see e.g. Refs. Royer & Dieulesaint [2000] and Nye [1985]): monoclinic system (except class m), orthorhombic system, tetragonal system, hexagonal system (except classes $\overline{6}$ and $\overline{6}m2$), and cubic system.*

Let us now define the following closed subspaces of the Sobolev space of order one $\left[H^1(\Omega^h)\right]^m$, $m \geq 1$:

$$\begin{cases} V^h_{0,w} = V^h_{0,w}(\Omega^h) = \{\boldsymbol{v}^h \in \left[H^1(\Omega^h)\right]^3 : \langle \boldsymbol{v}^h \rangle = \mathbf{0} \text{ on } \Gamma^h_{dD}\}, \\ \Psi^h_0 = \Psi^h_0(\Omega^h) = \{\psi^h \in H^1(\Omega^h) : \psi^h = 0 \text{ on } \Gamma^h_{eD}\}. \end{cases} \tag{4.3}$$

Since $\text{meas}(\Gamma^h_{dD}) > 0$ and $\text{meas}(\Gamma^h_{eD}) > 0$, it follows from Korn's and Poincaré-Friedrichs' inequalities that the following norms are equivalent to the standard Sobolev norms on $V^h_{0,w}$ and Ψ^h_0, respectively,

$$\left\|\boldsymbol{v}^h\right\|_{V^h_{0,w}} = \left|\boldsymbol{e}^h(\boldsymbol{v}^h)\right|_{\Omega^h}, \quad \left\|\varphi^h\right\|_{\Psi^h} = \left|\nabla^h \varphi^h\right|_{\Omega^h}.$$

Defining $\bar{\varphi}^h = \varphi^h - \hat{\varphi}^h$, where $\hat{\varphi}^h$ is a trace lifting in $H^1(\Omega^h)$ of the boundary potential acting on Γ^h_{eD}, we obtain that the solution $(\boldsymbol{u}^h, \varphi^h)$ is derived from $\varphi^h = \bar{\varphi}^h + \hat{\varphi}^h$ with $(\boldsymbol{u}^h, \bar{\varphi}^h)$ satisfying:

$$\begin{aligned} &\text{Find } (\boldsymbol{u}^h, \bar{\varphi}^h) \in V^h_{0,w} \times \Psi^h_0 \text{ such that} \\ &a((\boldsymbol{u}^h, \bar{\varphi}^h), (\boldsymbol{v}^h, \psi^h)) = l_2(\boldsymbol{v}^h, \psi^h) \qquad \forall (\boldsymbol{v}^h, \psi^h) \in V^h_{0,w} \times \Psi^h_0 \end{aligned} \tag{4.4}$$

where the bilinear form $a(\cdot,\cdot)$ and $l_2(\cdot)$ are defined in Chapter 2 as follow

$$\begin{aligned} a(\left(\boldsymbol{u}^h, \varphi^h\right), \left(\boldsymbol{v}^h, \psi^h\right)) &= \int_{\Omega^h} \left[C_{ijkl} e^h_{kl}(\boldsymbol{u}^h) - P_{mij}\, E^h_m\left(\varphi^h\right)\right] e^h_{ij}(\boldsymbol{v}^h) d\boldsymbol{x}^h \\ &+ \int_{\Omega^h} \left[P_{mij}\, e^h_{ij}(\boldsymbol{u}^h) + \varepsilon_{mi}\, E^h_i\left(\varphi^h\right)\right] E^h_m\left(\psi^h\right) d\boldsymbol{x}^h, \end{aligned} \tag{4.5}$$

$$\begin{aligned} l_2(\boldsymbol{v}^h, \psi^h) &= \int_{\Omega^h} \left[f^h_i v^h_i + P_{kij} E^h_k(\hat{\varphi}^h) e^h_{ij}(\boldsymbol{v}^h) - \varepsilon_{kl} E^h_l(\hat{\varphi}^h) E^h_k(\psi^h)\right] d\boldsymbol{x}^h \\ &+ \int_{\Gamma^h_{dN}} g^h_i v^h_i d\Gamma^h. \end{aligned} \tag{4.6}$$

The existence and uniqueness is ensured from Lax-Milgram Lemma.

Remark 9 *The derivation of (4.4) follows closely the steps presented in [Bernadou & Haenel, 2003]'s work for the strong clamping case with minor changing to accommodate the weak clamping condition adopted here.*

We now consider that the displacement field $\boldsymbol{u}^h$, the electric potential φ^h, the stress tensor field $\boldsymbol{\sigma}^h$ and the electric displacement field $\boldsymbol{D}^h$ satisfy the following problem, consisting of the principle of virtual work (cf. (3.13)):

$$
\begin{aligned}
&(\boldsymbol{u}^h, \varphi^h) \in \boldsymbol{X}_2^h = V_{0,w}^h \times \Psi_2^h \\
&(\boldsymbol{\sigma}^h, \boldsymbol{D}^h) \in \boldsymbol{X}_1^h = \left\{(\boldsymbol{\tau}, d) = (\tau_{ij}, d_k) \in [L^2(\Omega^h)]^3 \times L^2(\Omega^h) : \ \tau_{ij} = \tau_{ji}\right\}, \\
&\int_{\Omega^h} \sigma_{ij}^h e_{ij}^h(\boldsymbol{v}^h)\, d\boldsymbol{x}^h + \int_{\Omega^h} D_k^h E_k^h(\psi^h)\, d\boldsymbol{x}^h = \int_{\Omega^h} f_i^h v_i^h\, d\boldsymbol{x}^h + \int_{\Gamma_{dN}^h} g_i^h v_i^h\, d\Gamma^h \\
&\text{for all } (\boldsymbol{v}^h, \psi^h) \in \boldsymbol{X}_{0,w}^h = V_{0,w}^h \times \Psi_0^h.
\end{aligned}
\tag{4.7}
$$

4.2 Transformation into a problem posed over a domain independent of h; fundamental scalings of the unknowns and assumptions on the data

As in Section 3.2, we need to transform problem (4.7) into a problem posed over a set that does not depend on h. Accordingly, we let

$$\Omega := \omega \times (0, L),$$

and with each point $\boldsymbol{x} \in \overline{\Omega}$, we associate the point $\boldsymbol{x}^h \in \overline{\Omega}^h$ through the bijection Π^h, defined in Chapter 3 by (3.28). Then, one has

$$
\begin{aligned}
&\Gamma_{dN}^h = \Pi^h(\Gamma_{dN}), \quad \Gamma_0^h = \Pi^h(\Gamma_0), \quad \Gamma_L^h = \Pi^h(\Gamma_L), \\
&\boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{n}^h(\boldsymbol{x}^h), \quad \Gamma_{eN}^h = \Pi^h(\Gamma_{eN}), \quad \gamma_{eD}^h = \Pi^h(\gamma_{eD}), \quad \Gamma_{eD}^h = \Pi^h(\Gamma_{eD}),
\end{aligned}
$$

where $\boldsymbol{n}$ is the unit outer normal to the set $\partial\Omega = \Gamma_{dD} \cup \Gamma_{dN} = \Gamma_{eD} \cup \Gamma_{eN}$ and

$$\Gamma_{eN} = \Gamma_0 \cup \Gamma_L, \tag{4.8}$$

$$\Gamma_{eD} = \gamma_{eD} \times (0, L), \qquad \text{meas}(\gamma_{eD}) > 0. \tag{4.9}$$

We define the spaces

$$V_{0,w} = V_{0,w}(\Omega) = \left\{ \boldsymbol{v} \in [H^1(\Omega)]^3 : \langle \boldsymbol{v} \rangle|_{\Gamma_D} = \boldsymbol{0} \right\},$$
$$\Psi_0 = \Psi_0(\Omega) = \left\{ \psi \in H^1(\Omega) : \psi|_{\Gamma_{eD}} = 0 \right\}, \tag{4.10}$$

equipped with the following norms equivalent to the usual Sobolev norms:

$$\|\boldsymbol{v}\|_V = |\boldsymbol{e}(\boldsymbol{v})|_{0,\Omega}, \quad \|\psi\|_\Psi = |\boldsymbol{\nabla}\psi|_{0,\Omega},$$

where

$$\boldsymbol{e}(\boldsymbol{v}) = (e_{ij}(\boldsymbol{v})), \quad e_{ij}(\boldsymbol{v}) = \frac{1}{2}(\partial_i v_j + \partial_j v_i).$$

Here, $\langle \boldsymbol{v} \rangle|_{\Gamma_{dD}} = \boldsymbol{0}$ stands for the boundary condition (cf. (3.30))

$$\int_{\omega\times\{a\}} u_i \, d\omega = 0, \quad \int_{\omega\times\{a\}} (x_i u_j - x_j u_i) \, d\omega = 0, \quad a = 0, L. \tag{4.11}$$

With the unknown and test displacement fields $\boldsymbol{u}^h$, $\boldsymbol{v}^h$ in $V_{0,w}^h$, we associate the (unknown and test) scaled displacement fields $\boldsymbol{u}(h) = (u_i(h))$ and $\boldsymbol{v}(h) = (v_i(h))$ in $V_{0,w}$ defined in (3.36)-(3.37). In this chapter, the electric potential $\bar{\varphi}^h$ and the test function ψ^h in Ψ_0^h are associated to the scaled potential $\bar{\varphi}(h)$ and the scaled (test) function $\psi(h)$ in Ψ_0 using the following scaling for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \overline{\Omega}$:

$$\bar{\varphi}(h)(\boldsymbol{x}) = h^{-1}\bar{\varphi}^h(\boldsymbol{x}^h), \qquad \psi(h)(\boldsymbol{x}) = h^{-1}\psi^h(\boldsymbol{x}^h). \tag{4.12}$$

We make the following assumptions on the data: We assume that the constants of the material, the applied body force density, the applied surface force density and the applied electric potential are the following form:

1. There exist functions $f_i \in L^2(\Omega)$ and $g_i \in L^2(\Gamma_{dN})$, independent of h, such that:

$$\begin{cases} f_\alpha^h(\boldsymbol{x}^h) = h f_\alpha(\boldsymbol{x}), \quad f_3^h(\boldsymbol{x}^h) = f_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h, \\ g_\alpha^h(\boldsymbol{x}^h) = h^2 g_\alpha(\boldsymbol{x}), \quad g_3^h(\boldsymbol{x}^h) = h g_3(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Gamma_{dN}^h. \end{cases} \tag{4.13}$$

2. There exists a function $\hat{\varphi} \in H^1(\Omega)$, independent of h, such that:

$$\hat{\varphi}^h(\boldsymbol{x}^h) = h\hat{\varphi}(\boldsymbol{x}), \quad \text{for all } \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}) \in \Omega^h. \tag{4.14}$$

We also denote by $\hat{\varphi} \in H^1(\Omega)$ a trace lifting of φ_0 and define $\varphi(h) = \bar{\varphi}(h) + \hat{\varphi}$.

The scaled strain tensor $\boldsymbol{e}(\boldsymbol{v}(h))(\boldsymbol{x})$ is defined as in (3.44); while that the electric field $\boldsymbol{E}(\psi(h))(\boldsymbol{x})$ is given by, for all $\psi^h \in \Psi_0^h$ and $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$:

$$\begin{cases} E_\alpha^h(\psi^h)(\boldsymbol{x}^h) = -\partial_\alpha^h \psi^h(\boldsymbol{x}^h) = -\partial_\alpha(\psi(h))(\boldsymbol{x}) = E_\alpha(\psi(h))(\boldsymbol{x}), \\ E_3^h(\psi^h)(\boldsymbol{x}^h) = -\partial_3^h \psi^h(\boldsymbol{x}^h) = -h\partial_3(\psi(h))(\boldsymbol{x}) = hE_3(\psi(h))(\boldsymbol{x}). \end{cases} \tag{4.15}$$

Motivated by these identities we define for any $(\boldsymbol{v}, \psi) \in V_{0,w}^h \times H^1(\Omega^h)$ the tensor $\boldsymbol{\kappa}(h; \boldsymbol{v})$ and the vector $\boldsymbol{\vartheta}(h; \psi)$ as follows:

$$\kappa_{\alpha\beta}(h; \boldsymbol{v}) := h^{-2} e_{\alpha\beta}(\boldsymbol{v}), \quad \kappa_{3\beta}(h; \boldsymbol{v}) := h^{-1} e_{3\beta}(\boldsymbol{v}), \quad \kappa_{33}(h; \boldsymbol{v}) := e_{33}(\boldsymbol{v}), \tag{4.16}$$

$$\vartheta_3(h; \psi) = h\partial_3 \psi, \quad \vartheta_\alpha(h; \psi) = \partial_\alpha \psi. \tag{4.17}$$

For simplicity we abbreviate the notation for the special cases $\boldsymbol{v} = \boldsymbol{u}(h)$, $\psi = \overline{\varphi}(h)$, $\psi = \varphi_0$, writing

$$\boldsymbol{\kappa}(h) := \boldsymbol{\kappa}(\boldsymbol{u}(h)), \quad \overline{\boldsymbol{\vartheta}}(h) := \boldsymbol{\vartheta}(\bar{\varphi}(h)), \quad \widehat{\boldsymbol{\vartheta}} := \boldsymbol{\vartheta}(\hat{\varphi}). \tag{4.18}$$

We remark that

$$\boldsymbol{\kappa}(h) = \boldsymbol{e}^h(\boldsymbol{u}^h), \quad \bar{\boldsymbol{\vartheta}}(h) = \boldsymbol{\nabla}^h \bar{\varphi}^h,$$

and define

$$\boldsymbol{\vartheta}(h) = \boldsymbol{\vartheta}(h)(\bar{\varphi}(h) + \varphi_0) = \bar{\boldsymbol{\vartheta}}(h) + \widehat{\boldsymbol{\vartheta}}(h).$$

Using the scalings of the unknown and the assumptions on the data, we reformulate, in the next two theorems, the problems (4.4) and (4.7) now posed over the set $\overline{\Omega}$.

The variational problem (4.4) becomes

$$\begin{aligned} &\text{Find } (\boldsymbol{u}(h), \bar{\varphi}(h)) \in V_{0,w} \times \Psi_0 \text{ such that} \\ &a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{v}, \psi)) = l_2(\boldsymbol{v}, \psi), \qquad \forall (\boldsymbol{v}, \psi) \in V_{0,w} \times \Psi_0, \end{aligned} \tag{4.19}$$

where the bilinear form a and the linear form l_2 are defined by

$$\begin{aligned} a((\boldsymbol{u}, \bar{\varphi}), (\boldsymbol{v}, \psi)) = &\int_\Omega C_{ijkl}\, \kappa_{kl}(h; \boldsymbol{u})\, \kappa_{ij}(h; \boldsymbol{v})\, dx + \int_\Omega \varepsilon_{ij}\, \overline{\vartheta}_i(h; \psi)\, \overline{\vartheta}_j(h; \psi)\, dx \\ &+ \int_\Omega P_{mij}(\vartheta_m(h; \bar{\varphi})\kappa_{ij}(h; \boldsymbol{v}) - \kappa_{ij}(h; \bar{\boldsymbol{u}})\vartheta_m(h; \psi))\, d\boldsymbol{x} \end{aligned} \tag{4.20}$$

$$l_2(\boldsymbol{v},\psi) = \int_\Omega f_i\, v_i\, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma - \int_\Omega \varepsilon_{ij}\, \widehat{\vartheta}_i\, \vartheta_j(h;\psi)\, d\boldsymbol{x}$$
$$- \int_\Omega P_{kij}\, \widehat{\vartheta}_k\, \kappa_{ij}(h;\boldsymbol{v})\, d\boldsymbol{x} \qquad (4.21)$$

Proposition 5 *Assume that* $(\boldsymbol{u}^h,\varphi^h) \in V^h_{0,w} \times H^1(\Omega^h)$ *is solution of problem (4.7).*

(a) The scaled unknowns $(\boldsymbol{u}(h),\varphi(h)) \in V_{0,w} \times H^1(\Omega)$ *satisfy the following variational problem, called the scaled three-dimensional problem of a piezoelectric clamped beam:*

$$(\boldsymbol{u}(h),\varphi(h)) \in X_{2,w},$$
$$\int_\Omega \sigma_{ij}(h) e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_k(h) E_k(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i\, d\Gamma, \qquad (4.22)$$

for all $(\boldsymbol{v},\psi) \in X_{0,w}$,

where

$$\begin{cases} \sigma_{\alpha\beta}(h) = h^{-4} C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}(h)) + h^{-2} C_{\alpha\beta33} e_{33}(\boldsymbol{u}(h)) - h^{-1} P_{3\alpha\beta} E_3(\varphi(h)), \\ \sigma_{3\alpha}(h) = 2h^{-2} C_{3\alpha3\beta} e_{3\beta}(\boldsymbol{u}(h)) - h^{-1} P_{\beta3\alpha} E_\beta(\varphi(h)), \\ \sigma_{33}(h) = h^{-2} C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}(h)) + C_{3333} e_{33}(\boldsymbol{u}(h)) - h P_{333} E_3(\varphi(h)), \\ D_\alpha(h) = 2h^{-1} P_{\alpha3\beta} e_{3\beta}(\boldsymbol{u}(h)) + \varepsilon_{\alpha\beta} E_\beta(\varphi(h)), \\ D_3(h) = h^{-1} P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}(h)) + h P_{333} e_{33}(\boldsymbol{u}(h)) + h^2 \varepsilon_{33} E_3(\varphi(h)). \end{cases} \qquad (4.23)$$

(b) The functions $(\sigma_{ij}(h)) \in L^2(\Omega)$ *and* $(D_i(h)) \in L^2(\Omega)$ *defined in (a) are also related to the components* $(\sigma_{ij}(h)) \in L^2(\Omega)$ *and* $(D_i(h)) \in L^2(\Omega)$ *by (cf. (3.39))*

$$\begin{cases} \sigma^h_{\alpha\beta}(\boldsymbol{x}^h) = h^2 \sigma_{\alpha\beta}(h)(\boldsymbol{x}), \quad \sigma^h_{3\alpha}(\boldsymbol{x}^h) = h\sigma_{3\alpha}(h)(\boldsymbol{x}), \quad \sigma^h_{33}(\boldsymbol{x}^h) = \sigma_{33}(h)(\boldsymbol{x}), \\ D^h_\alpha(\boldsymbol{x}^h) = D_\alpha(h)(\boldsymbol{x}), \quad D^h_3(\boldsymbol{x}^h) = h^{-1} D_3(h)(\boldsymbol{x}). \end{cases} \qquad (4.24)$$

Proof. The proof of (a) reduces to simple computations, based on the displacement, the assumptions on the data, and the formulas $\partial^h_\alpha = h^{-1}\partial_\alpha$, $\partial^h_3 = \partial_3$, and

$$\int_{\Omega^h} \theta^h d\boldsymbol{x}^h = h^2 \int_\Omega \left(\theta^h \circ \Pi^h\right) d\boldsymbol{x}, \qquad \int_{\Gamma^h_{dN}} \theta^h d\Gamma^h = h^2 \int_{\Gamma_{dN}} \left(\theta^h \circ \Pi^h\right) d\Gamma.$$

the formulas in part (b) follow from the definitions of the functions $\sigma_{ij}(h)$ and $D_i(h)$ given in part (a). ■

Alternatively, we could also have used the scaled inverted constitutive equation defined by (cf. (3.8))

$$
\begin{aligned}
e_{3\alpha}(\boldsymbol{u}(h)) &= 2h^2\bar{C}_{3\alpha3\theta}\,\sigma_{3\theta}(h) + h\bar{P}_{\beta3\alpha}\,D_\beta(h),\\
E_\alpha(\varphi(h)) &= -2h\bar{P}_{\alpha3\beta}\,\sigma_{3\beta}(h) + \bar{\varepsilon}_{\alpha\beta}\,D_\beta(h),\\
e_{\alpha\beta}(\boldsymbol{u}(h)) &= h^2\bar{C}_{\alpha\beta33}\,\sigma_{33}(h) + h^4\bar{C}_{\alpha\beta\theta\rho}\,\sigma_{\theta\rho}(h) + h\bar{P}_{3\alpha\beta}\,D_3(h),\\
e_{33}(\boldsymbol{u}(h)) &= \bar{C}_{3333}\,\sigma_{33}(h) + h^2\bar{C}_{33\theta\rho}\,\sigma_{\theta\rho}(h) + h\bar{P}_{333}\,D_3(h),\\
E_3(\varphi(h)) &= -h\bar{P}_{3\alpha\beta}\,\sigma_{\alpha\beta}(h) - h^{-1}\bar{P}_{333}\,\sigma_{33}(h) + h^{-2}\bar{\varepsilon}_{33}\,D_3(h).
\end{aligned}
\tag{4.25}
$$

As in the *displacement - electric potential* approach, the dependence on parameter h is now explicit and “polynomial”: more specifically, problem (2.67) reads

$$
\begin{aligned}
&h^{-2}a_{-2}\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right) + h^{-1}a_{-1}\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right)\\
&\quad + a_0\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right) + ha_1\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right)\\
&\quad + h^2a_2\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right) + h^4a_4\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right)\\
&\quad + b_H\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{u}(h),\varphi(h))\right) = 0, \quad \text{for all } \ (\boldsymbol{\tau},\boldsymbol{d}) \in \boldsymbol{X}_1,
\end{aligned}
\tag{4.26}
$$

$$
b_H\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{v},\psi)\right) = l_H(\boldsymbol{v},\psi), \qquad \text{for all } \ (\boldsymbol{v},\psi) \in \boldsymbol{X}_0,
\tag{4.27}
$$

where the bilinear forms a_{-2}, a_{-1}, a_0, a_1, a_2, a_4 and b_H, and the linear form l_H are defined by

$$
a_{-2}\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = \int_\Omega \bar{\varepsilon}_{33} D_3 d_3 d\boldsymbol{x},
\tag{4.28}
$$

$$
a_{-1}\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = \int_\Omega \bar{P}_{333} D_3 \tau_{33}\, d\boldsymbol{x} - \int_\Omega \bar{P}_{333}\sigma_{33} d_3 d\boldsymbol{x},
\tag{4.29}
$$

$$
a_0\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = \int_\Omega \bar{C}_{3333}\sigma_{33}\tau_{33}\, d\boldsymbol{x} + \int_\Omega \bar{\varepsilon}_{\alpha\beta} D_\beta d_\alpha d\boldsymbol{x},
\tag{4.30}
$$

$$
\begin{aligned}
a_1\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = &2\int_\Omega \bar{P}_{33\alpha} D_3 \tau_{3\alpha}\, d\boldsymbol{x} + \int_\Omega \bar{P}_{3\alpha\beta} D_3 \tau_{\alpha\beta}\, d\boldsymbol{x}\\
&- \int_\Omega \bar{P}_{3\alpha\beta}\sigma_{\alpha\beta} d_3 d\boldsymbol{x} - 2\int_\Omega \bar{P}_{\alpha3\theta}\sigma_{3\theta} d_\alpha d\boldsymbol{x},
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
a_2\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = &\int_\Omega \bar{C}_{33\alpha\beta}\sigma_{\alpha\beta}\tau_{33}\, d\boldsymbol{x} + 4\int_\Omega \bar{C}_{3\alpha3\beta}\sigma_{3\beta}\tau_{3\alpha}\, d\boldsymbol{x}\\
&+ \int_\Omega \bar{C}_{\alpha\beta33}\sigma_{33}\tau_{\alpha\beta}\, d\boldsymbol{x},
\end{aligned}
\tag{4.32}
$$

$$
a_4\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{\tau},\boldsymbol{d})\right) = \int_\Omega \bar{C}_{\alpha\beta\theta\rho}\sigma_{\theta\rho}\tau_{\alpha\beta}\, d\boldsymbol{x},
\tag{4.33}
$$

$$l_H(\boldsymbol{v},\psi) = -\int_\Omega f_i v_i d\boldsymbol{x} - \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma, \tag{4.34}$$

$$b_H\left((\boldsymbol{\sigma},\boldsymbol{D}),(\boldsymbol{v},\psi)\right) = -\int_\Omega \sigma_{ij}\, e_{ij}(\boldsymbol{v}) d\boldsymbol{x} - \int_\Omega D_k E_k(\psi)\, d\boldsymbol{x}. \tag{4.35}$$

4.3 The method of formal asymptotic expansions: the displacement-electric potential approach

In this section we intend to study the behavior of the solution to the problem (4.22)-(4.24) when $h \to 0$. This study is based on formal asymptotic expansions method.

We now assume that the solution of the problem (4.19)-(4.21) can be expressed in the form

$$\boldsymbol{u}(h) = \boldsymbol{u}^0 + h\boldsymbol{u}^1 + h^2\boldsymbol{u}^2 + ..., \quad \boldsymbol{u}^i \in V_{0,w}, \tag{4.36}$$

$$\bar{\varphi}(h) = \bar{\varphi}^0 + h\bar{\varphi}^1 + h^2\bar{\varphi}^2 + ..., \quad \bar{\varphi}^i \in \Psi_0, \tag{4.37}$$

where the successive coefficients of the power of h are independent of h. Since $\bar{\varphi}(h) = \varphi(h) - \hat{\varphi}$, then

$$\varphi(h) = \varphi^0 + h\bar{\varphi}^1 + h^2\bar{\varphi}^2 + ...\ , \tag{4.38}$$

with $\varphi^0 = \hat{\varphi} + \bar{\varphi}^0$.

The asymptotic developments (4.36) and (4.38) induce the following formal expansions for tensors $\boldsymbol{\sigma}(h)$ and $\boldsymbol{D}(h)$ (cf. (4.23))

$$\begin{cases} \boldsymbol{\sigma}(h) = h^{-4}\boldsymbol{\sigma}^{-4} + h^{-3}\boldsymbol{\sigma}^{-3} + h^{-2}\boldsymbol{\sigma}^{-2} + ..., \\ \boldsymbol{D}(h) = h^{-1}\boldsymbol{D}^{-1} + \boldsymbol{D}^0 + h\boldsymbol{D}^1 + ..., \end{cases} \tag{4.39}$$

with $\boldsymbol{\sigma}^q = \left(\boldsymbol{\sigma}^q_{ij}\right)$ and $\boldsymbol{D}^q = (D^q_i)$ independent of h. Inserting developments (4.36)-(4.39) into problem (4.22) results in a set of variational equations that must be satisfied for all $h > 0$ and consequently the terms at the successive powers of h must be zero. This procedure yields the following problems at the successive powers of h for all $(\boldsymbol{v},\psi) \in V_{0,w} \times \Psi_0$:

$$(P^{-4}): \quad \int_\Omega \sigma^{-4}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \tag{4.40}$$

$$(P^{-3}): \quad \int_\Omega \sigma^{-3}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \tag{4.41}$$

$$(P^{-2}): \quad \int_\Omega \sigma_{ij}^{-2} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \tag{4.42}$$

$$(P^{-1}): \quad \int_\Omega \sigma_{ij}^{-1} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_k^{-1} E_k(\psi)\, d\boldsymbol{x} = 0, \tag{4.43}$$

$$(P^{0}): \quad \int_\Omega \sigma_{ij}^{0} e_{ij}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega D_k^{0} E_k(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i\, d\Gamma, \tag{4.44}$$

where

$$\sigma_{\alpha\beta}^{-4} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^0), \tag{4.45}$$

$$\sigma_{\alpha\beta}^{-3} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^1), \tag{4.46}$$

$$\sigma_{\alpha\beta}^{-2} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^0), \tag{4.47}$$

$$\sigma_{\alpha\beta}^{-1} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^3) + C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^1) - P_{3\alpha\beta} E_3(\varphi^0), \tag{4.48}$$

$$\sigma_{\alpha\beta}^{p} = C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^{p+4}) + C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^{p+2}) - P_{3\alpha\beta} E_3(\bar{\varphi}^{p+1}), \quad p \geq 0,$$

$$\sigma_{3\alpha}^{-2} = 2C_{3\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^0), \tag{4.49}$$

$$\sigma_{3\alpha}^{-1} = 2C_{3\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^1) - P_{\beta 3\alpha} E_\beta(\varphi^0), \tag{4.50}$$

$$\sigma_{3\alpha}^{p} = 2C_{3\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^{p+2}) - P_{\beta 3\alpha} E_\beta(\bar{\varphi}^{p+1}), \quad p \geq 0,$$

$$\sigma_{33}^{-2} = C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^0), \tag{4.51}$$

$$\sigma_{33}^{-1} = C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^1), \tag{4.52}$$

$$\sigma_{33}^{0} = C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^2) + C_{3333} e_{33}(\boldsymbol{u}^0), \tag{4.53}$$

$$\sigma_{33}^{1} = C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^3) + C_{3333} e_{33}(\boldsymbol{u}^1) - P_{333} E_3(\varphi^0), \tag{4.54}$$

$$\sigma_{33}^{p} = C_{33\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^{p+2}) + C_{3333} e_{33}(\boldsymbol{u}^p) - P_{333} E_3(\bar{\varphi}^{p-1}), \quad p \geq 2,$$

$$D_\alpha^{-1} = 2P_{\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^0), \tag{4.55}$$

$$D_\alpha^{0} = 2P_{\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^1) + \varepsilon_{\alpha\beta} E_\beta(\varphi^0), \tag{4.56}$$

$$D_\alpha^{p} = 2P_{\alpha 3\beta} e_{3\beta}(\boldsymbol{u}^{p+1}) + \varepsilon_{\alpha\beta} E_\beta(\bar{\varphi}^p), \quad p \geq 1,$$

$$D_3^{-1} = P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^0), \tag{4.57}$$

$$D_3^{0} = P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^1), \tag{4.58}$$

$$D_3^{1} = P_{3\alpha\beta} e_{\alpha\beta}(\boldsymbol{u}^2) + P_{333} e_{33}(\boldsymbol{u}^0), \tag{4.59}$$

$$D_3^2 = P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^3) + P_{333}e_{33}(\boldsymbol{u}^1) + \varepsilon_{33}E_3(\varphi^0),$$

$$D_3^p = P_{3\alpha\beta}e_{\alpha\beta}(\boldsymbol{u}^{p+1}) + P_{333}e_{33}(\boldsymbol{u}^{p-1}) + \varepsilon_{33}E_3(\bar{\varphi}^{p-2}), \quad p \geq 3.$$

4.3.0.1 Cancelation of the factors of h^q, $-4 \leq q \leq 0$.

Let V_{BN} be the space of Bernoulli-Navier displacements defined by

$$V_{BN} = V_{BN}(\Omega) = \{\boldsymbol{v} \in V_{0,w} : e_{\alpha\beta}(\boldsymbol{v}) = e_{3\beta}(\boldsymbol{v}) = 0\}, \tag{4.60}$$

equipped with the norm $\|\boldsymbol{v}\|_{V_{0,w}} = |\boldsymbol{e}(\boldsymbol{v})|_\Omega$. The space V_{BN} can be equivalently defined by (see e.g. Ref. Trabucho & Viaño [1996])

$$\begin{aligned} V_{BN} =\{\boldsymbol{v} \in V_{0,w} : v_\alpha(x_1,x_2,x_3) = \chi_\alpha(x_3), \quad \chi_\alpha \in H_0^2(0,L), \\ v_3(x_1,x_2,x_3) = \chi_3(x_3) - x_\beta\chi'_\beta(x_3), \quad \chi_3 \in H_0^1(0,L)\}. \end{aligned} \tag{4.61}$$

Now, in order to obtain some properties of $\boldsymbol{u}^0$ and $\boldsymbol{u}^2$, we will show that some terms of the developments (4.39) are null. We start with problem (P^{-4}) and prove that $\sigma_{\alpha\beta}^{-4} = 0$. For such, we consider (4.40) and (4.45), and take $\boldsymbol{v} = \boldsymbol{u}^0$ yielding

$$\int_\Omega C_{\alpha\beta\theta\rho}e_{\theta\rho}(\boldsymbol{u}^0)e_{\alpha\beta}(\boldsymbol{u}^0)\, d\boldsymbol{x} = 0.$$

The coerciveness of $\boldsymbol{C}$ leads straightforwardly to

$$e_{\alpha\beta}(\boldsymbol{u}^0) = 0, \tag{4.62}$$

and therefore (cf. (4.45), (4.51) and (4.57))

$$\sigma_{\alpha\beta}^{-4} = \sigma_{33}^{-2} = 0, \quad D_3^{-1} = 0. \tag{4.63}$$

We turn now to problem (P^{-2}). Taking $\boldsymbol{v} = \boldsymbol{u}^0$ in (4.42) and bearing in mind (4.62) and (4.63) we obtain

$$\int_\Omega C_{3\alpha3\beta}e_{3\beta}(\boldsymbol{u}^0)e_{3\alpha}(\boldsymbol{u}^0)\, d\boldsymbol{x} = 0,$$

which implies

$$e_{3\alpha}(\boldsymbol{u}^0) = 0, \tag{4.64}$$

and therefore (cf. (4.49) and (4.55))

$$\sigma_{3\alpha}^{-2} = 0, \quad D_\alpha^{-1} = 0. \tag{4.65}$$

As we have seen previously, conditions (4.62) and (4.64) are equivalent to (4.60) and (4.61), i.e. $\boldsymbol{u}^0$ is an element of the space of Bernoulli-Navier displacements and consequently

$$\begin{cases} u^0_\alpha(x_1,x_2,x_3) = \xi_\alpha(x_3), \quad u^0_3(x_1,x_2,x_3) = \xi_3(x_3) - x_\beta \xi'_\beta(x_3), \\ \xi_\alpha \in H^2_0(0,L), \quad \xi_3 \in H^1_0(0,L). \end{cases} \tag{4.66}$$

In the following we will establish necessary and sufficient conditions for $\sigma^{-2}_{\alpha\beta} = 0$. This will allow to characterize the transverse components of the second order displacement $\boldsymbol{u}^2$. As a result, we will be able to obtain variational problems that have ξ_α and ξ_3 as unique solutions.

Theorem 8 *A necessary and sufficient condition for* $\sigma^{-2}_{\alpha\beta} = 0$ *is that there exists* $\widetilde{\boldsymbol{u}} \in V_{0,w}$ *such that*

$$C_{\alpha\beta\theta\rho} e_{\theta\rho}(\widetilde{\boldsymbol{u}}) + C_{\alpha\beta33} e_{33}(\boldsymbol{u}^0) = 0. \tag{4.67}$$

Proof. From (4.63) and (4.65) problem (P^{-2}) can be written

$$\int_\Omega \sigma^{-2}_{\alpha\beta} e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in V_{0,w}.$$

In the view of (4.47) the previous equality becomes

$$\int_\Omega \left(C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^2) + C_{\alpha\beta33} e_{33}(\boldsymbol{u}^0)\right) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0,$$

equivalent to

$$\int_\Omega \left(C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^2) - C_{\alpha\beta\theta\rho} e_{\theta\rho}(\widetilde{\boldsymbol{u}}) + C_{\alpha\beta\theta\rho} e_{\theta\rho}(\widetilde{\boldsymbol{u}}) + C_{\alpha\beta33} e_{33}(\boldsymbol{u}^0)\right) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0.$$

If $\widetilde{\boldsymbol{u}} \in V_{0,w}$ verifies (4.67), then the previous equation reads

$$\int_\Omega C_{\alpha\beta\theta\rho} \left(e_{\theta\rho}(\boldsymbol{u}^2 - \widetilde{\boldsymbol{u}})\right) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0,$$

and therefore taking $\boldsymbol{v} = \boldsymbol{u}^2 - \widetilde{\boldsymbol{u}} \in V_{0,w}$ in this equation leads to $\widetilde{\boldsymbol{u}} = \boldsymbol{u}^2$ by the coerciveness of $\boldsymbol{C}$, implying $\sigma^{-2}_{\alpha\beta} = 0$. Inversely, if $\sigma^{-2}_{\alpha\beta} = 0$ then (4.67) holds with $\widetilde{\boldsymbol{u}} = \boldsymbol{u}^2$. ■

Remark 10 *From Theorem 8 and (4.47) we conclude that if equation (4.67) has solutions then* u^2 *is a solution of (4.67). Next we charaterize the solutions of (4.67)*

4.3.0.2 Characterization of $\boldsymbol{u}^0$ and $\boldsymbol{u}^2$

Firstly, we define the following reduced symmetric matrices:

$$\boldsymbol{M} = \begin{pmatrix} C_{1111} & C_{1211} & C_{1122} \\ C_{1211} & C_{1212} & C_{1222} \\ C_{1122} & C_{1222} & C_{2222} \end{pmatrix}, \qquad \boldsymbol{N} = \begin{pmatrix} C_{3131} & C_{3132} \\ C_{3132} & C_{3232} \end{pmatrix},$$

$$\boldsymbol{C} = \begin{pmatrix} C_{1111} & C_{1211} & C_{1122} & 0 & 0 & C_{1133} \\ C_{1211} & C_{1212} & C_{1222} & 0 & 0 & C_{1233} \\ C_{1122} & C_{1222} & C_{2222} & 0 & 0 & C_{2233} \\ 0 & 0 & 0 & C_{3131} & C_{3132} & 0 \\ 0 & 0 & 0 & C_{3132} & C_{3232} & 0 \\ C_{1133} & C_{1233} & C_{2233} & 0 & 0 & C_{3333} \end{pmatrix}.$$

The hypothesis ($\mathbf{H}^c_{21}$) and ($\mathbf{H}^c_{22}$), introduced in Section 2.1.3, imply that these matrices are definite positive and therefore $\det \boldsymbol{C} > 0$, $\det \boldsymbol{M} > 0$ and $\det \boldsymbol{N} > 0$.

Before proceeding, we now define some functions that will be useful in the characterization of the elements of $V_{0,w}$ that verify (4.67).
We define the geometry functions $\overline{\overline{\Lambda}}_\alpha(x_1, x_2)$ and $\boldsymbol{\Phi} = (\Phi_{\alpha\beta})$ defined by

$$\begin{aligned}
&\overline{\overline{\Lambda}}_\alpha(x_1, x_2) = -\frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}} x_\beta, \\
&\Phi_{11} = \frac{1}{2 \det \boldsymbol{M}} (\det \boldsymbol{M}_{11} x_1^2 - \det \boldsymbol{M}_{22} x_2^2), \\
&\Phi_{12} = \frac{\det \boldsymbol{M}_{1\theta}}{\det \boldsymbol{M}} x_\theta x_2, \qquad \Phi_{21} = \frac{\det \boldsymbol{M}_{2\theta}}{\det \boldsymbol{M}} x_\theta x_1, \\
&\Phi_{22} - \frac{1}{2 \det \boldsymbol{M}} (\det \boldsymbol{M}_{11} x_1^2 - \det \boldsymbol{M}_{22} x_2^2)
\end{aligned}$$

where

$$\boldsymbol{M}_{11} = \begin{pmatrix} C_{1133} & C_{1112} & C_{1122} \\ C_{1233} & C_{1212} & C_{1222} \\ C_{2233} & C_{1222} & C_{2222} \end{pmatrix}, \quad \boldsymbol{M}_{22} = \begin{pmatrix} C_{1111} & C_{1112} & C_{1133} \\ C_{1112} & C_{1212} & C_{1233} \\ C_{1122} & C_{1222} & C_{2233} \end{pmatrix},$$

$$\boldsymbol{M}_{12} = \boldsymbol{M}_{21} = \frac{1}{2} \begin{pmatrix} C_{1111} & C_{1133} & C_{1122} \\ C_{1112} & C_{1233} & C_{1222} \\ C_{1122} & C_{2233} & C_{2222} \end{pmatrix}.$$

We also introduce the following constants that depend only on the geometry of the cross section,

$$I_\alpha = \int_\omega x_\alpha^2 \, d\omega, \quad \overline{X}_{\alpha\beta} = \int_\omega \Phi_{\alpha\beta} \, d\omega, \quad \overline{Y}_\alpha = \int_\omega \Phi_{\alpha\beta}\delta_\beta \, d\omega, \quad \overline{Z} = \int_\omega \overline{\overline{\Lambda}}_\alpha \delta_\alpha \, d\omega.$$

Theorem 9 *Let $\boldsymbol{u}^0 \in V_{BN}$ be given by (4.66). Then every element $(\widetilde{u}_\alpha) \in [L^2(\Omega)]^2$ is a solution of (4.67) if and only if has the following form:*

$$\widetilde{u}_\alpha = s_\alpha + \delta_\alpha s + \Phi_{\alpha\beta}\xi''_\beta + \overline{\overline{\Lambda}}_\alpha \xi'_3, \tag{4.68}$$

with s, $s_\alpha \in L^2(0, L)$ depending only on x_3.

