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Hugo Morais Martins **Partial classical propositional logic**

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Abstract

Kochen and Specker developed in the 1960s an alternative to Birkhoff and von Neumann's quantum logic based on partial Boolean algebras, called partial classical propositional logic, which has been recently revisited in studies of contextuality. Unlike more common quantum logics, in the language of the logic studied here, a new symbol is added to express a relation of commeasurability or compatibility. Semantically, the binary connectives are partial functions, with the logical value of a connective defined only for compatible propositions.

This dissertation explores partial algebras, partial Boolean algebras and the concept of validity that they originate, comparing the notions of validity in this logic with those in classical propositional logic. The logical calculus of Kochen and Specker, which axiomatizes validity in partial classical propositional logic, is also studied. The theorems of soundness and completeness are proven, establishing an equivalence between both ways of characterizing the validity of this logic.

Keywords partial classical propositional logic, partial algebras, partial Boolean algebras, quantum logic

Resumo

Kochen e Specker desenvolveram nos anos 60 uma alternativa à lógica quântica de Birkhoff e von Neumann baseada em álgebras Booleanas parciais, a lógica clássica proposicional parcial, recentemente revisitada em estudos de contextualidade. Contrariamente às lógicas quânticas mais comuns, à linguagem da lógica aqui estudada adiciona-se um novo símbolo, para exprimir uma relação de comensurabilidade ou compatibilidade. A nível semântico, os conetivos binários são funções parciais, estando o valor lógico de um conetivo definido apenas para proposições compatíveis.

Nesta dissertação estudam-se as álgebras parciais, as álgebras Booleanas parciais e a noção de validade que originam e comparam-se as noções de validade desta lógica com a noção de validade da lógica clássica proposicional. Estuda-se também o cálculo lógico de Kochen e Specker que axiomatiza a validade na lógica clássica proposicional parcial. Demonstram-se os teoremas da correção e da completude, o que estabelece uma equivalência entre ambas as formas de caracterizar a validade desta lógica.

Palavras-chave lógica clássica proposicional parcial, álgebras parciais, álgebras Booleanas parciais, lógica quântica

Contents

Chapter 1

Introduction

The term “quantum logic” first appeared in the book “Mathematische Grundlagen der Quantenmechanik” (Mathematical Foundations of Quantum Mechanics), released in 1932 by John von Neumann, but only in 1936, in a paper published by Garret Birkhoff and von Neumann, this ideia was fully established, where a systematic attempt is made to propose a propositional calculus for quantum logic [5]. Approximately a decade passed before mathematicians and philosophers began showing interest in quantum logic, as this concept was very difficult to understand solely based on the 1936 Birkhoff and von Neumann paper [5].

There are multiple “quantum logics” that have been studied, with some of the most well known ones being the orthologic and the orthomodular quantum logic. The algebraic semantics of the orthologic is based on ortholattices and of the orthomodular quantum logic on orthomodular lattices [4]. For the purpose of this dissertation, both of these quantum logics will consist of a set of atomic formulas and of two primitive connectives: $\neg$ (not) and $\vee$ (or). The notion of formula of the language is defined in the expected way (which will be formally defined in the preliminaries) and the connective conjunction is supposed defined via de Morgan's law: $\alpha \wedge \beta = \neg(\neg\alpha \vee \neg\beta)$ [4].[1]

During the 1960s, Kochen and Specker [9, 8] developed alternatives to the quantum logic proposed by Birkhoff and von Neumann [3, 4] based on partial Boolean algebras. So, in a general overview, this dissertation will focus on a specific type of quantum logic, named *partial* classical propositional logic, because it operates with a relation of compatibility of sentences. This means that the operations involved may not always be defined for all formulas, leading to situations where sentence meanings are not necessarily always defined. This logic is termed *partial classical propositional* logic due to the existence of certain sentence (from propositional calculus) groupings with defined meanings (compatibilities) exhibiting classical behavior (this concept will be further explained in the second and third chapters). Consequently, some classical laws, such as distributivity, violated in more common quantum logics, become logical truths within this context [4]. This new quantum logic is distinct from the most common ones as those logics pos-

[1] In the cited book, the two primitive connectives used were $\neg$ (not) and $\wedge$ (conjunction). Therefore, via de Morgan's law, we have: $\alpha \vee \beta = \neg(\neg\alpha \wedge \neg\beta)$.

sess a language that is closed under logical connectives, ensuring every sentence holds meaning within the algebraic semantics. The used connectives and the way formulas are constructed is similar to the ones of orthologic and orthomodular quantum logic. However, in order to construct them, compatibilities are essential to ensure the meaningfulness of the formulas.

Partial Boolean algebras have been recently revisited in the context of studying contextuality [1], which is an ingredient associated to the advantages of quantum computing [7]. Their appearance is related to the need of modelling situations in which the traditional laws of Boolean algebras don't fully apply, that is, they were developed to handle cases where the principles of classical Boolean algebra are not true. This new approach generalizes the classical Boolean algebra to include partial information.

In order to fully understand the previous concepts, it is crucial to know the most basic structures from which these originated. So, it is essential to study partial algebras, which are amongst the primary mathematical structures implemented, for instance, on computers [2]. Partial functions are being used in mathematics for a long time, such as partial subtraction for natural numbers, partial division for integers, partial multiplicative inversion in arbitrary fields, partial recursive functions in computability theory, etc [2].

The impulse to investigate partial algebras was strengthened within the context of the software crisis in computer science [2]. Simultaneously, computer scientists began to recognize the potency of universal algebra as a good language and theoretical framework in computer science for dealing, for example, with abstract data types and with programming languages and their semantics [2]. It was also noted that numerous structures, perhaps even the majority, in the scope of computer science, are partial. Specifically, due to the fact that computers are only capable of realizing and processing information of finite parts of structures (which are usually infinite), almost every implementation of a computer program represents a partial algebraic structure [2].

So, the research undertaken in this thesis centered on the exploration of partial algebras as well as partial Boolean algebras and the notion of validity they originate. Additionally, it was investigated the logical calculus of Kochen and Specker that axiomatizes this notion of validity [4].

This dissertation has the following structure:

The second chapter, titled “Preliminaries”, will present fundamental definitions and theorems which are going to be useful to demonstrate important results in subsequent chapters.

The third chapter, “Partial algebras”, will elaborate on the nature of partial algebras. Within this context, we will introduce the concept of identity, illustrating certain identities holding in all partial algebras and contrasting them with those that do not. Introducing the concept of “commeasurability” (or compatibility), we will understand its significance as it is crucial to enable operations on the elements in the partial algebras through specific functions. We will also explore the meaning of polynomials and their domain within this context.

In the fourth chapter, titled "Partial Boolean algebras", we will present two ways of approaching partial Boolean algebras. One of them involves the use of partial algebras and the other one is independently of them. Within this context, we will explore the meaning of Boolean polynomials and their respective domain, and understand the relation between the polynomials (of a partial algebra) and the Boolean polynomials (of a partial Boolean algebra).

In the fifth chapter, "Partial classical propositional logic", we are going to start by explaining the meaning of Q-valid formulas and present some examples and counterexamples, and demonstrate a result involving Q-valid and C-valid formulas. Additionally, through the introduction of a new syntactic system as well as new definitions in order to understand the concept of a Q-proof of a formula, we will be able to state and prove the soundness and completeness theorems.

Finally, in the last chapter, we will draw some conclusions of these investigations and outline future research that could be undertaken in order to complement the existing studies.

Chapter 2

Preliminaries

The following section will be essential to demonstrate and/or understand the theory presented in the next chapters. Therefore, this chapter will involve classical logic and algebraic concepts and basic results as well as graph theory.

2.1 Algebra over a field and its properties

For the subsequent definitions, we will be following the book [11].

Definition 2.1.1. *Let $\mathbb{K}$ be a field and V be a nonempty set equipped with two binary functions:*

$$+ \colon V \times V \longrightarrow V \qquad \bullet \colon \mathbb{K} \times V \longrightarrow V$$
$$(x, y) \longmapsto x + y \qquad (a, y) \longmapsto a \bullet y$$

We will call the first function addition and the second one scalar multiplication. In order to simplify the notation, we will denote $a \bullet y$ as ay.

We say that V, along with the two previous operations, is a vector space over the field $\mathbb{K}$ if the following properties are satisfied:

1. *For all $x, y \in V$, $x + y = y + x$ (Commutativity of vector addition)*
2. *For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ (Associativity of vector addition)*
3. *There exists* **0** *$\in V$ such that, for all $x \in V$,* **0** *$+ x = x +$* **0** *$= x$ (Identity element of vector addition)*
4. *For all $x \in V$, there exists $-x \in V$ such that $x + (-x) = -x + x =$* **0** *(Symmetric elements of vector addition)*

5. *For all $a, b \in \mathbb{K}$ and for all $x \in V$, $(a+b)x = ax + bx$ (Distributivity of scalar multiplication with respect to field addition)*

6. *For all $a \in \mathbb{K}$ and for all $x, y \in V$, $a(x+y) = ax + ay$ (Distributivity of scalar multiplication with respect to vector addition)*

7. *For all $a, b \in \mathbb{K}$ and for all $x \in V$, $(a \times b)x = a(bx)$, where the operation $\times$ is defined in the field $\mathbb{K}$ (Compatibility of scalar multiplication with field multiplication)*

8. *For all $x \in V$, $1x = x$, where 1 is the multiplicative identity element of the field $\mathbb{K}$ (Identity element of scalar multiplication)*

We will denote vectores as x, y and z. Some direct consequences from the previous properties are:

- For all $x \in V$, $0x = \mathbf{0}$, where $0 \in \mathbb{K}$ and $\mathbf{0} \in V$
- For all $a \in \mathbb{K}$, $a\mathbf{0} = \mathbf{0}$, where $\mathbf{0} \in V$
- For all $x \in V$, $(-1)x = -x$, where $-1 \in \mathbb{K}$
- For all $a \in \mathbb{K}$ and for all $x \in V$, if $ax = \mathbf{0}$, then $a = 0$ or $x = \mathbf{0}$, where $0 \in \mathbb{K}$ and $\mathbf{0} \in V$

Definition 2.1.2. *Let V be a vector space over a field $\mathbb{K}$ equipped with an additional operation $\cdot$ defined from $V \times V$ to V. V is an algebra over $\mathbb{K}$ if it is a vector space that satisfies:*

1. *For all $x, y, z \in V$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Associativity)*

2. *For all $x, y, z \in V$, $(x + y) \cdot z = x \cdot z + y \cdot z$ (Right distributivity)*

3. *For all $x, y, z \in V$, $z \cdot (x + y) = z \cdot x + z \cdot y$ (Left distributivity)*

4. *For all $x, y \in V$ and for all $a, b \in \mathbb{K}$, $(ax) \cdot (by) = (a \times b)(x \cdot y)$, where $\times$ is defined in the field $\mathbb{K}$ (Compatibility with scalars)*

A binary operation is bilinear if it satisfies the last three properties.

Observation: Some authors do not consider the associativity property as being part of the definition of an algebra over a field.

Definition 2.1.3. *Let $\mathbb{K}$ be a field. V is a commutative algebra over $\mathbb{K}$ if V is an algebra over $\mathbb{K}$ and the operation $\cdot$ is commutative, that is,*

$$\text{For all } x, y \in V,\ x \cdot y = y \cdot x.$$

2.2 Lattices and their properties

Definition 2.2.1. *A lattice is a structure* $(B, \vee, \wedge)$ *where* B *is a nonempty set and the operations* $\vee$ *and* $\wedge$ *are defined from* $B \times B$ *to* B *and the following properties are satisfied: For all* $a, b, c \in B$*,*

(i) $a \vee a = a \wedge a = a$ *(Idempotency)*

(ii) $a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$ *(Commutativity)*

(iii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *and* $(a \vee b) \vee c = a \vee (b \vee c)$ *(Associativity)*

(iv) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ *(Absorption)*

We call $\vee$ *as supremum and* $\wedge$ *as infimum.*

Definition 2.2.2. *Let* A *be a set and* $\leq$ *a binary relation in* A*. One says that* $\leq$ *is a partial order relation in* A *if the following properties are satisfied: For all* $a, b, c \in A$*,*

(i) $a \leq a$ *(Reflexivity)*

(ii) ($a \leq b$ *and* $b \leq a$*)* $\Rightarrow a = b$ *(Antisymmetry)*

(iii) ($a \leq b$ *and* $b \leq c$*)* $\Rightarrow a \leq c$ *(Transitivity)*

We call the pair $(A, \leq)$ *a partially ordered set (poset).*

Definition 2.2.3. *A lattice* $(B, \vee, \wedge)$ *is distributive iff one of the following properties hold:*

For all $a, b, c \in B$

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \wedge c)$

Observations: Let us consider a lattice $(B, \vee, \wedge)$. For all $a, b \in B$, we have the following statements:

(i) $a \wedge b = a$ iff $a \vee b = b$

(ii) The relation $\leq$ is defined such that $a \leq b$ iff $a \wedge b = a$

(iii) $\leq$ is a partial order relation and, consequently, we have that $(B, \leq)$ is a poset

(iv) Let us consider a poset $(B, \leq)$ such that, for all $a, b \in B$, there exists $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$, where $\bigvee\{a, b\}$ denotes the supremum of $\{a, b\}$ and $\bigwedge\{a, b\}$ denotes the infimum of $\{a, b\}$. Then, $(B, \leq)$ is a lattice, where $\bigvee\{a, b\} = a \vee b$ and $\bigwedge\{a, b\} = a \wedge b$.

2.3 Classical Logic

2.3.1 Boolean algebra and its properties

Definition 2.3.1. *A Boolean algebra is a structure $\mathcal{B} = (B, \vee, \wedge, \neg, \mathbf{1}, \mathbf{0})$, where $\vee$ and $\wedge$ are two binary operations on B, $\neg$ is a unary operation on B, $\mathbf{1} \in B$ and $\mathbf{0} \in B$, such that:*

(i) $(B, \vee, \wedge)$ *is a distributive lattice*

(ii) $a \wedge \mathbf{0} = \mathbf{0}$ *and* $a \vee \mathbf{1} = \mathbf{1}$*, for all* $a \in B$ *(which is equivalent to* $a \vee \mathbf{0} = a$ *and* $a \wedge \mathbf{1} = a$*, respectively)*

(iii) $a \wedge \neg a = \mathbf{0}$ *and* $a \vee \neg a = \mathbf{1}$*, for all* $a \in B$

We call the operation $\neg$ complement and for $a \in B$ we say that $\neg a$ is the complement of a.

Lemma 2.3.2. *Let $\mathcal{B} = (B, \vee, \wedge, \neg, \mathbf{1}, \mathbf{0})$ be a Boolean algebra. Then, we have the following properties: For all $a, b \in B$,*

1. $\neg\neg a = a$
2. $\neg\mathbf{0} = \mathbf{1}$ *and* $\neg\mathbf{1} = \mathbf{0}$
3. $\neg(a \vee b) = \neg a \wedge \neg b$ *and* $\neg(a \wedge b) = \neg a \vee \neg b$

It is convenient to rephrase the usual truth-table semantics of classical propositional formulas, defined so that each connective is seen as an operation acting on the set $\{0, 1\}$ [13], as a semantics based on the following specific Boolean algebra:

Definition 2.3.3. *Let $\mathbb{B} = (\{0, 1\}, \vee, \wedge, \neg, \mathbf{1}, \mathbf{0})$ be a Boolean algebra, where the operations are defined in the usual way by the truth tables. One calls $\mathbb{B}$ the Boolean algebra of truth values. [12]*

2.3.2 Propositional Logic

Definition 2.3.4. *Let $n \in \mathbb{N}$. Σ_n is the set of formulas of the propositional calculus in the variables $x_1, \cdots, x_n$ and the connectives $\vee$ and $\neg$, defined inductively by:*

1. $x_i \in \Sigma_n$*, for* $i \in \{1, \cdots, n\}$
2. *If* $\alpha \in \Sigma_n$*, then* $(\neg\alpha) \in \Sigma_n$

3. *If $\alpha, \beta \in \Sigma_n$, then $(\alpha \vee \beta) \in \Sigma_n$*

We will call $\mathcal{X}$ to the set of all propositional variables x_i, that is, $\mathcal{X} = \{x_i \colon i \in \mathbb{N}\}$.

The set of all formulas of propositional calculus is $\Sigma = \bigcup_{n \in \mathbb{N}} \Sigma_n$.

Observations:

- In general, parentheses will be omitted in a formula when it does not cause ambiguity.
- The lower case letters of the Greek alphabet α, β, γ and θ will be used to denote formulas of the propositional calculus.

Definition 2.3.5. *A valuation in a Boolean algebra $\mathcal{B} = (B, \vee, \wedge, \neg, \mathbf{1}, \mathbf{0})$ is any map v from the set of propositional variables $\mathcal{X}$ to B.*

The value of a formula $\alpha \in \Sigma_n$ with respect to a valuation v, $\overline{\alpha}(v)$, is defined by recursion:

(i) $\overline{x_i}(v) = v(x_i)$, *for all $i \in \mathbb{N}$*

(ii) $\overline{\neg\alpha}(v) = \neg\overline{\alpha}(v)$, *for all $\alpha \in \Sigma_n$*

(iii) $\overline{\alpha \vee \beta}(v) = \overline{\alpha}(v) \vee \overline{\beta}(v)$, *for all $\alpha, \beta \in \Sigma_n$*

One writes:

- $\mathcal{B}, v \models \alpha$ when $\overline{\alpha}(v) = \mathbf{1}$ in the Boolean algebra $\mathcal{B}$;
- $\mathcal{B} \models \alpha$ when $\mathcal{B}, v \models \alpha$, for all v valuation in the Boolean algebra $\mathcal{B}$.