Proof. Let us consider (4.67) in the following form

$$C_{\alpha\beta\theta\rho} e_{\theta\rho}(\widetilde{\boldsymbol{u}}) = -C_{\alpha\beta 33} e_{33}(\boldsymbol{u}^0),$$

or, equivalently,

$$\begin{cases} \partial_1 \widetilde{u}_1 = \dfrac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}} \left(x_\alpha \xi''_\alpha - \xi'_3\right), \\ \partial_2 \widetilde{u}_2 = \dfrac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}} \left(x_\alpha \xi''_\alpha - \xi'_3\right), \\ \partial_1 \widetilde{u}_2 + \partial_2 \widetilde{u}_1 = \dfrac{\det \boldsymbol{M}_{12} + \det \boldsymbol{M}_{21}}{\det \boldsymbol{M}} \left(x_\alpha \xi''_\alpha - \xi'_3\right). \end{cases} \tag{4.69}$$

A direct integration of the first two equations gives

$$\begin{cases} \widetilde{u}_1 = \dfrac{\det \boldsymbol{M}_{11}}{2 \det \boldsymbol{M}} (x_1^2 \xi''_1 + 2x_1 x_2 \xi''_2 - 2x_1 \xi'_3) + k_1(x_2, x_3), \\ \widetilde{u}_2 = \dfrac{\det \boldsymbol{M}_{22}}{2 \det \boldsymbol{M}} (x_2^2 \xi''_2 + 2x_1 x_2 \xi''_1 - 2x_2 \xi'_3) + k_2(x_1, x_3). \end{cases} \tag{4.70}$$

Substituting these expressions in the third equation of (4.69), we obtain

$$\begin{aligned} &\frac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}} x_2 \xi''_1 - \frac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}} \left(2x_2 \xi''_2 - \xi'_3\right) + \partial_2 k_1(x_2, x_3) \\ &\qquad = -\frac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}} x_1 \xi''_2 + \frac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}} \left(2x_1 \xi''_1 - \xi'_3\right) - \partial_1 k_2(x_1, x_3). \end{aligned}$$

Since the left (resp. right) hand term depends only on variables x_2 (resp. x_1)and x_3, we conclude that

$$\begin{aligned} \partial_2 k_1(x_2, x_3) &= s - \frac{\det \boldsymbol{M}_{22}}{\det \boldsymbol{M}} x_2 \xi''_1 + \frac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}} \left(2x_2 \xi''_2 - \xi'_3\right), \\ \partial_1 k_2(x_1, x_3) &= -s - \frac{\det \boldsymbol{M}_{11}}{\det \boldsymbol{M}} x_1 \xi''_2 + \frac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}} \left(2x_1 \xi''_1 - \xi'_3\right). \end{aligned}$$

Thus,

$$\left\{\begin{aligned} k_1(x_2,x_3) &= s_1 + x_2 s - \frac{\det \boldsymbol{M}_{22}}{2\det \boldsymbol{M}} x_2^2 \xi_1'' + \frac{\det \boldsymbol{M}_{12}}{\det \boldsymbol{M}} \left(x_2^2 \xi_2'' - x_2 \xi_3'\right), \\ k_2(x_1,x_3) &= s_2 - x_1 s - \frac{\det \boldsymbol{M}_{11}}{2\det \boldsymbol{M}} x_1^2 \xi_2'' + \frac{\det \boldsymbol{M}_{21}}{\det \boldsymbol{M}} \left(x_1^2 \xi_1'' - x_1 \xi_3'\right), \end{aligned}\right. \tag{4.71}$$

where s_α are arbitrary functions depending only on x_3. Combining (4.70) with (4.71) leads to (4.68).

Let $\widetilde{u}_\alpha \in L^2(\Omega)$. Then $\int_\omega \widetilde{u}_\alpha \, d\omega$, $\int_\omega \delta_\beta \widetilde{u}_\beta \, d\omega$, ξ_α'', $\xi_3' \in L^2(0,L)$ and (3.29) together with (4.68) lead to s_α, $s \in L^2(0,L)$. ■

Corollary 7 *Let $(\widetilde{u}_\alpha) \in [L^2(\Omega)]^2$ a solution of (4.67) using the form (4.68). Then, a necessary and sufficient condition for problem (4.67) to have solutions $\widetilde{\boldsymbol{u}} \in V_{0,w}$ is that $\boldsymbol{u}^0 \in V_{BN}$ be such that $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$, and that s, $s_\alpha \in H^1(0,L)$ verify*

$$\left\{\begin{aligned} s_\alpha(a) &= -\frac{1}{A(\omega)} \overline{X}_{\alpha\beta} \xi_\beta''(a), \\ s(a) &= -\frac{1}{I_1 + I_2} \left(\overline{Y}_\beta \xi_\beta''(a) + \overline{Z} \xi_3'(a)\right), \end{aligned}\right. \tag{4.72}$$

with $a = 0, L$.

Proof. Let $\widetilde{u}_\alpha^2$, the transverse components of $\widetilde{\boldsymbol{u}} \in V_{0,w}$, be the solution of (4.67). From Theorem 9 it follows that $\widetilde{u}_\alpha$ is of the form (4.68), verifying (4.69). Therefore, using (3.29) we obtain

$$\begin{aligned} &\int_\omega \partial_\alpha \widetilde{u}_\alpha \, d\omega = -\frac{\det \boldsymbol{M}_{\alpha\alpha}}{\det \boldsymbol{M}} A(\omega) \xi_3' \qquad \text{(no sum on } \alpha), \\ &\int_\omega x_\alpha \partial_\alpha \widetilde{u}_\alpha \, d\omega = \frac{\det \boldsymbol{M}_{\alpha\alpha}}{\det \boldsymbol{M}} I_\alpha \xi_\alpha'' \qquad \text{(no sum on } \alpha). \end{aligned}$$

Since $\widetilde{u}_\alpha \in H^1(\Omega)$, then $\int_\omega \partial_\alpha \widetilde{u}_\alpha \, d\omega$, $\int_\omega x_\alpha \partial_\alpha \widetilde{u}_\alpha \, d\omega \in H^1(0,L)$ (no sum on α) and consequently $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$. On the other hand, identity (4.68) allows to write

$$\begin{aligned} &\int_\omega \widetilde{u}_\alpha \, d\omega = A(\omega) s_\alpha + \overline{X}_{\alpha\beta} \xi_\beta'' \in H^1(0,L), \\ &\int_\omega \delta_\beta \widetilde{u}_\beta \, d\omega = (I_1 + I_2)\, s + \overline{Y}_\theta \xi_\theta'' + \overline{Z} \xi_3' \in H^1(0,L). \end{aligned}$$

Hence, the previous regularity results lead to s, $s_\alpha \in H^1(0,L)$. Furthermore, the boundary condition for $\widetilde{u}_\alpha$ (cf. (4.11)) leads to (4.72).

Conversely, taking $\boldsymbol{u}^0 \in V_{BN}$ such that $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$, and s, $s_\alpha \in H^1(0,L)$ of the form (4.72), then (4.68) implies that $u_\alpha^2 \in H^1(\Omega)$ verifies the boundary condition (4.11). ■

Corollary 8 *The components* u_α^2 *of the displacement* $\boldsymbol{u}^2 \in V_{0,w}$ *are of the form (4.68) with* $s, s_\alpha \in H^1(0,L)$ *satisfying (4.72) (and consequently* $\sigma_{\alpha\beta}^{-2} = 0$*) if and only if* $\boldsymbol{u}^0 \in V_{BN}$ *is such that* $\xi_\alpha \in H^3(0,L)$*,* $\xi_3 \in H^2(0,L)$*.*

Remark 11 *From (4.68) it follows that in general* u_α^2 *is not null at the beam ends, for that would require that both* ξ_β'' *and* ξ_3' *vanish there. Therefore, if we had chosen strong boundary conditions in the definition of* $V_{0,w}$*, then we would not have been able to ensure that* $\boldsymbol{u}^2 \in V$ *(the well-known "boundary layer phenomenon") and consequently that* $\sigma_{\alpha\beta}^{-2} = 0$*.*

For now we can only ensure that $\xi_\alpha \in H_0^2(0,L)$, $\xi_3 \in H_0^1(0,L)$. The additional regularity required by the previous corollary will follow from the variational problems to which ξ_α, ξ_3 obey. We have the following results.

Theorem 10 *Let* $\boldsymbol{u}^0 \in V_{BN}$ *given by (4.66) and* $f_i \in L^2(\Omega)$*,* $g_i \in L^2(\Gamma_{dN})$*. Then* $\boldsymbol{u}^0$ *is the unique solution of the following variational problem:*

$$\int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{v})\, d\boldsymbol{x} = \int_\Omega f_i v_i \, d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i v_i \, d\Gamma, \quad \text{for all } \boldsymbol{v} \in V_{BN}, \tag{4.73}$$

where σ_{33}^0 *is given by*

$$\sigma_{33}^0 = Y e_{33}(\boldsymbol{u}^0) = Y(\xi_3' - x_\alpha \xi_\alpha''), \tag{4.74}$$

and

$$Y = \frac{\det \boldsymbol{C}}{\det \boldsymbol{M} \ \det \boldsymbol{N}} \tag{4.75}$$

is positive and bounded.

Proof. Problem (4.73) follows from the variational equation (4.44) taking $\boldsymbol{v} \in V_{BN}$ and $\psi = 0$. From Corollary 8 we get that u_α^2 is of the form (4.68), and consequently (4.53) becomes (4.74). ■

Corollary 9 *The transverse components* ξ_α *and the stretch component* ξ_3 *of the zeroth order displacement field are respectively the unique solutions of the following variational problems (no sum on* α*):*

$$\xi_\alpha \in H_0^2(0,L),$$

$$\int_0^L Y I_\alpha \xi_\alpha'' \chi_\alpha''\, dx_3 = \int_0^L F_\alpha \chi_\alpha \, dx_3 - \int_0^L M_\alpha \chi_\alpha' \, dx_3, \quad \text{for all } \chi_\alpha \in H_0^2(0,L), \tag{4.76}$$

and

$$\begin{aligned}
&\xi_3 \in H_0^1(0,L),\\
&\int_0^L YA(\omega)\xi_3'\chi_3'\,dx_3 = \int_0^L F_3\chi_3\,dx_3, \qquad \textit{for all } \chi_3 \in H_0^1(0,L),
\end{aligned} \tag{4.77}$$

with

$$F_i(x_3) = \int_\omega f_i\,d\omega + \int_{\gamma_{dN}} g_i\,d\gamma \in L^2(0,L), \tag{4.78}$$

$$M_\alpha(x_3) = \int_\omega x_\alpha f_3\,d\omega + \int_{\gamma_{dN}} x_\alpha g_3\,d\gamma \in L^2(0,L). \tag{4.79}$$

Proof. We consider (4.73) with the test function $\boldsymbol{v} \in V_{BN}$ of the form (4.61), implying $e_{33}(\boldsymbol{v}) = \chi_3'(x_3) - x_\theta\chi_\theta''(x_3)$. Then, taking successively $\chi_\theta = 0$ and $\chi_3 = 0$ in the resulting equation we obtain (4.76) and (4.77), respectively. ■

The following regularity result (see e.g. Ref. Brezis [1983]) ensures the regularity requirements for ξ_α and ξ_3 in Theorem 10.

Corollary 10 *The solutions of (4.76) and (4.77) are unique and* $\xi_\alpha \in H^3(0,L)$, $\xi_3 \in H^2(0,L)$.

Corollary 11 *The components* u_α^2 *of* $\boldsymbol{u}^2$ *verify (4.68) and consequently* $\sigma_{\alpha\beta}^{-2} = 0$.

Corollary 12 *The component* D_3^1 *of* $\boldsymbol{D}^1$ *is given by*

$$D_3^1 = \overline{P}e_{33}(\boldsymbol{u}^0) = \overline{P}\,(\xi_3' - x_\alpha\xi_\alpha'') \in H^1(\Omega),$$

with

$$\overline{P} = P_{333} - \frac{\det \boldsymbol{M}_{\alpha\beta}}{\det \boldsymbol{M}}P_{3\alpha\beta}. \tag{4.80}$$

Proof. This result follows straightforwardly from (4.59) taking into account that $\boldsymbol{u}^2$ verifies (4.69) and that (4.74) holds. ■

4.3.0.3 Characterization of $\boldsymbol{u}^1$ and φ^0

Let V_1 be the space

$$V_1 = V_1(\Omega) = \{\boldsymbol{v} \in V : e_{\alpha\beta}(\boldsymbol{v}) = 0\}\,.$$

We have, as in Trabucho & Viaño [1996], some equivalent expressions for V_1 .

Lemma 3 *It may verified that:*

$$V_1 = \{\boldsymbol{v} \in V_{0,w} : v_\alpha(x_1, x_2, x_3) = \zeta_\alpha(x_3) + \delta_\alpha \zeta(x_3), \quad \zeta_\alpha, \zeta \in H_0^1(0, L)\}. \tag{4.81}$$

We can now introduce the spaces

$$Q = Q(\omega) = \{q \in H^1(\omega) : \textstyle\int_\omega q \, d\omega = 0\} \tag{4.82}$$

$$S = S(\omega) = \{\psi \in H^1(\omega) : \psi = 0 \text{ on } \gamma_{eD}\}. \tag{4.83}$$

with its natural norms

$$\|\rho\|_Q = \left(|\rho|_\omega^2 + |\partial_1\rho|_\omega^2 + |\partial_2\rho|_\omega^2\right)^{1/2}, \qquad \|\psi\|_S \quad = \left(|\psi|_\omega^2 + |\partial_1\psi|_\omega^2 + |\partial_2\psi|_\omega^2\right)^{1/2},$$

which are equivalent (thank to the Poincaré-Friedrichs inequality in ω) to the following norms

$$\|\rho\|_Q = \left(|\partial_1\rho|_\omega^2 + |\partial_2\rho|_\omega^2\right)^{1/2}, \qquad \|\psi\|_S = \left(|\partial_1\psi|_\omega^2 + |\partial_2\psi|_\omega^2\right)^{1/2}.$$

We equip the product space

$$T := T(\omega) = Q(\omega) \times S(\omega) \tag{4.84}$$

with the norm

$$\|(\rho, \psi)\|_T^2 = \|\rho\|_Q^2 + \|\psi\|_S^2 = |\partial_1\rho|_\omega^2 + |\partial_2\rho|_\omega^2 + |\partial_1\psi|_\omega^2 + |\partial_2\psi|_\omega^2.$$

Finally, let Ψ_l be the space

$$\Psi_l = \Psi_l(\Omega) = \{\psi \in L^2(\Omega) : \partial_\alpha\psi \in L^2(\Omega)\} \equiv L^2(0, L; H^1(\omega)), \tag{4.85}$$

and let us define its subspaces

$$R = R(\Omega) = L^2(0, L; Q(\omega)), \quad \text{and} \quad \Psi_{l0} = L^2(0, L; S(\omega)). \tag{4.86}$$

We now define the warping function $w(x_1, x_2)$ as the unique solution of the variational problem:

$$\begin{aligned} & w \in Q \text{ such that} \\ & \int_\omega C_{3\alpha3\beta}\partial_\beta w \partial_\alpha v \, d\omega = \int_\omega C_{3\alpha3\beta}\delta_\beta\partial_\alpha v \, d\omega, \quad \text{for all } v \in H^1(\omega), \end{aligned} \tag{4.87}$$

as well as the torsion constant $J > 0$:

$$J = \int_\omega C_{3\alpha3\beta}(\delta_\beta - \partial_\beta w)(\delta_\alpha - \partial_\alpha w)\, d\omega = \int_\omega C_{3\alpha3\beta}(\delta_\beta - \partial_\beta w)\delta_\alpha\, d\omega. \tag{4.88}$$

Finally, let $r \in R$ be the unique solution of

$$\begin{aligned} & r(x_3) \in Q \text{ such that } \forall x_3 \in [0, L] : \\ & \int_\omega C_{3\alpha3\beta}\partial_\beta r(x_3)\partial_\alpha \upsilon\, d\omega = \int_\omega P_{\beta3\alpha}\partial_\beta \varphi^0(x_3)\partial_\alpha \upsilon\, d\omega, \quad \text{for all } \upsilon \in Q. \end{aligned} \tag{4.89}$$

Having the previous definitions in mind, we turn to the characterization of $\boldsymbol{u}^1$ and φ^0. For that, we start considering problem (P^{-3}). Combining (4.41) with (4.46) we get

$$\int_\Omega C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^1) e_{\alpha\beta}(\boldsymbol{v})\, d\boldsymbol{x} = 0 \quad \text{for all } \boldsymbol{v} \in V_{0,w}. \tag{4.90}$$

The previous equation for $\boldsymbol{v} = \boldsymbol{u}^1 \in V_{0,w}$ is written:

$$\int_\Omega C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^1) e_{\alpha\beta}(\boldsymbol{u}^1)\, d\boldsymbol{x} = 0,$$

then $e_{\alpha\beta}(\boldsymbol{u}^1) = 0$. Hence, $\boldsymbol{u}^1 = (u_i^1) \in V_1$ and we write (cf. (4.81))

$$u_\alpha^1 = z_\alpha + \delta_\alpha z, \quad z_\alpha, z \in H_0^1(0, L). \tag{4.91}$$

Thus (cf. (4.46), (4.52) and (4.58))

$$\sigma_{\alpha\beta}^{-3} = \sigma_{33}^{-1} = 0, \quad D_3^0 = 0.$$

Equation (4.43) evaluated in $\boldsymbol{v} \in V_1$ and equation (4.44) for $\boldsymbol{v} = \boldsymbol{0}$ in (4.44), and considering (4.50) and (4.56) are given by

$$\int_\Omega \left(2C_{3\alpha3\beta} e_{3\beta}(\boldsymbol{u}^1) + P_{\beta3\alpha}\partial_\beta \varphi^0\right) e_{3\alpha}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in V_1, \tag{4.92}$$

$$\int_\Omega \left(2P_{\beta3\alpha} e_{3\alpha}(\boldsymbol{u}^1) - \varepsilon_{\alpha\beta}\partial_\alpha \varphi^0\right) \partial_\beta \psi\, d\boldsymbol{x} = 0, \quad \text{for all } \psi \in \Psi_0, \tag{4.93}$$

which show that $\boldsymbol{u}^1$ and φ^0 are coupled. In fact, we have the following result.

Theorem 11 *Let $\varphi^0 \in \Psi_l$ and $\widehat{\boldsymbol{u}} \in [L^2(\Omega)]^3$ be of the form (4.91). Then $\widehat{\boldsymbol{u}}$ is a solution of (4.92) if and only if*

$$\widehat{u}_3 = -r - z_3 - x_\alpha z_\alpha' - w z', \tag{4.94}$$

where $r \in R$ solves (4.89), w is the warping function and $z_3 \in L^2(0,L)$ is an arbitrary function of x_3.

Proof. Assuming that $\widehat{\boldsymbol{u}}$ is a solution of (4.92) and in view of (4.91) we conclude that $\widehat{u}_3$ verifies

$$\int_\Omega \left(C_{3\alpha 3\beta}\left(\partial_\beta \widehat{u}_3 + z'_\beta + \delta_\beta z'\right) + P_{\beta 3\alpha}\partial_\beta \varphi^0\right) e_{3\alpha}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in V_1. \tag{4.95}$$

We restrict (4.95) to $\boldsymbol{v} = (0,0,\chi(x_3)\upsilon(x_1,x_2))$, $\chi \in H^1_0(0,L)$, $\upsilon \in Q$ and combining it with (4.87), we get *a.e.* $x_3 \in (0,L)$

$$\int_\omega C_{3\alpha 3\beta}\partial_\beta \widehat{u}_3(x_3)\partial_\alpha \upsilon\, d\omega = -\int_\omega C_{3\alpha 3\beta}\partial_\beta(x_\theta z'_\theta(x_3) + wz'(x_3))\partial_\alpha \upsilon\, d\omega$$
$$-\int_\omega P_{\beta 3\alpha}\partial_\beta \varphi^0(x_3)\partial_\alpha \upsilon\, d\omega, \quad \text{for all } \upsilon \in Q.$$

Therefore,

$$U = \widehat{u}_3 + x_\theta z'_\theta + wz' \tag{4.96}$$

is a solution a.e. $x_3 \in (0,L)$ of

$$\int_\omega C_{3\alpha 3\beta}\partial_\beta U(x_3)\partial_\alpha \upsilon\, d\omega = -\int_\omega P_{\beta 3\alpha}\partial_\beta \varphi^0(x_3)\partial_\alpha \upsilon\, d\omega, \qquad \forall \upsilon \in Q. \tag{4.97}$$

Hence, if r is the solution of (4.89) then

$$U = -r - z_3, \quad a.e.\ x_3 \in (0,L), \tag{4.98}$$

where z_3 is an arbitrary function of x_3, leading to (4.94). From (4.94) we conclude that since $r \in L^2(0,L;H^1(\omega))$, $z_\alpha, z \in H^1_0(0,L)$ and $w \in H^1(\omega)$, then $z_3 \in L^2(0,L)$. The converse result is immediate. ∎

Corollary 13 *Let u^1_α, the transverse component of $\boldsymbol{u}^1$, be of the form (4.91) and let $\boldsymbol{u}^1$ verify (4.92). Then the axial component u^1_3 is such that*

$$u^1_3 = -r - z_3 - x_\alpha z'_\alpha - wz' \in \Psi_l, \tag{4.99}$$

where $z_3 \in H^1_0(0,L)$ is an arbitrary function of x_3. Furthermore, the component D^0_α of $\boldsymbol{D}^0$ is given by

$$D^0_\alpha = -P_{\alpha 3\beta}(\partial_\beta r + z'\left(\partial_\beta w - \delta_\beta\right)) - \varepsilon_{\alpha\beta}\partial_\beta \varphi^0 \in L^2(\Omega).$$

Proof. The expression for D^0_α is obtained from (4.56) taking (4.99) into account. ∎

In view of the previous regularity result we conclude that although (4.91) implies that $u^1_\alpha \in H^1(\Omega)$, we can only ensure that $u^1_3 \in L^2(0,L;H^1(\omega))$. Furthermore, $\int_\omega u^1_3\,d\omega$ vanishes at the beam ends as long as $z_3 \in H^1_0(0,L)$, but in general this is not the case for $\int_\omega x_\alpha u^1_3\,d\omega$. Therefore, we cannot ensure that $\boldsymbol{u}^1 \in V_{0,w}$, traducing a "boundary layer phenomenon". On the other hand, the regularity result obtained for u^1_3 implies that $U \in L^2(0,L;H^1(\omega))$ - cf. (4.96) - and therefore from (4.97) we can only guarantee that φ^0 is in Ψ_l not in Ψ_0. Hence, $\bar{\varphi}^0 = \varphi^0 - \varphi_0 \in \Psi_{l0}$.

Still, the characterization of $\boldsymbol{u}^1$ done so far is valid, in particular as far as equations (4.91) and (4.99) are concerned. This will allow to use equations (4.92) and (4.93) to derive a variational problem for $\bar{\varphi}^0$ and r posed in each cross section $\omega \times \{x_3\}$. In this context it will be useful to consider for a generic function $\eta : \overline{\Omega} \to \mathbb{R}$ its average along the x_3-axis, that will be denoted

$$\underline{\eta}(x_1,x_2) = \frac{1}{L}\int_0^L \eta(x_1,x_2,s)\,ds, \tag{4.100}$$

as well as the corresponding deviation

$$\eta^D(x_1,x_2,x_3) = \eta(x_1,x_2,x_3) - \underline{\eta}(x_1,x_2). \tag{4.101}$$

We have a first variational equation relating the unknowns $\overline{\varphi}^0$, r and z'.

Theorem 12 *Let $\boldsymbol{u}^1$ be such that (4.91) and (4.99) hold and let $(r,\varphi^0) \in R \times \Psi_l$ verify (4.89) and (4.93). Then, $\bar{\varphi}^0$, r and z verify a.e. $x_3 \in (0,L)$*

$$\begin{aligned}
&(r(x_3), \bar{\varphi}^0(x_3)) \in T,\ \ z(x_3) \in H^1_0(0,L) \ \textit{such that}\\
&\int_\omega C_{3\alpha3\beta}\partial_\beta r(x_3)\partial_\alpha\rho\,d\omega + \int_\omega \varepsilon_{\alpha\beta}\partial_\alpha\bar{\varphi}^0(x_3)\partial_\beta\psi\,d\omega + z'(x_3)\int_\omega P_{\beta3\alpha}\left(\partial_\alpha w - \delta_\alpha\right)\partial_\beta\psi\,d\omega\\
&\qquad + \int_\omega P_{\beta3\alpha}(\partial_\alpha r(x_3)\partial_\beta\psi - \partial_\beta\overline{\varphi}^0(x_3)\partial_\alpha\rho)\,d\omega \qquad\qquad (4.102)\\
&\quad = \int_\omega P_{\beta3\alpha}\partial_\beta\varphi_0(x_3)\partial_\alpha\rho\,d\omega - \int_\omega \varepsilon_{\alpha\beta}\partial_\alpha\varphi_0(x_3)\partial_\beta\psi\,d\omega, \quad \textit{for all } (\rho,\psi) \in T.
\end{aligned}$$

Proof. Starting from (4.91) and (4.99) we obtain

$$2e_{3\alpha}(\boldsymbol{u}^1) = -\partial_\alpha r - z'\left(\partial_\alpha w - \delta_\alpha\right), \tag{4.103}$$

which plugged into (4.93) yields

$$\int_\Omega \left(P_{\beta3\alpha}(\partial_\alpha r + z'\left(\partial_\alpha w - \delta_\alpha\right)) + \varepsilon_{\alpha\beta}\partial_\alpha\varphi^0\right)\partial_\beta\widetilde{\psi}\,d\boldsymbol{x} = 0, \quad \text{for all } \widetilde{\psi} \in \Psi.$$

Taking $\widetilde{\psi} = \psi(x_1, x_2)\,\chi(x_3)$, $\psi \in S$, $\chi \in H^1(0, L)$ in the previous equation we have a.e. $x_3 \in (0, L)$

$$\int_\omega \left(P_{\beta 3\alpha}(\partial_\alpha r(x_3) + z'(x_3)\,(\partial_\alpha w - \delta_\alpha)) + \varepsilon_{\alpha\beta}\partial_\alpha \varphi^0(x_3)\right)\partial_\beta \psi \, d\omega = 0, \quad \text{for all } \psi \in S,$$

which combined with (4.89) and $\varphi^0 = \bar{\varphi}^0 + \varphi_0$ leads to (4.102). ■

Corollary 14 *Let us consider $(\underline{r}, \underline{\bar{\varphi}}^0) \in T$ the element obtained from $(r, \bar{\varphi}^0) \in T$ using the definition(4.100). If the material the beam is such that the coefficients $C_{3\alpha3\beta}$, $P_{\beta3\alpha}$ and $\varepsilon_{\alpha\beta}$ do not depend on x_3, then $(\underline{r}, \underline{\bar{\varphi}}^0)$ is the unique solution of the following 2D variational problem,*

$$\begin{aligned}
&(\underline{r}, \underline{\bar{\varphi}}^0) \in T \text{ such that} \\
&\int_\omega C_{3\alpha3\beta}\partial_\beta \underline{r}\partial_\alpha \rho \, d\omega + \int_\omega \varepsilon_{\alpha\beta}\partial_\alpha \underline{\bar{\varphi}}^0 \partial_\beta \psi \, d\omega + \int_\omega P_{\beta3\alpha}(\partial_\alpha \underline{r}\partial_\beta \psi - \partial_\beta \underline{\bar{\varphi}}^0 \partial_\alpha \rho) \, d\omega \\
&\quad = \int_\omega P_{\beta3\alpha}\partial_\beta \underline{\varphi}_0 \partial_\alpha \rho \, d\omega - \int_\omega \varepsilon_{\alpha\beta}\partial_\alpha \underline{\varphi}_0 \partial_\beta \psi \, d\omega, \quad \text{for all } (\rho, \psi) \in T.
\end{aligned} \tag{4.104}$$

Proof. The variational problem is obtained straightforwardly from (4.102) integrating both sides of this equation over $(0, L)$ and taking into account that $z \in H_0^1(0, L)$. The uniqueness of solution follows from Lax-Milgram Lemma since properties $(\mathbf{H}_{22}^c)$ imply that there exists positive constants c_1, c_2 and c_3 such that

$$\int_\omega C_{3\alpha3\beta}\partial_\beta \rho \partial_\alpha \rho \, d\omega + \int_\omega \varepsilon_{\alpha\beta}\partial_\alpha \psi \partial_\beta \psi \, d\omega \geq c_1 \|\rho\|_Q^2 + c_2 \|\psi\|_S^2 \geq c_3 \|(\rho, \psi)\|_T^2,$$

for all $(\rho, \psi) \in T$. ■

We are now in conditions to express z' as a function of the electric potential.

Theorem 13 *Let $\boldsymbol{u}^1$ be such that (4.91) and (4.99) - and therefore (4.103) - hold. Then $z \in H_0^1(0, L)$ is such that*

$$z'(x_3) = \frac{1}{J}\int_\omega P_{\beta3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta(\varphi^{0D}(x_3)) \, d\omega, \quad a.e. \ x_3 \in (0, L), \tag{4.105}$$

or

$$z'(x_3) = \frac{1}{J}\int_\omega P_{\beta3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta(\bar{\varphi}^{0D}(x_3) + \varphi_0^D(x_3)) \, d\omega, \quad a.e. \ x_3 \in (0, L). \tag{4.106}$$

Proof. Taking (4.103) into account, equation (4.91) can be rewritten as

$$\int_\Omega \left(C_{3\alpha 3\beta}(z'\,(\delta_\beta - \partial_\beta w) - \partial_\beta r) + P_{\beta 3\alpha}\partial_\beta \varphi^0\right) e_{3\alpha}(\boldsymbol{v})\, d\boldsymbol{x} = 0, \quad \text{for all } \boldsymbol{v} \in V_1.$$

Taking $\boldsymbol{v} = (\chi(x_3)x_2, -\chi(x_3)x_1, 0)$, $\chi \in H_0^1(0,L)$ in the previous equation we get

$$\int_\Omega \left(C_{3\alpha 3\beta}(z'\,(\delta_\beta - \partial_\beta w) - \partial_\beta r) + P_{\beta 3\alpha}\partial_\beta \varphi^0\right) \delta_\alpha \chi'\, d\boldsymbol{x} = 0.$$

Since the coefficients $C_{3\alpha 3\beta}$ do not depend on x_3, we can reduce previous equation to

$$\begin{aligned} z''(x_3) &\int_\omega C_{3\alpha 3\beta}(\delta_\beta - \partial_\beta w)\delta_\alpha\, d\omega \\ &= \frac{d}{dx_3}\int_\omega \left(C_{3\alpha 3\beta}\partial_\beta r(x_3) - P_{\beta 3\alpha}\partial_\beta \varphi^0(x_3)\right)\delta_\alpha\, d\omega, \quad \text{a.e. } x_3 \in (0,L), \end{aligned}$$

or, in view of (4.87)-(4.89),

$$Jz''(x_3) = \frac{d}{dx_3}\int_\omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta \varphi^0(x_3)\, d\omega, \quad \text{a.e. } x_3 \in (0,L). \tag{4.107}$$

Integrating both sides of (4.107) and taking into account that the coefficients $P_{\beta 3\alpha}$ not depend on x_3 we obtain

$$Jz'(x_3) = \int_\omega P_{\beta 3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta \varphi^0(x_3)\, d\omega + c_z, \quad \text{a.e.. } x_3 \in (0,L).$$

The expression for the constant c_z is obtained integrating both sides of the previous equation over $(0,L)$ and taking into account that z vanishes at the beam ends. This leads to (4.105). The alternative expression (4.106) is obtained from (4.105) taking into account that $\varphi^0 = \bar{\varphi}^0 + \varphi_0$ implies (cf. (4.101)) $\varphi^{0D} = \overline{\varphi}^{0D} + \varphi_0^D$. ■

Remark 12 *From the previous result we conclude that in general it is not possible to obtain the elasticity result* $z = 0$ *(see Álvarez-Dios [1992]).*

Equation (4.106) allows to obtain the following set of 2D variational problems yielding the electric potential.

Theorem 14 *Let*

$$(r, \bar{\varphi}^0) = (\underline{r}, \underline{\bar{\varphi}}^0) + (r^D, \bar{\varphi}^{0D}) \in R \times \Psi_{l0}, \tag{4.108}$$

where $(\underline{r}, \underline{\bar{\varphi}}^0)$ *is the unique solution of the variational problem (4.104). Then element*

$(r^D, \bar{\varphi}^{0D}) \in R \times \Psi_{l0}$ *is the unique solution of the following variational problem*

$$\begin{aligned}&(r^D(x_3), \bar{\varphi}^{0D}(x_3)) \in T, \text{ such that a.e. } x_3 \in (0, L),\\ &\widetilde{a}((r^D(x_3), \bar{\varphi}^{0D}(x_3)), (\rho, \psi)) = \widetilde{l}\,(\rho, \psi)\,,\\ &\text{for all } (\rho, \psi) \in T,\end{aligned} \tag{4.109}$$

where the bilinear form $\widetilde{a}$ *and the linear functional* $\widetilde{l}$ *are given by*

$$\begin{aligned}\widetilde{a}((\widetilde{r}, \widetilde{\varphi}), (\rho, \psi)) &= \int_\omega C_{3\alpha3\beta}\partial_\beta\widetilde{r}\partial_\alpha\rho\, d\omega + \int_\omega \varepsilon_{\alpha\beta}\partial_\beta\widetilde{\varphi}\partial_\alpha\psi\, d\omega\\ &+ \frac{1}{J}\,(\int_\omega P_{\theta3\nu}(\partial_\nu w - \delta_\nu)\partial_\theta\widetilde{\varphi}\, d\omega)\,(\int_\omega P_{\beta3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta\psi\, d\omega)\\ &+ \int_\omega P_{\beta3\alpha}(\partial_\alpha\widetilde{r}\partial_\beta\psi - \partial_\beta\widetilde{\varphi}\partial_\alpha\rho)\, d\omega\end{aligned} \tag{4.110}$$

and

$$\begin{aligned}\widetilde{l}\,(\rho, \psi) &= \int_\omega P_{\beta3\alpha}\partial_\beta(\varphi_0^D(x_3))\partial_\alpha\rho\, d\omega - \int_\omega \varepsilon_{\alpha\beta}\partial_\beta(\varphi_0^D(x_3))\partial_\alpha\psi\, d\omega\\ &- \frac{1}{J}\,(\int_\omega P_{\theta3\nu}(\partial_\nu w - \delta_\nu)\partial_\theta(\varphi_0^D(x_3))\, d\omega)\,(\int_\omega P_{\beta3\alpha}(\partial_\alpha w - \delta_\alpha)\partial_\beta\psi\, d\omega)\end{aligned} \tag{4.111}$$

has unique solution.

Proof. The variational problem (4.109)-(4.111) is obtained substituting (4.105) in (4.102) and combining the resulting variational equation with (4.104). Both $\widetilde{a}$ and $\widetilde{l}$ are continuous due to the boundness of $\boldsymbol{C}$, $\boldsymbol{P}$ and $\boldsymbol{\varepsilon}$. The ellipticity of $\widetilde{a}$ is obtained straightforwardly if we take into account the proof of Corollary 14 and the fact that

$$\frac{1}{J}\,(\int_\omega P_{\beta3\alpha}(\partial_\alpha w - \delta_\alpha)\partial\psi_\beta\, d\omega)^2 \geq 0.$$

Hence, the uniqueness of solution of the variational problem (4.109)-(4.111) follows from Lax-Milgram Lemma. ■

Remark 13 *It is worth noting that some of the terms appearing in the bilinear form and linear functional that define problem (4.109)-(4.111) coincide with the corresponding ones appearing in (4.104). On the other hand, the set of problems (4.109)-(4.111) has two important particularities: cross sections having the same electric potential applied on the boundary have the same solution* $\bar{\varphi}^{0M}$*. These properties can be rather useful when solving the whole problem numerically (e.g. finite elements). Finally, the "decomposition"*

$\bar{\varphi}^0 = \underline{\bar{\varphi}}^0 + \bar{\varphi}^{0D}$, *allows to see the electric potential* $\bar{\varphi}^0$ *as the sum of two components: a global one* $\underline{\bar{\varphi}}^0$ *and a local one* $\bar{\varphi}^{0D}$.

4.3.0.4 Model for a beam belonging to the class 6mm of piezoelectric crystals

In this section we will to derive a model for a special case in which the beam is made of an anisotropic homogeneous material of class *6mm*. The crystal class *6mm* represents an important class of piezo-materials (Ceramic PZT-4, Zinc oxide (ZnO) and Cadmium sulphide (CdS) are examples of piezoelectric materials belonging to the *6mm* class) and it is contained in crystal class 2. The interest in obtaining this model, is due to the fact that this type of material (class *6mm*) is widely used in engineering applications.

Theorem 15 *Let us consider that the beam is made of an anisotropic homogeneous material of class 6mm where the piezo-elastic-dielectric coefficients satisfy (2.21)-(2.23). Then, equations (4.44) determine in a unique way* $(\boldsymbol{u}^0, \varphi^0) \in V_{0,w} \times \Psi_0$, *through the following expressions:*

1. *The displacement components* u_i *are of the form:*

$$u_\alpha^0(x_1, x_2, x_3) = \xi_\alpha(x_3), \tag{4.112}$$

$$u_3^0(x_1, x_2, x_3) = \xi_3(x_3) - x_\alpha \xi_\alpha'(x_3), \tag{4.113}$$

where functions ξ_α *and* ξ_3 *are determine by:*

(a) *The additional bending component* ξ_α *depends on* x_3 *only and they are the unique solution to the following problem*

$$\begin{aligned} &\xi_\alpha \in H_0^2(0, L), \\ &\int_0^L Y I_\alpha \xi_\alpha'' \chi_\alpha''\, dx_3 = \int_0^L F_\alpha \chi_\alpha\, dx_3 - \int_0^L M_\alpha \chi_\alpha'\, dx_3, \quad \textit{for all } \chi_\alpha \in H_0^2(0, L), \end{aligned} \tag{4.114}$$

(b) *The additional stretching* ξ_3 *depends only on* x_3, *and is the unique solution to the following problem*

$$\begin{aligned} &\xi_3 \in H_0^1(0, L), \\ &\int_0^L Y A(\omega) \xi_3' \chi_3'\, dx_3 = \int_0^L F_3 \chi_3\, dx_3, \qquad \textit{for all } \chi_3 \in H_0^1(0, L), \end{aligned} \tag{4.115}$$

with

$$Y = \frac{\det \boldsymbol{C}}{\det \boldsymbol{M} \ \det \boldsymbol{N}}, \tag{4.116}$$

$$F_i(x_3) = \int_\omega f_i \, d\omega + \int_{\gamma_{dN}} g_i \, d\gamma \in L^2(0,L), \tag{4.117}$$

$$M_\alpha(x_3) = \int_\omega x_\alpha f_3 \, d\omega + \int_{\gamma_{dN}} x_\alpha g_3 \, d\gamma \in L^2(0,L). \tag{4.118}$$

2. The first-order displacement $\boldsymbol{u}^1$ *is of the form*

$$u^1_\alpha(x_1,x_2,x_3) = z_\alpha(x_3), \tag{4.119}$$

$$u^1_3(x_1,x_2,x_3) = z_3(x_3) - x_\beta z'_\beta(x_3) - \frac{P_{14}}{C_{44}}\varphi^0(x_3)(x_1,x_2), \tag{4.120}$$

where $z_3 \in H^1_0(0,L)$ *and the electric potential* $\varphi^0(x_1,x_2,x_3) = \varphi^0(x_3)(x_1,x_2)$ *is the only solution of*

$\bar{\varphi}^0 \in S(\omega)$ *such that a.e.* $x_3 \in [0,L]$,

$$\int_\omega \left(P_{14}\frac{P_{14}}{C_{44}} + \varepsilon_{11} \right) \partial_\beta \bar{\varphi}^0 \partial_\beta \psi d\omega = -\int_\omega \left(P_{14}\frac{P_{14}}{C_{44}} + \varepsilon_{11} \right) \partial_\beta \hat{\varphi} \partial_\beta \psi d\omega, \tag{4.121}$$

for all $\psi \in S(\omega)$.