Definition 2.3.6 (C-validity)**.** *Let $\mathbb{B}$ be the Boolean algebra of truth values. A propositional formula α is a tautology in classical logic if $\mathbb{B} \models \alpha$.*

Theorem 2.3.7. *A propositional formula α is C-valid iff $\mathcal{B} \models \alpha$, for all Boolean algebras $\mathcal{B}$.*

Proof. See [6, 12]. □

Notation: Let $\beta \in \Sigma_n$. We write $\beta(\alpha_1, \cdots, \alpha_n)$ to denote the simultaneous substitution in β of each formula α_i for the corresponding variable x_i, for $i \in \{1, \cdots, n\}$.

Theorem 2.3.8 (Principle of substitution for tautologies)**.** *Let α_1, α_2, $\cdots$, α_n be n formulas of propositional logic in the variables $x_1, \cdots, x_n$, $n \in \mathbb{N}$, and β a tautology (in the same n variables). Then, $\beta(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is also a tautology in the same n variables.*

2.4 Graphs and equivalence relations

Definition 2.4.1. *A graph $\mathcal{G}$ is a structure (G, R), where G is a nonempty set whose elements are called vertices and $R \subseteq G^2$ is a binary symmetric and irreflexive relation on G. We are going to read $R(a, b)$ as "a and b are connected", for all $a, b \in G$.*

Observation: We will use both the notations $R(a, b)$ and $(a, b) \in R$ to represent that the element (a, b) is in the relation R.

Definition 2.4.2. *Let A be a set and R be a binary relation on A. R is an equivalence relation if:*

1. *R is reflexive, i.e, for all $a \in A$, $(a, a) \in R$*
2. *R is symmetric, i.e., for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$*
3. *R is transitive, i.e., for all $a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$*

Definition 2.4.3. *Let A be a set and R be a binary equivalence relation on A. The equivalence class of $a \in A$ is defined as the set $[a]_R = \{x \in A : (a, x) \in R\}$, which represents the elements that are related to a under the relation R. The quotient set is represented by $A/R = \{[a]_R : a \in A\}$, which contains all the equivalence classes of the elements in A.*

Chapter 3

Partial algebras

In this chapter, we will introduce partial algebras along with associated definitions and propositions. Polynomials within the context of partial algebras, including their domain and associated mappings, will be discussed. The concept of identities in partial algebras will also be introduced, with examples illustrating both identities holding in all partial algebras and others that do not. Considerable space will be dedicated to one example of the latter kind, involving the construction of a partial algebra of functions associated with special graphs.

3.1 Partial algebra and its properties

Definition 3.1.1. *A partial algebra $\mathcal{A} = (A, \midcir, +, \cdot, \circ, \mathbf{1})$ is defined by a nonempty set A, a binary relation $\midcir$ on A, called compatibility or commeasurability, two partial binary operations on A, $+$ and $\cdot$, called sum and product, respectively, a function $\circ$ defined from $\mathbb{R} \times A$ to A and the identity element for the operation product of A, called* **1***, with the following properties:*

1. *The relation $\midcir$ is reflexive and symmetric*
2. *For all $q \in A$, $q \midcir \mathbf{1}$ (i.e.,* **1** *is compatible with all elements in A)*
3. *The partial binary functions are defined exactly for those pairs $(q_1, q_2) \in A \times A$ for which $q_1 \midcir q_2$*
4. *If any two of q_1, q_2 and q_3 are commeasurable (i.e., for all $i, j \in \{1, 2, 3\}, q_i \midcir q_j$), then $(q_1 + q_2) \midcir q_3$, $(q_1 \cdot q_2) \midcir q_3$ and $(a \circ q_1) \midcir q_2$ (a is a real number)*
5. *If any two of q_1, q_2 and q_3 are commeasurable, then the algebra of the polynomials in q_1, q_2 and q_3 (defined in the observation below) is a commutative algebra over the field of real numbers*

Following the article [9], we call the elements of A observables.

Observations:

- It is important to note that the concept "partial algebra" is an abbreviation of "partial commutative algebra over the field of real numbers", and the latter generalizes the concept of "commutative algebra over the field of real numbers", as introduced in the previous definition.

- Let us assume $(A, \downharpoonleft, +, \cdot, \circ, \mathbf{1})$ defined as in the previous definition. If q_1, q_2, and q_3 are pairwise commeasurable, then the algebra of the polynomials in q_1, q_2 and q_3 is the structure $(A', +', \cdot', \circ', \mathbf{1})$, where:

  - $A' \subseteq A$ is inductively defined:
    1. q_1, q_2, $q_3 \in A'$
    2. $\mathbf{1} \in A'$
    3. If $x, y \in A'$ and $x \downharpoonleft y$, then $x + y, x \cdot y \in A'$
    4. If $x \in A'$ and $a \in \mathbb{R}$, then $a \circ x \in A'$

  - The operations $+'$, $\cdot'$ and $\circ'$ are a restriction of the original ones $+, \cdot$ and $\circ$, respectively, that is:

    $+' = +|_{A' \times A'}$, $\cdot' = \cdot|_{A' \times A'}$, $\circ' = \circ|_{\mathbb{R} \times A'}$

Now, we will verify that the previous structure $(A', +', \cdot', \circ', \mathbf{1})$ constitutes an algebraic structure, that is, the operations $+'$, $\cdot'$ and $\circ'$ are total functions and that A' is closed under these operations.

Lemma 3.1.2. *Any two elements of A' are compatible.*

Proof. Let us suppose that $x \in A'$, $q_1 \downharpoonleft q_2, q_1 \downharpoonleft q_3$ and $q_2 \downharpoonleft q_3$. Let us consider $P(x)$ the property: for all $z \in A'$, $x \downharpoonleft z$. The proof follows by induction on $x \in A'$.

1. We want to show $P(q_1)$, that is, for all $z \in A'$, $q_1 \downharpoonleft z$. Let $z \in A'$ and let us consider the following property: for all $y \in A'$, $Q(y)$ iff $y \downharpoonleft q_1$.

   (i) $Q(q_1)$ iff $q_1 \downharpoonleft q_1$. Since $\downharpoonleft$ is reflexive, then $q_1 \downharpoonleft q_1$ holds.

   (ii) $Q(q_2)$ iff $q_1 \downharpoonleft q_2$, which is true based on the hypothesis.

   (iii) $Q(q_3)$ iff $q_1 \downharpoonleft q_3$, which is also true based on the hypothesis.

   (iv) $Q(\mathbf{1})$ iff $\mathbf{1} \downharpoonleft q_1$. Since $q_1 \in A'$, $\mathbf{1}$ is commeasurable with all elements of A and $A' \subseteq A$, then $\mathbf{1} \downharpoonleft q_1$.

(v) Let us suppose $Q(x)$, $Q(y)$ and $x \downspoon y$, $x, y \in A'$. We want to show $Q(x+y)$ and $Q(x \cdot y)$, that is, $(x+y) \downspoon q_1$ and $(x \cdot y) \downspoon q_1$, respectively. Since $x \downspoon q_1$, $y \downspoon q_1$ and $x \downspoon y$, i.e., any two of x, y, q_1 are commeasurable then, by definition 3.1.1, $(x+y) \downspoon q_1$ and $(x \cdot y) \downspoon q_1$.

(vi) Let us suppose $Q(x)$, $x \in A'$. We want to show $Q(a \circ x)$, $a \in \mathbb{R}$. Since, by $Q(x)$, $x \downspoon q_1$ then, by definition 3.1.1, $(a \circ x) \downspoon q_1$, $a \in \mathbb{R}$.

The proofs of $P(q_2)$ and $P(q_3)$ are analogous.

2. We want to show $P(\mathbf{1})$, that is, for all $z \in A'$, $\mathbf{1} \downspoon z$. Let $z \in A'$. Since **1** is compatible with all the observables in A and $A' \subseteq A$, then $\mathbf{1} \downspoon z$.

3. Let us suppose $P(x)$, $P(y)$ and $x \downspoon y$, for $x, y \in A'$. We want to show $P(x+y)$ and $P(x \cdot y)$, that is, for all $z \in A'$, $(x+y) \downspoon z$ and $(x \cdot y) \downspoon z$, respectively. Let $z \in A'$. By the hypothesis $P(x)$ and $P(y)$, we have, respectively, that $x \downspoon z$ and $y \downspoon z$, and by the fact that $x \downspoon y$, we have that any two of x, y, z are commeasurable. So, by definition 3.1.1, $(x+y) \downspoon z$ and $(x \cdot y) \downspoon z$.

4. Let us suppose $P(x)$, for $x \in A'$. We want to show $P(a \circ x)$, that is, for all $z \in A'$, $(a \circ x) \downspoon z$. Let $z \in A'$. By the hypothesis $P(x)$, $x \downspoon z$. So, by definition 3.1.1, $(a \circ x) \downspoon z$, for all $a \in \mathbb{R}$.

So $P(x)$, for all $x \in A'$. □

Proposition 3.1.3. *The operations $+'$, $\cdot'$ and $\circ'$ are total functions and A' is closed under these operations.*

Proof. Let us consider $x, y \in A'$. We want to show that $x +' y \in A'$, $x \cdot' y \in A'$ and $a \circ' x \in A'$, for all $a \in \mathbb{R}$. By lemma 3.1.2, $x \downspoon y$. Consequently, the elements $x + y$, $x \cdot y$ and $a \circ y$ belong to A' (due to the statements 3 and 4 of the definition of A'). By definition of restriction of a function, the value of $x +' y$ is $x + y$, the value of $x \cdot' y$ is $x \cdot y$ and the value of $a \circ' y$ is $a \circ y$. So, $x +' y, x \cdot' y, a \circ' x \in A'$. □

Observation: Sometimes, in order to clarify certain results, it can be useful to write $(q_i, q_j) \in \downspoon$ instead of $q_i \downspoon q_j$.

Proposition 3.1.4. *We can generalize the statement 5 of the definition of partial algebra (3.1.1) to any number of observables, that is, if any two of $q_1, \cdots, q_n$ are commeasurable, $n \in \mathbb{N}$, then the algebra of the polynomials in $q_1, \cdots, q_n$ is a commutative algebra over the field of real numbers.*

3.2 Polynomials in the context of partial algebras

Definition 3.2.1. *Let $n \in \mathbb{N}$. Let P_n, the set of polynomials in $x_1, \cdots, x_n$, be defined as:*

(i) $1 \in P_n$

(ii) $x_i \in P_n$*, for all* $1 \leq i \leq n$

(iii) If $\varphi \in P_n$*, then* $a \circ \varphi \in P_n$*, for all* $a \in \mathbb{R}$

(iv) If $\varphi, \psi \in P_n$*, then* $\varphi + \psi \in P_n$ *and* $\varphi \cdot \psi \in P_n$

The set of all polynomials is $\bigcup\limits_{n \in \mathbb{N}} P_n$.

Observations:

- The polynomials of P_n are expressions over the alphabet $\{x_1, \cdots, x_n\} \cup \{1, \circ, +, \cdot\} \cup \mathbb{R}$.
- The lower case letters of the Greek alphabet φ, ψ and χ will be used to denote polynomials.

Definition 3.2.2. *Let $\mathcal{A} = (A, \downarrow, +, \cdot, \circ, \mathbf{1})$ be a partial algebra and P_n be the set of polynomials previously defined. We define recursively on a polynomial $\varphi \in P_n$ the set $D_{\varphi,n} \subseteq A^n$ and the map $\varphi^* : D_{\varphi,n} \to A$ as follows:*

1. *If* $\varphi = 1$*, then* $D_{\varphi,n} = A^n$ *and* $\varphi^*(\vec{q}) = \mathbf{1}$ [1]
2. *If* $\varphi = x_i, 1 \leq i \leq n$*, then* $D_{\varphi,n} = A^n$ *and* $\varphi^*(\vec{q}) = \varphi^*((q_1, \cdots, q_n)) = q_i$
3. *If* $\varphi = a \circ \psi$*, then* $D_{\varphi,n} = D_{\psi,n}$ *and* $\varphi^*(\vec{q}) = a \circ \psi^*(\vec{q})$
4. *If* $\varphi = \psi \otimes \chi$*, where* $\otimes \in \{+, \cdot\}$*, then* $D_{\varphi,n} = \{\vec{q} \in A^n : \vec{q} \in D_{\psi,n} \cap D_{\chi,n}$ *and* $\psi^*(\vec{q}) \downarrow \chi^*(\vec{q})\}$ *and* $\varphi^*(\vec{q}) = \psi^*(\vec{q}) \otimes \chi^*(\vec{q})$

$D_{\varphi,n}$ and $\varphi^*(\vec{q})$ are, respectively, the domain and the map associated to the polynomial φ relative to A.

[1] In order to simplify the notation, we will write $\vec{q}$ to denote $(q_1, \cdots, q_n)$.

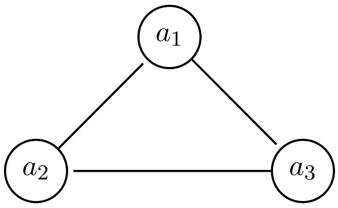
3.3 A partial algebra in a graph context

Definition 3.3.1. *A graph $\mathcal{G} = (G, R)$ satisfies condition C if it satisfies the following two properties:*

1. *For all $a, b \in G$, if $R(a, b)$ then there exists exactly one $c \in G$ such that $R(a, c)$ and $R(b, c)$, that is, any two connected vertices belong to exactly one triangle.*

2. *$\mathcal{G}$ contains, at least, one pair of connected vertices.*

The following graph satisfies the condition C: $\mathcal{G} = (G, R)$, where $G = \{a_1, a_2, a_3\}$ and $R(a, b)$ iff $a \neq b$, for all $a, b \in G$. In fact, $\mathcal{G}$ is a triangle of the type:



Definition 3.3.2. *F is a class of functions associated with a graph $\mathcal{G}$ satisfying the condition C when any $f \in F$ is a function whose values are real numbers and the domain, dom_f, is a set of three vertices of G any two of which are connected.*

Definition 3.3.3. *Let F be a class of functions associated with a graph $\mathcal{G}$. E is the binary relation defined on F such that $E(f, g)$ holds iff one of the following conditions is satisfied:*

1. *$f = g$*

2. *The sets dom_f and dom_g have one element in common, say $dom_f = \{a, b, c\}$, $dom_g = \{a, b', c'\}$ and $f(a) = g(a)$ and $f(b) = f(c) = g(b') = g(c')$*

3. *$f(a) = g(b) = r$, $r \in \mathbb{R}$, for all $a \in dom_f, b \in dom_g$ (f and g are both constant functions with the same constant value)*

Observation: From now on, when we refer to relation E, we assume that a graph $\mathcal{G} = (G, R)$ that satisfies the condition C and a class of functions, F, associated with the graph $\mathcal{G}$, are implicitly understood.

Lemma 3.3.4. *E is an equivalence relation.*

Proof. To prove that E is an equivalence relation we need to show that:

(i) E is reflexive:

Let $f \in F$. By the first statement of the definition 3.3.3, $E(f, f)$ holds.

(ii) E is symmetric:

Let $f, g \in F$ such that $E(f, g)$. We want to prove $E(g, f)$. Then, we have one of three cases:

1. Case $f = g$: Then, by hypothesis, we have that $E(f, f)$ holds.
2. Case $dom_f = \{a, b, c\}$, $dom_g = \{a, b', c'\}$, $f(a) = g(a)$ and $f(b) = f(c) = g(b') = g(c')$ (dom_f and dom_g only have the element a in common). Since the relation $=$ is symmetric, then $g(a) = f(a)$ and $g(b') = g(c') = f(b) = f(c)$, that is, $E(g, f)$.
3. Case $f(a) = g(b)$, for all $a \in dom_f$ and $b \in dom_g$. Since the relation $=$ is symmetric, then $g(b) = f(a)$, for all $b \in dom_g$ and $a \in dom_f$, that is, $E(g, f)$.

(iii) E is transitive:

Let $f, g, h \in F$ such that $E(f, g)$ and $E(g, h)$ hold. We want to prove $E(f, h)$.

1. Case $f = g$ or $g = h$: Then, $E(f, h)$ holds from one of the hypothesis.
2. Case $E(f, g)$ and $E(g, h)$ come from the second statement of the definition 3.3.3:
   - Case dom_f and dom_g have one element in common, say $dom_f = \{a, b, c\}, dom_g = \{a, b', c'\}$ and $f(a) = g(a)$, $f(b) = f(c) = g(b') = g(c')$, and dom_g and dom_h have also one element in common but it is different from the common element of dom_f and dom_g, say $dom_h = \{a'', b', c''\}$ and $g(b') = h(b')$, $g(a) = g(c') = h(a'') = h(c'')$. Then, we have that for all $x \in dom_f$, $y \in dom_g$, $z \in dom_h$, $f(x) = g(y) = h(z)$. Since $f(x) = h(z)$ for all $x \in dom_f$, $z \in dom_h$, $E(f, h)$ holds.
   - Case dom_f, dom_g and dom_h have the same element in common (and the only one), say $dom_f = \{a, b, c\}, dom_g = \{a, b', c'\}, dom_h = \{a, b'', c''\}$ and $f(a) = g(a)$, $f(b) = f(c) = g(b') = g(c')$ (this comes from $E(f, g)$) and $g(a) = h(a)$, $g(b') = g(c') = h(b'') = h(c'')$ (this comes from $E(g, h)$). Then, $f(a) = h(a)$ and $f(b) = f(c) = h(b'') = h(c'')$, that is, $E(f, h)$ holds.
3. Case $E(f, g)$ comes from the second statement of the definition 3.3.3, that is, dom_f and dom_g have one element in common, say $dom_f = \{a, b, c\}, dom_g = \{a, b', c'\}$, $f(a) = g(a)$ and $f(b) = f(c) = g(b') = g(c')$, and $E(g, h)$ comes from the third statement of the same definition, i.e., $g(x) = h(y)$, for all $x \in dom_g$ and $y \in dom_h$. Then, g and

h are constant functions and, therefore, $f(a) = f(b) = f(c) = g(x) = h(y)$, for all $x \in dom_g, y \in dom_h$. So, $f(z) = h(y)$, for all $z \in dom_f, y \in dom_h$, that is, $E(f, h)$ holds.