Proof. Let us consider the equation (4.92) for $\boldsymbol{v} = \boldsymbol{u}^1$, we obtain

$$\int_\Omega C_{\alpha\beta\theta\rho} e_{\theta\rho}(\boldsymbol{u}^1) e_{\alpha\beta}(\boldsymbol{u}^1) \, d\boldsymbol{x} = 0, \tag{4.122}$$

and, therefore,

$$u^1_\alpha\,(x_1,x_2,x_3) = z_\alpha(x_3) + \delta_\alpha z(x_3), \quad z_\alpha, z \in H^1_0(0,L),$$

that is, $\boldsymbol{v} \in V_1$. Now, taking $\boldsymbol{v} \in V_1$ in equation (4.122) leads to

$$\int_\Omega \left(C_{3\beta 3\theta} \left(\partial_\theta u^1_3 + z'_\theta + \delta_\theta z' \right) + P_{\theta 3\beta} \partial_\theta \varphi^0 \right) e_{3\beta}(\boldsymbol{v}) \, d\boldsymbol{x} = 0. \tag{4.123}$$

Taking $\boldsymbol{v} = (0, 0, \chi(x_3)\psi(x_1, x_2))$, $\chi \in H_0^1(0, L)$, $\psi \in Q(\omega)$ in the previous equation we get

$$\int_\omega C_{3\beta 3\theta}\partial_\theta u_3^1\,\partial_\beta\psi\,d\omega = -\int_\omega \left(C_{3\beta 3\theta}\left(z_\theta' + \delta_\theta z'\right) + P_{\theta 3\beta}\partial_\theta\varphi^0\right)\partial_\beta\psi\,d\omega, \quad \text{for all } \psi \in Q, \tag{4.124}$$

valid a.e. $x_3 \in (0, L)$. For a homogeneous anisotropic piezoelectric beam whose constants of the materials satisfy (2.21)-(2.23), the equation (4.124) takes the following form

$$\int_\omega \partial_\theta u_3^1\,\partial_\theta\psi\,d\omega = -\int_\omega \left(z_\theta' + \delta_\theta z'\right)\partial_\theta\psi\,d\omega - \frac{P_{14}}{C_{44}}\int_\omega \partial_\theta\varphi^0\,\partial_\theta\psi\,d\omega, \quad \text{for all } \psi \in Q, \tag{4.125}$$

Therefore, (4.124) and the fact that $u_3^1 \in L^2(0, L; H^1(\omega))$ lead to

$$u_3^1(x_1, x_2, x_3) = z_3(x_3) - x_\beta z_\beta'(x_3) - w(x_1, x_2)\,z'(x_3) - \frac{P_{14}}{C_{44}}\varphi^0(x_3)(x_1, x_2), \tag{4.126}$$

where $z_3 \in H_0^1(0, L)$ and the warping function $w(x_3)(x_1, x_2)$ of ω as the unique solution of the problem

$$\begin{aligned} & w \in H^1(\omega), \\ & \int_\omega \partial_\alpha w \partial_\alpha \chi d\omega = \int_\omega \delta_\alpha \partial_\alpha \chi d\omega, \quad \text{for all } \chi \in H^1(\omega), \\ & \int_\omega w\,d\omega = 0. \end{aligned} \tag{4.127}$$

where $\delta_1 = x_2$, $\delta_2 = -x_1$.

Taking $\boldsymbol{v} = (\chi(x_3)x_2, -\chi(x_3)x_1, 0)$, $\chi \in H_0^1(0, L)$ in (4.123) we get

$$C_{44}\int_\Omega \left(z'\left(\delta_\theta - \partial_\theta w\right) - \frac{P_{14}}{C_{44}}\partial_\theta\varphi^0\right)\delta_\theta\chi'\,d\boldsymbol{x} = -P_{14}\int_\Omega \partial_\theta\varphi^0\,\delta_\theta\chi'\,d\boldsymbol{x},$$

that is

$$\int_\Omega \left(\delta_\theta - \partial_\theta w\right)\delta_\theta\,z'\chi'\,d\boldsymbol{x} = 0, \quad \chi \in H_0^1(0, L)$$

or, equivalently,

$$\int_0^L J z'\chi'\,d\boldsymbol{x} = 0, \quad \chi \in H_0^1(0, L) \tag{4.128}$$

where

$$J = -\int_\omega \left(\delta_\theta - \partial_\theta w\right)\delta_\theta d\omega = -\int_\omega \left(x_2\partial_2\theta + x_1\partial_1\theta\right)d\omega = 2\int_\omega \theta d\omega.$$

Therefore, we deduce that $z = 0$ and

$$2e_{3\alpha}(\boldsymbol{u}^1) = -P_{14}\frac{P_{14}}{C_{44}}\partial_\alpha\varphi^0. \tag{4.129}$$

Hence we may also write equation (4.93) as follows

$$\int_\Omega \left(P_{14}\frac{P_{14}}{C_{44}}\partial_\alpha\varphi^0) + \varepsilon_{11}\partial_\alpha\varphi^0\right)\partial_\alpha\widetilde{\psi}\,d\boldsymbol{x} = 0, \quad \text{for all } \widetilde{\psi} \in \Psi_0.$$

Taking $\widetilde{\psi} = \psi\,(x_1, x_2)\,\chi(x_3)$, $\psi \in S(\omega)$, $\chi \in H_0^1(0, L)$ in the previous equation we have

$$\int_\omega \left(P_{14}\frac{P_{14}}{C_{44}} + \varepsilon_{11}\right)\partial_\alpha\varphi^0(x_3)\partial_\beta\psi\,d\omega = 0, \quad \text{a.e. in } (0, L),$$

which combined with (4.127) and $\varphi^0 = \bar{\varphi}^0 + \varphi_0$ leads to (4.121).

■

4.4 Convergence of the scaled unknowns as $h \to 0$

In this section we are going to establish the convergence results justifying the formal asymptotic expansions.

4.4.1 A priory estimations and weak convergence

Following Sene [2001], we establish that the scaled displacement and the scaled electric potential are bounded. Its first published proof for piezoelectric beam appeared in Figueiredo & Leal [2006].

Taking $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \bar{\varphi}(h))$ in problem (4.19)-(4.21) we obtain

$$\begin{aligned} a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))) &= \int_\Omega C_{ijkl}\,\kappa_{kl}(h)\,\kappa_{ij}(h)\,dx + \int_\Omega \varepsilon_{ij}\,\overline{\vartheta}_i(h)\,\overline{\vartheta}_j(h)\,dx \\ = \int_\Omega f_i\,u_i(h)\,dx &+ \int_{\Gamma_{dN}} g_i\,u_i(h)\,d\Gamma - \int_\Omega \varepsilon_{ij}\,\widehat{\vartheta}_i(h)\,\overline{\vartheta}_j(h)\,dx \\ &- \int_\Omega P_{mij}\,\widehat{\vartheta}_m(h)\,\kappa_{ij}(h)\,dx = l_2(\boldsymbol{u}(h), \bar{\varphi}(h)). \end{aligned} \tag{4.130}$$

Lemma 4 *There exists $c > 0$ such that*

$$a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))) \geq c\left(|\boldsymbol{\kappa}(h)|_\Omega^2 + \left|\overline{\boldsymbol{\vartheta}}(h)\right|_\Omega^2\right), \quad \textit{for all } h > 0. \tag{4.131}$$

Proof. From condition ($\mathbf{H}_{22}^c$), i.e., the ellipticity properties of $\boldsymbol{C}$ and $\boldsymbol{\varepsilon}$, we get

$$a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))) = \int_\Omega C_{ijkl}\, \kappa_{ij}(h)\, \kappa_{kl}(h)\, dx + \int_\Omega \varepsilon_{ij}\, \overline{\vartheta}_i(h)\, \overline{\vartheta}_j(h)\, dx$$

$$\geq c_1 \int_\Omega \sum_{i,j=1}^{3} (\kappa_{ij}(h))^2\, dx + c_2 \int_\Omega \sum_{i=1}^{3} (\overline{\vartheta}_i(h))^2\, dx$$

$$\geq c \left(|\boldsymbol{\kappa}(h)|_\Omega^2 + \left|\overline{\boldsymbol{\vartheta}}(h)\right|_\Omega^2 \right),$$

where $c = \min(c_1, c_2) > 0$. ■

In the following lemmas we will use the inequality:

$$2ab \leq ma^2 + \frac{1}{m} b^2, \quad \text{for all } (a, b, m) \in \mathbb{R}^2 \times \mathbb{R}^+.$$

Lemma 5 *There exists a constant $c > 0$ such that*

$$\left| \int_\Omega f_i\, u_i(h)\, dx + \int_{\Gamma_{dN}} g_i\, u_i(h)\, d\Gamma \right| \leq m \left(|\boldsymbol{f}|_\Omega^2 + |\boldsymbol{g}|_{\Gamma_{dN}}^2 \right) + \frac{c}{m} \|\boldsymbol{u}(h)\|_{V_{0,w}}^2, \tag{4.132}$$

for all h, $m > 0$.

Proof. Here we invoke the continuity of the trace operator, the fact that $\boldsymbol{f}$ and $\boldsymbol{g}$ belong to $[L^2(\Omega)]^3$ and $[L^2(\Gamma_{dN})]^3$, respectively, and the fact that $\|\cdot\|_{V_{0,w}}$ and $\|\cdot\|_{[H^1(\Omega)]^3}$ are equivalent norms, to obtain

$$\left| \int_\Omega f_i\, u_i(h)\, dx + \int_{\Gamma_{dN}} g_i\, u_i(h)\, d\Gamma \right| \leq \left| \int_\Omega f_i\, u_i(h)\, dx \right| + \left| \int_{\Gamma_{dN}} g_i\, u_i(h)\, d\Gamma \right|$$

$$\leq \frac{m}{2} |\boldsymbol{f}|_\Omega^2 + \frac{1}{2m} |\boldsymbol{u}(h)|_\Omega^2 + \frac{m}{2} |\boldsymbol{g}|_{\Gamma_{dN}}^2 + \frac{1}{2m} |\boldsymbol{u}(h)|_{\Gamma_{dN}}^2$$

$$\leq \frac{m}{2} \left(|\boldsymbol{f}|_\Omega^2 + |\boldsymbol{g}|_{\Gamma_{dN}}^2 \right) + \frac{c_1}{m} \|\boldsymbol{u}(h)\|_{[H^1(\Omega)]^3}^2$$

$$\leq m \left(|\boldsymbol{f}|_\Omega^2 + |\boldsymbol{g}|_{\Gamma_{dN}}^2 \right) + \frac{c}{m} \|\boldsymbol{u}(h)\|_{V_{0,w}}^2,$$

for all $h, m > 0$. ■

Lemma 6 *There exists $c > 0$ such that*

$$\left| \int_\Omega \varepsilon_{ij}\, \widehat{\theta}_i(h)\, \overline{\theta}_j(h)\, dx \right| \leq cm + \frac{3}{2m} \left|\overline{\boldsymbol{\vartheta}}(h)\right|_\Omega^2, \quad \textit{for all } h, m > 0. \tag{4.133}$$

Proof. Let $r = \sup_{1\le i,j\le 3} |\varepsilon_{ij}|$. Since $\widehat{\vartheta}(h)$ is bounded in $[L^2(\Omega)]^3$, one has

$$\begin{aligned}\left|\int_\Omega \varepsilon_{ij}\,\widehat{\vartheta}_i(h)\,\overline{\vartheta}_j(h)\,dx\right| &\le \sum_{i,j=1}^{3}\left(\frac{m}{2}r^2\,|\widehat{\vartheta}_i(h)|^2_\Omega + \frac{1}{2m}\,|\overline{\theta}_j(h)|^2_\Omega\right)\\ &= \frac{3m}{2}r^2\sum_{j=1}^{3}|\widehat{\vartheta}_i(h)|^2_\Omega + \frac{3}{2m}\sum_{j=1}^{3}|\overline{\vartheta}_j(h)|^2_\Omega\\ &= \frac{3m}{2}r^2\,|\widehat{\boldsymbol{\vartheta}}(h)|^2_\Omega + \frac{3}{2m}\,\left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \le cm + \frac{3}{2m}\,\left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega,\end{aligned}$$

for all $h, m > 0$. ■

Lemma 7 *There exists $c > 0$ such that*

$$\left|\int_\Omega P_{mij}\,\widehat{\vartheta}_m(h)\,\kappa_{ij}(h)\,dx\right| \le cm + \frac{3}{2m}\,|\boldsymbol{\kappa}(h)|^2_\Omega, \quad \textit{for all } h, m > 0. \tag{4.134}$$

Proof. Let $p = \sup_{1\le i,j,l\le 3} |P_{lij}|$. Since $\widehat{\boldsymbol{\vartheta}}(h)$ is bounded in $[L^2(\Omega)]^3$, one has

$$\begin{aligned}\left|\int_\Omega P_{lij}\,\widehat{\theta}_l(h)\,\kappa_{ij}(h)\,dx\right| &\le \sum_{i,j,l=1}^{3}\left(\frac{m}{2}p^2\,|\widehat{\vartheta}_l(h)|^2_\Omega + \frac{1}{2m}\,|\kappa_{ij}(h)|^2_\Omega\right)\\ &= \frac{9m}{2}p^2\sum_{l=1}^{3}|\widehat{\vartheta}_l(h)|^2_\Omega + \frac{3}{2m}\sum_{i,j=1}^{3}|\kappa_{ij}(h)|^2_\Omega\\ &= \frac{9m}{2}p^2\,|\widehat{\boldsymbol{\vartheta}}(h)|^2_\Omega + \frac{3}{2m}\,|\boldsymbol{\kappa}(h)|^2_\Omega \le cm + \frac{3}{2m}\,|\boldsymbol{\kappa}(h)|^2_\Omega,\end{aligned}$$

for all $h, m > 0$. ■

Proposition 6 *There exists $c > 0$ such that*

$$\|\boldsymbol{u}(h)\|^2_{V_{0,w}} + |\boldsymbol{\kappa}(h)|^2_\Omega + \left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \le c, \quad \textit{for all } h \in (0,1). \tag{4.135}$$

Proof. Korn's inequality and the fact that $0 < h < 1$ allows to write

$$c_1\,\|\boldsymbol{u}(h)\|^2_{[H^1(\Omega)]^3} \le |\boldsymbol{e}(h)|^2_\Omega \le |\boldsymbol{\kappa}(h)|^2_\Omega,$$

or, since $\|\cdot\|_{V_{0,w}}$ and $\|\cdot\|_{[H^1(\Omega)]^3}$ are equivalent norms,

$$c_2\,\|\boldsymbol{u}(h)\|^2_{V_{0,w}} \le |\boldsymbol{e}(h)|^2_\Omega \le |\boldsymbol{\kappa}(h)|^2_\Omega.$$

This result and the estimate (4.131) lead to

$$c_3 \|\boldsymbol{u}(h)\|^2_{V_{0,w}} + c_4 |\boldsymbol{\kappa}(h)|^2_\Omega + c_5 \left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \leq a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))). \tag{4.136}$$

On the other hand, (4.130) and (4.132) - (4.134) can be combined to yield

$$a((\boldsymbol{u}(h), \bar{\varphi}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h))) \leq c_6 m + \frac{c_7}{m} \|\boldsymbol{u}(h)\|^2_{V_{0,w}} + \frac{3}{2m} \left(|\boldsymbol{\kappa}(h)|^2_\Omega + \left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \right),$$

which, together with (4.136), lead to

$$\left(c_3 - \frac{c_7}{m}\right) \|\boldsymbol{u}(h)\|^2_{V_{0,w}} + \left(c_4 - \frac{3}{2m}\right) |\boldsymbol{\kappa}(h)|^2_\Omega + \left(c_5 - \frac{3}{2m}\right) \left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \leq cm.$$

Taking m large enough in this expression we obtain the estimate

$$\|\boldsymbol{u}(h)\|^2_{V_{0,w}} + |\boldsymbol{\kappa}(h)|^2_\Omega + \left|\overline{\boldsymbol{\vartheta}}(h)\right|^2_\Omega \leq c,$$

for all $h \in (0, 1)$. ■

Corollary 15 *There exists $c > 0$ such that*

$$\|\boldsymbol{u}(h)\|^2_{V_{0,w}} + |\boldsymbol{\kappa}(h)|^2_\Omega + |\boldsymbol{\vartheta}(h)|^2_\Omega \leq c, \quad \textit{for all } h \in (0, 1).$$

Proof. The previous result follows from (4.135) and (cf. 4.18)

$$\boldsymbol{\vartheta}(h) = \overline{\boldsymbol{\vartheta}}(h) + \hat{\boldsymbol{\vartheta}} = \overline{\boldsymbol{\vartheta}}(h) + (\partial_1 \hat{\varphi}, \partial_2 \hat{\varphi}, h\, \partial_3 \hat{\varphi}),$$

in a straightforward way, since $\hat{\varphi} \in H^1(\Omega)$. ■

Corollary 16 *Let $\bar{\varphi}(h) \in \Psi_0$. Then there exists $c > 0$ such that*

$$|\bar{\varphi}(h)|_\Omega \leq c, \quad \textit{for all } h \in (0, 1). \tag{4.137}$$

Proof. The Poincaré–Friedrichs inequality allows to write, for any s fixed in $(0, L)$

$$\begin{aligned} \int_\omega [\bar{\varphi}(h)(\cdot, \cdot, s)]^2\, d\omega &= \|\bar{\varphi}(h)(\cdot, \cdot, s)\|^2_{L^2(\omega)} \\ &\leq C(\omega) \left[\|\partial_1 \bar{\varphi}(h)(\cdot, \cdot, s)\|^2_{L^2(\omega)} + \|\partial_2 \bar{\varphi}(h)(\cdot, \cdot, s)\|^2_{L^2(\omega)} \right]. \end{aligned}$$

Hence,

$$\begin{aligned}
|\bar\varphi(h)|^2_\Omega &= \int_\Omega [\bar\varphi(h)(x_1,x_2,x_3)]^2\,dx = \int_0^L \left\{ \int_\omega [\bar\varphi(h)(\cdot,\cdot,s)]^2\,d\omega \right\} ds \\
&\le C(\omega) \int_0^L \left[\|\partial_1 \bar\varphi(h)(\cdot,\cdot,s)\|^2_{L^2(\omega)} + \|\partial_2 \bar\varphi(h)(\cdot,\cdot,s)\|^2_{L^2(\omega)} \right] ds \\
&= C(\omega) \left[|\partial_1 \bar\varphi(h)|^2_\Omega + |\partial_2 \bar\varphi(h)|^2_\Omega \right].
\end{aligned}$$

Now, the estimation (4.137) is obtained from (4.135). ■

Lemma 8 *There exists a subsequence of* $(\boldsymbol{u}(h), \boldsymbol{\kappa}(h), \bar\varphi(h), \overline{\boldsymbol{\vartheta}}(h))_{0<h<1}$*, still parameterized by* h*, and there exist* $\boldsymbol{u} \in V_{0,w}$*,* $\boldsymbol{\kappa} \in [L^2(\Omega)]^9$*,* $\bar\varphi \in [L^2(\Omega)]$ *and* $\overline{\boldsymbol{\vartheta}} \in [L^2(\Omega)]^3$*, such that the following weak convergence results hold when* h *tends to zero:*

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u} \quad \text{in } \left[H^1(\Omega)\right]^3, \tag{4.138}$$
$$\boldsymbol{\kappa}(h) \rightharpoonup \boldsymbol{\kappa} \quad \text{in } \left[L^2(\Omega)\right]^9, \tag{4.139}$$
$$\bar\varphi(h) \rightharpoonup \bar\varphi \quad \text{in } L^2(\Omega), \tag{4.140}$$
$$\overline{\boldsymbol{\vartheta}}(h) \rightharpoonup \overline{\boldsymbol{\vartheta}} \quad \text{in } \left[L^2(\Omega)\right]^3. \tag{4.141}$$

Proof. Due to the Lemma 15 we show that there exists subsequences $(\boldsymbol{u}(h))_{h>0}$, $(\boldsymbol{\kappa}(h))_{h>0}$, $(\bar\varphi(h))_{h>0}$, $\left(\overline{\boldsymbol{\vartheta}}(h)\right)_{h>0}$ and functions $\boldsymbol{u}$, $\boldsymbol{\kappa}$, $\bar\varphi$ and $\overline{\boldsymbol{\vartheta}}$ satisfying (4.138)-(4.141). ■

Lemma 9 *There exists a subsequence of* $(\varphi(h), \boldsymbol{\vartheta}(h))_{0<h<1}$*, still parameterized by* h*, and there exist* $\varphi \in L^2(\Omega)$*,* $\boldsymbol{\vartheta} \in [L^2(\Omega)]^3$*, such that*

$$\begin{aligned}
\varphi(h) &\rightharpoonup \varphi \quad \text{in } L^2(\Omega), \\
\boldsymbol{\vartheta}(h) &\rightharpoonup \boldsymbol{\vartheta} \quad \text{in } \left[L^2(\Omega)\right]^3,
\end{aligned}$$

as h *tends to zero. Moreover,*

$$\varphi = \bar\varphi^0 + \varphi_0, \quad \boldsymbol{\vartheta} = \bar{\boldsymbol{\vartheta}} + \hat{\boldsymbol{\vartheta}}, \quad \text{where} \quad \hat{\boldsymbol{\vartheta}} = (\partial_1\hat\varphi, \partial_2\hat\varphi, 0).$$

Proof. These results follow straightforwardly from (4.140)-(4.141) taking into account that $\varphi(h) = \bar\varphi(h) + \varphi_0$ and $\boldsymbol{\vartheta}(h) = \overline{\boldsymbol{\vartheta}}(h) + \hat{\boldsymbol{\vartheta}}$. ■

We define (cf. 4.24)

$$\widetilde{\boldsymbol{\sigma}}(h) = (h^2\sigma_{\alpha\beta}(h), h\sigma_{3\alpha}(h), \sigma_{33}(h)), \tag{4.142}$$
$$\widetilde{\boldsymbol{D}}(h) = (D_\alpha(h), h^{-1}D_3(h)), \tag{4.143}$$

that is (cf. (4.16)-(4.17)

$$\widetilde{\sigma}_{\alpha\beta}(h) = h^2\sigma_{\alpha\beta}(h) = C_{\alpha\beta\theta\rho}\kappa_{\theta\rho}(h) + C_{\alpha\beta 33}\kappa_{33}(h) + P_{3\alpha\beta}\vartheta_3(h), \tag{4.144}$$

$$\widetilde{\sigma}_{3\alpha}(h) = h\sigma_{3\alpha}(h) = 2C_{3\alpha 3\theta}\kappa_{3\theta}(h) + P_{\theta 3\alpha}\vartheta_\theta(h), \tag{4.145}$$

$$\widetilde{\sigma}_{33}(h) = \sigma_{33}(h) = C_{33\theta\rho}\kappa_{\theta\rho}(h) + C_{3333}\kappa_{33}(h) + P_{333}\vartheta_3(h), \tag{4.146}$$

$$\widetilde{D}_3(h) = h^{-1}D_3(h) = P_{3\theta\rho}\kappa_{\theta\rho}(h) + P_{333}\kappa_{33}(h) - \varepsilon_{33}\vartheta_3(h), \tag{4.147}$$

$$\widetilde{D}_\beta(h) = D_\beta(h) = 2P_{\beta 3\theta}\kappa_{3\theta}(h) - \varepsilon_{\theta\beta}\vartheta_\theta(h). \tag{4.148}$$

From equalities (4.142)-(4.148) and Lemmas 8 and 9, we conclude that the sequences $(\widetilde{\boldsymbol{D}}(h))_{0<h<1}$ and $(\widetilde{\boldsymbol{\sigma}}(h))_{0<h<1}$ are bounded in $[L^2(\Omega)]^3$ and $[L^2(\Omega)]^9$, respectively. Therefore, one has the following result.

Corollary 17 *There exists a subsequence of $(\widetilde{\boldsymbol{D}}(h), \widetilde{\boldsymbol{\sigma}}(h))_{0<h<1}$, still parameterized by h, and there exist $\mathfrak{D} \in [L^2(\Omega)]^3$ and $\boldsymbol{\Sigma} \in [L^2(\Omega)]^9_s$, such that the following weak convergence hold in $L^2(\Omega)$ when h tends to zero:*

$$\begin{cases} \tilde{\sigma}_{\alpha\beta}(h) = h^2\sigma_{\alpha\beta}(h) \rightharpoonup \Sigma_{\alpha\beta} = C_{\alpha\beta\theta\rho}\kappa_{\theta\rho} + C_{\alpha\beta 33}\kappa_{33}, \\ \tilde{\sigma}_{3\alpha}(h) = h\sigma_{3\alpha}(h) \rightharpoonup \Sigma_{3\alpha} = 2C_{3\alpha 3\theta}\kappa_{3\theta} + P_{\theta 3\alpha}\vartheta_\theta, \\ \tilde{\sigma}_{33}(h) = \sigma_{33}(h) \rightharpoonup \Sigma_{33} = C_{33\theta\rho}\kappa_{\theta\rho} + C_{3333}\kappa_{33}, \\ \tilde{D}_\beta(h) = D_\beta(h) \rightharpoonup \mathfrak{D}_\beta = 2P_{\beta 3\theta}\kappa_{3\theta} - \varepsilon_{\theta\beta}\vartheta_\theta, \\ \tilde{D}_3(h) = h^{-1}D_3(h) \rightharpoonup \mathfrak{D}_3 = P_{3\theta\rho}\kappa_{\theta\rho} + P_{333}\kappa_{33} - \varepsilon_{33}\vartheta_3. \end{cases} \tag{4.149}$$

Proof. Properties (4.149) are obtained from (4.142)-(4.148) taking the limit as h tends to zero.

■

4.4.2 Identification of the limit problem and the weak convergence of the scaled displacement and electric potential to the leading terms

The aim of this section is to derive the limit models for a homogeneous anisotropic material of class 2 and class *6mm* and to compare with the equations found in Section 4.3 .

4.4.2.1 For a beam belonging to the class 2 of piezoelectric crystals

In order to identify the limit model, we consider all the notation defined in Section 4.3, and denote by $V_{0,w}(\Omega) = W_1(\Omega) \times W_2(\Omega)$ the space of scaled displacements as follow

$$W_1(\Omega) = \left\{ \eta \in H^1(\Omega) : \int_{\omega\times\{a\}} \eta = \int_{\omega\times\{0\}} x_\alpha \eta = 0, \quad \mathrm{a} = 0, L \right\} \tag{4.150}$$

$$W_2(\Omega) = \left\{ \widehat{\rho} = (\rho_\alpha) \in \left[H^1(\Omega)\right]^2 : \int_{\omega\times\{a\}} \rho_\alpha = \int_{\omega\times\{a\}} \rho_\alpha \delta_\alpha = 0, \ \mathrm{a} = 0, L \right\} \tag{4.151}$$

with $\delta_1 = -x_2$ and $\delta_2 = x_1$. We know that V_{BN} can be equally defined by

$$\begin{aligned} V_{BN} =& \{ \boldsymbol{v} = (v_i) \in V : v_\alpha(x_1, x_2, x_3) = \chi_\alpha(x_3), \quad \chi_\alpha \in H_0^2(0, L), \\ & v_3(x_1, x_2, x_3) = \chi_3(x_3) - x_\beta \chi'_\beta(x_3), \quad \chi_3 \in H_0^1(0, L)\}. \end{aligned} \tag{4.152}$$

Corollary 18 *Let us that the beam is made of an anisotropic piezoelectric material of class 2 material whose coefficients $A^c_{33} A^c_{\alpha\beta}$ do not depend on x_1 and x_2. Then, the following properties hold:*

$$\int_\Omega \Sigma_{3\alpha} \partial_\alpha v^0 \, d\boldsymbol{x} = 0, \quad \forall v^0 \in W_1(\Omega), \tag{4.153}$$

$$\int_\Omega \Sigma_{\alpha\beta} \partial_\alpha v_\beta \, d\boldsymbol{x} = 0, \quad \forall v_\beta \in W_2(\Omega), \tag{4.154}$$

$$\int_\omega x_\gamma \Sigma_{\alpha\beta} = \int_\omega \Sigma_{\alpha\beta} = 0, \tag{4.155}$$

$$\boldsymbol{u} \in V_{BN} : \qquad e_{\alpha\beta}(\boldsymbol{u}) = e_{3\beta}(\boldsymbol{u}) = 0, \tag{4.156}$$

$$\kappa_{33}(\boldsymbol{u}) = e_{33}(\boldsymbol{u}), \tag{4.157}$$

$$\bar{\vartheta} = (\partial_1 \bar{\varphi}, \partial_2 \bar{\varphi}, 0)^T, \tag{4.158}$$

$$\vartheta = (\partial_1 \varphi, \partial_2 \varphi, 0)^T, \tag{4.159}$$

$$\Sigma_{33} = A^c_{33} \left[\bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) - A^c_{\alpha\beta} \Sigma_{\alpha\beta} \right], \tag{4.160}$$

$$E_\alpha(\varphi) = -\bar{P}_{\alpha 3\theta} \Sigma_{3\theta} + \bar{\varepsilon}_{\alpha\beta} \mathfrak{D}_\beta, \tag{4.161}$$

$$A^c_{33} \bar{\varepsilon}_{33} = \frac{\bar{\varepsilon}_{33}}{\bar{\varepsilon}_{33} \bar{C}_{3333} + \bar{P}_{333} \bar{P}_{333}}, \quad A^c_{\alpha\beta} = \bar{\varepsilon}_{33} \bar{C}_{33\alpha\beta} + \bar{P}_{333} \bar{P}_{3\alpha\beta} \tag{4.162}$$

where $\boldsymbol{u} \in V_{BN}$ is the solution of the following problem:

$$\int_\Omega A^c_{33} \bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i \, v_i \, d\Gamma, \quad \forall \boldsymbol{v} \in V_{BN}. \tag{4.163}$$

Proof. From the weak convergence $\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}$ in $[H^1(\Omega)]^3$, since $\boldsymbol{u}(h) \in V_{0,w}$, we deduce that $\boldsymbol{u} \in V_{0,w}$. Moreover,

$$e_{ij}(\boldsymbol{u}(h)) \rightharpoonup e_{ij}(\boldsymbol{u}) \quad \text{in } L^2(\Omega). \tag{4.164}$$

In addition, from identities (4.16) and from weak convergence of Lemma 8 we have

$$\begin{cases} e_{\alpha\beta}(\boldsymbol{u}(h)) = h^2\, \kappa_{\alpha\beta}(h) \rightharpoonup 0 \quad \text{in } L^2(\Omega), \\ e_{3\beta}(\boldsymbol{u}(h)) = h\, \kappa_{3\beta}(h) \rightharpoonup 0 \quad \text{in } L^2(\Omega), \\ e_{33}(\boldsymbol{u}(h)) = \kappa_{33}(h) \rightharpoonup \ \kappa_{33} \quad \text{in } L^2(\Omega). \end{cases} \tag{4.165}$$

The uniqueness of the limits appearing in (4.164)-(4.165) implies (4.157) besides $e_{\alpha\beta}(\boldsymbol{u}) = e_{3\alpha}(\boldsymbol{u}) = 0$. Now, from (4.140) we conclude that

$$\partial_i \bar{\varphi}(h) \rightharpoonup \partial_i \bar{\varphi} = \partial_i \bar{\varphi} \quad \text{in } H^{-1}(\Omega). \tag{4.166}$$

On the other hand, in the view of (4.17) and (4.141), we get

$$(\overline{\vartheta}_\alpha(h), \overline{\vartheta}_3(h)) = (\partial_\alpha \bar{\varphi}(h), h\partial_3 \bar{\varphi}(h)) \rightharpoonup (\overline{\vartheta}_\alpha, \overline{\vartheta}_3) \quad \text{in } [L^2(\Omega)]^3.$$

Therefore, given (4.166) and the uniqueness of the limit appearing in the right-hand side of (4.141) we conclude that

$$(\overline{\vartheta}_\alpha, \overline{\vartheta}_3) = (\partial_\alpha \bar{\varphi}, 0),$$

that is (4.158), which implies (4.159).

In order to establish (4.156)-(4.157), we consider the following equation obtained from (4.26) by setting $d_i = 0$ and passing to the limit when $h \to 0$:

$$\int_\Omega \left(\bar{C}_{3333}\Sigma_{33} + \bar{C}_{33\alpha\beta}\Sigma_{\alpha\beta} \ + \bar{P}_{333}\mathfrak{D}_3\right) \tau_{33}\, d\boldsymbol{x} = \int_\Omega e_{ij}(\boldsymbol{u})\tau_{ij}\, d\boldsymbol{x}, \tag{4.167}$$

which implies

$$e_{33}(\boldsymbol{u}) = \bar{C}_{3333}\Sigma_{33} + \bar{C}_{33\alpha\beta}\Sigma_{\alpha\beta} \ + \bar{P}_{333}\mathfrak{D}_3, \tag{4.168}$$

$$e_{3\alpha}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) = 0. \tag{4.169}$$

Letting $\boldsymbol{\sigma} = \boldsymbol{0}$ in (4.26) and multiplying by h, we likewise find that we arrive at

$$\int_\Omega \left(-\bar{P}_{3\alpha\beta}\Sigma_{\alpha\beta} - \bar{P}_{333}\Sigma_{33} + \bar{\varepsilon}_{33}\mathfrak{D}_3\right) d_3 d\boldsymbol{x} = 0,$$

and thus

$$\mathfrak{D}_3 = \frac{1}{\bar{\varepsilon}_{33}}\left(\bar{P}_{3\alpha\beta}\Sigma_{\alpha\beta} + \bar{P}_{333}\Sigma_{33}\right).$$

Substituting the last expression in (4.168), we deduce

$$e_{33}(\boldsymbol{u}) = \bar{C}_{3333}\Sigma_{33} + \bar{C}_{33\alpha\beta}\Sigma_{\alpha\beta} + \frac{\bar{P}_{333}}{\bar{\varepsilon}_{33}}\left(\bar{P}_{3\alpha\beta}\Sigma_{\alpha\beta} + \bar{P}_{333}\Sigma_{33}\right),$$

from which we arrive at (4.160) with

$$A^c_{33} = \frac{1}{\bar{\varepsilon}_{33}\bar{C}_{3333} + \bar{P}_{333}\bar{P}_{333}}, \quad A^c_{\alpha\beta} = \bar{\varepsilon}_{33}\bar{C}_{33\alpha\beta} + \bar{P}_{333}\bar{P}_{3\alpha\beta}.$$

Setting $\boldsymbol{v} = (v_1, v_2, 0)$ and $d_i = 0$ in (4.27) and multiplying by h^2 one obtains

$$h^2\int_\Omega \sigma_{\alpha\beta}(h)\, e_{\alpha\beta}(\boldsymbol{v})d\boldsymbol{x} + 2h^2\int_\Omega \sigma_{3\alpha}(h)\, e_{3\alpha}(\boldsymbol{v})d\boldsymbol{x} = h^2\left[\int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma\right].$$

Hence passing to the limit $h \to 0$ gives

$$\int_\Omega \Sigma_{\alpha\beta}\partial_\alpha v_\beta d\boldsymbol{x} = 0, \qquad \text{for all} \;\; v_\beta \in W_2 \tag{4.170}$$

which implies (for $v_\beta = x_\beta v^0$, $v_\beta = \frac{1}{2}x_\gamma^2 v^0$, $v_\beta = x_1x_2v^0$ with $v^0 \in H_0^1(0, L)$) that

$$\int_\omega x_\gamma\Sigma_{\alpha\beta}\, d\omega = \int_\omega \Sigma_{\alpha\beta}\, d\omega = 0. \tag{4.171}$$

In a similar way, taking $\boldsymbol{v} = (0, 0, v_3)$ and $d_i = 0$ in (4.27) and multiplying by h one has

$$h\int_\Omega \sigma_{33}(h)\, e_{33}(\boldsymbol{v})d\boldsymbol{x} + 2h\int_\Omega \sigma_{3\alpha}(h)\, e_{3\alpha}(\boldsymbol{v})d\boldsymbol{x} = h\left[\int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma\right].$$

Taking the limit $h \to 0$ gives (4.153).

Now we put $\boldsymbol{\tau} = 0$ and $d_3 = 0$ into (4.26). Passing to the limit as $h \to 0$, we thus find

$$\int_\Omega \left(-\bar{P}_{\alpha 3\theta}\Sigma_{3\theta} + \bar{\varepsilon}_{\alpha\beta}\mathfrak{D}_\beta\right) d_\alpha d\boldsymbol{x} = \int_\Omega E_\alpha(\varphi) d_\alpha d\boldsymbol{x}$$

and we infer that

$$E_\alpha(\varphi) = -\bar{P}_{\alpha 3\theta}\Sigma_{3\theta} + \bar{\varepsilon}_{\alpha\beta}\mathfrak{D}_\beta. \tag{4.172}$$

Let us now consider the restriction of equation (4.26) to $\boldsymbol{v} \in V_{BN}$. Then since $e_{3\alpha}(\boldsymbol{u}) = e_{\alpha\beta}(\boldsymbol{u}) = 0$ we obtain

$$\int_\Omega \sigma_{33}(h)\, e_{33}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega D_k(h) E_k(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma,$$

Passing to the limit when $h \to 0$ gives

$$\int_\Omega \Sigma_{33}\, e_{33}(\boldsymbol{v}) d\boldsymbol{x} + \int_\Omega \mathfrak{D}_\alpha E_\alpha(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma, \tag{4.173}$$

for all $(\boldsymbol{u}, \psi) \in V_{BN} \times \Psi_0$. Taking $d_i = 0$ and substituting (4.160) in the previous equation we deduce

$$\int_\Omega A^c_{33} \left[\bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) - A^c_{\alpha\beta} \Sigma_{\alpha\beta}\right]\, e_{33}(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma, \tag{4.174}$$

for all $\boldsymbol{u} \in V_{BN}$. For homogeneous anisotropic material, the coefficient $A^c_{33} A^c_{\alpha\beta}$ does not depend either on x_1 or x_2 and therefore equation (4.174) becomes, by property (4.171),

$$\int_\Omega A^c_{33} \bar{\varepsilon}_{33} e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma,\ \forall \boldsymbol{v} \in V_{BN}. \tag{4.175}$$

Using algebraic software tools we prove that

$$Y = \frac{\det \boldsymbol{C}}{\det \boldsymbol{M} \det \boldsymbol{N}} = A^c_{33} \bar{\varepsilon}_{33},$$

and consequently the unicity of solution of the problem (4.175) follows.

Setting $\boldsymbol{v} = \boldsymbol{0}$ in (4.173) we get

$$\int_\Omega \mathfrak{D}_\alpha E_\alpha(\psi)\, d\boldsymbol{x} = 0, \text{for all } \psi \in \Psi_0. \tag{4.176}$$

■

Corollary 19 *Sequences* $(\boldsymbol{u}(h))_{h>0}$ *satisfy as* $h \to 0$

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}^0 \qquad \text{in } [H^1(\Omega)]^3, \tag{4.177}$$

Proof. From uniqueness of solution of (4.114) and comparing with (4.163) we conclude that $u_\alpha = u^0_\alpha$ and $u_3 = u^0_3$ and therefore $\boldsymbol{u} = \boldsymbol{u}^0$. In section 4.4.1 it was showed that $(\boldsymbol{u}(h))$ is weakly convergent in $V_{0,w}(\Omega)$. By unicity of the limit this implies that the whole sequence $(\boldsymbol{u}(h))_{h>0}$ converges to $\boldsymbol{u}^0$. ■

Remark 14 *We note that the variational problem (4.176) cannot be expressed in order* *to φ and r. Consequently, we cannot guarantee the weak convergence of the sequence $(\varphi(h))_{h>0}$ to φ^0.*

4.4.2.2 For the homogeneous transversely isotropic case - 6mm symmetry class

We shall now establish the weak convergence of sequence $(\boldsymbol{u}(h), \varphi(h))_{h>0}$ to the first term of the asymptotic expansion $(\boldsymbol{u}^0, \varphi^0) \in V_{0,w} \times \Psi_0$, for the anisotropic homogeneous material of class *6mm*.

Theorem 16 *Let us consider that the beam is made of a material anisotropic of class 6mm. Sequences $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ satisfy as $h \to 0$*

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}^0 \qquad in\ [H^1(\Omega)]^3, \tag{4.178}$$

$$\varphi(h) \rightharpoonup \varphi^0 \qquad in\ L^2(\Omega) \tag{4.179}$$

where $\boldsymbol{u}^0 \in V_{BN}$ is the Bernoulli-Navier characterized by equations (4.113) and which is the first term in the asymptotic expansion of the scaled displacement field (4.36), and $\varphi^0 \in \Psi_0$ satisfies (4.121) and is the first term in the asymptotic expansion of the scaled electric potential (4.37).