4. Case both $E(f, g)$ and $E(g, h)$ come from the third statement of the definition 3.3.3, that is, for all $a \in dom_f, b \in dom_g$ $f(a) = g(b) = r$, $r \in \mathbb{R}$, and for all $x \in dom_g, y \in dom_h$ $g(x) = h(y) = p$, $p \in \mathbb{R}$. Then, this is just possible if $p = r$, that is, for all $a \in dom_f, y \in dom_h$, $f(a) = h(y) = r$. So, $E(f, h)$.

□

Observations:

- The equivalence classes of E, denoted as $[f]_E = \{g \in F : E(f, g)\}$ for any $f \in F$, are referred to as "observables".
- We will denote the set of all equivalence classes (the quotient set F/E) as Q.

Definition 3.3.5. *Given $q_1, q_2 \in Q$, they are said to be commeasurable (one writes $q_1 \downarrow\!\!\circ\, q_2$) if there exist functions $f_i \in q_i, i \in \{1, 2\}$, such that $dom_{f_1} = dom_{f_2}$. Sum and product of commeasurable observables are defined as follows: $q_1 + q_2$ is the equivalence class of the functions $f_1 + f_2$; $q_1 \cdot q_2$ is the equivalence class of the functions $f_1 \cdot f_2$, where $f_i \in q_i$ and $dom_{f_1} = dom_{f_2}$. Let $a \circ q = \{af : f \in q\}$. If q is an observable and a is a real number, then all the functions af for $f \in q$ belong to the same equivalence class which is, by definition, the class $a \circ q$. One can write $a \circ q$ as $a \circ [f]_E$, for $f \in q$.*

Proposition 3.3.6. *Let us consider $q_1, q_2 \in Q$ and two functions $f_1 \in q_1, f_2 \in q_2$ such that $dom_{f_1} = dom_{f_2}$. We have the following statements:*

1. *If there exist functions $f_1' \in q_1$ and $f_2' \in q_2$ such that $dom_{f_1'} = dom_{f_2'}$ and $E(f_1', f_1)$ and $E(f_2', f_2)$, then $E(f_1 + f_2, f_1' + f_2')$ and $E(f_1 \cdot f_2, f_1' \cdot f_2')$ hold (We will only prove that $E(f_1 + f_2, f_1' + f_2')$ holds, as the case of $E(f_1 \cdot f_2, f_1' \cdot f_2')$ is analogous).*
2. *If there exists a function $f_1' \in q_1$ such that $E(f_1, f_1')$, then $E(af_1, af_1')$ holds, for all $a \in \mathbb{R}$.*

Therefore, the operations of sum, product and scalar multiplication on Q are well defined.

Proof. Let $q_1, q_2 \in Q$ and $f_1 \in q_1, f_2 \in q_2$ such that $dom_{f_1} = dom_{f_2}$.

1. Let us consider that there exist functions $f_1' \in q_1$ and $f_2' \in q_2$ such that $dom_{f_1'} = dom_{f_2'}$ and $E(f_1', f_1)$ and $E(f_2', f_2)$ hold. We want to show that $E(f_1 + f_2, f_1' + f_2')$ holds.

(a) Case $E(f_1', f_1)$ or $E(f_2', f_2)$ come from the first statement of the definition 3.3.3:

Let us suppose that $f_1' = f_1$ ($E(f_1', f_1)$). By hypothesis, $dom_{f_1} = dom_{f_2}$ and $dom_{f_1'} = dom_{f_2'}$. So, since $f_1' = f_1$, $dom_{f_1} = dom_{f_1'} = dom_{f_2} = dom_{f_2'}$.

- Case $E(f_2', f_2)$ comes from the first statement of the definition 3.3.3, that is, $f_2' = f_2$: Then, $E(f_1 + f_2, f_1' + f_2') = E(f_1 + f_2, f_1 + f_2)$ and, due to the reflexive property of the relation E, $E(f_1 + f_2, f_1 + f_2)$ holds.
- The case in which $E(f_2', f_2)$ comes from the second statement of the definition 3.3.3 is not possible because f_2 and f_2' have the same domain.
- Case $E(f_2', f_2)$ comes from the third statement of the definition 3.3.3:
  Then, $f_2'(a) = f_2(b) = r$, $r \in \mathbb{R}$, for all $a \in dom_{f_2'}, b \in dom_{f_2}$. Since $dom_{f_2'} = dom_{f_2}$, $f_2' = f_2$ and due to the reflexive property of E, $E(f_1 + f_2, f_1' + f_2') = E(f_1 + f_2, f_1 + f_2)$ holds.

(b) Case $E(f_1', f_1)$ or $E(f_2', f_2)$ come from the second statement of the definition 3.3.3:

Let us suppose that $dom_{f_1} = \{a, b, c\}$, $dom_{f_1'} = \{a, b', c'\}$ (the only element in common is a), $f_1(a) = f_1'(a)$ and $f_1(b) = f_1(c) = f_1'(b') = f_1'(c')$. Since, by hypothesis, $dom_{f_1} = dom_{f_2}$ and $dom_{f_1'} = dom_{f_2'}$, then $dom_{f_1+f_2} = dom_{f_1}$ and $dom_{f_1'+f_2'} = dom_{f_1'}$ have the element a in common.

- Let us consider that $E(f_2', f_2)$ comes from the second statement of the definition mentioned above, that is, $f_2(a) = f_2'(a)$ and $f_2(b) = f_2(c) = f_2'(b') = f_2'(c')$. Then, $(f_1 + f_2)(a) = f_1(a) + f_2(a) = f_1'(a) + f_2'(a) = (f_1' + f_2')(a)$ and for all $x \in \{b, c\}, y \in \{b', c'\}$, $(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_1'(y) + f_2'(y) = (f_1' + f_2')(y)$. Since $dom_{f_1+f_2}$ and $dom_{f_1'+f_2'}$ have only one element in common, a, $(f_1 + f_2)(a) = (f_1' + f_2')(a)$ and $(f_1 + f_2)(x) = (f_1' + f_2')(y)$, for all $x \in dom_{f_1+f_2} \setminus \{a\}, y \in dom_{f_1'+f_2'} \setminus \{a\}$, we conclude that $E(f_1 + f_2, f_1' + f_2')$ holds.
- Let us consider that $E(f_2', f_2)$ comes from the third statement of the definition mentioned above, i.e., for all $x \in dom_{f_2}, y \in dom_{f_2'}$, $f_2(x) = f_2'(y)$. Then, we have $(f_1 + f_2)(a) = f_1(a) + f_2(a) = f_1'(a) + f_2'(a) = (f_1' + f_2')(a)$ and for all $a \in \{b, c\}, y \in \{b', c'\}$, $(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_1'(y) + f_2'(y) = (f_1' + f_2')(y)$. Since $dom_{f_1+f_2}$ and $dom_{f_1'+f_2'}$ have only one element in common, a, $(f_1 + f_2)(a) = (f_1' + f_2')(a)$ and $(f_1 + f_2)(x) = (f_1' + f_2')(y)$, for all

$x \in dom_{f_1+f_2} \setminus \{a\}, y \in dom_{f'_1+f'_2} \setminus \{a\}$, we conclude that $E(f_1 + f_2, f'_1 + f'_2)$ holds.

(c) Case both $E(f'_1, f_1)$ and $E(f'_2, f_2)$ come from the third statement of the definition 3.3.3:

Then, for all $a \in dom_{f_1}, b \in dom_{f'_1}, f_1(a) = f'_1(b)$ and for all $a \in dom_{f_2}, b \in dom_{f'_2}, f_2(a) = f'_2(b)$. So, $(f_1 + f_2)(a) = f_1(a) + f_2(a) = f'_1(b) + f'_2(b) = (f'_1 + f'_2)(b)$, for all $a \in dom_{f_1+f_2}, b \in dom_{f'_1+f'_2}$. Therefore, $E(f_1 + f_2, f'_1 + f'_2)$ holds.

Obs: $dom_{f_1+f_2} = dom_{f_1} = dom_{f_2}$ and $dom_{f'_1+f'_2} = dom_{f'_1} = dom_{f'_2}$.

2. Let us consider that there exists a function $f'_1 \in q_1$ such that $E(f_1, f'_1)$ holds. We want to show that $E(af_1, af'_1)$ holds, for all $a \in \mathbb{R}$.

Trivially, the product of a function by a real constant will not change its domain. So, regardless of the domain of f_1 and f'_1, the domain of the functions af_1 and af'_1 will remain the same as f_1 and f'_1, respectively. Therefore, $E(af_1, af'_1)$ holds, for all $a \in \mathbb{R}$.

□

Observation: We will define the element **1** as the set $\{f \in F : \text{for all } a \in dom_f, f(a) = 1\}$.
Let us consider a function $f \in F$ defined as follows:

$$f : \{0, 1, 2\} \to \mathbb{R}$$
$$f(a) = 1$$

Then, $[f]_E =$ **1** and, consequently, **1** $\in Q$.

Proof. We want to prove that $[f]_E =$ **1**.

- $[f]_E \subseteq$ **1**: Let $f_1 \in [f]_E$, that is, $f_1 \in F$ and $E(f_1, f)$. We want to show that $f_1 \in$ **1**.

  Since $E(f_1, f)$ holds and f is a constant function which yields the real number 1, then $f_1 = f$ or for all $a \in dom_f, b \in dom_{f_1}$, $f_1(a) = f(b) = 1$. In both situations, f_1 always returns the value 1. So, due to the fact that $f_1 \in F$ and for all $a \in dom_{f_1}, f_1(a) = 1$, we conclude that $f_1 \in$ **1**.

- **1** $\subseteq [f]_E$: Let $f_1 \in$ **1**, that is, $f_1 \in F$ and for all $a \in dom_{f_1}, f_1(a) = 1$. We want to show that $f_1 \in [f]_E$.

  By definition, $f_1 \in F$. It remains to show that $E(f_1, f)$ holds. Since for all $a \in dom_{f_1}, f_1(a) = 1$ and for all $b \in dom_f, f(b) = 1$, then $E(f_1, f)$ holds. Therefore, $f_1 \in [f]_E$.

So, given that $[f]_E \subseteq \mathbf{1}$ and $\mathbf{1} \subseteq [f]_E$, we conclude that $[f]_E = \mathbf{1}$, and, consequently, $\mathbf{1} \in Q$. $\square$

Theorem 3.3.7. *The structure $\mathcal{A} = (Q, \downspoon, +, \cdot, \circ, \mathbf{1})$ is a partial algebra, for $Q, \downspoon, +, \cdot, \circ$ and $\mathbf{1}$ given as in the previous definition and observation.*

Proof. Let us consider the structure $\mathcal{A} = (Q, \downspoon, +, \cdot, \circ, \mathbf{1})$.

1. We want to show that the relation $\downspoon$ is reflexive and symmetric. Let us consider an observable q_1. Trivially, there exists $f_1 \in q_1$ with domain dom_{f_1}. Since $dom_{f_1} = dom_{f_1}$, then $q_1 \downspoon q_1$, i.e., $\downspoon$ is reflexive.

   Now, let us consider two observables q_1 and q_2 and two functions $f_1 \in q_1$ and $f_2 \in q_2$ such that $dom_{f_1} = dom_{f_2}$. Since $dom_{f_2} = dom_{f_1}$, then $\downspoon$ is symmetric.

2. We want to prove that $\mathbf{1} \downspoon q$, for all $q \in Q$. Let us consider $q \in Q$ and $f_1 \in q$. We want to show that there exists $f \in \mathbf{1}$ such that $dom_f = dom_{f_1}$. By the observation above, $\mathbf{1} = \{f \in F : \text{for all } a \in dom_f, f(a) = 1\}$. Let us choose a function $f \in F$ such that $dom_f = dom_{f_1}$ and for all $a \in dom_f$, $f(a) = 1$. Clearly, $f \in \mathbf{1}$. Therefore, since $dom_f = dom_{f_1}$, $\mathbf{1} \downspoon q$.

   Moreover, since we already verified that $\mathbf{1}$ is commeasurable with all elements in Q, we shall now demonstrate its identity property with respect to the $\cdot$ operation for all elements in Q. Let us consider an observable q. Our objective is to prove that $q \cdot \mathbf{1} = \mathbf{1} \cdot q = q$. Let f be an element of q and g be the function whose domain is the same as that of f and be defined such that for all $a \in dom_g$, $g(a) = 1$. Then, for all $a \in dom_g$,

   $$(f \cdot g)(a) = f(a) \cdot g(a) = f(a) \cdot 1 = f(a)$$

   and

   $$(g \cdot f)(a) = g(a) \cdot f(a) = 1 \cdot f(a) = f(a)$$

   So, $\mathbf{1}$ is the identity element of Q for the operation $\cdot$.

3. By definition, sum and product are defined exactly for commeasurable observables.

4. Let us consider that any two of $q_1, q_2, q_3 \in Q$ are commeasurable. We want to prove that $(q_1 + q_2) \downspoon q_3$, $(q_1 \cdot q_2) \downspoon q_3$ and $(a \circ q_1) \downspoon q_2$.

   (a) Case $(q_1 + q_2) \downspoon q_3$ (The $(q_1 \cdot q_2) \downspoon q_3$ case is analogous):

We want to show the existence of functions $g \in q_1 + q_2$ and $f_3 \in q_3$ such that $dom_g = dom_{f_3}$. Since any two of the observables q_1, q_2, q_3 are commeasurable, meaning that there are functions $f_1 \in q_1, f_2 \in q_2, f_3 \in q_3$ such that $dom_{f_1} = dom_{f_2}$, $dom_{f_1} = dom_{f_3}$ and $dom_{f_2} = dom_{f_3}$, we deduce that $dom_{f_1} = dom_{f_2} = dom_{f_3} = dom_{f_1+f_2}$. By definition, $q_1+q_2 = [f_1 + f_2]_E = \{g \in F : E(f_1+f_2, g)\}$. Let us consider the function $g = f_1 + f_2$. Since $f_1 + f_2 \in q_1 + q_2$, $f_3 \in q_3$ and $dom_g = dom_{f_3}$, we conclude that $(q_1 + q_2)\mathrel{\rfloor_{\circ}} q_3$.

(b) Case $(a \circ q_1)\mathrel{\rfloor_{\circ}} q_2$, where a is a real number.

We want to show the existence of functions $g \in a \circ q_1$ and $f_2 \in q_2$ such that $dom_g = dom_{f_2}$. Since the observables q_1 and q_2 are commeasurable, then there are functions $f_1 \in q_1$, $f_2 \in q_2$ such that $dom_{f_1} = dom_{f_2}$. By definition, $a \circ q_1 = \{af_1 : f_1 \in q_1\}$. Let us consider $g = af_1$. So, due to the fact that $g \in a \circ q_1$, $dom_g = dom_{f_1}$ and, by hypothesis, $dom_{f_1} = dom_{f_2}$, we conclude that $(a \circ q_1)\mathrel{\rfloor_{\circ}} q_2$.

5. Let us consider that any two of q_1, q_2 and q_3 are commeasurable. We want to prove that the algebra of the polynomials on q_1, q_2, q_3, that is, $\mathcal{A}' = (Q', +', \cdot', \circ', \mathbf{1})$, is a commutative algebra, where:

- $Q' \subseteq Q$ is inductively defined:
  (a) $q_1, q_2, q_3 \in Q'$
  (b) $\mathbf{1} \in Q'$
  (c) If $x, y \in Q'$ and $x\mathrel{\rfloor_{\circ}} y$, then $x + y, x \cdot y \in Q'$
  (d) If $x \in Q'$ and $a \in \mathbb{R}$, then $a \circ x \in Q'$
- The operations $+'$, $\cdot'$ and $\circ'$ are defined as:
  $+' : \; + |_{Q' \times Q'}, \; \cdot' : \; \cdot |_{Q' \times Q'}, \; \circ' : \; \circ |_{\mathbb{R} \times Q'}$

Observation: From the definition 3.1.1, we know that all the elements in Q' are compatible and that Q' is closed under the operations $+'$, $\cdot'$ and $\circ'$. Now, we will prove that $\mathcal{A}' = (Q', +', \cdot', \circ', \mathbf{1})$ is a commutative algebra over the field of real numbers, that is, $\mathcal{A}'$ is an algebra over the field of real numbers and the operation $\cdot'$ is commutative.

- First of all, in order to prove that $\mathcal{A}'$ is an algebra over the field of real numbers, we need to demonstrate that Q', along with the operations $+'$ and $\circ'$, is a vector space over the field of real numbers.