Proof. From (2.21)-(2.23) and using some algebraic software tools, we show that

$$\begin{aligned}
&\bar{C}_{11} = \bar{C}_{1111} = \bar{C}_{2222}, \quad \bar{C}_{13} = \bar{C}_{1122}, \quad \bar{C}_{16} = \bar{C}_{1133} = \bar{C}_{2233},\\
&\bar{C}_{44} = \bar{C}_{1313} = \bar{C}_{2323}, \quad \bar{C}_{1212} = \frac{\bar{C}_{11} - \bar{C}_{13}}{2}, \quad \bar{C}_{66} = \bar{C}_{3333},\\
&\bar{P}_{14} = \bar{P}_{131} = \bar{P}_{232}, \quad \bar{P}_{31} = \bar{P}_{311} = \bar{P}_{322}, \quad \bar{P}_{36} = \bar{P}_{333}, \qquad \bar{\varepsilon}_{11} = \bar{\varepsilon}_{22}.
\end{aligned}$$

Consequently, equation (4.163) may now be written as

$$\int_\Omega \frac{\bar{\varepsilon}_{33}}{\bar{\varepsilon}_{33}\bar{C}_{66} + \bar{P}_{36}\bar{P}_{36}} e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) d\boldsymbol{x} = \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_{dN}} g_i\, v_i\, d\Gamma. \tag{4.180}$$

and relations (4.161) verify

$$E_1(\varphi) = -\bar{P}_{131}\Sigma_{31} + \bar{\varepsilon}_{11}\mathfrak{D}_1, \qquad E_2(\varphi) = -\bar{P}_{232}\Sigma_{32} + \bar{\varepsilon}_{22}\mathfrak{D}_2 \tag{4.181}$$

or equivalently,

$$\mathfrak{D}_1 = \frac{1}{\bar{\varepsilon}_{11}}(E_1(\varphi) + \bar{P}_{14}\Sigma_{31}), \qquad \mathfrak{D}_2 = \frac{1}{\bar{\varepsilon}_{11}}(E_2(\varphi) + \bar{P}_{14}\Sigma_{32}). \tag{4.182}$$

Substituting these expressions into (4.173) we obtain

$$\int_\Omega [\frac{1}{\bar{\varepsilon}_{11}}(E_1(\varphi) + \bar{P}_{131}\Sigma_{31})]E_1(\psi)\,d\boldsymbol{x} + \int_\Omega [\frac{1}{\bar{\varepsilon}_{22}}(E_2(\varphi) + \bar{P}_{232}\Sigma_{32})]E_2(\psi)\,d\boldsymbol{x} = 0, \quad (4.183)$$

for all $\psi \in \Psi_0$. If, in the previous situation, we consider the 6mm material case, or, equivalently

$$\int_\Omega \frac{1}{\bar{\varepsilon}_{11}}E_\alpha(\varphi)E_\alpha(\psi)\,d\boldsymbol{x} + \int_\Omega \bar{P}_{14}\Sigma_{3\alpha}E_\alpha(\psi)\,d\boldsymbol{x} = 0, \quad \text{for all } \psi \in \Psi_0. \quad (4.184)$$

Choosing in (4.184) test functions of the form $\psi = \phi(x_1, x_2)\sigma(x_3)$ with $\phi \in H^1_{0,\gamma_{eD}}(\omega)$ and $\sigma \in H^1_0(0, L)$, we have

$$\int_0^L \left(\int_\omega \frac{1}{\bar{\varepsilon}_{11}}E_\alpha(\varphi)\partial_\alpha\phi\,d\omega\right)\sigma\,dx_3 + \int_0^L \bar{P}_{14}\left(\int_\omega \Sigma_{3\alpha}\partial_\alpha\phi\,d\omega\right)\sigma dx_3 = 0, \quad (4.185)$$

and consequently we find that φ is a solution of the following problem a.e. in [0,L]

$$\int_\omega \frac{1}{\bar{\varepsilon}_{11}}E_\alpha(\varphi)\partial_\alpha\phi\,d\omega + \bar{P}_{14}\int_\omega \Sigma_{3\alpha}\partial_\alpha\phi\,d\omega = 0, \quad (4.186)$$

for all $\phi \in H_{0,\gamma_{eD}}(\omega)$. For the homogeneous case, the coefficient $\bar{P}_{14}$ is independent on x_1 and x_2, then from property (4.153) satisfied by function $\Sigma_{3\alpha}$ we obtain the second term of (4.186) is identically zero and the equation reads

$$\int_\omega \frac{1}{\bar{\varepsilon}_{11}}\partial_\alpha\varphi\partial_\alpha\phi\,d\omega = 0, \quad (4.187)$$

for all $\phi \in H_{0,\gamma_{eD}}(\omega)$. We prove, using algebraic manipulation software tools, that

$$\frac{1}{\bar{\varepsilon}_{11}} = \varepsilon_{11} + \frac{P_{14}P_{14}}{C_{66}},$$

and therefore the previous equation coincides with (4.121).

From the unicity of the limits $u_\alpha = u^0_\alpha$, $u_3 = u^0_3$ and $\varphi = \varphi^0$ we conclude that the weak convergence (4.178) and (4.178) hold for the whole sequence $\boldsymbol{u}(h)$ and $\varphi(h)$, respectively.
■

4.5 The limit model on the actual beam Ω^h

This section is subdivided in two section in which models for materials of class 2 and *6mm* will be written for the original beam Ω^h.

4.5.1 Model for a beam belonging to the class 2 of piezoelectric crystals

We now return to the actual beam Ω^h and define the following spaces (cf. (4.82)-(4.85)):

$$Q^h = Q^h(\omega^h) = \{\rho \in H^1(\omega^h) : \int_{\omega^h} \rho \, d\omega^h = 0\},$$

$$S^h = S^h(\omega^h) = \{\psi \in H^1(\omega^h) : \psi = 0 \text{ on } \gamma_{eD}^h\},$$

$$T^h = T^h(\omega^h) = Q^h \times S^h,$$

$$\Psi_l^h = \Psi_l^h(\Omega^h) = L^2(0, L; H^1(\omega^h)),$$

$$R^h = R^h(\Omega^h) = L^2(0, L; Q^h(\omega^h)),$$

$$\Psi_{l0}^h = \Psi_{l0}^h(\Omega^h) = L^2(0, L; S^h(\omega^h)).$$

Given the scalings (3.36), (4.12) and (4.24), the developments (4.36), (4.38 and (4.39) induce formal developments on $\boldsymbol{u}^h$, $\bar{\varphi}^h$, $\boldsymbol{\sigma}^h$ and $\boldsymbol{D}^h$, respectively, whose leading terms we will identify and characterize in the following. For that we will undo the change of variable $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$ and accordingly define the de-scaled quantities:

$$\xi_\alpha^h(x_3^h) = h^{-1}\xi_\alpha(x_3), \qquad \xi_3^h(x_3^h) = \xi_3(x_3),$$

$$\bar{\varphi}^{0h}(\boldsymbol{x}^h) = h\bar{\varphi}^0\left(\boldsymbol{x}\right), \qquad r^h(\boldsymbol{x}^h) = hr\left(\boldsymbol{x}\right),$$

as well as the warping function

$$w^h = w^h(x_1^h, x_2^h) = h^2 w(x_1, x_2),$$

which is the unique solution of (cf. (4.87))

$$\begin{aligned} & w^h \in Q^h \text{ such that} \\ & \int_{\omega^h} C_{3\alpha 3\beta} \partial_\beta^h w^h \partial_\alpha^h \upsilon^h \, d\omega^h = \int_{\omega^h} C_{3\alpha 3\beta} \delta_\beta^h \partial_\alpha^h \upsilon^h \, d\omega^h, \quad \text{for all } \upsilon^h \in H^1(\omega^h), \end{aligned} \tag{4.188}$$

the torsion constant (cf. (4.88))

$$J^h = h^4 J = \int_{\omega^h} C_{3\alpha 3\beta}(\delta_\beta^h - \partial_\beta^h w^h)(\delta_\alpha^h - \partial_\alpha^h w^h) \, d\omega^h = \int_{\omega^h} C_{3\alpha 3\beta}(\delta_\beta^h - \partial_\beta^h w^h)\delta_\alpha^h \, d\omega^h, \tag{4.189}$$

and the second moment of area of the cross section with respect to axis Ox^h_β ($\alpha \neq \beta$)

$$I^h_\alpha = \int_{\omega^h} (x^h_\alpha)^2 \, d\boldsymbol{\omega}^h,$$

where $\delta^h_1(x^h_1, x^h_2) = x^h_2$, $\delta^h_2(x^h_1, x^h_2) = -x^h_1$. We also define the resultant of the applied loads and moments in each cross section (cf. (4.78)-(4.79)):

$$F^h_\alpha(x^h_3) = h^3 F_\alpha(x_3) = \int_{\omega^h} f^h_\alpha(x^h_1, x^h_2, x^h_3)\, d\omega^h + \int_{\gamma^h_{dN}} g^h_\alpha(x^h_1, x^h_2, x^h_3)\, d\gamma^h \in L^2(0, L),$$

$$F^h_3(x^h_3) = h^2 F_3(x_3) = \int_{\omega^h} f^h_3(x^h_1, x^h_2, x^h_3)\, d\omega^h + \int_{\gamma^h_{dN}} g^h_3(x^h_1, x^h_2, x^h_3)\, d\gamma^h \in L^2(0, L),$$

$$M^h_\alpha(x^h_3) = h^3 M_\alpha(x_3) = \int_{\omega^h} x^h_\alpha f^h_3(x^h_1, x^h_2, x^h_3)\, d\omega^h + \int_{\gamma^h_{dN}} x^h_\alpha g^h_3(x^h_1, x^h_2, x^h_3)\, d\gamma^h \in L^2(0, L).$$

For a generic function $\eta^h : \overline{\Omega}^h \to \mathbb{R}$ we define its average along the x^h_3-axis (cf. (4.100)),

$$\underline{\eta}^h(x^h_1, x^h_2) = \frac{1}{L} \int_0^L \eta^h(x^h_1, x^h_2, s)\, ds,$$

and consider the deviation (cf. (4.101))

$$\eta^{D,h}(x^h_1, x^h_2, x^h_3) = \eta^h(x^h_1, x^h_2, x^h_3) - \underline{\eta}^h(x^h_1, x^h_2).$$

Finally, we define $z^h = z^h(x^h_3) = h^{-1} z\,(x_3)$, implying (cf. (4.106))

$$(z^h)'(x^h_3) = \frac{1}{J^h} \int_{\omega^h} P_{\beta 3\alpha}(\partial^h_\alpha w^h - \delta^h_\alpha)\partial^h_\beta(\Delta^h \bar{\varphi}^{0h}(x^h_3) + \Delta^h \varphi^h_0(x^h_3))\, d\omega^h, \quad \text{a.e. } x^h_3 \in (0, L),$$

and also $(\underline{r}^h, \underline{\bar{\varphi}}^{0h}) \in T^h$, the unique solution of the variational problem

$$\begin{aligned}
&(\underline{r}^h, \underline{\bar{\varphi}}^{0h}) \in T^h \text{ such that} \\
&\int_{\omega^h} C_{3\alpha 3\beta}\partial^h_\beta \underline{r}^h \partial^h_\alpha \rho\, d\omega^h + \int_{\omega^h} \varepsilon_{\alpha\beta}\partial^h_\alpha \underline{\bar{\varphi}}^{0h} \partial^h_\beta \psi\, d\omega^h \\
&\qquad + \int_\omega P_{\beta 3\alpha}(\partial^h_\alpha \underline{r}^h \partial^h_\beta \psi - \partial^h_\beta \underline{\bar{\varphi}}^{0h} \partial_\alpha \rho)\, d\omega^h \\
&= \int_{\omega^h} P_{\beta 3\alpha}\partial^h_\beta \underline{\varphi}^h_0 \partial^h_\alpha \rho\, d\omega^h - \int_{\omega^h} \varepsilon_{\alpha\beta}\partial^h_\alpha \underline{\varphi}^h_0 \partial^h_\beta \psi\, d\omega^h, \\
&\text{for all } (\rho, \psi) \in T^h.
\end{aligned} \tag{4.190}$$

4.5.2 For a beam belonging to the class 6mm of piezoelectric crystals

We assume that the piezoelectric material is anisotropic and of a type of transversely isotropic (hexagonal crystal system, class *6mm*).

Firstly, we introduce auxiliary functions $\overline{\overline{\Lambda}}^h_\alpha$, $\Phi^h_{\alpha\beta}$ depending on the geometry of the cross section ω^h. Then, the following properties hold

$$\overline{\overline{\Lambda}}^h_\alpha(x_1^h, x_2^h) = h\overline{\overline{\Lambda}}_\alpha(x_1, x_2), \tag{4.191}$$

$$\Phi^h_{\alpha\beta}(x_1^h, x_2^h) = h^2\Phi_{\alpha\beta}(x_1, x_2), \tag{4.192}$$

where

$$\begin{aligned}
\overline{\overline{\Lambda}}_1(x_1, x_2) &= -\frac{C_{16}}{C_{11}+C_{13}}x_1, \\
\overline{\overline{\Lambda}}_2(x_1, x_2) &= \frac{C_{16}}{C_{11}+C_{13}}x_2 \\
\boldsymbol{\Phi}(x_1, x_2) &= (\Phi_{\alpha\beta})(x_1, x_2) = \frac{C_{16}}{(C_{11}+C_{13})}\begin{pmatrix} \frac{(x_1^2-x_2^2)}{2} & x_1x_2 \\ x_1x_2 & \frac{(x_2^2-x_1^2)}{2} \end{pmatrix}.
\end{aligned}$$

The warping function warping function $w^h(x_1^h, x_2^h)$ as the unique solution of the variational problem:

$$\begin{aligned}
&w^h \in H^1(\omega^h), \quad \int_{\omega^h} w^h = 0 \\
&\int_{\omega^h} \partial_\beta w^h \partial_\alpha v^h \, d\omega^h = \int_{\omega^h} (x_2^h \partial_1^h v^h - x_1^h \partial_2^h v^h)\, d\omega^h, \quad \text{for all } v^h \in H^1(\omega^h).
\end{aligned} \tag{4.193}$$

Corollary 20 *The approximations* $(\boldsymbol{u}^{h,0}, \bar{\varphi}^{h,0}, \sigma_{33}^{h,0}, D_\alpha^{h,0})$ *are uniquely characterized as follows:*

$$u_\alpha^{h,0} = \xi_\alpha^h(x_3), \quad \xi_\alpha^h \in H_0^2(0, L), \tag{4.194}$$

$$u_3^{h,0} = \xi_\alpha^h - x_\alpha(\xi_\alpha^h)'(x_3), \quad \xi_3^h \in H_0^1(0, L), \tag{4.195}$$

$$u_\alpha^{h,1} = z_\alpha^h, \quad z_\alpha^h \in H_0^1(0, L), \tag{4.196}$$

$$u_3^{h,1} = z_3^h - x_\alpha^h(z_\alpha^h)' - \frac{P_{14}}{C_{14}}\varphi^{h,0}, \tag{4.197}$$

$$\bar{\varphi}^{h,0} = \varphi^{h,0} - \hat{\varphi}^h, \quad \varphi^{h,0} \in \Psi_l^h, \tag{4.198}$$

$$\sigma^{h,0}_{\alpha\beta} = \sigma^{h,0}_{3\alpha} = 0, \tag{4.199}$$

$$\sigma^{h,0}_{33} = Y\, e_{33}(\boldsymbol{u}^{h,0}) = Y[(\xi^h_\alpha)' - x_\alpha(\xi^h_\alpha)''(x_3)], \tag{4.200}$$

$$D^{h,0}_\alpha = -R_{11}\, \partial_\beta \varphi^{h,0}, \tag{4.201}$$

where $(\xi^h, \varphi^{h,0})$ is the solution of the following boundary value problem:

$$\begin{cases} -Y\, A(\omega^h)\, \left(\xi^h_3\right)'' = F^h_3 \quad \textit{in } (0,L), \\ \xi^h_3(0) = \xi^h_3(L) = 0, \\ Y\, I^h_\beta\, \left(\xi^h_\beta\right)^{(4)} = F^h_\beta + \left(M^h_\beta\right)' \quad \textit{in } (0,L), \\ \xi^h_\beta(0) = \xi^h_\beta(L) = 0,\ \left(\xi^h_\beta\right)'(0) = \left(\xi^h_\beta\right)'(L) = 0, \end{cases} \tag{4.202}$$

$$\begin{cases} \varphi^{h,0} \in H^1(\omega) \textit{ such that a.e. in } (0,L) \\ -R_{11}\, \partial^h_{\beta\beta}\varphi^{h,0} = 0 \quad \textit{in } \omega^h, \\ R_{11}\, \partial^h_\beta \varphi^{h,0}\, n^h_\beta = 0 \quad \textit{on } \gamma^h_{eN}, \\ \varphi^{h,0} = \varphi^h_0 \quad \textit{on } \gamma^h_{eD}, \end{cases} \tag{4.203}$$

where Young's modulus $Y > 0$ is given by (cf. Subsection 2.1.3.3)

$$Y = \frac{1}{C_{11} + C_{13}}(-2C^2_{16} + C_{11}C_{66} + C_{13}C_{66}) = \frac{\bar{\varepsilon}_{33}}{\bar{\varepsilon}_{33}\bar{C}_{66} + \bar{P}_{36}\bar{P}_{36}}, \tag{4.204}$$

$$R_{11} = P_{14}\frac{P_{14}}{C_{44}} + \varepsilon_{11} = \frac{1}{\bar{\varepsilon}_{11}}. \tag{4.205}$$

We note that for the homogeneous transversely isotropic beam model, the electrical and mechanical phenomena are decoupled. The boundary problems found in (4.194)-(4.202) of Corollary 20 are respectively called the one-dimensional bending equations and the one-dimensional stretching equations of a linearly elastic beam. Together with the property that $u(0)$ is a Bernoulli-Navier displacement field (Theorem 15), they constitute the linear Bernoulli-Navier model of a linearly elastic beam. Furthermore, the electric potential can be obtained through a two-dimensional Laplace's equation, expressed by the boundary value problem (4.203).

Chapter 5

Shallow arch theory with an electric potential applied at both ends

In many applications "weakly" curved beams are used instead of straight beams. In this chapter, a "weakly" curved beams is a beam which the length of the centreline much greater than the diameter of the cross section and the curvature is of the order of the diameter of the cross section. This type of curved beams are also known as *shallow arches*. We present a zeroth-order model for a transversely isotropic - *6mm* symmetric class - piezoelectric shallow arch under the influence of an applied electric potential on both end faces, obtained by asymptotic methods.

Let us briefly outline the content of this chapter, which closely follows Álvarez-Dios & Viaño [1996]. In the next two sections we introduce the notation and present the piezoelectricity problem in its Hellinger-Reissner variational principle. In Section 5.3 we rescale the three-dimensional problem posed in a straight reference rod. In Sections 5.5 and 5.5.2, using the same results of Álvarez-Dios & Viaño [1998], we study the limit behavior of the unknowns (displacement, stress, electric potential and electric displacement) when the cross-sectional diameter of the beam tends to zero. Based on the asymptotic expansion method, we prove in Section 5.5.3 that, when the cross-sectional diameter tends to zero the scaled solution of the three-dimensional problem strongly converges to the leading term of its asymptotic expansion. A key idea to prove the strong convergence is to take for the limits the first terms of the developments identified in Section 5.5.2. The one-dimensional equations of the coupled mechanical and electrical field in the original domain are established in Section 5.7 and they are written as a BVP.

5.1 Geometry of shallow arch

As before, let h and L be two positive scalars and let ω^h denote an open bounded, simply connected subset of $\mathbb{R}^2$, with Lipschitz continuous boundary bounded γ^h having area $A(\omega^h) = h^2$. We suppose that system $Ox_1^h x_2^h x_3$ is a principal system of inertia associated to ω^h, therefore axis Ox_3 passes through the mass center of $\omega^h \times \{x_3\}$, which means that

$$\int\limits_{\omega^h \times \{x_3\}} x_\alpha^h \, d\omega^h = \int\limits_{\omega^h \times \{x_3\}} x_1^h x_2^h \, d\omega^h = 0, \quad x_3 \in [0, L]. \tag{5.1}$$

A curved rod can be represented by a space curve

$$C^h = \left\{ \boldsymbol{\phi}^h(x_3) = \left(\phi_1^h(x_3), \phi_2^h(x_3), x_3\right) \in \mathbb{R}^3 : x_3 \in [0, L] \right\},$$

parameterized by its arc length $s^h(x_3)$, $x_3 \in [0, L]$. The Frenet trihedron $(\boldsymbol{t}^{*,h}, \boldsymbol{n}^{*,h}, \boldsymbol{b}^{*,h})$ is formed by the tangent, normal and binormal vectors of the curve,

$$\boldsymbol{t}^{*,h} = (t_i^{*,h}) = \frac{1}{\sqrt{\mathcal{A}^h}} (\phi_i^{h\prime}), \tag{5.2}$$

$$\boldsymbol{n}^{*,h} = \begin{pmatrix} n_1^{*,h} \\ n_2^{*,h} \\ n_3^{*,h} \end{pmatrix} = \frac{1}{\sqrt{\mathcal{A}^h \mathcal{B}^h}} \begin{pmatrix} \mathcal{A}^h \phi_1^{h\prime\prime} - \phi_1^{h\prime} \phi_\alpha^{h\prime} \phi_\alpha^{h\prime\prime} \\ \mathcal{A}^h \phi_2^{h\prime\prime} - \phi_2^{h\prime} \phi_\alpha^{h\prime} \phi_\alpha^{h\prime\prime} \\ -\phi_\alpha^{h\prime} \phi_\alpha^{h\prime\prime} \end{pmatrix}, \tag{5.3}$$

$$\boldsymbol{b}^{*,h} = \begin{pmatrix} b_1^{*,h} \\ b_2^{*,h} \\ b_3^{*,h} \end{pmatrix} = \frac{1}{\sqrt{\mathcal{B}^h}} \begin{pmatrix} -\phi_2^{h\prime\prime} \\ \phi_1^{h\prime\prime} \\ \phi_1^{h\prime} \phi_2^{h\prime\prime} - \phi_2^{h\prime} \phi_1^{h\prime\prime} \end{pmatrix}, \tag{5.4}$$

where

$$\mathcal{A}^h = |\boldsymbol{\phi}^{h\prime}| = \phi_\alpha^{h\prime} \phi_\alpha^{h\prime} + 1, \tag{5.5}$$

$$\mathcal{B}^h = |\boldsymbol{\phi}^{h\prime} \times \boldsymbol{\phi}^{h\prime\prime}|^2 = \phi_\beta^{h\prime\prime} \phi_\beta^{h\prime\prime} + (\phi_1^{h\prime} \phi_2^{h\prime\prime} - \phi_1^{h\prime\prime} \phi_2^{h\prime})^2. \tag{5.6}$$

The trihedron is given by the Frenet equations ($s^h = s^h(x_3)$):

$$\frac{d\boldsymbol{t}^{*,h}}{ds^h} = \kappa^h \boldsymbol{n}^{*,h}, \tag{5.7}$$

$$\frac{d\boldsymbol{n}^{*,h}}{ds^h} = -\kappa^h \boldsymbol{t}^{*,h} + \tau^h \boldsymbol{b}^{*,h}, \tag{5.8}$$

$$\frac{d\boldsymbol{b}^{*,h}}{ds^h} = -\tau^h \boldsymbol{n}^{*,h}, \tag{5.9}$$

where $k^h(s^h(x_3)) \equiv \kappa^h(x_3)$ and $\tau^h(s^h(x_3)) \equiv \tau^h(x_3)$ are the centreline's curvature and torsion. The curvature and torsion are extracted from the curve parametrization as follows

$$\kappa^h(x_3) = \frac{|\boldsymbol{\phi}^{h'} \times \boldsymbol{\phi}^{h''}|}{|\boldsymbol{\phi}^{h'}|^3} = \sqrt{\frac{\mathcal{B}^h}{(\mathcal{A}^h)^3}},$$
$$\tau^h(x_3) = \frac{\boldsymbol{\phi}^{h'''} \cdot (\boldsymbol{\phi}^{h'} \times \boldsymbol{\phi}^{h''})}{\left|\boldsymbol{\phi}^{h'} \times \boldsymbol{\phi}^{h''}\right|^2} = \frac{1}{\mathcal{B}^h}(\phi_1^{h''}\phi_2^{h'''} - \phi_2^{h''}\phi_1^{h'''}).$$

As assumed in Álvarez-Dios & Viaño [1998], the family of curves C^h, $0 < h < 1$, satisfies the following hypothesis:

(**HC1**) $\phi^h_\alpha \in \mathcal{C}^3[0, L]$.

(**HC2**) For all $x_3 \in [0, L]$, $(\boldsymbol{t}^{*,h}, \boldsymbol{n}^{*,h}, \boldsymbol{b}^{*,h})$ is a positive oriented orthonormal basis of $\mathbb{R}^3$.

(**HC3**) Frenet equations (5.7)-(5.9) hold for curvature κ^h and τ^h belonging to $\mathcal{C}[0, L]$.

Now we define the map $\boldsymbol{\Theta}^h : \bar{\Omega}^h \to \boldsymbol{\Theta}^h(\bar{\Omega}^h) \subset \mathbb{R}^3$ in the following manner (Figure 5.1)

$$\boldsymbol{\Theta}^h\left(\boldsymbol{x}^h\right) = \left((\phi_1^h(x_3), \phi_2^h(x_3), x_3) + x_1^h \boldsymbol{n}^{*,h}(x_3) + x_2^h \boldsymbol{b}^{*,h}(x_3)\right) \tag{5.10}$$

which is a C^1-diffeomorphism [Álvarez-Dios & Viaño, 1998, see Theorem 1.1-1].

For each $h > 0$ and each $\boldsymbol{x}^h \in \Omega^h$, let $\boldsymbol{\nabla}^h\boldsymbol{\Theta}^h(\boldsymbol{x}^h)$ denote the Jacobian matrix $(\partial^h_j \Theta^h_i(\boldsymbol{x}^h))$ and let

$$b^h_{ij}(\boldsymbol{x}^h) := \left(\boldsymbol{\nabla}^h\boldsymbol{\Theta}^h(\boldsymbol{x}^h))^{-1}\right)_{ij} \quad \text{for all } \boldsymbol{x}^h \in \Omega^h,$$
$$o^h(\boldsymbol{x}^h) := \det\left\{\boldsymbol{\nabla}^h\boldsymbol{\Theta}^h(\boldsymbol{x}^h)\right\} \quad \text{for all } \boldsymbol{x}^h \in \Omega^h,$$

where the scalar $o^h(\boldsymbol{x}^h)$ and the vectors $b^h_{ij}(\boldsymbol{x}^h)$, $\boldsymbol{x}^h \in \Omega^h$ are of the form (see Álvarez-Dios & Viaño [1998])

$$o^h(\boldsymbol{x}^h) = \sqrt{\mathcal{A}^h}(1 - x_1^h\kappa^h) = \sqrt{\mathcal{A}^h} - x_1^h\frac{\sqrt{\mathcal{B}^h}}{\mathcal{A}^h}, \tag{5.11}$$
$$b^h_{1j}(\boldsymbol{x}^h) = n_j^{*,h} + \frac{\tau^h\sqrt{\mathcal{A}^h}}{o^h}x_2^h t_j^{*,h}, \tag{5.12}$$
$$b^h_{2j}(\boldsymbol{x}^h) = b_j^{*,h} - \frac{\tau^h\sqrt{\mathcal{A}^h}}{o^h}x_1^h t_j^{*,h}, \tag{5.13}$$
$$b^h_{3j}(\boldsymbol{x}^h) = \frac{1}{o^h}t_j^{*,h}, \tag{5.14}$$

where the map $\boldsymbol{\Theta}^h$ is assumed to be an orientation-preserving map, that is,

$$o^h(\boldsymbol{x}^h) > 0, \text{ for all } \boldsymbol{x}^h \in \Omega^h.$$

The beam in study occupies the volume $\{\breve{\Omega}^h\}^- = \boldsymbol{\Theta}(\bar{\Omega}^h)$ which is well-known as a weakly curved rod of axis C^h.

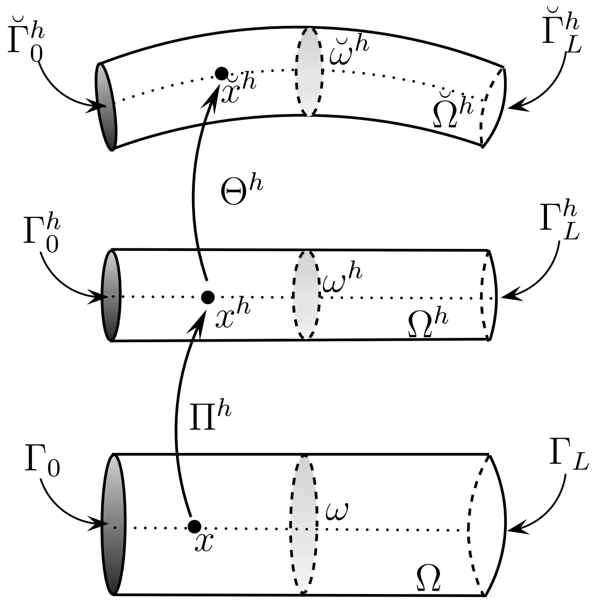


Figure 5.1: The reference configuration of the shallow arch described in curvilinear and Cartesian coordinates, and in a fixed domain.

The boundary of $\breve{\Omega}^h$, $\partial^h\breve{\Omega}^h$, is the union of the end faces $\breve{\Gamma}_0^h = \boldsymbol{\Theta}^h(\omega^h \times \{0\})$, $\breve{\Gamma}_L^h = \boldsymbol{\Theta}^h(\omega^h \times \{L\})$ and the lateral surface $\breve{\Gamma}_N^h = \boldsymbol{\Theta}^h(\gamma^h \times (0, L))$.

This boundary is the disjoint union of $\breve{\Gamma}_{dD}^h$ and Γ_{dN}^h on the one hand and on the other hand, it is the union of $\breve{\Gamma}_{eD}^h$ and $\breve{\Gamma}_{eN}^h$ where

$$\breve{\Gamma}_{dD}^h = \boldsymbol{\Theta}^h(\Gamma_0^h), \quad \breve{\Gamma}_{dN}^h = \boldsymbol{\Theta}^h(\Gamma_{dN}^h), \quad \breve{\Gamma}_{eD}^h = \boldsymbol{\Theta}^h(\Gamma_{dN}^h), \quad \breve{\Gamma}_{eN}^h = \boldsymbol{\Theta}^h(\Gamma_N^h),$$

with

$$\Gamma_{dN}^h = \Gamma_N^h \cup \Gamma_L^h, \quad \Gamma_{eD}^h = \Gamma_0^h \cup \Gamma_L^h.$$

The reference configuration of the shallow arch can thus be described in terms of the three-dimensional curvilinear coordinates, or in terms of the Cartesian coordinates $\breve{x}_1^h, \breve{x}_2^h, \breve{x}_3$, of the same point $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}(\boldsymbol{x}^h) \in \{\breve{\Omega}^h\}^-$. Let $\partial_i^h := \partial/\partial x_i^h$ (hence $\partial_\alpha^h := \partial/\partial x_\alpha^h$)

and $\breve{\partial}_i^h$ will stand for the differential operator $\partial/\partial\breve{x}_i^h$. In an analogous way, we denote by $\breve{\eta}^h$ a function defined in $\breve{\Omega}^h$, η^h a function defined in Ω^h and η a function defined in Ω, related by

$$\breve{\eta}^h(\breve{\boldsymbol{x}}^h) = \eta^h(\boldsymbol{x}^h) = \eta(\boldsymbol{x}), \qquad \breve{\boldsymbol{x}} = \boldsymbol{\Theta}(\boldsymbol{x}^h);\ \ \boldsymbol{x}^h = \Pi^h(\boldsymbol{x}).$$

5.2 Three-dimensional equations of a linearly piezoelectric clamped Shallow Arch in Cartesian coordinates

We now review the formulation of the linearly piezoelectric problem in the set $\{\breve{\Omega}^h\}^-$. We assume that the shallow arch, whose reference configuration is $\bar{\breve{\Omega}}^h$, is clamped on the portion $\breve{\Gamma}_{dD}^h = \breve{\Gamma}_0^h$ and submitted to a mechanical volume force of density $\breve{\boldsymbol{f}}^h$ in $\breve{\Omega}^h$, to a mechanical surface force $\breve{\boldsymbol{g}}^h$ on $\breve{\Gamma}_N^h$ and to a surface force $\breve{\boldsymbol{p}}^h$ on the end $\breve{\Gamma}_L^h$. We denote the outward unit normal to $\partial\breve{\Omega}^h$ by $\breve{\boldsymbol{n}}^h$. An electric potential is applied on the left end, with value $\breve{\varphi}_0^{0,h}$, and on the right end, with value $\breve{\varphi}_0^{L,h}$, which means that $\breve{\Gamma}_{eD}^h = \breve{\Gamma}_0^h \cup \breve{\Gamma}_L^h$. The left end of the beam is weakly clamped, and therefore the boundary condition (3.2), introduced in Chapter 3, is also assumed.

Then, the body undergoes a mechanical displacement field $\breve{\boldsymbol{u}}^h = (\breve{u}_i^h) : \bar{\breve{\Omega}}^h \to \mathbb{R}^3$ and an electrical potential $\breve{\varphi}^h : \bar{\breve{\Omega}}^h \to \mathbb{R}$ satisfying the following equilibrium equations:

$$\left\{\begin{array}{rcll} -\breve{\partial}_i^h \breve{\sigma}_{ij}^h(\breve{\boldsymbol{u}}^h, \breve{\varphi}^h) & = & \breve{f}_i & \text{in} \quad \breve{\Omega}^h \\ \breve{\sigma}_{ij}^h(\breve{\boldsymbol{u}}^h, \breve{\varphi}^h)\ \breve{n}_j^h & = & \breve{g}_i^h & \text{on} \quad \breve{\Gamma}_N^h \\ \breve{\sigma}_{i3}^h(\breve{\boldsymbol{u}}^h, \breve{\varphi}^h) & = & \breve{p}_i^h & \text{on} \quad \breve{\Gamma}_L^h \end{array}\right. , \tag{5.15}$$

$$\left\{\begin{array}{rcll} \breve{\partial}_k^h \breve{D}_k^h(\breve{\boldsymbol{u}}^h, \breve{\varphi}^h) & = & 0 & \text{in} \quad \breve{\Omega}^h \\ \breve{D}_k^h(\breve{\boldsymbol{u}}^h, \breve{\varphi}^h)\ \breve{n}_k & = & 0 & \text{on} \quad \breve{\Gamma}_{eN}^h \\ \breve{\varphi}^h & = & \breve{\varphi}_0^h & \text{on} \quad \breve{\Gamma}_{eD}^h \end{array}\right. , \tag{5.16}$$

where we denote the second-order strain tensor $\breve{e}_{ij}^h(\breve{\boldsymbol{u}}^h) = \frac{1}{2}(\breve{\partial}_i^h \breve{u}_j^h + \breve{\partial}_j^h \breve{u}_i^h)$ and electric vector $\breve{E}_k^h(\breve{\varphi}^h) = -\breve{\partial}_k^h \breve{\varphi}^h$ are linearly related to the piezoelectric stress tensor $\breve{\boldsymbol{\sigma}}^h = (\breve{\sigma}_{ij}^h)$

and electric displacement field $\breve{\boldsymbol{D}}^h = (\breve{D}_i^h)$ by the constitutive law :

$$\begin{aligned}
\breve{E}_3^h &= -\breve{\bar{P}}_{3\theta\theta}^h\,\breve{\sigma}_{\theta\theta}^h - \bar{P}_{333}^h\,\breve{\sigma}_{33}^h + \breve{\bar{\varepsilon}}_{33}^h\,\breve{D}_3^h \quad \text{in } \breve{\Omega}^h,\\
\breve{E}_1^h &= -2\breve{\bar{P}}_{131}^h\,\breve{\sigma}_{31}^h + \breve{\bar{\varepsilon}}_{11}^h\,\breve{D}_1^h \quad \text{in } \breve{\Omega}^h,\\
\breve{E}_2^h &= -2\breve{\bar{P}}_{232}^h\,\breve{\sigma}_{32}^h + \breve{\bar{\varepsilon}}_{22}^h\,\breve{D}_2^h \quad \text{in } \breve{\Omega}^h,
\end{aligned} \tag{5.17}$$

and (no sum on α)

$$\begin{aligned}
\breve{e}_{\alpha\alpha}^h &= \breve{\bar{C}}_{\alpha\alpha 33}^h\,\breve{\sigma}_{33}^h + \breve{\bar{C}}_{\alpha\alpha\theta\theta}^h\,\breve{\sigma}_{\theta\theta}^h + \breve{\bar{P}}_{3\alpha\alpha}^h\,\breve{D}_3^h \quad \text{in } \breve{\Omega}^h,\\
\breve{e}_{12}^h &= 2\breve{\bar{C}}_{1212}^h\,\breve{\sigma}_{12}^h \quad \text{in } \breve{\Omega}^h,\\
\breve{e}_{31}^h &= 2\breve{\bar{C}}_{3131}^h\,\breve{\sigma}_{31}^h + \breve{\bar{P}}_{131}^h\,\breve{D}_1^h \quad \text{in } \breve{\Omega}^h,\\
\breve{e}_{32}^h &= 2\breve{\bar{C}}_{3232}^h\,\breve{\sigma}_{32}^h + \breve{\bar{P}}_{232}^h\,\breve{D}_2^h \quad \text{in } \breve{\Omega}^h,\\
\breve{e}_{33}^h &= \breve{\bar{C}}_{3333}^h\,\breve{\sigma}_{33}^h + \bar{C}_{33\theta\theta}\,\breve{\sigma}_{\theta\theta}^h + \breve{\bar{P}}_{333}^h\,\breve{D}_3^h \quad \text{in } \breve{\Omega}^h.
\end{aligned} \tag{5.18}$$

These constitutive equations describe piezoelectric materials of *crystal class 6mm* in which the components of the elastic, piezoelectric and dielectric material satisfy (cf. (3.8) in Chapter 3)

$$\begin{aligned}
&\breve{\bar{C}}_{iijj}^h \neq 0, \quad \breve{\bar{C}}_{klkl}^h \neq 0 \qquad \text{for} \quad k \neq l\\
&\breve{\bar{P}}_{3\alpha\alpha}^h \neq 0 \quad \breve{\bar{P}}_{\alpha 3\alpha}^h \neq 0, \quad \breve{\bar{P}}_{333}^h \neq 0, \quad \breve{\bar{\varepsilon}}_{ii}^h \neq 0,
\end{aligned} \tag{5.19}$$

(no summation over repeated indices). The remaining coefficients

$$\breve{\bar{C}}_{ijkl}^h, \quad \breve{\bar{P}}_{kij}^h \quad \text{and} \quad \breve{\bar{\varepsilon}}_{ij}^h \quad \text{vanish.} \tag{5.20}$$

Remark 15 *The condition (5.20) is used only when strictly required.*

The hypothesis ($\mathbf{H}_1^m$) and ($\mathbf{H}_2^m$), established in Section 2.3.2.3, are also assumed in this chapter.

Now, we define the following spaces for the admissible displacements and admissible electric potentials

$$\breve{\Psi}_0^h := \breve{\Psi}_0^h(\breve{\Omega}^h) = \left\{\breve{\psi}^h \in H^1(\breve{\Omega}^h) : \breve{\psi}^h = 0 \text{ on } \breve{\Gamma}_{eD}^h = \breve{\Gamma}_0^h \cup \breve{\Gamma}_L^h\right\},$$

$$\breve{V}_{0,w}^h := \breve{V}_{0,w}^h(\breve{\Omega}^h) = \left\{\breve{\boldsymbol{v}}^h \in [H^1(\breve{\Omega}^h)]^3 : \int_{\breve{\omega}^h\times\{0\}} \breve{\boldsymbol{v}}^h d\breve{\Gamma} = \int_{\breve{\omega}^h\times\{0\}} \left(\breve{x}_i^h\breve{v}_j^h - \breve{x}_j^h\breve{v}_i^h\right) d\breve{\Gamma} = 0\right\}.$$

We have already defined the spaces $\breve{\boldsymbol{X}}^h_{0,w} := \breve{V}^h_{0,w} \times \breve{\Psi}^h_0$ and $\breve{\boldsymbol{X}}^h_1 := [L^2(\breve{\Omega}^h)]^9_s \times L^2(\breve{\Omega}^h)$ equipped with the norms,

$$\|(\breve{\boldsymbol{v}}^h, \breve{\psi}^h)\|_{\breve{\boldsymbol{X}}^h_{0,w}} = \left(\|\breve{\boldsymbol{v}}^h\|^2_{\breve{V}^h_{0,w}} + \|\breve{\psi}^h\|^2_{H^1(\breve{\Omega}^h)}\right)^{1/2},$$

$$\|(\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h)\|_{\breve{\boldsymbol{X}}^h_1} = \left(|\breve{\boldsymbol{\tau}}^h|^2_{L^2(\breve{\Omega}^h)} + |\breve{\boldsymbol{d}}^h|^2_{L^2(\breve{\Omega}^h)}\right)^{1/2},$$

respectively, and $\|\breve{\boldsymbol{v}}^h\|_{\breve{V}^h_{0,w}} = |\breve{\boldsymbol{e}}^h(\breve{\boldsymbol{v}}^h)|_{0,\breve{\Omega}^h}$.