(i) Let $x, y \in Q'$. We want to prove that $x +' y = y +' x$. Let us consider $f \in x$ and $g \in y$ such that $dom_f = dom_g$. Then, $x +' y \overset{(a)}{=} [f]_E +' [g]_E \overset{(b)}{=} [f + g]_E \overset{(c)}{=} [g + f]_E \overset{(d)}{=} [g]_E +' [f]_E \overset{(e)}{=} y +' x$

(a) f and g are representatives of the classes x and y, respectively

(b) Definition of sum of commeasurable observables in $\mathcal{A}'$

(c) Sum of functions is commutative

(d) Definition of sum of commeasurable observables in $\mathcal{A}'$

(e) f and g are representatives of the classes x and y, respectively

(ii) Let $x, y, z \in Q'$. We want to prove that $(x +' y) +' z = x +' (y +' z)$. Let us consider $f \in x, g \in y$ and $h \in z$ such that $dom_f = dom_g = dom_h$. Then, $(x +' y) +' z \overset{(a)}{=} ([f]_E +' [g]_E) +' [h]_E \overset{(b)}{=} ([f + g]_E) +' [h]_E \overset{(c)}{=} [(f + g) + h]_E \overset{(d)}{=} [f + (g + h)]_E \overset{(e)}{=} [f]_E +' [g + h]_E \overset{(f)}{=} [f]_E +' ([g]_E +' [h]_E) \overset{(g)}{=} x +' (y +' z)$

(a) f, g and h are representatives of the classes x, y and z, respectively

(b) Definition of sum of commeasurable observables in $\mathcal{A}'$

(c) Definition of sum of commeasurable observables in $\mathcal{A}'$

(d) Sum of functions is associative

(e) Definition of sum of commeasurable observables in $\mathcal{A}'$

(f) Definition of sum of commeasurable observables in $\mathcal{A}'$

(g) f, g and h are representatives of the classes x, y and z, respectively

(iii) We want to prove that there exists an identity element for the sum operation, that is, that there exists **0** $\in Q'$ such that, for all $x \in Q'$, **0** $+' x = x +'$ **0** $= x$. Let $x \in Q'$ and $f \in x$. Let us consider **0** as the set $\{g \in F :$ for all $a \in dom_g, g(a) = 0\}$ and the function $g \in F$ such that $dom_g = dom_f$ and for all $a \in dom_g, g(a) = 0$. Clearly, $g \in$ **0**. Then, for all $a \in dom_g$,

$$(g + f)(a) = g(a) + f(a) = 0 + f(a) = f(a)$$

and

$$(f + g)(a) = f(a) + g(a) = f(a) + 0 = f(a)$$

Observation: The proof that **0** is in Q is similar to the proof that **1** is in Q, as previously showed. In fact, we just need to change the value that the function f yields to 0 (instead of 1).

(iv) We want to prove that there exists a symmetric element for all the observables in Q', that is, for all $x \in Q'$, there exists $y \in Q'$ such that $x +' y = y +' x =$ **0**, where **0** is the identity element for the operation $+'$. Let us consider $x \in Q'$ and $f \in x$ and the function g defined such that $dom_g = dom_f$ and for all $a \in dom_g, g(a) = -f(a)$. Then, for all $a \in dom_g$,

$$(f+g)(a) = f(a) + g(a) = f(a) - f(a) = 0$$

and

$$(g+f)(a) = g(a) + f(a) = -f(a) + f(a) = 0$$

Since $f+g, g+f \in F$ and for all $a \in dom_{f+g} = dom_{g+f}, (f+g)(a) = (g+f)(a) = 0$, then $f+g, g+f \in$ **0**. So, $x +' y = y +' x =$ **0**.

(v) Let $a, b \in \mathbb{R}$ and $x \in Q'$. We want to prove that $(a+b) \circ' x = a \circ' x +' b \circ' x$. Let $f \in x$. Then, $(a+b) \circ' x \overset{(a)}{=} (a+b) \circ' [f]_E \overset{(b)}{=} [(a+b) \times f]_E \overset{(c)}{=} [a \times f + b \times f]_E \overset{(d)}{=} [a \times f]_E +' [b \times f]_E \overset{(e)}{=} a \circ' [f]_E +' b \circ' [f]_E \overset{(f)}{=} a \circ' x + b \circ' x$, where $+$ and $\times$ are the sum and the product in the field of real numbers, respectively.

  (a) f is a representative of the class x
  (b) Definition of scalar product in $\mathcal{A}'$
  (c) Distributivity of $+$ with respect to $\times$, where $+$ and $\times$ are defined in the field of real numbers
  (d) Definition of sum of commeasurable observables
  (e) Definition of scalar product in $\mathcal{A}'$
  (f) f is a representative of the class x

(vi) Let $a \in \mathbb{R}$ and $x, y \in Q'$. We want to prove that $a \circ' (x +' y) = a \circ' x +' a \circ' y$. Let $f \in x$ and $g \in y$ such that $dom_f = dom_g$. Then, $a \circ' (x +' y) \overset{(a)}{=} a \circ' ([f]_E +' [g]_E) \overset{(b)}{=} a \circ' [f+g]_E \overset{(c)}{=} [a \times (f+g)]_E \overset{(d)}{=} [a \times f + a \times g]_E \overset{(e)}{=} [a \times f]_E +' [a \times g]_E \overset{(f)}{=} a \circ' [f]_E +' a \circ' [g]_E \overset{(g)}{=} a \circ' x +' a \circ' y$, where $+$ and $\times$ are the sum and the product in the field of real numbers, respectively.

  (a) f and g are representatives of the classes x and y, respectively
  (b) Definition of sum of commeasurable observables
  (c) Definition of scalar product in $\mathcal{A}'$
  (d) Distributivity of $+$ with respect to $\times$, where $+$ and $\times$ are defined in the field of real numbers

(e) Definition of sum of commeasurable observables

(f) Definition of scalar product in $\mathcal{A}'$

(g) f and g are representatives of the classes x and y, respectively

(vii) Let $a, b \in \mathbb{R}$ and $x \in Q'$. We want to prove that $(a \times b) \circ' x = a \circ' (b \circ' x)$. Let us consider $f \in x$. Then, $(a \times b) \circ' x \overset{\text{(a)}}{=} (a \times b) \circ' [f]_E \overset{\text{(b)}}{=} [(a \times b) \times f]_E \overset{\text{(c)}}{=} [a \times (b \times f)]_E \overset{\text{(d)}}{=} a \circ' [b \times f]_E \overset{\text{(e)}}{=} a \circ' (b \circ' [f]_E) \overset{\text{(f)}}{=} a \circ' (b \circ' x)$

(a) f is a representative of the class x

(b) Definition of scalar product in $\mathcal{A}'$

(c) The operation $\times$, in the field of real numbers, is associative

(d) Definition of scalar product in $\mathcal{A}'$

(e) Definition of scalar product in $\mathcal{A}'$

(f) f is a representative of the class x

(viii) Let $x \in Q'$. We want to show that $1 \circ' x = x$, where 1 is the multiplicative identity of the field of real numbers. Let us consider $f \in x$. Then, $1 \circ' x \overset{\text{(a)}}{=} 1 \circ' [f]_E \overset{\text{(b)}}{=} [1 \times f]_E \overset{\text{(c)}}{=} [f]_E \overset{\text{(d)}}{=} x$, where $\times$ is the product in the field of real numbers.

(a) f is a representative of the class x

(b) Definition of scalar product in $\mathcal{A}'$

(c) The real number 1 is the identity of $\circ$ in the field of real numbers

(d) f is a representative of the class x

So, Q' is a vector space over the field of real numbers.

- Considering the previous fact, now we need to prove that the operation $\cdot'$ satisfies the following four properties: associativity, right distributivity, left distributivity and compatibility with scalars. Once this is proved, $\mathcal{A}'$ will be an algebra over the field of real numbers.

(i) Let $x, y, z \in Q'$. We want to prove that $(x \cdot' y) \cdot' z = x \cdot' (y \cdot' z)$. Let us consider $f \in x$, $g \in y$ and $h \in z$ such that $dom_f = dom_g = dom_h$. Then, $(x \cdot' y) \cdot' z \overset{\text{(a)}}{=} ([f]_E \cdot' [g]_E) \cdot' [h]_E \overset{\text{(b)}}{=} [f \times g]_E \cdot' [h]_E \overset{\text{(c)}}{=} [(f \times g) \times h]_E \overset{\text{(d)}}{=} [f \times (g \times h)]_E \overset{\text{(e)}}{=} [f]_E \cdot' [g \times h]_E \overset{\text{(f)}}{=} [f]_E \cdot' ([g]_E \cdot' [h]_E) \overset{\text{(g)}}{=} x \cdot' (y \cdot' z)$, where $\times$ is the product in the field of real numbers.