As in Remark 4 established in Section 3.1.2, we admit the existence of a function $\hat{\breve{\varphi}}^h \in H^1(\breve{\Omega}^h)$ such that $\hat{\breve{\varphi}}^h = \breve{\varphi}^h_0$ on $\breve{\Gamma}^h_{eD}$.

In addition, the closed convex set on $H^1(\breve{\Omega}^h)$ is defined by

$$\breve{\Psi}^h_2 := \breve{\Psi}^h_2(\breve{\Omega}^h) = \left\{\breve{\varphi}^h \in H^1(\breve{\Omega}^h) : \breve{\varphi}^h - \hat{\breve{\varphi}}^h \in \breve{\Psi}^h_0\right\},$$

and consequently we have $\breve{\boldsymbol{X}}^h_{2,w} = \breve{V}^h_{0,w} \times \breve{\Psi}^h_2$.

Finally we assume the following regularity assumptions on the data:

$$\breve{\boldsymbol{f}}^h = (\breve{f}^h_i) \in \left[L^2(\breve{\Omega}^h)\right]^3, \quad \breve{\boldsymbol{g}}^h = (\breve{g}^h_i) \in \left[L^2(\breve{\Gamma}^h_N)\right]^3, \quad \breve{\boldsymbol{p}}^h = (\breve{p}^h_i) \in \left[L^2(\breve{\Gamma}^h_L)\right]^3.$$

To obtain the mixed problem, we multiply by $\breve{d}^h_k \in L^2(\breve{\Omega}^h)$ the equations (5.17) and by $\breve{\tau}^h_{ij} \in L^2(\breve{\Omega}^h)$ in (5.18). If we multiply the first equation of (5.15) and (5.16) by $\breve{\boldsymbol{v}}^h \in \breve{V}^h_{0,w}$ and $\breve{\psi}^h \in \breve{\Psi}^h_0$, respectively, and integrate over $\breve{\Omega}^h$ and apply the divergence theorem, then we obtain the Hellinger-Reissner mixed variational formulation of problem (5.15)-(5.18): *Find* $((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{u}}^h, \breve{\varphi}^h))$ *such that*

$$\left\{\begin{array}{l} ((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{u}}^h, \breve{\varphi}^h)) \in \breve{\boldsymbol{X}}^h_1 \times \breve{\boldsymbol{X}}^h_{2,w}, \\ \breve{a}^h_H((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h)) + \breve{b}^h_H((\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h), (\breve{\boldsymbol{u}}^h, \breve{\varphi}^h)) = 0, \qquad \forall (\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h) \in \breve{\boldsymbol{X}}^h_1, \\ \breve{b}^h_H((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{v}}^h, \breve{\psi}^h)) = \breve{l}^h_H(\breve{\boldsymbol{v}}^h, \breve{\psi}^h), \qquad \forall (\breve{\boldsymbol{v}}^h, \breve{\psi}^h) \in \breve{\boldsymbol{X}}^h_{0,w}, \end{array}\right. \quad (5.21)$$

where

$$\breve{a}_H^h((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h)) = \int_{\breve{\Omega}^h} \left(\breve{C}_{ijkl}^h \breve{\sigma}_{kl}^h + \breve{P}_{kij}^h \breve{D}_k^h\right) \breve{\tau}_{ij}^h \, d\breve{\boldsymbol{x}}^h$$
$$+ \int_{\breve{\Omega}^h} \left(-\breve{P}_{kij}^h \breve{\sigma}_{ij}^h + \breve{\varepsilon}_{kl}^h \breve{D}_l^h\right) \breve{d}_k^h \, d\breve{\boldsymbol{x}}^h, \tag{5.22}$$

$$\breve{b}_H^h((\breve{\boldsymbol{\tau}}^h, \breve{\boldsymbol{d}}^h), (\breve{\boldsymbol{v}}^h, \breve{\psi}^h)) = -\int_{\breve{\Omega}^h} \breve{\tau}_{ij}^h \, \breve{e}_{ij}^h(\breve{\boldsymbol{v}}^h) \, d\breve{\boldsymbol{x}}^h - \int_{\breve{\Omega}^h} \breve{d}_k^h \, \breve{E}_k^h(\breve{\psi}^h) \, d\breve{\boldsymbol{x}}^h, \tag{5.23}$$

$$\breve{l}_H^h(\breve{\boldsymbol{v}}^h, \breve{\psi}^h) = -\int_{\breve{\Omega}^h} \breve{f}_i^h \, \breve{v}_i^h \, d\breve{\boldsymbol{x}}^h - \int_{\breve{\Gamma}_N^h} \breve{g}_i^h \, \breve{v}_i^h \, d\breve{\Gamma}^h - \int_{\breve{\Gamma}_L^h} \breve{p}_i^h \, \breve{v}_i^h \, d\breve{\Gamma}^h. \tag{5.24}$$

As we mentioned in Chapter 3, the pair $((\breve{\boldsymbol{\sigma}}^h, \breve{\boldsymbol{D}}^h), (\breve{\boldsymbol{u}}^h, \breve{\varphi}^h)) \in \breve{\boldsymbol{X}}_1^h \times \breve{\boldsymbol{X}}_{2,w}^h$ is the unique solution of the problem (5.21)-(5.24).

5.2.1 Formulation in curvilinear coordinates

Now our objective is to transform the problem (5.21)-(5.24) expressed in Cartesian coordinates into another problem in curvilinear coordinates. We define the transformation from the old to the new test functions,

$$\breve{\boldsymbol{v}}^h \in [H^1(\breve{\Omega}^h)]^3 \to \boldsymbol{v}^h = \breve{\boldsymbol{v}}^h \circ \boldsymbol{\Theta}^h \in [H^1(\Omega^h)]^3,$$
$$\breve{\psi}^h \in H^1(\breve{\Omega}^h) \to \psi^h = \breve{\psi}^h \circ \boldsymbol{\Theta}^h \in H^1(\Omega^h),$$

which in turn induce a bijection between the spaces $\breve{V}_{0,w}^h(\breve{\Omega}^h)$, $\breve{\Psi}_0^h(\breve{\Omega}^h)$ and $\breve{\Psi}_2^h(\breve{\Omega}^h)$, and the respectively spaces defined by

$$V_{0,w}^h = V_{0,w}^h(\Omega^h) = \left\{\boldsymbol{v}^h \in \left[H^1(\Omega^h)\right]^3 : \langle \boldsymbol{v}\rangle^h |_{\Gamma_0^h} = 0\right\},$$
$$\Psi_0^h = \Psi_0^h(\Omega^h) = \left\{\psi^h \in H^1(\Omega^h) : \psi^h = 0 \text{ on } \Gamma_{eD}^h\right\},$$
$$\Psi_2^h = \Psi_2^h(\Omega^h) = \left\{\psi^h \in H^1(\Omega^h) : \psi^h - \hat{\varphi}^h \in \Psi_0^h\right\}.$$

We define the spaces $\boldsymbol{X}_{0,w}^h = V_{0,w}^h \times \Psi_0^h$, $\boldsymbol{X}_1^h = [L^2(\Omega^h)]_s^9 \times L^2(\Omega^h)$ and $\boldsymbol{X}_{2,w}^h = V_{0,w}^h \times \Psi_2^h$.

The transformation $\boldsymbol{\Theta}^h$ implies the following relations:

(a) The volume element $d\breve{\boldsymbol{x}}^h$ at $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h) \in \breve{\Omega}^h$ is given in terms of the volume element $d\boldsymbol{x}^h$ at $\boldsymbol{x}^h \in \Omega^h$ by

$$d\breve{\boldsymbol{x}}^h = o^h(\boldsymbol{x}^h) \, d\boldsymbol{x}^h. \tag{5.25}$$

(b) The area of elements along $\partial\breve{\Omega}^h$ and $\partial\Omega^h$ is given by

$$d\breve{\Gamma}^h = o^h(\boldsymbol{x}^h)\tilde{o}^h(\boldsymbol{x}^h)d\Gamma^h \qquad \text{on } \Gamma^h_N, \tag{5.26}$$

$$d\breve{\Gamma}^h = o^h(\boldsymbol{x}^h)\sqrt{b^h_{3i}(\boldsymbol{x}^h)b^h_{3i}(\boldsymbol{x}^h)}d\Gamma^h \qquad \text{on } \Gamma^h_0 \cup \Gamma^h_L, \tag{5.27}$$

where

$$\tilde{o}^h(\boldsymbol{x}^h) = \sqrt{b^h_{\alpha i}(\boldsymbol{x}^h)n^h_\alpha(\boldsymbol{x}^h)b^h_{\beta i}(\boldsymbol{x}^h)n^h_\beta(\boldsymbol{x}^h)}. \tag{5.28}$$

Consequently, we have from (5.14) the relations (see Álvarez-Dios & Viaño [1998])

$$d\breve{\Gamma}^h = o^h(\boldsymbol{x}^h)\tilde{o}^h(\boldsymbol{x}^h)d\Gamma^h \qquad \text{on } \Gamma^h_N, \tag{5.29}$$

$$d\breve{\Gamma}^h = d\Gamma^h \qquad \text{on } \Gamma^h_0 \cup \Gamma^h_L. \tag{5.30}$$

(c) Using the formulas

$$\breve{\partial}^h_k \breve{\psi}^h(\breve{\boldsymbol{x}}^h) = b^h_{kj}\,\partial^h_j\psi^h(\boldsymbol{x}^h) \qquad \breve{\partial}^h_j \breve{v}^h_i(\breve{\boldsymbol{x}}^h) = b^h_{kj}(\boldsymbol{x}^h)\,\partial^h_k v^h_i(\boldsymbol{x}^h), \tag{5.31}$$

and the following definitions

$$e^h_{ij}(\boldsymbol{v}^h) = \frac{1}{2}(b^h_{ki}\partial^h_k v^h_j + b^h_{kj}\partial^h_k v^h_i), \quad \boldsymbol{v}^h \in V_{0,w}(\Omega^h), \tag{5.32}$$

$$E^h_i(\psi^h) = b^h_{kj}\,\partial^h_k\psi^h, \quad \psi^h \in \Psi_0(\Omega^h), \tag{5.33}$$

we obtain $\breve{e}^h_{ij}(\breve{\boldsymbol{v}}^h) = e^h_{ij}(\boldsymbol{v}^h)$ and $\breve{E}^h_i(\breve{\psi}^h) = E^h_i(\psi^h)$

From all these above, the next theorem follows, which gives a new formulation of the piezoelectric problem in curvilinear coordinates (cf. Álvarez-Dios & Viaño [1998]).

Theorem 17 $\boldsymbol{\Theta}^h : \bar{\Omega}^h \to \{\Omega^h\}^-$ *be a $\mathcal{C}^1$-diffeomorphism satisfying the orientation preserving condition*

$$o^h(\boldsymbol{x}^h) > 0,$$

for all $\boldsymbol{x}^h \in \Omega^h$. Then the field $\left((\boldsymbol{\sigma}^h, \boldsymbol{D}^h), (\boldsymbol{u}^h, \varphi^h)\right) \in \boldsymbol{X}^h_1 \times \boldsymbol{X}^h_{2,w}$ defined by

$$\boldsymbol{u}^h(\boldsymbol{x}^h) = \breve{\boldsymbol{u}}(\breve{\boldsymbol{x}}^h), \quad \varphi^h(\boldsymbol{x}^h) = \breve{\varphi}^h(\breve{\boldsymbol{x}}^h), \quad \boldsymbol{\sigma}^h(\boldsymbol{x}^h) = \breve{\boldsymbol{\sigma}}^h(\breve{\boldsymbol{x}}^h), \quad \boldsymbol{D}^h(\boldsymbol{x}^h) = \breve{\boldsymbol{D}}^h(\breve{\boldsymbol{x}}^h),$$

for all $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h) \in \{\breve{\Omega}^h\}^-$, *satisfies the following variational problem:*

$$\text{Find } \left(\left(\boldsymbol{\sigma}^h, \boldsymbol{D}^h\right), (\boldsymbol{u}^h, \varphi^h)\right) \in \boldsymbol{X}_1^h \times \boldsymbol{X}_{2,w}^h \text{ such that}$$

$$\begin{cases} a_H^h\left(\left(\boldsymbol{\sigma}^h, \boldsymbol{D}^h\right), (\boldsymbol{\tau}^h, \boldsymbol{d}^h)\right) + b_H^h\left(\left(\boldsymbol{\tau}^h, \boldsymbol{d}^h\right), \left(\boldsymbol{u}^h, \varphi^h\right)\right) = 0, \\ \text{for all } \left(\boldsymbol{\tau}^h, \boldsymbol{d}^h\right) \in \boldsymbol{X}_1^h, \end{cases} \tag{5.34}$$

$$\begin{cases} b_H^h\left(\left(\boldsymbol{\sigma}^h, \boldsymbol{D}^h\right), \left(\boldsymbol{v}^h, \psi^h\right)\right) = l_H^h(\boldsymbol{v}^h, \psi^h), \\ \text{for all } \left(\boldsymbol{v}^h, \psi^h\right) \in \boldsymbol{X}_{0,w}^h, \end{cases} \tag{5.35}$$

where

$$\begin{aligned} a_H^h((\bar{\boldsymbol{\tau}}^h, \bar{\boldsymbol{d}}^h), (\boldsymbol{\tau}^h, \boldsymbol{d}^h)) = \int_{\Omega^h} \left(\bar{C}_{ijkl}^h\, \bar{\tau}_{kl}^h + \bar{P}_{kij}^h \bar{d}_k^h\right)\, \tau_{ij}^h\, o^h(\boldsymbol{x}^h)\, d\boldsymbol{x}^h \\ + \int_{\Omega^h} \left(-\bar{P}_{kij}^h \bar{\tau}_{ij}^h + \bar{\varepsilon}_{kl}^h\, \bar{d}_l^h\right)\, d_k^h\, o^h(\boldsymbol{x}^h)\, d\boldsymbol{x}^h, \end{aligned} \tag{5.36}$$

$$b_H^h((\bar{\boldsymbol{\tau}}^h, \bar{\boldsymbol{d}}^h), (\boldsymbol{v}^h, \psi^h)) = -\int_{\Omega^h} \bar{\tau}_{ij}^h e_{ij}^h(\boldsymbol{v}^h) o^h(\boldsymbol{x}^h) d\boldsymbol{x}^h - \int_{\Omega^h} \bar{d}_k^h E_k^h(\psi^h) o^h(\boldsymbol{x}^h) d\boldsymbol{x}^h, \tag{5.37}$$

$$\begin{aligned} l_H^h(\boldsymbol{v}^h, \psi^h) = -\int_{\Omega^h} f_i^h\, v_i^h\, o^h(\boldsymbol{x}^h) d\boldsymbol{x}^h - \int_{\Gamma_N^h} g_i^h\, v_i^h\, o^h(\boldsymbol{x}^h) \tilde{o}^h(\boldsymbol{x}^h) d\Gamma^h \\ - \int_{\Gamma_L^h} p_i^h\, v_i^h o^h(\boldsymbol{x}^h) d\Gamma^h, \end{aligned} \tag{5.38}$$

where $\boldsymbol{f}^h = (f_i^h) : \Omega^h \to \mathbb{R}^3$, $\boldsymbol{g}^h = (g_i^h) : \Gamma_N^h \to \mathbb{R}^3$ *and* $\boldsymbol{p}^h = (p_i^h) : \Gamma_L^h \to \mathbb{R}^3$ *are given by*

$$f_i^h(\boldsymbol{x}^h) = \breve{f}_i^h(\breve{\boldsymbol{x}}^h) \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h) \in \breve{\Omega}^h,$$

$$g_i^h(\boldsymbol{x}^h) = \breve{g}_i^h(\breve{\boldsymbol{x}}^h) \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h) \in \breve{\Gamma}_N^h,$$

$$p_i^h(\boldsymbol{x}^h) = \breve{p}_i^h(\breve{\boldsymbol{x}}^h) \quad \text{for all } \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h) \in \breve{\Gamma}_L^h,$$

and the constants characterizing the material satisfy (see (5.19)

$$\begin{cases} \bar{C}_{klij}^h(\boldsymbol{x}^h) = \breve{\bar{C}}_{klij}^h(\breve{\boldsymbol{x}}^h), \\ \bar{P}_{kij}^h(\boldsymbol{x}^h) = \breve{\bar{P}}_{kij}^h(\breve{\boldsymbol{x}}^h), \qquad \forall \breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\boldsymbol{x}^h),\ \boldsymbol{x}^h \in \Omega^h. \\ \bar{\varepsilon}_{ij}^h(\boldsymbol{x}^h) = \breve{\bar{\varepsilon}}_{ij}^h(\breve{\boldsymbol{x}}^h), \end{cases}$$

5.3 Transformation into a problem posed over a domain independent of h; fundamental scalings of the unknowns and assumptions on the data

As usual, our first task is to define a problem equivalent to problem (5.34)-(5.38), but now posed over a domain that does not depend on h. For such, we use the transformation (3.28) introduced in Chapter 3, and consequently the scalings (3.36)-(3.39) to the unknowns and the assumptions (3.40)-(3.43) on the data. The scalings for the functions $b_{ij}(h) : \bar{\Omega} \to \mathbb{R}$ and $o(h) : \bar{\Omega} \to \mathbb{R}$ are defined by:

$$b_{ij}(h)(\boldsymbol{x}) := b_{ij}^h(\boldsymbol{x}^h), \quad o(h)(\boldsymbol{x}) := o^h(\boldsymbol{x}^h), \quad \tilde{o}(h)(\boldsymbol{x}) := \tilde{o}^h(\boldsymbol{x}^h), \tag{5.39}$$

for all $\boldsymbol{x}^h = \Pi^h(\boldsymbol{x})$, $\boldsymbol{x} \in \bar{\Omega}$. Let the function ϕ_α^h be such that

$$\phi_\alpha^h(x_3) = h\phi_\alpha(x_3) \text{ for all } x_3 \in [0, L] \tag{5.40}$$

where $\phi_\alpha \in \mathcal{C}^3[0, L]$ is independent of h.

Using the scalings and the assumptions on the data, we can recast the variational problem (5.34)-(5.38) in the following equivalent form.

Find a pair $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w}$ *such that*

$$\left\{\begin{array}{l} a_{H,0}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + h^2 a_{H,2}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ + h^4 a_{H,4}\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})\right) + b_H\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))\right) = 0, \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{array}\right. \tag{5.41}$$

$$\left\{\begin{array}{l} b_H\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{v}, \psi)\right) = l_H(\boldsymbol{v}, \psi), \\ \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}, \end{array}\right. \tag{5.42}$$

where $a_{H,i}(\cdot, \cdot) : \boldsymbol{X}_1 \times \boldsymbol{X}_1 \to \mathbb{R}$ *and* $b(\cdot, \cdot) : \boldsymbol{X}_1 \times \boldsymbol{X}_{0,w} \to \mathbb{R}$ *are the following bilinear forms,*

$$b_H((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{v}, \psi)) = -\int_\Omega \tau_{ij}\, e_{ij}(\boldsymbol{v})\, o(h) d\boldsymbol{x} - \int_\Omega d_k\, E_k(\psi) o(h) d\boldsymbol{x},$$

$$a_{H,4}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) = \int_\Omega \bar{C}_{\alpha\beta\theta\rho}\, \bar{\tau}_{\theta\rho} \tau_{\alpha\beta}\, o(h) d\boldsymbol{x},$$

$$a_{H,2}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) = \int_{\Omega} \left(\bar{C}_{\alpha\beta 33}\, \bar{\tau}_{33} + \bar{P}_{3\alpha\beta}\bar{d}_3\right) \tau_{\alpha\beta}\, o(h) d\boldsymbol{x} + \int_{\Omega} \bar{C}_{33\theta\rho}\, \bar{\tau}_{\theta\rho}\tau_{33}\, o(h) d\boldsymbol{x}$$
$$+ 2\int_{\Omega} \left(2\bar{C}_{3\alpha 3\theta}\, \bar{\tau}_{3\theta} + \bar{P}_{\theta 3\alpha}\bar{d}_{\theta}\right) \tau_{3\alpha}\, o(h) d\boldsymbol{x}$$
$$- \int_{\Omega} \bar{P}_{3\alpha\beta}\, \bar{\tau}_{\alpha\beta}\, d_3\, o(h) d\boldsymbol{x} + \int_{\Omega} \left(-2\bar{P}_{\theta 3\alpha}\, \bar{\tau}_{3\alpha} + \bar{\varepsilon}_{\theta\alpha}\, \bar{d}_{\alpha}\right) d_{\theta}\, o(h) d\boldsymbol{x},$$
$$a_{H,0}((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\boldsymbol{\tau}, \boldsymbol{d})) = \int_{\Omega} \left(\bar{C}_{3333}\, \bar{\tau}_{33} + \bar{P}_{333}\bar{d}_3\right) \tau_{33}\, o(h) d\boldsymbol{x}$$
$$+ \int_{\Omega} \left(-\bar{P}_{333}\bar{\tau}_{33} + \bar{\varepsilon}_{33}\, \bar{d}_3\right) d_3\, o(h) d\boldsymbol{x},$$

and the linear form $l_H(\cdot) : \boldsymbol{X}_{0,w} \to \mathbb{R}$ *reads*

$$l_H(\boldsymbol{v}, \psi) = -\int_{\Omega} f_i\, v_i\, o(h)\, d\boldsymbol{x} - \int_{\Gamma_N} g_i\, v_i\, o(h)\, \tilde{o}(h)\, d\Gamma - \int_{\Gamma_L} p_i\, v_i\, o(h)\, d\Gamma, \tag{5.43}$$

and

$$\Psi_0(\Omega) = \left\{\psi \in H^1(\Omega) : \psi = 0 \text{ on } \Gamma_{eD}\right\}, \tag{5.44}$$

$$\Psi_2(\Omega) = \left\{\psi \in H^1(\Omega) : \psi - \varphi_0 \in \Psi_0(\Omega)\right\}, \tag{5.45}$$

$$V_{0,w}(\Omega) = \left\{\boldsymbol{v} \in \left[H^1(\Omega)\right]^3 : \langle \boldsymbol{v} \rangle = 0 \text{ on } \Gamma_{dD}\right\}. \tag{5.46}$$

The pair $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ can also be characterized as the unique solution of this problem thanks to the Lions-Stampachia's theorem and the Lax-Milgram's Lemma.

In the next three theorems there will be established some geometrical, mechanical and electrical preliminaries needed in the sequel for the asymptotic analysis of linearly piezoelectric shallow arches. The geometrical preliminaries results are due to Álvarez-Dios & Viaño [1998] and are summarized in the next two Lemmas.

Lemma 10 (Álvarez-Dios & Viaño [1998]) *Let* $\boldsymbol{\phi}^h \in \mathrm{C}^3(0, L; \mathbb{R}^3)$ *be such that its Frenet trihedron* $(\boldsymbol{t}^{*,h}, \boldsymbol{n}^{*,h}, \boldsymbol{b}^{*,h})$ *is a positively oriented basis of* $\mathbb{R}^3$ *and satisfies Frenet equations for curvature* κ^h *and torsion* τ^h*. If* ϕ^h_α *satisfies (5.40) then, for all* $h > 0$*, we have:*

$$\{\mathcal{A}^h\}^r = 1 + h^2 s_0(h, \phi), \quad r \in \mathbb{R},$$

$$\kappa^h = h\{c + h^2 s_1(h, \phi)\}, \quad \tau^h = d + h^2 s_2(h, \phi),$$

$$n_1^{*,h} = b_1 + h^2 s_3(h, \phi), \qquad n_2^{*,h} = b_2 + h^2 s_4(h, \phi), \quad n_3^{*,h} = h\{h_1 + h^2 s_5(h, \phi)\}$$

$$b_1^{*,h} = -b_2 + h^2 s_6(h,\phi), \quad b_2^{*,h} = b_1 + h^2 s_7(h,\phi), \qquad b_3^{*,h} = h\{h_2 + h^2 s_8(h,\phi)\},$$

$$t_1^{*,h} = h\{t_1 + h^2 s_9(h,\phi)\} \quad t_2^{*,h} = h\{t_2 + h^2 s_{10}(h,\phi)\}, \quad t_3^{*,h} = t_3 + h^2 s_{11}(h,\phi),$$

where s_i, $i = 1,\ldots,11$, are uniformly bounded constants on $h > 0$:

$$\sup_{h>0} \max_{x_3\in[0,L]} |s_i(h,\phi)(x_3)| < +\infty, \tag{5.47}$$

and c, d, b_α, h_α and t_i are functions defined in $[0, L]$, independent of h and satisfying the following properties:

(H_1^b) $b_1^2 + b_2^2 = 1$,

(H_2^b) $b_1 h_1 - b_2 h_2 = -t_1 = -\phi_1'$,

(H_3^b) $b_1 h_2 + b_2 h_1 = -t_2 = -\phi_2'$,

(H_4^b) $b_1' = -d\, b_2, \quad b_2' = d\, b_1$.

Moreover, if $\phi_1''(x_3)\phi_2''(x_3) \neq 0$ for all $x_3 \in [0, L]$, then we have $\kappa(x_3) \neq 0$, for all $h > 0$ and for all $x_3 \in [0, L]$, and also

$$c = \sqrt{(\phi_1'')^2 + (\phi_2'')^2}, \tag{5.48}$$

$$b_\alpha = \phi_\alpha''/c, \tag{5.49}$$

$$h_1 = \frac{-\phi_1'\phi_1'' + \phi_2'\phi_2''}{c}, \qquad h_2 = \frac{\phi_1'\phi_2'' - \phi_2'\phi_1''}{c}, \tag{5.50}$$

$$t_\alpha = \phi_\alpha', \qquad\qquad t_3 = 1, \tag{5.51}$$

$$d = \frac{\phi_1''\phi_2''' - \phi_2''\phi_1'''}{c^2}. \tag{5.52}$$

We note that the limits of these functions for $h = 0$ are functions of $x_3 \in [0, L]$ only, i.e., the limits are independent of the transversal variable x_α.

Lemma 11 (Álvarez-Dios & Viaño [1998]) *There exists $h_0 = h_0(\phi, c, d, b_\alpha, h_\alpha)$ such that the Jacobian matrix $\nabla^h \boldsymbol{\Theta}^h(\boldsymbol{x}^h)$ is non-singular for all $\boldsymbol{x}^h \in \bar{\Omega}^h$ and for all $h \leq h_0$, and $\boldsymbol{\Theta}^h$ is an orientation-preserving map for all $h \leq h_0$. We have for all $\boldsymbol{x}^h \in \bar{\Omega}^h$ and for all $h \leq h_0$:*

$$o(h) = 1 + h^2 o^{\#}(h,\phi), \quad \tilde{o}(h) = 1 + h^2 \tilde{o}^{\#}(h,\phi), \tag{5.53}$$

$$\boldsymbol{b}(h)(\boldsymbol{x}) = \boldsymbol{C}_0(\boldsymbol{x}) + h\boldsymbol{C}_1(\boldsymbol{x}) + h^2 \boldsymbol{C}_2^{\#}(h,\phi)(\boldsymbol{x}) + h^3 \boldsymbol{C}_3^{\#}(h,\phi)(\boldsymbol{x}), \tag{5.54}$$

where

$$\boldsymbol{C}_0(\boldsymbol{x}) = \begin{pmatrix} b_1(x_3) & b_2(x_3) & 0 \\ -b_2(x_3) & b_1(x_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.55}$$

$$\boldsymbol{C}_1(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & h_1(x_3) + x_2 d(x_3) \\ 0 & 0 & h_2(x_3) - x_1 d(x_3) \\ t_1(x_3) & t_2(x_3) & 0 \end{pmatrix}, \tag{5.56}$$

$$\boldsymbol{C}_2^{\#}(h,\phi)(\boldsymbol{x}) = \begin{pmatrix} b_{11}^{\#}(h,\phi) & b_{12}^{\#}(h,\phi) & 0 \\ b_{21}^{\#}(h,\phi) & b_{22}^{\#}(h,\phi) & 0 \\ 0 & 0 & b_{33}^{\#}(h,\phi) \end{pmatrix}, \tag{5.57}$$

$$\boldsymbol{C}_3^{\#}(h,\phi)(\boldsymbol{x}) = \begin{pmatrix} 0 & 0 & b_{13}^{\#}(h,\phi) \\ 0 & 0 & b_{23}^{\#}(h,\phi) \\ b_{31}^{\#}(h,\phi)(\boldsymbol{x}) & b_{32}^{\#}(h,\phi) & 0 \end{pmatrix}, \tag{5.58}$$

and there exists a constant $C_0(\phi)$ *such that*

$$\sup_{0<h\leq h_0} \max_{i,j} \max_{\boldsymbol{x}\in\bar{\Omega}} |b_{ij}^{\#}(h,\phi)(\boldsymbol{x})| \leq C_0(\phi), \tag{5.59}$$

$$\sup_{0<h\leq h_0} \max_{\boldsymbol{x}\in\bar{\Omega}} |o^{\#}(h,\phi)(\boldsymbol{x})| \leq C_0(\phi), \tag{5.60}$$

$$\sup_{0<h\leq h_0} \max_{\boldsymbol{x}\in\bar{\Omega}} |\tilde{o}^{\#}(h,\phi)(\boldsymbol{x})| \leq C_0(\phi). \tag{5.61}$$

Using all these results we establish a relation for the electric vector field in $\breve{\Omega}^h$ with respect to the electric vector field in Ω^h and complete the Lemma 4.4. introduced by Álvarez-Dios & Viaño [1998].

Lemma 12 *Let the functions* $\breve{v}_i^h, \breve{\psi}^h \in H^1(\breve{\Omega}^h)$*, and* $\breve{v}_i^h, \psi(h) \in H^1(\Omega)$ *be related by the bijection* $v_i^h = \breve{v}_i^h \circ \boldsymbol{\Theta}^h$*,* $\psi^h = \breve{\psi}^h \circ \boldsymbol{\Theta}^h \in H^1(\Omega^h)$ *and the scalings*

$$v_\alpha(h)(\boldsymbol{x}) = h\, v_\alpha^h(\boldsymbol{x}^h), \quad v_3(h)(\boldsymbol{x}) = v_3^h(\boldsymbol{x}^h), \quad \psi(h)(\boldsymbol{x}) = \psi^h(\boldsymbol{x}^h),$$

for all $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}^h(\Pi^h(\boldsymbol{x})) \in \{\breve{\Omega}^h\}^-$. *Then*

$$\breve{\partial}_1\breve{\psi}^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-1}\left\{\begin{array}{c}(b_1\partial_1\psi - b_2\partial_2\psi) + h^2\phi_1'\partial_3\psi + h^2 b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha\psi \\ +h^4 b_{31}^{\#}(h,\phi)\partial_3\psi\end{array}\right\}(\boldsymbol{x}), \tag{5.62}$$

$$\breve{\partial}_2\breve{\psi}^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-1}\left\{\begin{array}{c}(b_2\partial_1\psi + b_1\partial_2\psi) + h^2\phi_2'\partial_3\psi + h^2 b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha\psi \\ +h^4 b_{32}^{\#}(h,\phi)\partial_3\psi\end{array}\right\}(\boldsymbol{x}), \tag{5.63}$$

$$\breve{\partial}_3\breve{\psi}^h\left(\breve{\boldsymbol{x}}^h\right) = \Big\{(h_1 + x_2 d)\partial_1\psi + (h_2 - x_1 d)\partial_2\psi + \partial_3\psi + h^2 b_{i3}^{\#}(h,\phi)\partial_i\psi\Big\}(\boldsymbol{x}), \tag{5.64}$$

$$\breve{\partial}_1\breve{v}_\beta^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-2}\left\{\begin{array}{c}(b_1\partial_1 v_\beta - b_2\partial_2 v_\beta) + h^2\phi_1'\partial_3 v_\beta + h^2 b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha v_\beta \\ +h^4 b_{31}^{\#}(h,\phi)\partial_3 v_\beta\end{array}\right\}(\boldsymbol{x}), \tag{5.65}$$

$$\breve{\partial}_1\breve{v}_3^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-1}\left\{\begin{array}{c}(b_1\partial_1 v_3 - b_2\partial_2 v_3) + h^2\phi_1'\partial_3 v_3 + h^2 b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha v_3 \\ +h^4 b_{31}^{\#}(h,\phi)\partial_3 v_3\end{array}\right\}(\boldsymbol{x}), \tag{5.66}$$

$$\breve{\partial}_2\breve{v}_\beta^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-2}\left\{\begin{array}{c}(b_2\partial_1 v_\beta + b_1\partial_2 v_\beta) + h^2\phi_2'\partial_3 v_\beta + h^2 b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha v_\beta \\ +h^4 b_{32}^{\#}(h,\phi)\partial_3 v_\beta\end{array}\right\}(\boldsymbol{x}), \tag{5.67}$$

$$\breve{\partial}_2\breve{v}_3^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-1}\left\{\begin{array}{c}(b_2\partial_1 v_3 + b_1\partial_2 v_3) + h^2\phi_2'\partial_3 v_3 + h^2 b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha v_3 \\ +h^4 b_{32}^{\#}(h,\phi)\partial_3 v_3\end{array}\right\}(\boldsymbol{x}), \tag{5.68}$$

$$\breve{\partial}_3\breve{v}_\beta^h\left(\breve{\boldsymbol{x}}^h\right) = h^{-1}\left\{\begin{array}{c}\partial_3 v_\beta + (h_1 + x_2 d)\partial_1 v_\beta + (h_2 - x_1 d)\partial_2 v_\beta \\ +h^2 b_{i3}^{\#}(h,\phi)\partial_i v_\beta\end{array}\right\}(\boldsymbol{x}), \tag{5.69}$$

$$\breve{\partial}_3\breve{v}_3^h\left(\breve{\boldsymbol{x}}^h\right) = \Big\{\partial_3 v_3 + (h_1 + x_2 d)\partial_1 v_3 + (h_2 - x_1 d)\partial_2 v_3 + h^2 b_{i3}^{\#}(h,\phi)\partial_i v_3\Big\}(\boldsymbol{x}). \tag{5.70}$$

Consequently, we obtain

$$\breve{E}_\alpha^h(\breve{\psi}^h)(\breve{\boldsymbol{x}}^h) = h^{-1}\{E_\alpha(h)(\psi)\}(\boldsymbol{x}), \qquad \breve{E}_3^h(\breve{\psi}^h)(\breve{\boldsymbol{x}}^h) = E_3(h)(\psi)(\boldsymbol{x}), \tag{5.71}$$

and

$$\left\{\begin{array}{l}\breve{e}_{\alpha\beta}^h(\breve{\boldsymbol{v}}^h)(\breve{\boldsymbol{x}}^h) = h^{-2}\{e_{\alpha\beta}(h)(\boldsymbol{v})\}(\boldsymbol{x}), \\ \breve{e}_{3\alpha}^h(\breve{\boldsymbol{v}}^h)(\breve{\boldsymbol{x}}^h) = h^{-1}\{e_{3\alpha}(h)(\boldsymbol{v})\}(\boldsymbol{x})\,(\boldsymbol{v})\,, \\ \breve{e}_{33}^h(\breve{\boldsymbol{v}}^h)(\breve{\boldsymbol{x}}^h) = e_{33}(h)(\boldsymbol{v})(\boldsymbol{x}),\end{array}\right. \tag{5.72}$$

where

$$e_{ij}(h)(\boldsymbol{v}) = e_{ij}^{\phi}(\boldsymbol{v}) + h^2 e_{ij}^{\#}(h,\phi;\boldsymbol{v}), \qquad E_i(h)(\psi) = E_i^{\phi}(\psi) + h^2 E_i^{\#}(h,\phi;\psi), \tag{5.73}$$

with $e_{ij}^{\phi}(\boldsymbol{v})$ *and* $E_i^{\phi}(\psi)$ *given by*

$$E_1^{\phi}(\psi) = -(b_1\partial_1\psi - b_2\partial_2\psi), \qquad E_2^{\phi}(\psi) = -(b_2\partial_1\psi + b_1\partial_2\psi), \tag{5.74}$$

$$E_3^{\phi}(\psi) = -\left[(h_1 + x_2 d)\partial_1\psi + (h_2 - x_1 d)\partial_2\psi + \partial_3\psi\right], \tag{5.75}$$

$$e_{11}^{\phi}(\boldsymbol{v}) = b_1\partial_1 v_1 - b_2\partial_2 v_1, \qquad e_{22}^{\phi}(\boldsymbol{v}) = b_2\partial_1 v_2 + b_1\partial_2 v_2, \tag{5.76}$$

$$e_{12}^{\phi}(\boldsymbol{v}) = \frac{1}{2}\left[b_1(\partial_1 v_2 + \partial_2 v_1) + b_2(\partial_1 v_1 - \partial_2 v_2)\right], \tag{5.77}$$

$$e_{13}^{\phi}(\boldsymbol{v}) = \frac{1}{2}\left[b_1\partial_1 v_3 - b_2\partial_2 v_3 + \partial_3 v_1 + (h_1 + x_2 d)\partial_1 v_1 + (h_2 - x_1 d)\partial_2 v_1\right], \tag{5.78}$$

$$e_{23}^{\phi}(\boldsymbol{v}) = \frac{1}{2}\left[b_2\partial_1 v_3 + b_1\partial_2 v_3 + \partial_3 v_2 + (h_1 + x_2 d)\partial_1 v_2 + (h_2 - x_1 d)\partial_2 v_2\right], \tag{5.79}$$

$$e_{33}^{\phi}(\boldsymbol{v}) = \partial_3 v_3 + (h_1 + x_2 d)\partial_1 v_3 + (h_2 - x_1 d)\partial_2 v_3, \tag{5.80}$$

and $e_{ij}^{\#}(h,\phi;\boldsymbol{v})$ *and* $E_i^{\#}(h,\phi;\psi)$ *defined by*

$$E_1^{\#}(h,\phi;\psi) = -\left(\phi_1'\partial_3\psi + b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha\psi + h^2 b_{31}^{\#}(h,\phi)\partial_3\psi\right), \tag{5.81}$$

$$E_2^{\#}(h,\phi;\psi) = -\left(\phi_2'\partial_3\psi + b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha\psi + h^2 b_{32}^{\#}(h,\phi)\partial_3\psi\right), \tag{5.82}$$

$$E_3^{\#}(h,\phi;\psi) = -b_{i3}^{\#}(h,\phi)\partial_i\psi, \tag{5.83}$$

$$e_{11}^{\#}(h,\phi;\boldsymbol{v}) = \phi_1'\partial_3 v_1 + b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha v_1 + h^2 b_{31}^{\#}(h,\phi)\partial_3 v_1, \tag{5.84}$$

$$e_{12}^{\#}(h,\phi;\boldsymbol{v}) = \frac{1}{2}\begin{pmatrix} \phi_1'\partial_3 v_2 + \phi_2'\partial_3 v_1 + b_{\alpha 1}^{\#}(h,\phi)\partial_\alpha v_2 + b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha v_1 \\ +h^2 b_{31}^{\#}(h,\phi)\partial_3 v_2 + h^2 b_{32}^{\#}(h,\phi)\partial_3 v_1 \end{pmatrix}, \tag{5.85}$$

$$e_{22}^{\#}(h,\phi;\boldsymbol{v}) = \phi_2'\partial_3 v_2 + b_{\alpha 2}^{\#}(h,\phi)\partial_\alpha v_2 + h^2 b_{32}^{\#}(h,\phi)\partial_3 v_2, \tag{5.86}$$

$$e_{3\alpha}^{\#}(h,\phi;\boldsymbol{v}) = \frac{1}{2}\left(\phi_\alpha'\partial_3 v_3 + b_{\beta\alpha}^{\#}(h,\phi)\partial_\beta v_3 + b_{i3}^{\#}(h,\phi)\partial_i v_\alpha + h^2 b_{3\alpha}^{\#}(h,\phi)\partial_3 v_3\right), \tag{5.87}$$

$$e_{33}^{\#}(h,\phi;\boldsymbol{v}) = b_{i3}^{\#}(h,\phi)\partial_i v_3. \tag{5.88}$$

Furthermore, there exists a constant $c_1(\phi)$ *such that*

$$\sup_{0<h\leq h_0} \max_{i,j} |e_{ij}^{\#}(h,\phi;\boldsymbol{v})|_{0,\Omega} \leq C_1(\phi)\|\boldsymbol{v}\|_{1,\Omega}, \tag{5.89}$$

$$\sup_{0<h\leq h_0} \max_{i} |E_i^{\#}(h,\phi;\psi)|_{0,\Omega} \leq C_1(\phi)\|\psi\|_{1,\Omega}. \tag{5.90}$$

Proof. From (5.62) we get

$$\breve{\partial}_1 \breve{\psi}^h(\boldsymbol{x}^h) = b_{k1}(\boldsymbol{x}^h)\partial_k^h \psi^h = b_{11}(\boldsymbol{x}^h)\partial_1^h \psi^h + b_{21}(\boldsymbol{x}^h)\partial_2^h \psi^h + b_{31}(\boldsymbol{x}^h)\partial_3^h \psi^h.$$

Using the scalings established in Chapter 3 to the unknowns as well as the two last Lemmas, we obtain (5.62)-(5.64), and consequently (5.71) where the tensors $\boldsymbol{E}^\phi$ and $\boldsymbol{E}^\#(h,\phi;\psi)$ are defined by (5.73), (5.74)-(5.75) and (5.81)-(5.83). Due to Lemma 11, in particular the inequality (5.59), we obtain

$$\breve{\partial}_1 \breve{\psi}^h(\breve{\boldsymbol{x}}^h) = h^{-1}\left\{(b_1\partial_1\psi - b_2\partial_2\psi) + h^2 \underline{r}_1^\#(h,\phi;\psi) + h^4 \underline{s}_1^\#(h,\phi;\psi)\right\}(\boldsymbol{x}), \tag{5.91}$$

$$\breve{\partial}_2 \breve{\psi}^h(\breve{\boldsymbol{x}}^h) = h^{-1}\left\{(b_2\partial_1\psi + b_1\partial_2\psi) + h^2 \underline{r}_2^\#(h,\phi;\psi) + h^4 \underline{s}_2^\#(h,\phi;\psi)\right\}(\boldsymbol{x}),$$

$$\breve{\partial}_3 \breve{\psi}^h(\breve{\boldsymbol{x}}^h) = \left\{(h_1 + x_2 d)\partial_1\psi + (h_2 - x_1 d)\partial_2\psi + \partial_3\psi + h^2 \underline{r}_3^\#(h,\phi;\psi)\right\}(\boldsymbol{x}),$$

and show that there exists a constant $C_1(\phi)$ such that

$$\sup_{0<h\leq h_1} \max_i |\underline{r}_i^\#(h,\phi;\psi)|_{0,\Omega} + \sup_{0<h\leq h_1} \max_\alpha |\underline{s}_\alpha^\#(h,\phi;\psi)|_{0,\Omega} \leq C_1(\phi)\|\psi\|_{1,\Omega}. \tag{5.92}$$

■

Consequently, we deduce a new formulation for the scaled three-dimensional problem of a linearly piezoelectric shallow arch.

Theorem 18 *Let the scalings to the unknown and to the data be as above. Then, for each $h \leq h_0$, the scaled unknown $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ satisfies the following variational problem (cf. (5.41)-(5.43)):*

Find a pair $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \boldsymbol{X}_1 \times \boldsymbol{X}_{2,w}$ *such that*

$$\left\{\begin{array}{l} h^4 a_{H,4}^\phi((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + h^2 a_{H,2}^\phi((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) \\ \quad + a_{H,0}^\phi((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) + h^2 a_H^\#(h,\phi)((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{\tau}, \boldsymbol{d})) \\ \quad + b_H^\phi((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))) + h^2 b_H^\#(h,\phi)((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h), \varphi(h))) = 0 \\ \textit{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{array}\right. \tag{5.93}$$

$$\left\{\begin{array}{l} b_H^\phi\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{v},\psi)\right)+h^2 b_H^\#(h,\phi)\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{v},\psi)\right) \\ =l_H^\phi(\boldsymbol{v},\psi)+h^2 l_H^\#(h,\phi)(\boldsymbol{v},\psi) \\ \textit{for all } (\boldsymbol{v},\psi)\in \boldsymbol{X}_0, \end{array}\right. \tag{5.94}$$

where the continuous bilinear and linear forms are expressed by

$$\begin{aligned} b_H^\phi((\boldsymbol{\tau},\boldsymbol{d}),(\boldsymbol{v},\psi)) &= -\int_\Omega \tau_{ij}\, e_{ij}^\phi(\boldsymbol{v})\,d\boldsymbol{x} - \int_\Omega d_k\, E_k^\phi(\psi)d\boldsymbol{x}, \\ a_{H,4}^\phi((\bar{\boldsymbol{\tau}},\bar{\boldsymbol{d}}),(\boldsymbol{\tau},\boldsymbol{d})) &= \int_\Omega \bar{C}_{\alpha\beta\theta\rho}\,\bar{\tau}_{\theta\rho}\,\tau_{\alpha\beta}\,d\boldsymbol{x}, \\ a_{H,2}^\phi((\bar{\boldsymbol{\tau}},\bar{\boldsymbol{d}}),(\boldsymbol{\tau},\boldsymbol{d})) &= \int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\bar{\tau}_{33}+\bar{P}_{3\alpha\beta}\bar{d}_3\right)\,\tau_{\alpha\beta}\,d\boldsymbol{x} + \int_\Omega \bar{C}_{33\theta\rho}\,\bar{\tau}_{\theta\rho}\,\tau_{33}\,d\boldsymbol{x} \\ &\quad +2\int_\Omega \left(2\bar{C}_{3\alpha 3\theta}\,\bar{\tau}_{3\theta}+\bar{P}_{\theta 3\alpha}\bar{d}_\theta\right)\,\tau_{3\alpha}\,d\boldsymbol{x} - \int_\Omega \bar{P}_{3\alpha\beta}\bar{\tau}_{\alpha\beta}\,d_3\,d\boldsymbol{x} \\ &\quad +\int_\Omega \left(-2\bar{P}_{\theta 3\alpha}\bar{\tau}_{3\alpha}+\bar{\varepsilon}_{\theta\alpha}\,\bar{d}_\alpha\right)\,d_\theta\,d\boldsymbol{x}, \end{aligned}$$

and

$$\begin{aligned} a_{H,0}^\phi((\bar{\boldsymbol{\tau}},\bar{\boldsymbol{d}}),(\boldsymbol{\tau},\boldsymbol{d})) &= \int_\Omega \left(\bar{C}_{3333}\,\bar{\tau}_{33}+\bar{P}_{333}\bar{d}_3\right)\tau_{33}d\boldsymbol{x} + \int_\Omega \left(-\bar{P}_{333}\bar{\tau}_{33}+\bar{\varepsilon}_{33}\,\bar{d}_3\right)d_3 d\boldsymbol{x}, \\ l_H^\phi(\boldsymbol{v},\psi) &= -\int_\Omega f_i\,v_i\,d\boldsymbol{x} - \int_{\Gamma_N} g_i\,v_i\,d\Gamma - \int_{\Gamma_L} p_i\,v_i\,d\Gamma. \end{aligned}$$

The bounded bilinear forms $b_H^\#(h,\phi)(\cdot,\cdot)$,$a_H^\#(\cdot,\cdot)$ and the linear form $l_H^\#(h,\phi)$ are defined by

$$b_H^\#(h,\phi)((\boldsymbol{\tau},\boldsymbol{d}),(\boldsymbol{v},\psi)) = -\int_\Omega \tau_{ij}\,e_{ij}(\boldsymbol{v})\,o^\#(h,\phi)d\boldsymbol{x} - \int_\Omega d_k\,E_k(\psi)o^\#(h,\phi)d\boldsymbol{x}, \tag{5.95}$$

$$\begin{aligned} a_H^\#((\bar{\boldsymbol{\tau}},\bar{\boldsymbol{d}}),(\boldsymbol{\tau},\boldsymbol{d})) &= h^4\int_\Omega \bar{C}_{\alpha\beta\theta\rho}\bar{\tau}_{\theta\rho}\tau_{\alpha\beta}\,o^\# d\boldsymbol{x} + h^2\int_\Omega \bar{P}_{3\alpha\beta}\bar{\tau}_{\alpha\beta}\,d_3\;o^\# d\boldsymbol{x} \\ &\quad + h^2\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\bar{\tau}_{33}-\bar{P}_{3\alpha\beta}\bar{d}_3\right)\tau_{\alpha\beta}\,o^\# d\boldsymbol{x} + 2h^2\int_\Omega \left(2\bar{C}_{3\alpha 3\theta}\,\bar{\tau}_{3\theta}-\bar{P}_{\theta 3\alpha}\bar{d}_\theta\right)\tau_{3\alpha}o^\# d\boldsymbol{x} \\ &\quad + h^2\int_\Omega \bar{C}_{33\theta\rho}\bar{\tau}_{\theta\rho}\tau_{33}\,o^\# d\boldsymbol{x} + h^2\int_\Omega \left(2\bar{P}_{\theta 3\alpha}\bar{\tau}_{3\alpha}+h^2\bar{\varepsilon}_{\theta\alpha}\,\bar{d}_\alpha\right)d_\theta\,o^\# d\boldsymbol{x} \\ &\quad + \int_\Omega \left(\bar{C}_{3333}\,\bar{\tau}_{33}-\bar{P}_{333}\bar{d}_3\right)\tau_{33}\;o^\# d\boldsymbol{x} + \int_\Omega \left(\bar{P}_{333}\bar{\tau}_{33}+\bar{\varepsilon}_{33}\,\bar{d}_3\right)d_3\,o^\# d\boldsymbol{x}, \end{aligned} \tag{5.96}$$

$$\begin{aligned} l_H^\#(h,\phi)(\boldsymbol{v},\psi) &= -\int_\Omega f_i\,v_i o^\#(h,\phi)d\boldsymbol{x} - \int_{\Gamma_N} g_i\,v_i[o^\#(h,\phi)+\tilde{o}^\#(h,\phi)]d\Gamma \\ &\quad -\int_{\Gamma_L} p_i\,v_i o^\#(h,\phi)d\Gamma - h^2\int_{\Gamma_N} g_i\,v_i o^\#(h,\phi)\tilde{o}^\#(h,\phi)d\Gamma. \end{aligned} \tag{5.97}$$

Applying inequalities (5.60) and (5.61), it is easy to verify that the last bilinear and linear forms are bounded.

5.4 Generalized Korn's and Poincaré's inequalities

Let us denote by $V^{\phi}_{BN}(\Omega)$ the space of generalized Bernoulli-Navier displacement defined by

$$V^{\phi}_{BN} = V^{\phi}_{BN}(\Omega) = \left\{ \boldsymbol{v} \in V_{0,w} : e^{\phi}_{\alpha\beta}(\boldsymbol{v}) = e^{\phi}_{3\beta}(\boldsymbol{v}) = 0 \right\},$$

which can be equivalently defined by (see Álvarez-Dios & Viaño [1998])

$$\begin{aligned} V^{\phi}_{BN} = \big\{ \boldsymbol{v} : \Omega \to \mathbb{R}^3 : v_\alpha(x_1,x_2,x_3) = \zeta_\alpha(x_3),\ \zeta_\alpha \in V^2_0(0,L), \\ v_3(x_1,x_2,x_3) = \zeta_3(x_3) - \chi^b_\alpha(x_1,x_2,x_3)\zeta'_\alpha(x_3),\ \zeta_3 \in V^1_0(0,L) \big\}, \end{aligned} \tag{5.98}$$

where

$$V^1_0(0,L) = \left\{ \eta \in H^1(0,L) : \eta(0) = 0 \right\}, \tag{5.99}$$

$$V^2_0(0,L) = \left\{ \eta \in H^2(0,L) : \eta(0) = \eta'(0) = 0 \right\}, \tag{5.100}$$

and

$$\begin{pmatrix} \chi^b_1(x_1,x_2,x_3) \\ \chi^b_2(x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} b_1(x_3) & -b_2(x_3) \\ b_2(x_3) & b_1(x_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.101}$$

$$b_\alpha = \frac{\phi''_\alpha}{\sqrt{(\phi''_1)^2 + (\phi''_2)^2}}, \tag{5.102}$$

$$b_1^2 + b_2^2 = 1. \tag{5.103}$$

This space is endowed with the norm $|\boldsymbol{v}|^{\phi}_{V^{\phi}_{BN}} := |e^{\phi}_{33}(\boldsymbol{v})|_{0,\Omega}$.

In Álvarez-Dios & Viaño [1998] we can see the proof of the following equivalence between the norms

$$|\boldsymbol{v}|^{\phi}_{V^{\phi}_{BN}} = \left| e^{\phi}_{33}(\boldsymbol{v}) \right|_{0,\Omega} \geq C_2(\phi) \left\| \boldsymbol{v} \right\|_{1,\Omega}, \quad \forall \boldsymbol{v} \in V^{\phi}_{BN}. \tag{5.104}$$

In addition, the admissible electric potential spaces are defined by

$$\Psi^{\phi} := \Psi^{\phi}(\Omega) = \left\{ \psi \in L^2(\Omega) : E^{\phi}_i(\psi) \in L^2(\Omega) \right\},$$

and

$$\Psi_0^\phi := \Psi_0^\phi(\Omega) = \left\{\psi \in \Psi^\phi(\Omega) : \ \psi = 0 \ \text{ on } \Gamma_{eD} = \Gamma_0 \cup \Gamma_L\right\},$$

respectively, equipped with norm

$$\|\psi\|_{\Psi^\phi}^\phi = \left\{|\psi|_\Omega^2 + \sum_k \left|E_k^\phi(\psi)\right|_{0,\Omega}^2\right\}^{1/2}.$$

Theorem 19 *Let Ω be a domain in $\mathbb{R}^3$ and let $\Gamma_{eD} \subset \partial\Omega$ be such that meas $(\Gamma_{eD}) > 0$. Then, there exists a constant $C_3(\phi)$ such that*

$$|\psi|_{\Psi_0^\phi}^\phi := \left|\boldsymbol{E}^\phi(\psi)\right|_{0,\Omega} = \left\{\sum_k \left|E_k^\phi(\psi)\right|_{0,\Omega}^2\right\}^{1/2} \geq C_3(\phi)\,\|\psi\|_{1,\Omega}, \qquad \forall\, \psi \in \Psi_0^\phi. \quad (5.105)$$

Proof. We proceed in several steps:

(i) Firstly, we need to prove that the spaces Ψ^ϕ and $H^1(\Omega)$ coincide. Clearly, $H^1(\Omega) \subset \Psi^\phi$. To establish the other inclusion, let $\psi \in \Psi^\phi$. Since $\psi, E_k^\phi(\psi) \in L^2(\Omega)$ and due to relations (5.74)-(5.75), then

$$\begin{pmatrix} \partial_1\psi \\ \partial_2\psi \\ \partial_3\psi \end{pmatrix} = \begin{pmatrix} b_1 & -b_2 & 0 \\ b_2 & b_1 & 0 \\ (h_1 + x_2 d) & (h_2 - x_1 d) & 1 \end{pmatrix}^{-1} \begin{pmatrix} -E_1^\phi(\psi) \\ -E_2^\phi(\psi) \\ -E_3^\phi(\psi) \end{pmatrix} \in \left[H^{-1}(\Omega)\right]^3,$$

$$\begin{pmatrix} \partial_{11}\psi \\ \partial_{12}\psi \\ \partial_{21}\psi \\ \partial_{22}\psi \end{pmatrix} = \begin{pmatrix} b_1 & -b_2 & 0 & 0 \\ 0 & 0 & b_2 & -b_2 \\ b_2 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_1 \end{pmatrix}^{-1} \begin{pmatrix} -\partial_1 E_1^\phi(\psi) \\ -\partial_2 E_1^\phi(\psi) \\ -\partial_1 E_2^\phi(\psi) \\ -\partial_2 E_2^\phi(\psi) \end{pmatrix} \in \left[H^{-1}(\Omega)\right]^4.$$

Given the regularity of $\boldsymbol{\phi}$ (the assumption $\boldsymbol{\phi} \in \mathcal{C}^3[0,L]$) we have that $\partial_{\alpha\beta}\psi \in H^{-1}(\Omega)$. Now, deriving (5.75) with respect to the variables x_1 and x_2, respectively, we deduce from previous conditions that

$$\partial_{31}\psi = (h_1 + x_2 d)\partial_{11}\psi + (h_2 - x_1 d)\partial_{22}\psi - \partial_1 E_3^\phi(\psi),$$

$$\partial_{32}\psi = (h_1 + x_2 d)\partial_{12}\psi + (h_2 - x_1 d)\partial_{22}\psi - \partial_2 E_3^\phi(\psi),$$

and consequently $\partial_{3\alpha}\psi \in H^{-1}(\Omega)$. Using similar arguments we deduce that

$$\partial_{33}\psi = (h_1 + x_2 d)\partial_{13}\psi + (h_2 - x_1 d)\partial_{23}\psi - \partial_3 E_3^{\phi}(\psi) \in H^{-1}(\Omega).$$

In work of Duvaut & Lions [1976], it is shown that any $\omega \in H^{-1}(\Omega)$ such that $\partial_i\omega \in H^{-1}(\Omega)$ in fact belongs to $L^2(\Omega)$, and so (i) is proved.

(ii) It is easy to show that the identity mapping from $H^1(\Omega)$ equipped with $\|\cdot\|_{1,\Omega}$ into Ψ^{ϕ} equipped with $\|\cdot\|_{\Psi^{\phi}}^{\phi}$ is continuous, since there clearly exists a constant d such that $\|\psi\|_{\Psi^{\phi}}^{\phi} \leq d_1 \|\psi\|_{1,\Omega}$ for all $\psi \in H^1(\Omega)$, and surjective, thanks to the step (i). The inverse mapping $\psi \in H^1(\Omega) \mapsto E^{\phi}(\psi) \in \Psi^{\phi}$ is continuous, that is

$$\|\psi\|_{1,\Omega} \leq d_2 \|\psi\|_{\Psi^{\phi}}^{\phi} \quad \text{for all } \psi \in \Psi^{\phi},$$

or equivalently, such that

$$|\psi|_{0,\Omega}^2 + |\boldsymbol{E}^{\phi}(\psi)|_{0,\Omega}^2 \geq d_2^{-1} \|\psi\|_{1,\Omega},$$

for all $\psi \in H^1(\Omega)$. The conclusions thus follows from the closed graph Theorem, since the space $\Psi^{\phi} = H^1(\Omega)$ is a Hilbert space when it is equipped with the norm $\|\cdot\|_{\Psi^{\phi}}^{\phi}$.

(iii) We establish that, if $\psi \in \Psi_0^{\phi}$ satisfies $\psi = 0$ on $\Gamma_0 \cup \Gamma_L$, then $\psi = 0$. Seminorm $|\cdot|^{\phi}$ is actually a norm over the space Ψ^{ϕ}. Let us consider any $\psi \in \Psi_0^{\phi}$ such that $|\psi|_{\Psi^{\phi}}^{\phi} = |\boldsymbol{E}^{\phi}(\psi)|_{\Omega} = 0$. From step (i), the condition $\boldsymbol{E}^{\phi}(\psi) = 0$ implies

$$\partial_{ij}\psi = 0 \quad \text{in } \Omega.$$

Solving $\partial_{\alpha\alpha}\psi = 0$ with boundary condition $\psi = 0$ on $\Gamma_0 \cup \Gamma_L$, we arrive to

$$\psi(x_1, x_2, x_3) = z(x_3), \quad z \in H_0^1(0, L).$$

The condition $\partial_{33}\psi = 0$ gives

$$z''(x_3) = 0,$$

and therefore $z(x_3) = c$. Since the boundary conditions $\psi(x_1, x_2, 0) = \psi(x_1, x_2, L) = 0$, we conclude that $c = 0$ and $\psi(x_1, x_2, x_3) = 0$ as required. ■

We introduce the space of admissible electric potentials

$$\Psi_3^{\phi} = \left\{\psi \in L^2(\Omega) : E_{\alpha}^{\phi}(\psi) = 0\right\},$$

which it is equivalently defined by (see next lemma):

$$\Psi_3^\phi = \left\{ \psi : \Omega \to \mathbb{R} : \psi(x_1, x_2, x_3) = z(x_3), \;\; z \in H_0^1\,(0, L) \right\}.$$

Lemma 13 *Let $\psi \in \Psi_0^\phi$. Then, the following conditions are equivalent:*

(i) $-E_\alpha^\phi(\psi) = 0$,

(ii) $\psi \in \Psi_3^\phi$.

Proof. It is clear that every element of Ψ_3^ϕ satisfies (i). Conversely, if $\psi \in \Psi_0^\phi$ satisfies (i), then from conditions (5.74) we deduce

$$b_1 \partial_1 \psi - b_2 \partial_2 \psi = 0, \tag{5.106}$$

$$b_2 \partial_1 \psi + b_1 \partial_2 \psi = 0. \tag{5.107}$$

Combining (5.106) and (5.107) we obtain

$$b_1(\partial_1 \psi - \partial_2 \psi) - b_2(\partial_2 \psi + \partial_1 \psi) = 0, \tag{5.108}$$

$$b_2(\partial_1 \psi - \partial_2 \psi) + b_1(\partial_1 \psi + \partial_2 \psi) = 0. \tag{5.109}$$

Multiplying equations (5.108) by b_1 (respectively by b_2) and (5.109) by b_2 (respectively by b_1) and after algebraic manipulations, we obtain

$$\partial_1 \psi - \partial_2 \psi = 0, \qquad \partial_1 \psi + \partial_2 \psi = 0. \tag{5.110}$$

Deriving with respect to x_1 and x_2, one obtain

$$\partial_{11} \psi - \partial_{12} \psi = 0, \qquad \partial_{11} \psi + \partial_{21} \psi = 0, \tag{5.111}$$

and

$$\partial_{21} \psi - \partial_{22} \psi = 0, \qquad \partial_{12} \psi + \partial_{22} \psi = 0. \tag{5.112}$$

Consequently one has

$$\partial_{11} \psi = \partial_{22} \psi = 0,$$

whose solution is given by

$$\psi\,(x_1, x_2, x_3) = z_1(x_3) + x_1 z_2(x_3) + x_2 z_3(x_3).$$

From (5.110), we deduce

$$z_2(x_3) = z_3(x_3), \qquad z_2(x_3) = -z_3(x_3).$$

Consequently there exists a function z depending only on variable x_3 such that

$$\psi(x_1, x_2, x_3) = z(x_3). \tag{5.113}$$

Since $\psi \in \Psi_0^\phi$, the boundary conditions give us $z \in H_0^1(0, L)$. ■

Corollary 21 *There exists a constant C such that*

$$|\psi|_{\Psi_3^\phi}^\phi := \left|E_3^\phi(\psi)\right|_{0,\Omega} \geq C \left\|\psi\right\|_{1,\Omega}, \quad \forall \psi \in \Psi_3^\phi. \tag{5.114}$$

Corollary 22 *There exists a constant $C > 0$ such that*

$$\sup_{(\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1} \frac{\left|-b_H^\phi\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{v}, \psi)\right)\right|}{|(\boldsymbol{\tau}, \boldsymbol{d})|} \geq C \left\{\|\boldsymbol{v}\|_{1,\Omega}^2 + \|\psi\|_{1,\Omega}^2\right\} \ \text{for all } (\boldsymbol{v}, \psi) \in V_{BN}^\phi \times \Psi_3^\phi.$$

5.5 Convergence of the scaled unknowns as $h \to 0$.

Repeating the argument used in Section 3.3, we can see that the scaled unknowns $(\boldsymbol{u}(h), \varphi(h))$ given in problem (5.41)-(5.43) converge in $[H^1(\Omega)]^3 \times H^1(\Omega)$ as $h \to 0$ toward a limit $(\boldsymbol{u}, \varphi)$ and this limit can be identified with the solution of a one-dimensional variational problem, which will be later identified.

5.5.1 Weak convergence

The following weak convergence hold.

Proposition 7 *There exists $C > 0$, independent of h, such that for all $0 < h \leq 1$ the solution $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h)))$ of problem (5.41)-(5.43) verifies*

$$|\sigma_{33}(h)|_{0,\Omega} \leq C(\phi), \quad h\,|\sigma_{\alpha 3}(h)|_{0,\Omega} \leq C(\phi), \quad h^2\,|\sigma_{\alpha\beta}(h)|_{0,\Omega} \leq C(\phi), \tag{5.115}$$

$$h\,|D_\alpha(h)|_{0,\Omega} \leq C(\phi), \quad |D_3(h)|_{0,\Omega} \leq C(\phi), \tag{5.116}$$

$$\|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq C(\phi), \quad \|(\boldsymbol{u}(h), \varphi(h))\|_{\boldsymbol{X}_{2,w}} \leq C(\phi). \tag{5.117}$$

Proof. Replacing $(\boldsymbol{\tau}, \boldsymbol{d})$ by $(\boldsymbol{S}(h), \boldsymbol{T}(h))$, with

$$S_{33}(h) = \sigma_{33}(h), \qquad S_{3\alpha}(h) = h\sigma_{3\alpha}(h), \qquad S_{\alpha\beta}(h) = h^2\sigma_{\alpha\beta}(h), \tag{5.118}$$

$$T_\beta(h) = hD_\alpha(h), \qquad T_3(h) = D_3(h), \tag{5.119}$$

in the first equation of (5.41), it reads

$$\begin{aligned} a^*\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) &:= a_{H,0}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) \\ &+ a_{H,2}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) + a_{H,4}\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) \\ &= -b_H\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{u}(h), \varphi(h))\right) \end{aligned}$$

or, equivalently, we have

$$\begin{aligned} -b_H\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{u}(h), \varphi(h))\right) &= \int_\Omega \left[(\bar{C}_{3333} S_{33}(h) S_{33}(h) + \bar{\varepsilon}_{33} T_3(h) T_3(h)\right] o(h) d\boldsymbol{x} \\ &+ \int_\Omega \left[\bar{C}_{33\theta\rho} S_{\theta\rho}(h) S_{33}(h) + 4\bar{C}_{3\alpha3\theta} S_{3\theta}(h) S_{3\alpha}(h)\right] o(h) d\boldsymbol{x} \\ &+ \int_\Omega \bar{C}_{\alpha\beta33} S_{33}(h) S_{\alpha\beta}(h) o(h) d\boldsymbol{x} + \int_\Omega \bar{\varepsilon}_{\alpha\theta} T_\theta(h) T_\alpha(h) o(h) d\boldsymbol{x} \\ &+ \int_\Omega \bar{C}_{\alpha\beta\theta\rho} S_{\theta\rho}(h) S_{\alpha\beta}(h) o(h) d\boldsymbol{x}. \end{aligned}$$

Using the estimate on $|o^\#(h,\phi)(\boldsymbol{x})|$ and the estimates on $|\tilde{o}^\#(h,\phi)(\boldsymbol{x})|$ established in Lemma 11, the properties of the coefficients of the material and the inequality

$$2o^\#(h,\phi) \geq -\frac{\left[o^\#(h,\phi)\right]^2}{h} - h, \tag{5.120}$$

we infer, from the two last equations, that

$$\begin{aligned} &a^*\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{S}(h), \boldsymbol{T}(h))\right) \\ &\geq \left(C_1 - \tfrac{1}{2}C_2 h^3\right) \int_\Omega \left[S_{ij}(h) S_{ij}(h) + T_k(h) T_k(h)\right] d\boldsymbol{x} \\ &-\tfrac{1}{2}C_2 h \int_\Omega \left[S_{ij}(h) S_{ij}(h) + T_k(h) T_k(h)\right] \left[o^\#(h,\phi)\right]^2 d\boldsymbol{x} \\ &\geq \left[C_1 - \tfrac{1}{2}C_2 h^3 - \tfrac{1}{2}C_2 h C_0(\phi)\right] \left|(\boldsymbol{S}(h), \boldsymbol{T}(h))\right|_{0,\Omega}^2. \end{aligned} \tag{5.121}$$

Setting $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \varphi(h) - \hat{\varphi}) \in \boldsymbol{X}_{0,w}$ in equation of (5.42), one obtain

$$-b_H\left((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{u}(h), \varphi(h)) + \hat{\varphi}(h))\right) = \int_\Omega f_i u_i(h)(1 + h^2 o^{\#}) d\boldsymbol{x}$$
$$+ \int_{\Gamma_N} g_i u_i(h)(1 + h^2 o^{\#})(1 + h^2 \tilde{o}^{\#}) d\Gamma + \int_{\Gamma_L} p_i u_i(h)(1 + h^2 o^{\#}) d\Gamma \tag{5.122}$$
$$+ \int_\Omega D_3(h) E_3(\hat{\varphi})(1 + h^2 o^{\#}) d\boldsymbol{x}.$$

Using the estimates (5.60) and (5.61) for $\tilde{o}^{\#}$ and $o^{\#}$, respectively, the Young's inequality $2ab \leq \frac{a^2}{m} + mb^2$ for $m > 0$ and the fact that $f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_N)$, $p_i \in L^2(\Gamma_L)$ and $\hat{\varphi} \in H^1(0, L)$, we obtain

$$\begin{aligned}
&- b_H((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{u}(h), \varphi(h))) \\
&\leq \left(C_3 + h^2 C_0(\phi)\right) \left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{0,\Gamma} + \|\boldsymbol{p}\|_{0,\Gamma}\right) \|\boldsymbol{u}(h)\|_{1,\Omega} \\
&\qquad + \left(C_4 + h^2 C_0(\phi)\right) \|\hat{\varphi}(h)\|_{1,(0,L)} \|D_3(h)\|_{0,\Omega} \\
&\leq (C_5 + h^2 C_5(\phi)) \|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} + \left(C_{46} + h^2 C_6(\phi)\right) \|D_3(h)\|_{0,\Omega} \\
&\leq (C_5 + h^2 C_5(\phi)) \|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} + \frac{(C_6 + h^2 C_6(\phi))^2}{2m} + m\|T_3(h)\|^2_{0,\Omega}.
\end{aligned}$$

Combining the previous inequality with (5.121), we deduce the existence of h_7 small enough, when m is large enough, and of constant $C_7(\phi)$ such that

$$\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|^2_{\mathbf{X}_1} \leq C_7(\phi) \|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}}, \quad \text{if } 0 < h \leq h_7. \tag{5.123}$$

Now, putting $(\boldsymbol{\tau}, \boldsymbol{d}) = \left(\boldsymbol{e}, \bar{\boldsymbol{E}}\right) \in \mathbf{X}_1$ in equation (5.41) we get

$$\begin{aligned}
&-b_H((\boldsymbol{e}^\phi, \bar{\boldsymbol{E}}), (\boldsymbol{u}(h), \bar{\varphi}(h))) = -b_H((\boldsymbol{e}^\phi, \bar{\boldsymbol{E}}), (\boldsymbol{u}(h), \varphi(h))) + b_H((\boldsymbol{e}^\phi, \bar{\boldsymbol{E}}), (\boldsymbol{0}, \hat{\varphi}(h))) \\
&= a^*((\boldsymbol{S}(h), \boldsymbol{T}(h)), (\boldsymbol{e}, \bar{\boldsymbol{E}})) + b_H((\boldsymbol{e}^\phi, \bar{\boldsymbol{E}}), (\boldsymbol{0}, \hat{\varphi}(h))),
\end{aligned} \tag{5.124}$$

and taking into account bound (5.60) for $o^{\#}(h, \phi)$ and the fact that $0 < h \leq 1$ we guarantee the existence of h_9 small enough and of constant $C_9(\phi)$ satisfying

$$\begin{aligned}
&\left|-b_H((\boldsymbol{e}, \bar{\boldsymbol{E}}), (\boldsymbol{u}(h), \bar{\varphi}(h)))\right| \leq C_8(\phi)(\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{\mathbf{X}_1} + 1)\|(\boldsymbol{e}, \bar{\boldsymbol{E}})\|_{0,\Omega} \\
&\leq (C_9(\phi) \|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|_{\mathbf{X}_1} + C_9(\phi)) \|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}},
\end{aligned} \tag{5.125}$$

thanks to inequalities (5.89), (5.90) and (5.105). On the other hand, we have

$$\begin{aligned} |-b_H((\boldsymbol{e},\bar{\boldsymbol{E}}),(\boldsymbol{u}(h),\bar{\varphi}(h)))| &= \int_\Omega e_{ij}(h)(\boldsymbol{u}(h))e_{ij}(h)(\boldsymbol{u}(h))o(h)d\boldsymbol{x} \\ &+ \int_\Omega E_k(\bar{\varphi}(h))E_k(\bar{\varphi}(h))o(h)d\boldsymbol{x} \\ &= \int_\Omega \left\{ e^\phi_{ij}(h)(\boldsymbol{u}(h)) + h^2 e^\#_{ij}(h,\phi;\boldsymbol{u}) \right\}^2 (1+h^2 o^\#(h,\phi))d\boldsymbol{x} \\ &+ \int_\Omega \left\{ E^\phi_k(\bar{\varphi}(h)) + h^2 E^\#_k(h,\phi;\bar{\varphi}(h)) \right\}^2 (1+h^2 o^\#(h,\phi))d\boldsymbol{x} \end{aligned} \tag{5.126}$$

By Álvarez-Dios & Viaño [1998] we have

$$\begin{aligned} |e_{ij}(h)(\boldsymbol{u}(h))|^2_{0,\Omega} &= \int_\Omega e_{ij}(h)(\boldsymbol{u}(h))e_{ij}(h)(\boldsymbol{u}(h))\,d\boldsymbol{x} \\ &= \int_\Omega \left\{ e^\phi_{ij}(h)(\boldsymbol{u}(h)) + h^2 e^\#_{ij}(h,\phi;\boldsymbol{u}) \right\}^2 d\boldsymbol{x} \geq C_{10}(\phi)\,\|\boldsymbol{u}(h)\|^2_{1,\Omega}\,. \end{aligned} \tag{5.127}$$

Using the same argument and applying (5.105), we prove the existence of h_{11} small enough and $C_{11}(\phi)$ such that

$$\begin{aligned} &|E_k(h)(\bar{\varphi}(h))|^2_{0,\Omega} = \int_\Omega \left\{ E^\phi_k(\bar{\varphi}(h)) + h^2 E^\#_k(h,\phi;\bar{\varphi}(h)) \right\}^2 d\boldsymbol{x} \\ &\geq \ |E^\phi_k(\bar{\varphi}(h))|^2_{0,\Omega} + h^4 |E^\#_k(h,\phi;\bar{\varphi}(h))|^2_{0,\Omega} - \frac{h}{2}|E^\phi_k(\bar{\varphi}(h))|^2_{0,\Omega} - \frac{h^3}{2}|E^\#_k(h,\phi;\bar{\varphi}(h))|^2_{0,\Omega} \\ &\geq \frac{1}{2}|E^\phi_k(\bar{\varphi}(h))|^2_{0,\Omega} - \frac{h^3}{2}|E^\#_k(h,\phi;\bar{\varphi}(h))|^2_{0,\Omega} \\ &\geq \left\{ \frac{1}{2}\left(\frac{1}{C_3(\phi)}\right)^2 - \frac{h^3}{2}\left[C_1(\phi)\right]^2 \right\} \|\bar{\varphi}(h)\|^2_{1,\Omega} \geq C_{11}(\phi)\|\bar{\varphi}(h)\|^2_{1,\Omega}, \end{aligned} \tag{5.128}$$

if $0 < h \leq h_{11}$. Therefore, we prove the existence of h_{13} small enough and $C_{13}(\phi)$ such that

$$\begin{aligned} &|-b_H((\boldsymbol{e},\bar{\boldsymbol{E}}),(\boldsymbol{u}(h),\bar{\varphi}(h)))| \geq C_{12}(\phi)\left(\int_\Omega (e^\phi_{ij}(\boldsymbol{u}(h)))^2 d\boldsymbol{x} + \int_\Omega (E^\phi_k(\bar{\varphi}(h)))^2 d\boldsymbol{x} \right) \\ &\geq C_{13}(\phi)\|(\boldsymbol{u}(h),\bar{\varphi}(h)\|^2_{\boldsymbol{X}_{0,w}}, \qquad \text{if } 0<h\leq h_{13}. \end{aligned} \tag{5.129}$$

Combining (5.125) with (5.129), and applying Babuška-Brezzi condition, we deduce the existence of h_{14} small enough and $C_{14}(\phi)$ such that

$$\|(\boldsymbol{u}(h),\bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq \frac{\left|-b_H((\boldsymbol{e},\bar{\boldsymbol{E}}),(\boldsymbol{u}(h),\bar{\varphi}(h)))\right|}{\|(\boldsymbol{u}(h),\varphi(h))\|_{\boldsymbol{X}_{0,w}}} \leq C_{14}(\phi)\,\|(\boldsymbol{S}(h),\boldsymbol{T}(h))\|_{\mathbf{X}_1}\,,$$

if $0 < h \leq h_{14}$. The previous inequality together with (5.123) allows to obtain,

$$\|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} \leq C(\phi), \tag{5.130}$$

$$\begin{aligned}\|(\boldsymbol{u}(h), \varphi(h))\|_{\boldsymbol{X}_{2,w}} &= \|(\boldsymbol{u}(h), \bar{\varphi}(h)) + (\boldsymbol{0}, \hat{\varphi}(h))\|_{\boldsymbol{X}_{2,w}} \\ &\leq \|(\boldsymbol{u}(h), \bar{\varphi}(h))\|_{\boldsymbol{X}_{0,w}} + \|\hat{\varphi}(h)\|_{1,(0,L)} \leq C(\phi)\end{aligned} \tag{5.131}$$

The weak convergence of $(\boldsymbol{S}(h), \boldsymbol{T}(h))$ follows by (5.123) and (5.131). ■

Corollary 23 *We assume hypothesis (5.40), conditions (***HC1***)-(***HC3***) for $\boldsymbol{\phi}^h$ and $(\boldsymbol{t}^{*,h}, \boldsymbol{n}^{*,h}, \boldsymbol{b}^{*,h})$ and that coefficients $A_{33}^d(\bar{\varepsilon}_{33}\bar{C}_{33\theta\rho} + \bar{P}_{333}\bar{P}_{3\theta\rho})$ and $A_{33}^d(-\bar{P}_{333}\bar{C}_{33\theta\rho} + \bar{\varepsilon}_{33}\bar{P}_{3\theta\rho})$ do not depend on x_α. Then, there exists a subsequence, still parameterized by h, and there exist $\boldsymbol{u} \in V_{0,w}$, $\boldsymbol{\Sigma} \in [L^2(\Omega)]_s^9$, $\varphi \in L^2(\Omega)$ and $\boldsymbol{\mathfrak{D}} \in [L^2(\Omega)]^3$, such that the following weak convergence hold when h tends to zero:*

$$\sigma_{33}(h) \rightharpoonup \Sigma_{33}, \qquad h\sigma_{\alpha 3}(h) \rightharpoonup \Sigma_{\alpha 3}, \qquad h^2\sigma_{\alpha\beta}(h) \rightharpoonup \Sigma_{\alpha\beta}, \qquad \textit{in } L^2(\Omega) \tag{5.132}$$

$$hD_\alpha(h) \rightharpoonup \mathfrak{D}_\alpha, \qquad D_3(h) \rightharpoonup \mathfrak{D}_3, \qquad \textit{in } L^2(\Omega) \tag{5.133}$$

$$\boldsymbol{u}(h) \rightharpoonup \boldsymbol{u}, \qquad \textit{in } V_{0,w}(\Omega), \tag{5.134}$$

$$\varphi(h) \rightharpoonup \varphi, \qquad \textit{in } \Psi_{2,w}(\Omega), \tag{5.135}$$

$$\bar{\varphi}(h) \rightharpoonup \varphi - \hat{\varphi}, \qquad \textit{in } \Psi_{0,w}(\Omega). \tag{5.136}$$

with the following properties:

$$\int_\omega \Sigma_{\alpha\beta} e_{\alpha\beta}^\phi(\boldsymbol{v}) d\omega = 0, \ \textit{ for all } \boldsymbol{v} = (v_1, v_2, 0) \in V_{0,w}(\Omega), \tag{5.137}$$

$$\int_\omega \Sigma_{\alpha\beta} d\omega = \int_\omega x_\gamma \Sigma_{\alpha\beta} d\omega = 0, \tag{5.138}$$

$$e_{33}^\phi(\boldsymbol{u}) = \bar{C}_{33\theta\rho}\Sigma_{\theta\rho} + \bar{C}_{3333}\Sigma_{33} + \bar{P}_{333}\mathfrak{D}_3, \tag{5.139}$$

$$e_{3\alpha}^\phi(\boldsymbol{u}) = e_{\alpha\beta}^\phi(\boldsymbol{u}) = 0 \tag{5.140}$$

$$E_3^\phi(\varphi) = -\bar{P}_{3\theta\rho}\Sigma_{\theta\rho} - \bar{P}_{333}\Sigma_{33} + \bar{\varepsilon}_{33}\mathfrak{D}_3, \tag{5.141}$$

$$E_\alpha^\phi(\varphi) = 0, \tag{5.142}$$

$$e_{\alpha\beta}^{\#}(h, \theta; \boldsymbol{u}) = \bar{C}_{\alpha\beta\theta\rho}\Sigma_{\theta\rho} + \bar{C}_{\alpha\beta 33}\Sigma_{33} + \bar{P}_{3\alpha\beta}\mathfrak{D}_3, \tag{5.143}$$

$$2\bar{C}_{3\alpha 3\theta}\Sigma_{3\theta} + \bar{P}_{\theta 3\alpha}\mathfrak{D}_\theta = 0, \qquad \bar{\varepsilon}_{\theta\alpha}\mathfrak{D}_\theta - 2\bar{P}_{\theta 3\alpha}\Sigma_{3\alpha} = 0, \tag{5.144}$$

and the limit variational equations read

$$\int_\Omega A^d_{33}\bar{\varepsilon}_{33}e^\phi_{33}(\boldsymbol{u})e^\phi_{33}(\boldsymbol{v}) + \int_\Omega A^d_{33}\bar{P}_{333}\left[e^\phi_{33}(\boldsymbol{u})E^\phi_3(\psi) - E^\phi_3(\varphi)e^\phi_{33}(\boldsymbol{v})\right] d\boldsymbol{x}$$

$$+ \int_\Omega A^d_{33}\bar{C}_{3333}E^\phi_3(\varphi)E^\phi_3(\psi)\ d\boldsymbol{x} - \int_\Omega \left(e^\phi_{33}(\boldsymbol{u})\tau_{33} + E^\phi_3(\varphi)d_3\right) d\boldsymbol{x} = 0, \quad (5.145)$$

for all $\tau_{33}, d_3 \in L^2(\Omega)$,

$$\int_\Omega \Sigma_{33}e^\phi_{33}(\boldsymbol{v}) + \int_\Omega \mathfrak{D}_3 E^\phi_3(\psi)\ d\boldsymbol{x} = \int_\Omega f_i\, v_i\ d\boldsymbol{x} + \int_{\Gamma_N} g_i\, v_i\, d\Gamma + \int_{\Gamma_L} p_i\, v_i\, d\Gamma, \quad (5.146)$$

for all $(\boldsymbol{v}, \psi) \in V^\phi_{BN} \times \Psi^\phi_3$,

where $A^d_{33} = \frac{1}{\bar{C}_{3333}\bar{\varepsilon}_{33} + \bar{P}_{333}\bar{P}_{333}}$.

Proof. From the previous estimates, we conclude the existence of a subsequence $(\boldsymbol{\Sigma}, \boldsymbol{\mathfrak{D}}, \boldsymbol{u}, \varphi)$ verifying (5.132) and (5.136).

On the other hand, using the same ideas as in proof of Proposition 4, and passing to the limit the first equation of (5.41), we obtain the relations (5.139)-(5.142) in $L^2(\Omega)$.

In particular, taking $\tau_{33} = d_3 = 0$ in equation (5.42), multiplying by h^{-2} and passing to the limit, we obtain

$$e^\#_{\alpha\beta}(h, \theta; \boldsymbol{u}) = \bar{C}_{\alpha\beta\theta\rho}\Sigma_{\theta\rho} + \bar{C}_{\alpha\beta33}\Sigma_{33} + \bar{P}_{3\alpha\beta}\mathfrak{D}_3,$$

since $e^\phi_{3\alpha}(\boldsymbol{u}) = e^\phi_{\alpha\beta}(\boldsymbol{u}) = E^\phi_\alpha(\varphi) = 0$. Setting $\tau_{33} = \tau_{\alpha\beta} = d_3 = 0$ in equation (5.42) and multiplying it by h^{-1} and passing to the limit we deduce (5.144).

We multiply equation (5.42) by h^2. Taking the limit when $h \to 0$ we obtain (5.137) when h goes to zero. Choosing now $\psi = 0$ and the test functions $(v_i) \in V_{0,w}$ as in Theorem 5.3. of Álvarez-Dios & Viaño [1998] in equation (5.42), we obtain immediately (5.138).

Taking now $\boldsymbol{v} \in V^\phi_{BN}$ and $\psi \in \Psi^\phi_3$ in (5.42) we obtain

$$\begin{aligned}
&\int_\Omega \sigma_{33}(h)\ \left[e^\phi_{33}(\boldsymbol{v}) + h^2 e^\#_{33}(h, \phi; \boldsymbol{v})\right] o(h) d\boldsymbol{x} + 2h^2 \int_\Omega \sigma_{3\alpha}(h)\, e^\#_{3\alpha}(h, \phi; \boldsymbol{v}) o(h) d\boldsymbol{x} \\
&\quad + h^2 \int_\Omega \sigma_{\alpha\beta}(h)\, e^\#_{\alpha\beta}(h, \phi; \boldsymbol{v}) o(h) d\boldsymbol{x} + h^2 \int_\Omega D_\alpha(h) E^\#_\alpha(h, \phi; \psi) o(h) d\boldsymbol{x} \\
&\quad + \int_\Omega D_3(h)\left[E^\phi_3(\psi) + h^2 E^\#_3(h, \phi; \psi)\right] o(h) d\boldsymbol{x} \\
&= \int_\Omega f_i v_i o(h) d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i o(h)\tilde{o}(h) d\Gamma + \int_{\Gamma_L} p_i v_i o(h)\tilde{o}(h) d\Gamma,
\end{aligned}$$

and passing to the limit when $h \to 0$, we get

$$\int_\Omega \Sigma_{33}\, e^{\phi}_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \Sigma_{\alpha\beta}\, e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3 E^{\phi}_3(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\, d\Gamma + \int_{\Gamma_L} p_i v_i\, d\Gamma,$$

which becomes (from (5.84) and (5.86))

$$\int_\Omega \Sigma_{33}\, e^{\phi}_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \Sigma_{\alpha\beta}\, \phi'_\alpha \zeta'_\beta\, d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3 E^{\phi}_3(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i\, d\Gamma + \int_{\Gamma_L} p_i v_i\, d\Gamma. \tag{5.147}$$

From (5.138) we obtain that the second term in the above equation vanishes, and therefore the previous equation reads.

$$\int_\Omega \Sigma_{33}\, e^{\phi}_{33}(\boldsymbol{v})\, d\boldsymbol{x} + \int_\Omega \mathfrak{D}_3 E^{\phi}_3(\psi)\, d\boldsymbol{x} = \int_\Omega f_i v_i\, d\boldsymbol{x} + \int_{\Gamma} g_i v_i\, d\Gamma + \int_{\Gamma_L} p_i v_i\, d\Gamma.$$

where

$$\begin{pmatrix} \Sigma_{33} \\ D_3 \end{pmatrix} = \begin{pmatrix} \bar{C}_{3333} & \bar{P}_{333} \\ -\bar{P}_{333} & \bar{\varepsilon}_{33} \end{pmatrix}^{-1} \begin{pmatrix} e^{\phi}_{33}(\boldsymbol{u}) - \bar{C}_{33\theta\rho}\Sigma_{\theta\rho} \\ E^{\phi}_3(\varphi) - \bar{P}_{3\theta\rho}\Sigma_{\theta\rho} \end{pmatrix}. \tag{5.148}$$

Applying properties (5.138) and taking into account that the coefficients $A^d_{33}(-\bar{\varepsilon}_{33}\bar{C}_{33\theta\rho} + \bar{P}_{333}\bar{P}_{3\theta\rho})$ and $A^d_{33}(\bar{P}_{333}\bar{C}_{33\theta\rho} + \bar{\varepsilon}_{33}\bar{P}_{3\theta\rho})$ do not depend on x_α, we obtain mixed variational problem (5.145)-(5.146). ■

5.5.2 The asymptotic expansion method

In order to be able to show that the coefficient $\Sigma_{\theta\rho}$ vanishes in expressions (5.139) and (5.141), we use now the *displacement-electric potential-stress-electric displacement* approach instead of the *displacement-electric potential* approach used in Section 3.4.

We assume that the solution of a problem (5.93)-(5.97) can be expressed as the asymptotic developments

$$\sigma(h) = h^{-4}\sigma^{-4} + h^{-2}\sigma^{-2} + \sigma^0 + h^2\sigma^2 + \dots \qquad \sigma_{ij} \in L^2(\Omega), \tag{5.149}$$

$$\boldsymbol{D}(h) = h^{-4}\boldsymbol{D}^{-4} + h^{-2}\boldsymbol{D}^{-2} + \boldsymbol{D}^0 + h^2\boldsymbol{D}^2 + \dots, \qquad D_k \in L^2(\Omega), \tag{5.150}$$

$$(\boldsymbol{u}(h), \varphi(h)) = (\boldsymbol{u}^0 + h^2\boldsymbol{u}^2 + \dots, \varphi^0 + h^2\varphi^2 + \dots), \tag{5.151}$$

where $(\boldsymbol{u}^0, \varphi^0) \in \mathbf{X}_{2,w}$, $(\boldsymbol{u}^p, \varphi^p) \in \boldsymbol{X}_{0,w}$, $p \geq 1$ and the successive coefficients of the powers of h are independent of h. The need to start the development with terms in h^{-4} comes from the scaling of (3.36)-(3.39) (see Chapter 3). Inserting these developments into problems (5.93)-(5.97) results in variational equations that must be satisfied whatever h, and consequently, the successive powers must be zero. The problems at the successive orders are

Problem $\mathbf{P}^{-4}$ is given by:

$$\begin{cases} a^{\phi}_{H,0}\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{\tau}, \boldsymbol{d})\right) = 0 \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{cases} \quad \begin{cases} b^{\phi}_{H}\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{v}, \psi)\right) = 0 \\ \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}. \end{cases} \tag{5.152}$$

Problem $\mathbf{P}^{-2}$ is specified by:

$$\begin{cases} a^{\phi}_{H,0}\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{d})\right) = -a^{\phi}_{H,2}\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ \qquad -a^{\#}_{H}(h,\phi)\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{\tau}, \boldsymbol{d})\right), \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{cases} \tag{5.153}$$

$$\begin{cases} b^{\phi}_{H}\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{v}, \psi)\right) = 0 \\ \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}. \end{cases} \tag{5.154}$$

Problem $\mathbf{P}^{0}$ reads:

$$\begin{cases} a^{\phi}_{H,0}\left((\boldsymbol{\sigma}^{0}, \boldsymbol{D}^{0}), (\boldsymbol{\tau}, \boldsymbol{D})\right) = -a^{\phi}_{H,2}\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ \quad -a^{\phi}_{H,4}\left((\boldsymbol{\sigma}^{-4}, \boldsymbol{D}^{-4}), (\boldsymbol{\tau}, \boldsymbol{d})\right) - b^{\phi}_{H}\left((\tau, \boldsymbol{d}), (\boldsymbol{u}^0, \varphi^0)\right) \\ \quad -a^{\#}_{H}(h,\phi)\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{d})\right), \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{cases} \tag{5.155}$$

$$\begin{cases} b^{\phi}_{H}\left((\boldsymbol{\sigma}^{0}, \boldsymbol{D}^{0}), (\boldsymbol{v}, \psi)\right) = l^{\phi}_{H}(\boldsymbol{v}, \psi), \\ \text{for all } (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}. \end{cases} \tag{5.156}$$

Problem $\mathbf{P}^{2}$ reads:

$$\begin{cases} a^{\phi}_{H,0}\left((\boldsymbol{\sigma}^{2}, \boldsymbol{D}^{2}), (\boldsymbol{\tau}, \boldsymbol{D})\right) = -a^{\phi}_{H,2}\left((\sigma^{0}, \boldsymbol{D}^{0}), (\boldsymbol{\tau}, \boldsymbol{d})\right) \\ \quad -a^{\phi}_{H,4}\left((\boldsymbol{\sigma}^{-2}, \boldsymbol{D}^{-2}), (\boldsymbol{\tau}, \boldsymbol{D})\right) - b^{\phi}_{H}\left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}^2, \varphi^2)\right) \\ \quad -a^{\#}_{H}(h,\phi)\left((\boldsymbol{\sigma}^{0}, \boldsymbol{D}^{0}), (\boldsymbol{\tau}, \boldsymbol{d})\right), \\ \text{for all } (\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1, \end{cases} \tag{5.157}$$

$$\left\{ b^{\phi}_{H}\left((\boldsymbol{\sigma}^{2}, \boldsymbol{D}^{2}), (\boldsymbol{v}, \psi)\right) = l^{\phi}_{H}(\boldsymbol{v}, \psi), \quad \forall (\boldsymbol{v}, \psi) \in \boldsymbol{X}_{0,w}. \right. \tag{5.158}$$

Furthermore, we have

$$\sigma_{\alpha\beta}^{-4} = C_{\alpha\beta\theta\rho} e_{\theta\rho}^{\phi}(\boldsymbol{u}^0), \tag{5.159}$$

$$\sigma_{\alpha\beta}^{-2} = C_{\alpha\beta\theta\rho} e_{\theta\rho}^{\phi}(\boldsymbol{u}^2) + C_{\alpha\beta33} e_{33}^{\phi}(\boldsymbol{u}^0) - P_{3\alpha\beta} E_3^{\phi}(\varphi^0) + C_{\alpha\beta\theta\rho} e_{\theta\rho}^{\#}(h,\phi;\boldsymbol{u}^0), \tag{5.160}$$

$$\begin{aligned}\sigma_{\alpha\beta}^{2p} = {} & C_{\alpha\beta\theta\rho} e_{\theta\rho}^{\phi}(\boldsymbol{u}^{2p+4}) + C_{\alpha\beta33} e_{33}^{\phi}(\boldsymbol{u}^{2p+2}) - P_{3\alpha\beta} E_3^{\phi}(\varphi^{2p+2}) \\ & + C_{\alpha\beta\theta\rho} e_{\theta\rho}^{\#}(h,\phi;\boldsymbol{u}^{2p+2}), \quad p \geq 0,\end{aligned} \tag{5.161}$$

$$\sigma_{3\alpha}^{-4} = 0, \tag{5.162}$$

$$\sigma_{3\alpha}^{-2} = 2C_{3\alpha3\beta} e_{3\beta}^{\phi}(\boldsymbol{u}^0) - P_{\theta3\alpha} E_{\theta}^{\phi}(\varphi^0), \tag{5.163}$$

$$\sigma_{3\alpha}^{0} = 2C_{3\alpha3\theta} e_{3\theta}^{\phi}(\boldsymbol{u}^2) - P_{\theta3\alpha} E_{\theta}^{\phi}(\varphi^2), \tag{5.164}$$

$$\sigma_{3\alpha}^{2p} = 2C_{3\alpha3\theta} e_{3\theta}^{\phi}(\boldsymbol{u}^{2p+2}) - P_{\theta3\alpha} E_{\theta}^{\phi}(\varphi^{2p+2}), \quad p \geq 1, \tag{5.165}$$

$$\sigma_{33}^{-4} = 0, \tag{5.166}$$

$$\sigma_{33}^{-2} = C_{33\alpha\beta} e_{\alpha\beta}^{\phi}(\boldsymbol{u}^0), \tag{5.167}$$

$$\sigma_{33}^{0} = C_{33\theta\rho} e_{\theta\rho}^{\phi}(\boldsymbol{u}^2) + C_{3333} e_{33}^{\phi}(\boldsymbol{u}^0) - P_{333} E_3^{\phi}(\varphi^0), \tag{5.168}$$

$$\sigma_{33}^{2p} = C_{33\alpha\beta} e_{\alpha\beta}^{\phi}(\boldsymbol{u}^{2p+2}) + C_{3333} e_{33}^{\phi}(\boldsymbol{u}^{2p}) - P_{333} E_3^{\phi}(\varphi^{2p}), \quad p \geq 1, \tag{5.169}$$

$$D_{\theta}^{-4} = 0, \tag{5.170}$$

$$D_{\theta}^{-2} = 2P_{\theta3\alpha} e_{3\alpha}^{\phi}(\boldsymbol{u}^0) + \varepsilon_{\theta\beta} E_{\beta}^{\phi}(\varphi^0), \tag{5.171}$$

$$D_{\theta}^{0} = 2P_{\theta3\alpha} e_{3\alpha}^{\phi}(\boldsymbol{u}^2) + \varepsilon_{\theta\alpha} E_{\alpha}^{\phi}(\varphi^2), \tag{5.172}$$

$$D_{\theta}^{2p} = 2P_{\theta3\alpha} e_{3\alpha}^{\phi}(\boldsymbol{u}^{2p+2}) + \varepsilon_{\theta\alpha} E_{\alpha}^{\phi}(\varphi^{2p+2}), \quad p \geq 1,$$

$$D_3^{-4} = 0, \tag{5.173}$$

$$D_3^{-2} = P_{3\alpha\beta} e_{\alpha\beta}^{\phi}(\boldsymbol{u}^0), \tag{5.174}$$

$$D_3^{0} = P_{3\alpha\beta} e_{\alpha\beta}^{\phi}(\boldsymbol{u}^2) + P_{333} e_{33}^{\phi}(\boldsymbol{u}^0) + \varepsilon_{33} E_3^{\phi}(\varphi^0), \tag{5.175}$$

$$D_3^{2p} = P_{3\alpha\beta} e_{\alpha\beta}^{\phi}(\boldsymbol{u}^{2p+2}) + P_{333} e_{33}^{\phi}(\boldsymbol{u}^{2p}) + \varepsilon_{33} E_3^{\phi}(\varphi^{2p}), \quad p \geq 1. \tag{5.176}$$

5.5.2.1 Cancelation of the factors of h^{-4}, $-4 \leq q \leq 0$, in the scaled tensors

In this section we show that the formal expansion of the tensor (5.149) and (5.150) induced by (5.151) do not contain any negative powers of h.

Choosing $\tau_{33} = \sigma_{33}^{-4}$ and $d_3 = D_3^{-4}$ for the test functions in the first equation of (5.152) we obtain

$$\sigma_{33}^{-4} = D_3^{-4} = 0 \text{ in } \; L^2(\Omega).$$

From (5.155) we deduce the following relations in $L^2(\Omega)$:

$$e^{\phi}_{\alpha\beta}(\boldsymbol{u}^0) = 0, \tag{5.177}$$

$$e^{\phi}_{3\alpha}(\boldsymbol{u}^0) = 0, \tag{5.178}$$

$$e^{\phi}_{33}(\boldsymbol{u}^0) = \bar{C}_{33\theta\rho}\,\sigma_{\theta\rho}^{-2} + \bar{C}_{3333}\sigma_{33}^{0} + \bar{P}_{333}D_3^0, \tag{5.179}$$

$$E^{\phi}_{\theta}(\varphi^0) = 0, \tag{5.180}$$

$$E^{\phi}_{3}(\varphi^0) = -\bar{P}_{3\alpha\beta}\sigma_{\alpha\beta}^{-2} - \bar{P}_{333}\sigma_{33}^{0} + \bar{\varepsilon}_{33}\,D_3^0, \tag{5.181}$$

and consequently

$$\sigma_{\alpha\beta}^{-4} = D_3^{-2} = \sigma_{33}^{-2} = 0, \quad \text{and} \quad \sigma_{3\alpha}^{-2} = D_\theta^{-2} = 0. \tag{5.182}$$

Choosing an appropriate test function in (5.155) we establish the following condition

$$e^{\phi}_{\alpha\beta}(\boldsymbol{u}^2) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0) = \bar{C}_{\alpha\beta\theta\rho}\,\sigma_{\theta\rho}^{-2} + \bar{C}_{\alpha\beta33}\,\sigma_{33}^{0} + \bar{P}_{3\alpha\beta}D_3^0, \quad \text{in } L^2(\Omega), \tag{5.183}$$

or, equivalently,

$$\bar{C}_{\alpha\beta\theta\rho}\,\sigma_{\theta\rho}^{-2} = e^{\phi}_{\alpha\beta}(\boldsymbol{u}^2) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0) - \bar{C}_{\alpha\beta33}\,\sigma_{33}^{0} - \bar{P}_{3\alpha\beta}D_3^0, \tag{5.184}$$

with $\bar{C}_{ii33} = \bar{P}_{312} = 0$ (see Remark 15).

Proceeding as in Chapter 3 we now establish that $\sigma_{\alpha\beta}^{-2} = 0$. For such, we need to prove the existence of $\bar{\boldsymbol{u}} \in V_{0,w}$ such that

$$\begin{cases} e^{\phi}_{11}(\bar{\boldsymbol{u}}) + e^{\#}_{11}(h,\phi;\boldsymbol{u}^0) - \left(\bar{C}_{1133}\,\sigma_{33}^{0} + \bar{P}_{311}D_3^0\right) = 0 \\ e^{\phi}_{12}(\bar{\boldsymbol{u}}) + e^{\#}_{12}(h,\phi;\boldsymbol{u}^0) = 0 \\ e^{\phi}_{22}(\bar{\boldsymbol{u}}) + e^{\#}_{22}(h,\phi;\boldsymbol{u}^0) - \left(\bar{C}_{2233}\,\sigma_{33}^{0} + \bar{P}_{322}D_3^0\right) = 0 \end{cases}, \tag{5.185}$$

to conclude that equation (5.184) becomes,

$$e^{\phi}_{\alpha\beta}(\bar{u} - \boldsymbol{u}^2) = \bar{C}_{\alpha\beta\theta\rho}\sigma^{-2}_{\theta\rho}. \tag{5.186}$$

Then choosing $\tau_{3i} = 0$ and $d_i = 0$ in (5.155) we get, after some algebraic manipulations,

$$\int_\Omega \left[\begin{array}{c} (\bar{C}_{\alpha\beta33}\,\sigma^0_{33} + \bar{P}_{3\alpha\beta}D^0_3) - \left(e^{\phi}_{\alpha\beta}(\bar{\boldsymbol{u}}) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0)\right) \\ - \left(e^{\phi}_{\alpha\beta}(\boldsymbol{u}^2) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0)\right) + \left(e^{\phi}_{\alpha\beta}(\bar{\boldsymbol{u}}) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0)\right) \end{array} \right] \tau_{\alpha\beta} d\boldsymbol{x} = 0,$$

for all $\tau_{\alpha\beta} \in L^2(\Omega)$. After some algebraic manipulations, the previous equation reads

$$\int_\Omega \left[- \left(e^{\phi}_{\alpha\beta}(\boldsymbol{u}^2) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0)\right) + \left(e^{\phi}_{\alpha\beta}(\bar{\boldsymbol{u}}) + e^{\#}_{\alpha\beta}(h,\phi;\boldsymbol{u}^0)\right) \right] \tau_{\alpha\beta} d\boldsymbol{x} = 0,$$

for all $\tau_{\alpha\beta} \in L^2(\Omega)$. Putting $\tau_{\alpha\beta} = e^{\phi}_{\alpha\beta}(\bar{\boldsymbol{u}} - \boldsymbol{u}^2)$ we obtain that $e^{\phi}_{\alpha\beta}(\bar{\boldsymbol{u}} - \boldsymbol{u}^2) = 0$, and therefore $\bar{\boldsymbol{u}} = \boldsymbol{u}^2$. Substituting $\bar{\boldsymbol{u}}$ by $\boldsymbol{u}^2$ in (5.186) we get

$$\sigma^{-2}_{\alpha\beta} = 0.$$

We assume then that $\sigma^{-2}_{\alpha\beta} = 0$, which is equivalent to saying that $\boldsymbol{u}^2$ satisfies (5.185). Then, combining the previous equation with the expressions (5.179) and (5.181), we obtain

$$e^{\phi}_{11}(\boldsymbol{u}^2) = -e^{\#}_{11}(h,\phi;\boldsymbol{u}^0) + A^d_{33} \left(\begin{array}{c} \left(\bar{C}_{1133}\,\bar{\varepsilon}_{33} + \bar{P}_{311}\bar{P}_{333}\right) e^{\phi}_{33}(\boldsymbol{u}^0) \\ + \left(-\bar{C}_{1133}\bar{P}_{333} + \bar{P}_{311}\bar{C}_{3333}\right) E^{\phi}_3(\varphi^0) \end{array} \right), \tag{5.187}$$

$$e^{\phi}_{12}(\boldsymbol{u}^2) = -e^{\#}_{12}(h,\phi;\boldsymbol{u}^0), \tag{5.188}$$

$$e^{\phi}_{22}(\boldsymbol{u}^2) = -e^{\#}_{22}(h,\phi;\boldsymbol{u}^0) + A^d_{33} \left(\begin{array}{c} \left(\bar{C}_{2233}\,\bar{\varepsilon}_{33} + \bar{P}_{322}\bar{P}_{333}\right) e^{\phi}_{33}(\boldsymbol{u}^0) \\ + \left(-\bar{C}_{2233}\bar{P}_{333} + \bar{P}_{322}\bar{C}_{3333}\right) E^{\phi}_3(\varphi^0) \end{array} \right), \tag{5.189}$$

with

$$A^d_{33} = \frac{1}{\bar{C}_{3333}\bar{\varepsilon}_{33} + \bar{P}_{333}\bar{P}_{333}}. \tag{5.190}$$

5.5.2.2 Identification of the leading term $(\boldsymbol{u}^0, \varphi^0)$

Corollary 24 *The displacement $\boldsymbol{u}^0$ and the electric potential φ^0 are given by*

$$\boldsymbol{u}^0 \in V^0_{BN} : \left\{ \begin{array}{ll} u^0_\alpha(x_1,x_2,x_3) = \xi_\alpha(x_3), & \xi_\alpha \in V^2_0(0,L), \\ u^0_3(x_1,x_2,x_3) = \xi_3(x_3) - \chi^b_\alpha \xi'_\alpha(x_3), & \xi_3 \in V^1_0(0,L), \end{array} \right. \tag{5.191}$$

$$\varphi^0 \in \hat{\varphi} + \Psi_3^\phi : \begin{cases} \varphi^0 = z_3, \quad z_3 \in H^1(0,L), \\ \varphi^0(x_1,x_2,0) = \hat{\varphi}(0) = \varphi_0^0, \\ \varphi^0(x_1,x_2,L) = \hat{\varphi}(L) = \varphi_0^L, \end{cases} \tag{5.192}$$

where

$$\begin{pmatrix} \chi_1^b(x_1,x_2,x_3) \\ \chi_2^b(x_1,x_2,x_3) \end{pmatrix} = \begin{pmatrix} b_1(x_3) & -b_2(x_3) \\ b_2(x_3) & b_1(x_3) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{5.193}$$

and

$$b_1^2 + b_2^2 = 1. \tag{5.194}$$

Proof. Taking into account relations (5.177)-(5.178) we get expressions (5.191) for the components of $\boldsymbol{u}^0$ such as in Álvarez-Dios & Viaño [1998]. Condition (5.192) follows from equation (5.180) taking into account the electric boundary potentials acting at both ends of the beam. ■

Substituting the relations (5.191)-(5.192) into (5.187)-(5.189) and taking into account the definitions of $e^\phi_{\alpha\beta}(\boldsymbol{u}^2)$, equation (5.76), we can evaluate u^2_α as a function of the field $(s, s_\alpha, \xi_i, z_3)$ that it is independent on variables x_1 and x_2. The following relationships are a consequence of previous corollary and they will be used in the next proof

$$u^0_\alpha = \xi_\alpha, \;\; u^0_3 = \xi_3 - \chi^b_\alpha \xi'_\alpha, \;\; \xi_\alpha \in V^2_0(0,L), \;\; \xi_3 \in V^1_0(0,L), \tag{5.195}$$

$$e^\phi_{33}(u^0_\alpha) = \xi'_3 - \chi^b_\alpha \xi''_\alpha + \phi'_\alpha \xi'_\alpha, \tag{5.196}$$

$$\varphi^0 = z_3, \;\; z_3 \in H^1(0,L) \text{ and satisfying } z_3(0) = \varphi^0_0, \;\; z_3(L) = \varphi^L_0, \tag{5.197}$$

$$E^\phi_3(\varphi^0) = -\partial_3 \varphi^0 = -z'_3, \tag{5.198}$$

$$E^\phi_3(\bar{\varphi}^0) = -\partial_3 \bar{\varphi}^0 = -z'_3 + \frac{1}{L}(\varphi^L_0 - \varphi^0_0), \tag{5.199}$$

$$\chi^b_1 = b_1 x_1 - b_2 x_2, \quad \chi^b_2 = b_2 x_1 + b_1 x_2. \tag{5.200}$$

Lemma 14 *Let $\boldsymbol{u}^0 \in V^\phi_{BN}$ and $\varphi^0 \in \hat{\varphi} + \Psi^\phi_3$ be given by (5.191) and (5.192), then every element $u^2_\alpha \in L^2(\Omega)$ satisfying (5.185) is such that*

$$u^2_1 = \begin{pmatrix} s_1(x_3) + \frac{1}{b_1}\chi^b_2 s(x_3) + \frac{1}{b_1} x_1 \left(C_m \xi'_3 - C_e z'_3\right) + x_1 r^0(x_3) \\ +x_1 b_1 \left(\phi'_2 \xi'_2 - \phi'_1 \xi'_1\right) - x_1 b_2 \left(\phi'_1 \xi'_2 + \phi'_2 \xi'_1\right) + \Phi_{1\beta} \xi''_\beta \end{pmatrix}, \tag{5.201}$$

$$u_2^2 = \begin{pmatrix} s_2(x_3) - \frac{1}{b_1}\chi_1^b s(x_3) + \frac{1}{b_1} x_2 \left(C_m \xi_3' - C_e z_3'\right) + x_2 r^0(x_3) \\ -x_1 b_2 \left(\phi_2' \xi_2' - \phi_1' \xi_1'\right) - x_1 b_1 \left(\phi_1' \xi_2' + \phi_2' \xi_1'\right) + \Phi_{2\beta} \xi_\beta'' \end{pmatrix}, \tag{5.202}$$

where $s_\alpha, s, z \in L^2(0, L)$ are arbitrary functions depending only on the variable x_3 and

$$r^0(x_3) = \frac{1}{b_1} C_m \left(\phi_2' \xi_2' + \phi_1' \xi_1'\right) - \frac{1}{b_1} \phi_1' \xi_1' - b_1 \left(\phi_2' \xi_2' - \phi_1' \xi_1'\right) + b_2 \left(\phi_1' \xi_2' + \phi_2' \xi_1'\right),$$

while $\boldsymbol{\Phi} = (\Phi_{\alpha\beta})$ denotes the symmetric matrix with components

$$\Phi_{11} = C_m \left[2 b_2 b_1 x_1 x_2 + \left(\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) \left(-b_1^2 + b_2^2\right)\right], \tag{5.203}$$

$$\Phi_{12} = C_m \left[x_1 x_2 \left(b_2^2 - b_1^2\right) - 2 b_1 b_2 \left(\frac{x_1^2}{2} - \frac{x_2^2}{2}\right)\right], \tag{5.204}$$

$$\Phi_{21} = C_m \left[2 b_2 b_1 \left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) + x_2 x_1 \left(-b_1^2 + b_2^2\right)\right], \tag{5.205}$$

$$\Phi_{22} = C_m \left[\left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) \left(b_2^2 - b_1^2\right) - 2 b_1 b_2 x_2 x_1\right]. \tag{5.206}$$

The constants C_m and C_e are given by

$$C_m = A_{33}^d \left(\bar{C}_{1133}\, \bar{\varepsilon}_{33} + \bar{P}_{311} \bar{P}_{333}\right),$$

$$C_e = A_{33}^d \left(-\bar{C}_{1133} \bar{P}_{333} + \bar{P}_{311} \bar{C}_{3333}\right).$$

Proof. Equations (5.187)-(5.189) become, from (5.76),

$$b_1 \partial_1 u_1^2 - b_2 \partial_2 u_1^2 = C_m e_{33}^\phi(\boldsymbol{u}^0) + C_e E_3^\phi(\varphi^0) - e_{11}^\#(h, \phi; \boldsymbol{u}^0), \tag{5.207}$$

$$[b_1(\partial_1 u_2^2 + \partial_2 u_1^2) + b_2(\partial_1 u_1^2 - \partial_2 u_2^2)] = -2 e_{12}^\#(h, \phi; \boldsymbol{u}^0), \tag{5.208}$$

$$b_2 \partial_1 u_2^2 + b_1 \partial_2 u_2^2 = C_m e_{33}^\phi(\boldsymbol{u}^0) + C_e E_3^\phi(\varphi^0) - e_{22}^\#(h, \phi; \boldsymbol{u}^0). \tag{5.209}$$

From (5.207) and (5.209) we get the following relations

$$b_1 \left(\partial_1 u_1^2 + \partial_2 u_2^2\right) + b_2 \left(\partial_1 u_2^2 - \partial_2 u_1^2\right) = 2 C_m e_{33}^\phi(\boldsymbol{u}^0) + 2 C_e E_3^\phi(\varphi^0) - e_{11}^\#(h, \phi; \boldsymbol{u}^0) - e_{22}^\#(h, \phi; \boldsymbol{u}^0), \tag{5.210}$$

$$b_1 \left(\partial_1 u_1^2 - \partial_2 u_2^2\right) - b_2 \left(\partial_2 u_1^2 + \partial_1 u_2^2\right) = e_{22}^\#(h, \phi; \boldsymbol{u}^0) - e_{11}^\#(h, \phi; \boldsymbol{u}^0). \tag{5.211}$$

Multiplying both equations (5.210) and (5.211) by b_1 and b_2, we get

$$b_1 b_2 \left(\partial_1 u_1^2 - \partial_2 u_2^2\right) - b_2^2 \left(\partial_2 u_1^2 + \partial_1 u_2^2\right) = b_2 \left(e_{22}^{\#}(h,\phi;\boldsymbol{u}^0) - e_{11}^{\#}(h,\phi;\boldsymbol{u}^0)\right), \tag{5.212}$$

$$b_1^2 \left(\partial_1 u_1^2 - \partial_2 u_2^2\right) - b_1 b_2 \left(\partial_2 u_1^2 + \partial_1 u_2^2\right) = b_1 \left(e_{22}^{\#}(h,\phi;\boldsymbol{u}^0) - e_{11}^{\#}(h,\phi;\boldsymbol{u}^0)\right), \tag{5.213}$$

and, by using equation (5.208), we arrive at the following relations

$$b_2 b_1 (\partial_1 u_2^2 + \partial_2 u_1^2) + b_2^2 (\partial_1 u_1^2 - \partial_2 u_2^2) = -2 b_2 e_{12}^{\#}(h,\phi;\mathbf{u}^0), \tag{5.214}$$

$$b_1^2 (\partial_1 u_2^2 + \partial_2 u_1^2) + b_1 b_2 (\partial_1 u_1^2 - \partial_2 u_2^2) = -2 b_1 e_{12}^{\#}(h,\phi;\mathbf{u}^0), \tag{5.215}$$

respectively. Adding up equation (5.213) with (5.214) and subtracting equation (5.215) to (5.212) we obtain

$$\partial_1 u_1^2 - \partial_2 u_2^2 = b_1 \left[e_{22}^{\#}(h,\phi;\boldsymbol{u}^0) - e_{11}^{\#}(h,\phi;\boldsymbol{u}^0)\right] - 2 b_2 e_{12}^{\#}(h,\phi;\boldsymbol{u}^0), \tag{5.216}$$

and

$$\partial_2 u_1^2 + \partial_1 u_2^2 = -2 b_1 e_{12}^{\#}(h,\phi;\boldsymbol{u}^0) - b_2 \left[e_{22}^{\#}(h,\phi;\boldsymbol{u}^0) - e_{11}^{\#}(h,\phi;\boldsymbol{u}^0)\right]. \tag{5.217}$$

Differentiating (5.216) with respect to x_1 and x_2, respectively, and (5.217) in order to x_1 and x_2 we obtain the following homogeneous system

$$\partial_{11} u_1^2 - \partial_{21} u_2^2 = 0, \tag{5.218}$$

$$\partial_{12} u_1^2 - \partial_{22} u_2^2 = 0, \tag{5.219}$$

$$\partial_{21} u_1^2 + \partial_{11} u_2^2 = 0, \tag{5.220}$$

$$\partial_{22} u_1^2 + \partial_{12} u_2^2 = 0. \tag{5.221}$$

Multiplying (5.207) by b_1 and deriving with respect to x_1, we have

$$b_1^2 \partial_{11} u_1^2 = b_1 b_2 \partial_{12} u_1^2 - b_1 \partial_1 \left(\phi_1' \xi_1' - C_m e_{33}^{\phi}(\boldsymbol{u}^0) - C_e E_3^{\phi}(\varphi^0)\right), \tag{5.222}$$

and applying (5.219), (5.209) and (5.198), we deduce

$$\begin{aligned} b_1^2 \partial_{11} u_1^2 &= b_2 \partial_2 \left(b_1 \partial_2 u_2^2\right) - b_1 \partial_1 \left(\phi_1' \xi_1' - C_m e_{33}^{\phi}(\boldsymbol{u}^0) - C_e E_3^{\phi}(\varphi^0)\right) \\ &= -b_2^2 \partial_{21} u_2^2 + b_2 C_m \partial_2 e_{33}^{\phi}(\boldsymbol{u}^0) + b_1 C_m \partial_1 e_{33}^{\phi}(\boldsymbol{u}^0). \end{aligned}$$

This expression can be written as follows, from (5.191) and (5.194),

$$\partial_{11} u_1^2 = C_m \left[\left(-b_1^2 + b_2^2 \right) \xi_1' - 2 b_1 b_2 \xi_2' \right]. \tag{5.223}$$

Consequently, there exist functions k_1 and k_2 such that

$$u_1^2 = k_2(x_2, x_3) + x_1 k_1(x_2, x_3) + \frac{x_1^2}{2} C_m \left[\left(-b_1^2 + b_2^2 \right) \xi_1' - 2 b_1 b_2 \xi_2' \right]. \tag{5.224}$$

To characterize the component u_2^2, we return to equation (5.209). Multiplying it by b_1 and deriving in order to x_2 we have

$$b_1 b_2 \partial_{12} u_2^2 + b_1^2 \partial_{22} u_2^2 = -b_1 \partial_2 \left(\phi_2' \xi_2' - C_m e_{33}^{\phi}(\boldsymbol{u}^0) - C_e E_3^{\phi}(\varphi^0) \right)$$

which, from (5.218)-(5.220), leads to

$$\begin{aligned} b_1^2 \partial_{22} u_2^2 &= -b_1 b_2 \partial_{12} u_2^2 - b_1 \partial_2 \left(\phi_2' \xi_2' - C_m e_{33}^{\phi}(\boldsymbol{u}^0) - C_e E_3^{\phi}(\varphi^0) \right) \\ &= -b_2 \partial_1 \left(b_1 \partial_1 u_1^2 \right) + b_1 C_m \partial_2 e_{33}^{\phi}(\boldsymbol{u}^0) \\ &= -b_2 \partial_1 \left(b_2 \partial_2 u_1^2 - \left(\phi_1' \xi_1' - C_m e_{33}^{\phi}(\boldsymbol{u}^0) - C_e E_3^{\phi}(\varphi^0) \right) \right) + b_1 C_m \partial_2 e_{33}^{\phi}(\boldsymbol{u}^0) \\ &= -b_2^2 \partial_{22} u_2^2 + C_m \left(-b_2 \partial_1 e_{33}^{\phi}(\boldsymbol{u}^0) + b_1 \partial_2 e_{33}^{\phi}(\boldsymbol{u}^0) \right). \end{aligned}$$

Using again the relations (5.191), (5.193) and (5.194) we conclude that

$$\partial_{22} u_2^2 = C_m \left[2 b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2 \right) \xi_2'' \right].$$

Solving this differential equation, we prove that there exist functions s_1 and s_2 such that

$$u_1^2 = k_2(x_2, x_3) + x_1 k_1(x_2, x_3) + \frac{x_1^2}{2} C_m \left[\left(-b_1^2 + b_2^2 \right) \xi_1'' - 2 b_1 b_2 \xi_2'' \right], \tag{5.225}$$

$$u_2^2 = s_2(x_1, x_3) + x_2 s_1(x_1, x_3) + \frac{x_2^2}{2} C_m \left[2 b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2 \right) \xi_2'' \right]. \tag{5.226}$$

Putting in (5.218) and (5.219) the components (5.225) and (5.226) we get

$$\partial_1 s_1(x_1, x_3) = C_m \left[\left(-b_1^2 + b_2^2 \right) \xi_1' - 2 b_1 b_2 \xi_2' \right], \tag{5.227}$$

$$\partial_2 k_1(x_2, x_3) = C_m \left[2 b_2 b_1 \xi_1' + \left(b_2^2 - b_1^2 \right) \xi_2' \right], \tag{5.228}$$

and therefore, there exist $s_3(x_3)$ and $k_3(x_3)$ such that

$$s_1(x_1, x_3) = s_3(x_3) + x_1 C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right], \tag{5.229}$$

$$k_1(x_2, x_3) = k_3(x_3) + x_2 C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right]. \tag{5.230}$$

Consequently, the expressions (5.225) and (5.226) become

$$u_1^2 = k_2(x_2, x_3) + x_1 k_3(x_3) + x_1 x_2 C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right] + \frac{x_1^2}{2} C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right], \tag{5.231}$$

$$u_2^2 = s_2(x_1, x_3) + x_2 s_3(x_3) + \frac{x_2^2}{2} C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right] + x_2 x_1 C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right]. \tag{5.232}$$

Substituting these expressions in (5.220) and (5.221), we obtain

$$\partial_{22} k_2(x_2, x_3) = -C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right], \tag{5.233}$$

$$\partial_{11} s_2(x_1, x_3) = -C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right], \tag{5.234}$$

and therefore, we prove the existence of the functions $k_4(x_3)$, $k_5(x_3)$, $s_4(x_3)$, $s_5(x_3)$ such that

$$k_2(x_2, x_3) = k_5(x_3) + x_2 k_4(x_3) - \frac{x_2^2}{2} C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right], \tag{5.235}$$

$$s_2(x_1, x_3) = s_5(x_3) + x_1 s_4(x_3) - \frac{x_1^2}{2} C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right]. \tag{5.236}$$

Hence

$$u_1^2 = k_5(x_3) + x_2 k_4(x_3) + x_1 k_3(x_3) + x_1 x_2 C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right] + \left(\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1 b_2 \xi_2''\right], \tag{5.237}$$

$$u_2^2 = s_5(x_3) + x_1 s_4(x_3) + x_2 s_3(x_3) + x_2 x_1 C_m \left[\left(-b_1^2 + b_2^2\right)\xi_1' - 2b_1 b_2 \xi_2'\right] + \left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) C_m \left[2b_2 b_1 \xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right]. \tag{5.238}$$

Applying in (5.216) and (5.217) the components u_α^2 we obtain

$$k_3(x_3) = s_3(x_3) + b_1\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) - b_2\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right), \tag{5.239}$$

$$s_4(x_3) = -k_4(x_3) - b_2\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) - b_1\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right), \tag{5.240}$$

so that we have

$$u_1^2 = k_5(x_3) + x_2 k_4(x_3) + x_1 s_3(x_3) + x_1 b_1\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) \tag{5.241}$$

$$- x_1 b_2\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right) + x_1 x_2 C_m\left[2b_2b_1\xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right]$$
$$+ \left(\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) C_m\left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1b_2\xi_2''\right], \tag{5.242}$$

$$\begin{aligned} u_2^2 = {} & s_5(x_3) - x_1 k_4(x_3) + x_2 s_3(x_3) - x_1 b_2\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) - x_1 b_1\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right) \\ & + \left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right) C_m\left[2b_2b_1\xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right] \\ & + x_2 x_1 C_m\left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1b_2\xi_2''\right]. \end{aligned} \tag{5.243}$$

Substituting these expressions into (5.210) we conclude

$$\begin{aligned} s_3(x_3) = {} & \frac{b_2}{b_1}k_4(x_3) + \frac{1}{b_1}\left(C_m\xi_3' - C_e z_3'\right) + \frac{1}{b_1}C_m\left(\phi_2'\xi_2' + \phi_1'\xi_1'\right) \\ & - \frac{1}{2b_1}\left(\phi_2'\xi_2' + \phi_1'\xi_1'\right) - \left(b_1^2 - b_2^2\right)\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) + 2b_1b_2\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right). \end{aligned}$$

Defining the function r^0 by

$$\begin{aligned} r^0(x_3) = {} & \frac{1}{b_1}C_m\left(\phi_2'\xi_2' + \phi_1'\xi_1'\right) - \frac{1}{2b_1}\left(\phi_2'\xi_2' + \phi_1'\xi_1'\right) \\ & - \left(b_1^2 - b_2^2\right)\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) + 2b_1b_2\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right), \end{aligned}$$

we may express u_1^2 and u_2^2 as follows

$$\begin{aligned} u_1^2 = {} & k_5(x_3) + \frac{1}{b_1}\left(x_2b_1 + x_1b_2\right)k_4(x_3) + \frac{1}{b_1}x_1\left(C_m\xi_3' - C_e z_3'\right) + x_1 r^0(x_3) \\ & + x_1 b_1\left(\phi_2'\xi_2' - \phi_1'\xi_1'\right) + x_1 x_2 C_m\left[2b_2b_1\xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right] \\ & - x_1 b_2\left(\phi_1'\xi_2' + \phi_2'\xi_1'\right) + \left(\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) C_m\left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1b_2\xi_2''\right], \end{aligned} \tag{5.244}$$

$$u_2^2 = s_5(x_3) - \frac{1}{b_1}(x_1b_1 - x_2b_2)\,k_4(x_3) + \frac{1}{b_1}x_2\,(C_m\xi_3' - C_e z_3') + x_2 r^0(x_3)$$
$$- x_1b_2\,(\phi_2'\xi_2' - \phi_1'\xi_1') + x_2x_1C_m\left[\left(-b_1^2 + b_2^2\right)\xi_1'' - 2b_1b_2\xi_2''\right]$$
$$- x_1b_1\,(\phi_1'\xi_2' + \phi_2'\xi_1') + \left(\frac{x_2^2}{2} - \frac{x_1^2}{2}\right)C_m\left[2b_2b_1\xi_1'' + \left(b_2^2 - b_1^2\right)\xi_2''\right]. \qquad (5.245)$$

■

We evaluate now the equation (5.156) for test functions in spaces $(\boldsymbol{v}, \psi) \in V_{BN}^{\phi} \times \Psi_3^{\phi}$. Then, there exists $\boldsymbol{u}^2$ satisfying (5.185), and consequently $\sigma_{\alpha\beta}^{-2} = 0$. Clearly, (5.179) and (5.181) become

$$e_{33}^{\phi}(\boldsymbol{u}^0) = \bar{C}_{3333}\sigma_{33}^0 + \bar{P}_{333}D_3^0, \qquad E_3^{\phi}(\varphi^0) = -\bar{P}_{333}\sigma_{33}^0 + \bar{\varepsilon}_{33}\,D_3^0, \qquad (5.246)$$

respectively. The positivity hypothesis $(\mathbf{H}_{22}^c)$ guarantees that

$$\begin{vmatrix} \bar{C}_{3333} & \bar{P}_{333} \\ \bar{P}_{333} & \bar{\varepsilon}_{33} \end{vmatrix} > 0, \qquad (5.247)$$

and therefore, we have

$$\sigma_{33}^0 = A_{33}^d\left(\bar{\varepsilon}_{33}e_{33}^{\phi}(\boldsymbol{u}^0) - \bar{P}_{333}E_3^{\phi}(\varphi^0)\right), \qquad (5.248)$$

$$D_3^0 = A_{33}^d\left(\bar{P}_{333}e_{33}^{\phi}(\boldsymbol{u}^0) + \bar{C}_{3333}E_3^{\phi}(\varphi^0)\right), \qquad (5.249)$$

where

$$A_{33}^d = \frac{1}{\bar{C}_{3333}\bar{\varepsilon}_{33} + \bar{P}_{333}\bar{P}_{333}}, \qquad (5.250)$$

$$e_{33}^{\phi}(\boldsymbol{u}^0) = \xi_3' - \chi_\alpha^b\xi_\alpha''(x_3) + \phi_\alpha'\xi_\alpha', \qquad E_3^{\phi}(\varphi^0) = -z_3'. \qquad (5.251)$$

Furthermore, the variational problem (5.155)-(5.156) becomes,

$$\int_\Omega \left(\bar{C}_{3333}\,\sigma_{33}^0 + \bar{P}_{333}D_3^0\right)\tau_{33}d\boldsymbol{x} + \int_\Omega \left(-\bar{P}_{333}\sigma_{33}^0 + \bar{\varepsilon}_{33}\,D_3^0\right)d_3d\boldsymbol{x}$$
$$- \int_\Omega \tau_{33}\,e_{33}^{\phi}(\boldsymbol{u}^0)d\boldsymbol{x} - \int_\Omega d_3\,E_3^{\phi}(\varphi^0)d\boldsymbol{x} = 0, \qquad \text{for all } (\tau_{33}, d_3) \in L^2(\Omega) \qquad (5.252)$$

$$\int_\Omega \sigma_{33}^0\,e_{33}^{\phi}(\boldsymbol{v})d\boldsymbol{x} + \int_\Omega D_3^0\,E_3^{\phi}(\psi)d\boldsymbol{x} = \int_\Omega f_iv_id\boldsymbol{x} + \int_{\Gamma_N} g_iv_id\Gamma$$
$$+ \int_{\Gamma_L} p_iv_id\Gamma, \qquad \text{for all } (\boldsymbol{v}, \psi) \in V_{BN}^{\phi} \times \Psi_3^{\phi}, \qquad (5.253)$$

5.5.3 Strong convergence

As in Section 3.4.1, and using similar techniques, we prove the following result.

Theorem 20 *For* $0 < h < 1$, *let* $((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \varphi(h))) \in \mathbf{X}_1 \times \mathbf{X}_2$ *be the solution of (5.41)-(5.43), and* $A^d_{33}(\bar{\varepsilon}_{33}\bar{C}_{33\theta\rho} + \bar{P}_{333}\bar{P}_{3\theta\rho})$ *and* $A^d_{33}(-\bar{P}_{333}\bar{C}_{33\theta\rho} + \bar{\varepsilon}_{33}\bar{P}_{3\theta\rho})$ *be independent on* x_α. *Then:*

$$\boldsymbol{u}(h) \to \boldsymbol{u}^0, \qquad \textit{strongly in } [H^1(\Omega)]^3, \tag{5.254}$$

$$\varphi(h) \to \varphi^0, \qquad \textit{strongly in } H^1(\Omega), \tag{5.255}$$

$$\sigma_{33}(h) \to \sigma^0_{33}, \quad h\sigma_{3\alpha}(h) \to 0, \quad h^2\sigma_{\alpha\beta}(h) \to 0, \qquad \textit{strongly in } L^2(\Omega), \tag{5.256}$$

$$hD_\alpha(h) \to 0, \quad D_3(h) \to D^0_3, \qquad \textit{strongly in } L^2(\Omega). \tag{5.257}$$

Proof. For any $(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}) \in \boldsymbol{X}_1$ one has

$$a_{H,0}\left((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}})\right) + a_{H,2}\left((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}})\right) + a_{H,4}\left((\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}}), (\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}})\right) \geq C\|(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}})\|^2_{\boldsymbol{X}_1}. \tag{5.258}$$

Replacing $(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{d}})$ in equation (5.41) by $(\bar{\boldsymbol{S}}(h), \bar{\boldsymbol{T}}(h))$, where

$$\bar{S}_{3\alpha} = hS_{3\alpha}, \quad \bar{S}_{\alpha\beta} = h^2 S_{\alpha\beta}, \quad \bar{T}_3 = T_3, \quad \bar{T}_\alpha = hT_\alpha,$$

with

$$\begin{aligned} \boldsymbol{S}(h) = \boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \qquad \boldsymbol{T}(h) = \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0 \\ \bar{\sigma}^0_{33} = \sigma^0_{33}, \quad \bar{\sigma}^0_{3\beta} = 0, \quad \bar{\sigma}^0_{\alpha\beta} = 0, \quad \bar{D}_3 = D^0_3, \quad \bar{D}_\alpha = 0, \end{aligned} \tag{5.259}$$

equality (5.258) reads

$$C\|(\boldsymbol{S}(h), \boldsymbol{T}(h))\|^2_{\boldsymbol{X}_1} = C\left\{\left|\sigma_{33}(h) - \sigma^0_{33}\right|^2_{0,\Omega} + h^2\left|\sigma_{3\alpha}(h)\right|^2_{0,\Omega}\right. \tag{5.260}$$

$$\left. + h^4\left|\sigma_{\alpha\beta}(h)\right|^2_{0,\Omega} + h^2\left|D_\alpha(h)\right|^2_{0,\Omega} + \left|D_3(h) - D^0_3\right|^2_{0,\Omega}\right\}$$

$$\leq a_{H,0}\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{T}(h) - \bar{\boldsymbol{D}}^0)\right)$$

$$- h^2 a_{H,2}\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0)\right)$$

$$- h^4 a_{H,4}\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0)\right) = \Lambda(h). \tag{5.261}$$

From problems (5.41), (5.42) and (5.155), and taking into account that $(\bar{\boldsymbol{\sigma}}^{-4}, \bar{\boldsymbol{D}}^{-4}) = (\bar{\boldsymbol{\sigma}}^{-2}, \bar{\boldsymbol{D}}^{-2}) = (\mathbf{0}, \mathbf{0})$, $\Lambda(h)$ may be written as

$$\begin{aligned}\Lambda(h) = &-b_H\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \varphi(h))\right)\\ &- a_{H,0}\left((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0)\right)\\ &- h^2 a_{H,2}\left((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0)\right)\\ &- h^4 a_{H,4}\left((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0)\right)\end{aligned}$$

or equivalently, it reads

$$\begin{aligned}\Lambda(h) = &-b_H\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \varphi(h))\right)\\ &+ b_H((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{u}^0, \varphi^0))\\ &- h^2\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\sigma_{33}^0 + \bar{P}_{3\alpha\beta} D_3^0\right)\sigma_{\alpha\beta}(h) o(h) d\boldsymbol{x},\end{aligned} \tag{5.262}$$

Taking $(\boldsymbol{v}, \psi) = (\boldsymbol{u}(h), \varphi(h) - \hat{\varphi})$ in equation (5.42), we have the following expression for the term

$$\begin{aligned}\Lambda_1(h) &= -b_H\left((\boldsymbol{\sigma}(h) - \bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h) - \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \bar{\varphi}(h) + \hat{\varphi})\right)\\ &= -b_H\left((\boldsymbol{\sigma}(h), \boldsymbol{D}(h)), (\boldsymbol{u}(h), \bar{\varphi}(h) + \hat{\varphi})\right) + b_H\left((\bar{\boldsymbol{\sigma}}^0, \bar{\boldsymbol{D}}^0), (\boldsymbol{u}(h), \bar{\varphi}(h) + \hat{\varphi})\right)\\ &= \int_\Omega f_i u_i(h) o(h) d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i(h) o(h)\tilde{o}(h) d\Gamma + \int_{\Gamma_L} p_i u_i(h) o(h)\tilde{o}(h) d\Gamma\\ &\quad + \int_\Omega D_3(h) E_3(\hat{\varphi}(h)) o(h) d\boldsymbol{x} - \int_\Omega \sigma_{33}^0 e_{33}(\boldsymbol{u}(h)) o(h) d\boldsymbol{x} - \int_\Omega D_3^0 E_3(\bar{\varphi} + \hat{\varphi})(h) o(h) d\boldsymbol{x}.\end{aligned}$$

Due to the weak convergence result, and to the assumed estimates for f_i, g_i, p_i and $o^\#$, one has, as $h \to 0$,

$$\begin{aligned}\Lambda_1 \to &\int_\Omega f_i u_i d\boldsymbol{x} + \int_{\Gamma_N} g_i u_i d\Gamma + \int_{\Gamma_L} p_i u_i d\Gamma + \int_\Omega \mathfrak{D}_3 E_3^\phi(\hat{\varphi}) d\boldsymbol{x}\\ &- \int_\Omega \sigma_{33}^0 e_{33}^\phi(\boldsymbol{u}^0) d\boldsymbol{x} - \int_\Omega D_3^0 E_3^\phi(\varphi^0 + \hat{\varphi}) d\boldsymbol{x}\\ = &\int_\Omega \mathfrak{D}_3 E_3^\phi(\hat{\varphi}) d\boldsymbol{x} - \int_\Omega D_3^0 E_3^\phi(\hat{\varphi}) d\boldsymbol{x}.\end{aligned} \tag{5.263}$$

On the other hand, the two last terms of (5.262) can be written in the following expansive way

$$\begin{aligned}\Lambda_2(h) &= b_H((\boldsymbol{\sigma}(h)-\bar{\boldsymbol{\sigma}}^0, \boldsymbol{D}(h)-\bar{\boldsymbol{D}}^0),(\boldsymbol{u}^0,\varphi^0)) - h^2\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\sigma_{33}^0 + \bar{P}_{3\alpha\beta}D_3^0\right)\sigma_{\alpha\beta}(h)o(h)d\boldsymbol{x}\\
&= -\int_\Omega \left(\sigma_{33}(h)-\sigma_{33}^0\right)e_{33}^\phi(\boldsymbol{u}^0)o(h)d\boldsymbol{x} - \int_\Omega \left(D_3(h)-D_3^0\right)E_3^\phi(\varphi^0)o(h)d\boldsymbol{x}\\
&\quad - h^2\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\sigma_{33}^0 + \bar{P}_{3\alpha\beta}D_3^0\right)\sigma_{\alpha\beta}(h)o(h)d\boldsymbol{x}.\end{aligned}$$

which become, as h goes to zero,

$$\begin{aligned}\Lambda_2 = &-\int_\Omega \left(\Sigma_{33}-\sigma_{33}^0\right)e_{33}^\phi(\boldsymbol{u}^0)d\boldsymbol{x} - \int_\Omega \left(\mathfrak{D}_3-D_3^0\right)E_3^\phi(\varphi^0)d\boldsymbol{x}\\
&-\int_\Omega \left(\bar{C}_{\alpha\beta 33}\,\sigma_{33}^0 + \bar{P}_{3\alpha\beta}D_3^0\right)\Sigma_{\alpha\beta}d\boldsymbol{x}.\end{aligned}$$

Choosing now $\boldsymbol{v} = \boldsymbol{u} \in V_{BN}^\phi$ and $\psi = \bar{\varphi} - \hat{\varphi} \in \Psi_3^\phi$ as test functions in (5.146) and (5.253) we conclude that

$$\begin{aligned}\Lambda_2 &= -\int_\Omega \Sigma_{33}\,e_{33}^\phi(\boldsymbol{u}^0)d\boldsymbol{x} - \int_\Omega \mathfrak{D}_3\,E_3^\phi(\varphi^0)d\boldsymbol{x} + \int_\Omega \sigma_{33}^0\,e_{33}^\phi(\boldsymbol{u}^0)d\boldsymbol{x} + \int_\Omega D_3^0\,E_3^\phi(\varphi^0)d\boldsymbol{x}\\
&= -\int_\Omega f_iu_i^0d\boldsymbol{x} - \int_{\Gamma_N} g_iu_i^0d\Gamma - \int_{\Gamma_L} p_iu_i^0d\Gamma - \int_\Omega \mathfrak{D}_3\,E_3^\phi(\hat{\varphi})d\boldsymbol{x}\\
&\quad + \int_\Omega f_iu_i^0d\boldsymbol{x} + \int_{\Gamma_N} g_iu_i^0d\Gamma + \int_{\Gamma_L} p_iu_i^0d\Gamma + \int_\Omega D_3^0\,E_3^\phi(\hat{\varphi})d\boldsymbol{x}\\
&= \int_\Omega \left(D_3^0-\mathfrak{D}_3\right)E_3^\phi(\hat{\varphi})d\boldsymbol{x},\end{aligned}$$

and consequently, by combining the two limits, we have

$$\Lambda = \Lambda_1 + \Lambda_2 = \int_\Omega \mathfrak{D}_3E_3^\phi(\hat{\varphi})d\boldsymbol{x} - \int_\Omega D_3^0E_3^\phi(\hat{\varphi})d\boldsymbol{x} + \int_\Omega \left(D_3^0-\mathfrak{D}_3\right)E_3^\phi(\hat{\varphi})d\boldsymbol{x} = 0.$$

From (5.41) one has

$$\begin{aligned}-b_H(((\boldsymbol{\tau},\boldsymbol{d}),(\boldsymbol{u}(h)-\boldsymbol{u}^0,\varphi(h)-\varphi^0)) &= a_{H,0}\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right)\\
&+h^2a_{H,2}\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right) + h^4a_{H,4}\left((\boldsymbol{\sigma}(h),\boldsymbol{D}(h)),(\boldsymbol{\tau},\boldsymbol{d})\right)\\
&-\int_\Omega \tau_{33}\,e_{33}(\boldsymbol{u}^0)o(h)\,d\boldsymbol{x} - \int_\Omega d_3\,E_3(\varphi^0)o(h)\,d\boldsymbol{x}.\end{aligned} \tag{5.264}$$

From inequalities (5.60), (5.89)-(5.90) and (5.117) we obtain

$$
\begin{aligned}
&- b_H \left(((\boldsymbol{\tau}, \boldsymbol{d}), \left(\boldsymbol{u}(h) - \boldsymbol{u}^0, \bar{\varphi}(h) - \bar{\varphi}^0\right)\right) \\
&\quad \leq (C_1 + h^2 C_0(\phi)) \left\{ \begin{array}{c} |D_3(h) - D_3^0|_{0,\Omega}^2 + |\sigma_{33}(h) - \sigma_{33}^0|_{0,\Omega}^2 \\ +h^2 |D_\alpha(h)|_{0,\Omega}^2 + h^2 |\sigma_{\alpha 3}(h)|_{0,\Omega}^2 + h^4 |\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \end{array} \right\}^{1/2} \|(\boldsymbol{\tau}, \boldsymbol{d})\|_{\mathbf{X}_1} \\
&\qquad + (C_2 + h^2 C_0(\phi)) \|(\boldsymbol{\tau}, \boldsymbol{d})\|_{\mathbf{X}_1},
\end{aligned}
$$

which together with the Babuška-Brezzi (inf-sup) condition allows us to prove the existence the existence of $D_1(\phi)$ such that

$$
\begin{aligned}
&\sup_{(\boldsymbol{\tau}, \boldsymbol{d}) \in \boldsymbol{X}_1} \frac{|-b_H \left((\boldsymbol{\tau}, \boldsymbol{d}), (\boldsymbol{u}(h) - \boldsymbol{u}^0, \varphi(h) - \varphi^0)\right)|}{\|(\boldsymbol{\tau}, \boldsymbol{d})\|_{\boldsymbol{X}_1}} \leq D_1(\phi) \left\{ \left|D_3(h) - D_3^0\right|_{0,\Omega}^2 \right. \\
&\quad \left. + \left|\sigma_{33}(h) - \sigma_{33}^0\right|_{0,\Omega}^2 + h^2 |D_\alpha(h)|_{0,\Omega}^2 + h^2 |\sigma_{\alpha 3}(h)|_{0,\Omega}^2 + h^4 |\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \right\} + D_1(\phi),
\end{aligned}
$$

if $0 < h \leq h_1$. Now, putting $(\boldsymbol{\tau}, \boldsymbol{d}) = (e_{33}^{\phi}(\boldsymbol{u}(h) - \boldsymbol{u}^0), E_3^{\phi}(\varphi(h) - \varphi^0))$ in the previous inequality and applying Korn's and Poincaré's inequalities, and evoking inequalities (5.127) and (5.128), we obtain

$$
\begin{aligned}
\left\|\left(\boldsymbol{u}(h) - \boldsymbol{u}^0, \varphi(h) - \varphi^0\right)\right\|_{\boldsymbol{X}_{2,w}} &\leq D_1(\phi) \left\{ \left|D_3(h) - D_3^0\right|_{0,\Omega}^2 + \left|\sigma_{33}(h) - \sigma_{33}^0\right|_{0,\Omega}^2 \right. \\
&\quad \left. + h^2 |D_\alpha(h)|_{0,\Omega}^2 + h^2 |\sigma_{\alpha 3}(h)|_{0,\Omega}^2 + h^4 |\sigma_{\alpha\beta}(h)|_{0,\Omega}^2 \right\} + D_1(\phi),
\end{aligned}
$$

when $h \to 0$. In this way, we finished to prove that $(\boldsymbol{u}(h))_{h>0}$ and $(\varphi(h))_{h>0}$ converge strongly. ■

5.6 The limit scaled one-dimensional problem: existence and uniqueness of the solution

Substituting equations (5.248) and (5.249) in the variational equation (5.253) we have

$$
\begin{aligned}
&\int_\Omega A_{33}^d \left(\bar{\varepsilon}_{33} e_{33}^{\phi}(\boldsymbol{u}^0) - \bar{P}_{333} E_3^{\phi}(\varphi^0)\right) e_{33}^{\phi}(\boldsymbol{v}) d\boldsymbol{x} \\
&\quad + \int_\Omega A_{33}^d \left(\bar{P}_{333} e_{33}^{\phi}(\boldsymbol{u}^0) + \bar{C}_{3333} E_3^{\phi}(\varphi^0)\right) E_3^{\phi}(\psi) d\boldsymbol{x} \\
&= \int_\Omega f_i v_i d\boldsymbol{x} + \int_{\Gamma_N} g_i v_i d\Gamma + \int_{\Gamma_L} p_i v_i d\Gamma, \qquad (\boldsymbol{v}, \psi) \in V_{BN}^{\phi} \times \Psi_3^{\phi},
\end{aligned} \tag{5.265}
$$

as the equation to be satisfied by $(\boldsymbol{u}^0, \varphi^0) \in V_{BN}^{\phi} \times (\hat{\varphi} + \Psi_3^{\phi})$

Lemma 15 *The problem (5.265) has one and only one solution.*

Proof. In order to prove the ellipticity of the bilinear form of the problem (5.265) we take into account Korn's inequality: there exists a constant $c > 0$ such that (cf. (5.104))

$$|e^{\phi}(\mathbf{v})|_{\Omega} = |e^{\phi}_{33}(\mathbf{v})|_{\Omega} \geq c(\phi)\|\mathbf{v}\|_{1,\Omega}, \quad \text{for all } \mathbf{v} \in V^{\phi}_{BN}. \tag{5.266}$$

The properties of the coefficients $\bar{\varepsilon}_{33}$ and $\bar{C}_{3333}$ and the inequality (5.114) together guarantee that

$$\int_{\Omega} A^d_{33}\bar{\varepsilon}_{33}(e^{\phi}_{33}(\mathbf{v}))^2\, d\mathbf{x} + \int_{\Omega} A^d_{33}\bar{C}_{3333}\,(E^{\phi}_3(\psi))^2 d\mathbf{x} \geq c_1|e^{\phi}_{33}(\mathbf{v})|^2_{\Omega} + c_2|E^{\phi}_3(\psi)|^2_{\Omega}$$

$$\geq c(\phi)\left(\|\mathbf{v}\|^2_{1,\Omega} + \|\psi\|^2_{1,\Omega}\right),$$

for all $(\mathbf{v}, \psi) \in V^{\phi}_{BN} \times \Psi^{\phi}_3$.

Using Stampacchia's Theorem and Lax-Milgramm Lemma, the existence and uniqueness of solution of problem (5.265) is guaranteed. ■

Putting now $\xi_3 = 0$, $\psi = 0$ and $\zeta_\alpha = 0$, $\psi = 0$ successively in (5.265) and taking into account the relations (5.195)-(5.200) we get the following result.

Corollary 25 *The element* $(\xi_1, \xi_2, \xi_3, z_3) \in [V^2_0(0,L)]^2 \times V^1_0(0,L) \times (\hat{\varphi} + H^1_0(0,L))$ *is the unique solution of the following coupled variational problem:*

$$\left\{\begin{array}{l} \textit{Find } \xi_\beta \in V^2_0(0,L), \textit{ such that} \\ \displaystyle\int_0^L A^d_{33}\bar{\varepsilon}_{33}I_{\alpha\beta}\xi''_\alpha\zeta''_\beta dx_3 + \int_0^L A^d_{33}\left\{\bar{\varepsilon}_{33}\left(\xi'_3 + \phi'_\beta\xi'_\beta\right) + \bar{P}_{333}z'_3\right\} A\phi'_\beta\zeta'_\beta dx_3 \\ \displaystyle = \int_0^L F_\alpha\zeta_\alpha dx_3 - \int_0^L M_\alpha\zeta'_\alpha dx_3 + F^L_\alpha\zeta_\alpha(L) - M^L_\alpha\zeta'_\alpha(L), \\ \textit{for all } \zeta_\beta \in V^2_0(0,L), \end{array}\right. \tag{5.267}$$

$$\left\{\begin{array}{l} \textit{Find } (\xi_3, z_3) \in V^1_0(0,L) \times (\hat{\varphi} + H^1_0(0,L)) \textit{ such that} \\ \displaystyle\int_0^L A^d_{33}\left\{\bar{\varepsilon}_{33}\left(\xi'_3 + \phi'_\beta\xi'_\beta\right) + \bar{P}_{333}z'_3\right\} A\zeta'_3 dx_3 \\ \displaystyle - \int_0^L A^d_{33}\left\{\bar{P}_{333}\left(\xi'_3 + \phi'_\beta\xi'_\beta\right) - \bar{C}_{3333}z'_3\right\} Aq'_3 dx_3 \\ \displaystyle = \int_0^L F_3\zeta_3 dx_3 + F^L_3\zeta_3(L), \\ \textit{for all } (\zeta_3, q_3) \in V^1_0(0,L) \times H^1_0(0,L), \end{array}\right. \tag{5.268}$$

where the transversal resultants are defined by

$$F_i = \int_\omega f_i d\omega + \int_{\gamma_N} g_i d\gamma, \quad M_\beta = \int_\omega \chi_\beta^b f_3 d\omega + \int_{\gamma_N} \chi_\beta^b g_3 d\gamma,$$
$$F_i^L = \int_\omega p_i d\omega, \quad M_\beta^L = \int_\omega \chi_\beta^b p_3 d\omega,$$
$$I_{\alpha\beta} = \int_\omega \chi_\alpha^b \chi_\beta^b d\omega.$$

5.7 The one-dimensional equations of a transversely anisotropic shallow arch; formulation as a boundary value problem

In this section our goal is to give an approach to the piezoelectric shallow arch occupying the volume $\{\Omega^h\}^-$. In view of the scalings

$$\zeta_\alpha^h(x_3) := h^{-1}\zeta_\alpha(x_3), \qquad \zeta_3^h(x_3) := \zeta_3(x_3), \qquad z_3^h(x_3) := z_3(x_3), \quad \text{in } \omega,$$

$$\breve{\boldsymbol{u}}^h(0)(\breve{\boldsymbol{x}}^h) := (h^{-1}\, u_\alpha^0(\boldsymbol{x}), u_3^0(\boldsymbol{x})),$$

$$\breve{\varphi}^h(0)(\breve{\boldsymbol{x}}^h) := \varphi^0(\boldsymbol{x}),$$

$$(\breve{\sigma}_{\alpha\beta}^h(0)(\breve{\boldsymbol{x}}^h), \breve{\sigma}_{3\alpha}^h(0)(\breve{\boldsymbol{x}}^h), \breve{\sigma}_{33}^h(0)(\boldsymbol{x}^h)) := (h^2\sigma_{\alpha\beta}^0(\boldsymbol{x}), h\sigma_{3\alpha}^0(\boldsymbol{x}), \sigma_{33}^0(\boldsymbol{x})),$$

$$(\breve{D}_\alpha^h(0)(\breve{\boldsymbol{x}}^h), \breve{D}_3^h(0)(\boldsymbol{x}^h)) := (hD_\alpha^0(\boldsymbol{x}), D_3^0(\boldsymbol{x})),$$

for all $\breve{\boldsymbol{x}}^h = (\breve{x}_1^h, \breve{x}_3^h, \breve{x}_3^h)$, where the mappings $\Phi^h : \bar{\Omega} \to \bar{\Omega}^h$ and $\boldsymbol{\Theta}^h : \bar{\Omega}^h \to \{\breve{\Omega}^h\}^{-1}$ are those defined in Sections 5.2.1 and 5.3, we obtain the following corollary.

Corollary 26 *(a) The de-scaled functions are given by*

$$\breve{u}_\alpha^h(0)(\breve{\boldsymbol{x}}^h) = \xi_\alpha^h, \quad \xi_\alpha^h \in V_0^2(0, L), \tag{5.269}$$

$$\breve{u}_3^h(0)(\breve{\boldsymbol{x}}^h) = \xi_3^h - \chi_\alpha^{b,h}\xi_\alpha^{h,\prime}, \quad \xi_3^h \in V_0^1(0, L), \tag{5.270}$$

$$\breve{\varphi}^h(0)(\breve{\boldsymbol{x}}^h) = z_3^h, \quad z_3^h \in H^1(0, L), \quad z_3^h(0) = \breve{\varphi}_0^{0,h} \quad z_3^h(L) = \breve{\varphi}_0^{L,h} \tag{5.271}$$

$$\breve{\sigma}^h_{\alpha\beta}(0) = \breve{\sigma}^h_{3\alpha}(0) = 0, \tag{5.272}$$

$$\breve{\sigma}^h_{33}(0) = A^{d,h}_{33}\left\{\bar{\varepsilon}^h_{33}\left[(\xi^h_3)' - \chi^{b,h}_\alpha(\xi^h_\alpha)'' + (\phi^h_\alpha)'(\xi^h_\alpha)'\right] + \bar{P}^h_{333} z^{h,\prime}_3\right\}, \tag{5.273}$$

$$\breve{D}^h_3(0) = A^{d,h}_{33}\left\{\bar{P}^h_{333}\left[(\xi^h_3)' - \chi^{b,h}_\alpha(\xi^h_\alpha)'' + (\phi^h_\alpha)'(\xi^h_\alpha)'\right] - \bar{C}^h_{3333} z^{h,\prime}_3\right\}, \tag{5.274}$$

for every point $\breve{\boldsymbol{x}}^h = \boldsymbol{\Theta}\left(\boldsymbol{x}^h\right)$.

(b) The mechanical and electrical field $(\xi^h_i, z^{h,0}_3)$ *is the solution of the following one-dimensional boundary problems (cf. (5.267)):*

$$\begin{cases} \left(m^{h,0}_\beta\right)'' - \left(n^{h,0}_3\phi'_\beta\right)' = F^h_\beta + \left(M^h_\beta\right)', & \textit{in } (0,L), \\ \xi^h_\beta(0) = (\xi^h_\beta)'(0) = 0, & \\ -m^{h,0}_\beta(L) = -M^{h,L}_\beta, & \\ -C^{h,*}_{33} I^h_{\alpha\beta}(\xi^h_\alpha)'''(L) + n^{h,0}_3(L)(\phi^h_\alpha)'(L) = F^h_\alpha - M^h_\alpha(L) & \end{cases}$$

$$\begin{cases} -\left(n^{h,0}_3\right)' + \left(d^{h,0}_3\right)' = F^h_3, & \textit{in } (0,L), \\ \xi^h_3(0) = 0, \quad z^h_3(0) = \breve{\varphi}^{0,h}_0, \quad z^h_3(L) = \breve{\varphi}^{L,h}_0, & \\ n^{h,0}_3(L) = F^{h,L}_3, \qquad d^{h,0}_3(0) = d^{h,0}_3(L) = 0 & \end{cases}$$

where

$$\begin{aligned}
n^{h,0}_3 &= \int_{\omega^h} \breve{\sigma}^h_{33}(0)\, d\omega^h = A^h\left\{C^{h,*}_{33}\left[(\xi^h_3)' + (\phi^h_\beta)'(\xi^h_\beta)'\right] + P^{h,*}_3 z^{h,\prime}_3\right\}, \\
m^{h,0}_\beta &= \int_{\omega^h} \chi^{b,h}_\beta\, \breve{\sigma}^h_{33}(0)\, d\omega^h = -C^{h,*}_{33} I^h_{\alpha\beta}(\xi^h_\alpha)'', \\
d^{h,0}_3 &= \int_{\omega^h} \breve{D}^h_3(0)\, d\omega^h = A^h\left\{P^{h,*}_3\left[(\xi^h_3)' + (\phi^h_\beta)'(\xi^h_\beta)'\right] - \varepsilon^{h,*}_3 z^{h,\prime}_3\right\}, \\
C^{h,*}_{33} &= \frac{\bar{\varepsilon}^h_{33}}{\bar{C}^h_{3333}\bar{\varepsilon}^h_{33} + \bar{P}^h_{333}\bar{P}^h_{333}} = A^{d,h}_{33}\bar{\varepsilon}^h_{33} \\
\varepsilon^{h,*}_3 &= \frac{\bar{C}^h_{3333}}{\bar{C}^h_{3333}\bar{\varepsilon}^h_{33} + \bar{P}^h_{333}\bar{P}^h_{333}} = A^{d,h}_{33}\bar{C}^h_{3333}, \\
P^{h,*}_3 &= \frac{\bar{P}^h_{333}}{\bar{C}^h_{3333}\bar{\varepsilon}^h_{33} + \bar{P}^h_{333}\bar{P}^h_{333}} = A^{d,h}_{33}\bar{P}^h_{333},
\end{aligned}$$

and

$$\begin{aligned}
&I^h_{\alpha\beta} = \int_{\omega^h} \chi^{b,h}_\alpha \chi^{b,h}_\beta \, d\omega^h, \\
&F^h_i(x^h_3) = \int_{\omega^h} f^h_i d\omega^h + \int_{\gamma^h_N} g^h_i d\gamma^h, \quad M^h_\alpha(x^h_3) = \int_{\omega^h} \chi^h_\alpha f^h_3 \, d\omega^h + \int_{\gamma^h_N} \chi^h_\alpha g^h_3 \, d\gamma^h, \\
&F^{L,h}_i(x^h_3) = \int_{\omega^h} p^h_i d\omega^h, \quad M^{L,h}_\alpha(x^h_3) = \int_{\omega^h} \chi^h_\alpha p^h_3 \, d\omega^h, \\
&\chi^{b,h}_1 = x^h_1 b_1(x_3) - x^h_2 b_2(x_3), \quad \chi^{b,h}_2 = x^h_1 b_2(x_3) + x^h_2 b_1(x_3).
\end{aligned}$$

A major conclusion is thus that we have been able to rigorously justify one-dimensional equations for piezoelectric shallow arches by showing that (up to appropriate scalings) their solution can be identified (in the sense of Corollary 26(a)) with the $[H^1(\Omega)]^3 \times H^1(\Omega)$-limit of the three-dimensional solution as the diameter of the cross section of the beam approaches zero.

We observe, in first place, that if we ignore the electric field considering a linearly elastic beam, whose material satisfies the conditions

$$\varepsilon^{h,*}_{33} = P^{h,*}_3 = 0, \qquad C^{h,*}_{33} = \frac{1}{\bar{C}^h_{3333}},$$

the one-dimensional model found here does indeed coincide with the linearly elastic shallow arch model derived by Álvarez-Dios & Viaño [1998]. On the other hand, we note that our asymptotic shallow arch model also coincides with asymptotic straight rod model derived in Chapter 3, taking the subclass *6mm* of the anisotropic piezoelectric material.

Chapter 6

Conclusions and future research

The primary goal of this research work was to develop mathematical lower-dimensional models for anisotropic piezoelectric beams. This chapter summarizes the conclusions of the work and suggests possible areas of future research.

6.0.1 Conclusions

The present research proposes three asymptotic models for anisotropic piezoelectric beams submitted to an electric potential. In the three cases, the procedure starts with a change of variable that is used to establish the original 3-D piezoelectricity problem in a fixed reference domain, which does not depend on the diameter of the beam cross section Then, the asymptotic method introduced by Lions (Lions [1973]) is used to perform a mathematically rigorous dimensional reduction.

The main contributions of the present work towards the development of reduced anisotropic beam models are the following:

- An interesting approach to the three-dimensional piezoelectricity problem for an anisotropic beam of class 2 with one end fixed and subjected to an electric potential applied on the right and left ends of the beam has been presented in the thesis. In this approach the piezoelectric beam is modeled as a one-dimensional domain. It is required that the displacements satisfy a weak clamping boundary condition to avoid the boundary layer phenomenon. The expression of the displacement second order term has been used to identify the limit model and to prove the strong convergence result, see Section 3.4.0.1 and Section 3.4.1.

- An asymptotically piezoelectric beam model for anisotropic materials of class 2 in response to an applied electric potential acting on the lateral surface has been constructed, see Chapter 4. The weak boundary condition has eliminated the difficulties commonly associated with the characterization of the higher-order terms of the displacements (see Theorem 9). Furthermore, two additional functions (the warping function is one of them) turned out to be necessary to write the expression of the first-order term of the axial displacement, according to Theorem 11.

- We have reduced the three-dimensional piezoelectricity problem to the zeroth-order model (Corollary 25) for a transversely isotropic piezoelectric shallow arch subjected to an electric potential applied at both ends of the beam by considering a *displacement - stress - electric potential - electric displacement* asymptotic expansion (see Section 5.7).

6.0.2 Some perfectives for future research

The present work can be considered as a starting point for the derivation of models for anisotropic piezoelectric beams using the Asymptotic Method as a tool. Moreover, there are still vast uncharted areas where the asymptotic method would be adequate to justify lower-dimensional models in a rigorous way. To complete and extend the present research, a number of future tasks could be performed:

- To complete our analysis, the most urgent work is to prove that, for the anisotropic piezoelectric beam of class 2, the displacement vector field and the electric potential weakly (and strongly) converge towards the leading terms of the *electric potential* expansions.

- Study the contribution of higher-order terms (correctors) of the asymptotic *displacement - electric potential* expansions in order to allow the construction of higher order models.

- To focus on the development of numerical experiments to validate the efficiency of the asymptotic models obtained and to compare the corresponding results with other existing approximations.

- There are various engineering applications that use piezoelectric layers for the vibration control in beams. The models used to describe the behavior of the materials are generally obtained by ad hoc assumptions. To justify these assumptions it is recommended to develop and justify in a rigorous way lower-dimensional models of sandwich piezoelectric beams.

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