(a) f, g and h are representatives of the classes x, y and z, respectively

(b) Definition of product of commeasurable observables in $\mathcal{A}'$

(c) Definition of product of commeasurable observables in $\mathcal{A}'$

(d) The operation $\times$, in the field of real numbers, is associative

(e) Definition of product of commeasurable observables in $\mathcal{A}'$

(f) Definition of product of commeasurable observables in $\mathcal{A}'$

(g) f, g and h are representatives of the classes x, y and z, respectively

(ii) Let $x, y, z \in Q'$. We want to prove that $(x +' y) \cdot' z = x \cdot' z +' y \cdot' z$. Let us consider $f \in x$, $g \in y$ and $h \in z$ such that $dom_f = dom_g = dom_h$. Then, $(x +' y) \cdot' z \overset{\text{(a)}}{=} ([f]_E +' [g]_E) \cdot' [h]_E \overset{\text{(b)}}{=} [f + g]_E \cdot' [h]_E \overset{\text{(c)}}{=} [(f + g) \times h]_E \overset{\text{(d)}}{=} [f \times h + g \times h]_E \overset{\text{(e)}}{=} [f \times h]_E +' [g \times h]_E \overset{\text{(f)}}{=} [f]_E \cdot' [h]_E +' [g]_E \cdot' [h]_E \overset{\text{(g)}}{=} x \cdot' z +' y \cdot' z$, where $\times$ is the product in the field of real numbers.

(a) f, g and h are representatives of the classes x, y and z, respectively

(b) Definition of sum of commeasurable observables in $\mathcal{A}'$

(c) Definition of product of commeasurable observables in $\mathcal{A}'$

(d) Distributivity of $\times$ with respect to $+$ in the field of real numbers

(e) Definition of sum of commeasurable observables in $\mathcal{A}'$

(f) Definition of product of commeasurable observables in $\mathcal{A}'$

(g) f, g and h are representatives of the classes x, y and z, respectively

(iii) This case is analogous to the previous one.

(iv) Let $x, y \in Q'$ and $a, b \in \mathbb{R}$. We want to prove that $(a \circ' x) \cdot' (b \circ' y) = (a \times b) \circ' (x \cdot' y)$. Let us consider $f \in x$ and $g \in y$ such that $dom_f = dom_g$. Then, $(a \circ' x) \cdot' (b \circ' y) \overset{\text{(a)}}{=} (a \circ' [f]_E) \cdot' (b \circ' [g]_E) \overset{\text{(b)}}{=} [a \times f]_E \cdot' [b \times g]_E \overset{\text{(c)}}{=} [(a \times f) \times (b \times g)]_E \overset{\text{(d)}}{=} [(a \times b) \times (f \times g)]_E \overset{\text{(e)}}{=} (a \times b) \circ' [f \times g]_E \overset{\text{(f)}}{=} (a \times b) \circ' ([f]_E \cdot' [g]_E) \overset{\text{(g)}}{=} (a \times b) \circ' (x \cdot' y)$, where $\times$ is the product in the field of real numbers.

(a) f and g are representatives of the classes x and y, respectively

(b) Definition of scalar product in $\mathcal{A}'$

(c) Definition of product of commeasurable observables in $\mathcal{A}'$

(d) The operation $\times$, in the field of real numbers, is commutative and associative

(e) Definition of scalar product in $\mathcal{A}'$

(f) Definition of product of commeasurable observables in $\mathcal{A}'$

(g) f and g are representatives of the classes x and y, respectively

So, since Q' is a vector space over the field of real numbers and the $\cdot'$ operation satisfies these four properties, then $\mathcal{A}'$ is an algebra over the field of real numbers.

- It remains to show that the operation $\cdot'$ is commutative.

  Let $x, y \in Q'$. We want to show that $x \cdot' y = y \cdot' x$. Let us consider $f \in x$ and $g \in y$ such that $dom_f = dom_g$. Then, $x \cdot' y \overset{\text{(a)}}{=} [f]_E \cdot' [g]_E \overset{\text{(b)}}{=} [f \times g]_E \overset{\text{(c)}}{=} [g \times f]_E \overset{\text{(d)}}{=} [g]_E \cdot' [f]_E \overset{\text{(e)}}{=} y \cdot' x$

  (a) f and g are representatives of the classes x and y, respectively

  (b) Definition of product of commeasurable observables in $\mathcal{A}'$

  (c) The operation $\times$, in the field of real numbers, is commutative

  (d) Definition of product of commeasurable observables in $\mathcal{A}'$

  (e) f and g are representatives of the classes x and y, respectively

So, since $\mathcal{A}'$ is an algebra over the field of real numbers and the operation $\cdot'$ is commutative, we conclude that $\mathcal{A}'$ is a commutative algebra over the field of real numbers. □

3.4 Identities in a partial algebra

Definition 3.4.1. *Let $\mathcal{A}$ be a partial algebra. One says "φ is identically 1 on $\mathcal{A}$" or, equivalently, "the identity $\varphi = 1$ holds in $\mathcal{A}$", if for all $q \in D_{\varphi,n}$, $\varphi^*(\vec{q}) = \mathbf{1}$.*

Observations: Let φ and ψ be two polynomials in n variables. Then, an identity $\varphi = \psi$ holding in $\mathcal{A}$ can be interpreted in two ways:

- If $\vec{q} \in D_{\varphi,n} \cap D_{\psi,n}$, then $\varphi^*(\vec{q}) = \psi^*(\vec{q})$ (the identity $\varphi = \psi$ holds strongly in $\mathcal{A}$)
- If $\vec{q} \in D_{\varphi,n} \cap D_{\psi,n}$ and $\varphi^*(\vec{q}) \downarrow\!\!\!\circ\, \psi^*(\vec{q})$, then $\varphi^*(\vec{q}) = \psi^*(\vec{q})$ (the identity $\varphi = \psi$ holds weakly in $\mathcal{A}$)

The first statement implies the second one but the converse is not necessarily true. If $\psi = 1$, then both statements are equivalent.

Now, we will give some examples of identities holding in all partial algebras and others that do not, which were taken from the article [9].

Example 1. Let us consider $\varphi = x_1 + x_2$ and $\psi = x_2 + x_1$ two polynomials in 2 variables. The identity $\varphi = \psi$ holds strongly (and, consequently, weakly) in all partial algebras.

Proof. Let $\mathcal{A} = (A, ⫰, +, \cdot, \circ, \mathbf{1})$ be a partial algebra and $\varphi = x_1 + x_2$ and $\psi = x_2 + x_1$ be two polynomials in 2 variables. We want to prove that the identity $\varphi = \psi$ holds in $\mathcal{A}$, that is, for all $\vec{q} \in D_{\varphi,2} \cap D_{\psi,2}, \varphi^*(\vec{q}) = \psi^*(\vec{q})$.

Let $\vec{q} = (q_1, q_2) \in D_{\varphi,2} \cap D_{\psi,2}$. By definition,

$$
\begin{aligned}
D_{x_1+x_2,2} &= \{\vec{q} \in A^2 : \vec{q} \in D_{x_1,2} \cap D_{x_2,2} \text{ and } {x_1}^*(\vec{q}) ⫰ {x_2}^*(\vec{q})\} \\
&= \{\vec{q} \in A^2 : q_1 ⫰ q_2\} \\
&= D_{x_2+x_1,2}
\end{aligned}
$$

So,

$$\varphi^*(\vec{q}) \overset{\text{(i)}}{=} q_1 + q_2 \overset{\text{(ii)}}{=} q_2 + q_1 \overset{\text{(iii)}}{=} \psi^*(\vec{q}).$$

(i) Definition of $\varphi^*(\vec{q})$

(ii) Since q_1 and q_2 are commeasurable (by definition of $D_{x_1+x_2,2}$) the algebra of the polynomials in q_1 and q_2 is a commutative algebra over the field of real numbers (by definition 3.1.1, statement 5). Therefore, we have commutativity for the operation $+$

(iii) Definition of $\psi^*(\vec{q})$

□

Example 2. Let us consider $\varphi = (x_1 + x_2) + x_3$ and $\psi = x_1 + (x_2 + x_3)$ two polynomials in 3 variables. The identity $\varphi = \psi$ holds strongly (and, consequently, weakly) in all partial algebras.

Proof. Let $\mathcal{A} = (A, ⫰, +, \cdot, \circ, \mathbf{1})$ be a partial algebra and $\varphi = (x_1+x_2)+x_3$ and $\psi = x_1+(x_2+x_3)$ be two polynomials in 3 variables. We want to prove that the identity $\varphi = \psi$ holds in $\mathcal{A}$, that is, for all $\vec{q} \in D_{\varphi,3} \cap D_{\psi,3}, \varphi^*(\vec{q}) = \psi^*(\vec{q})$.

Let $\vec{q} = (q_1, q_2, q_3) \in D_{\varphi,3} \cap D_{\psi,3}$. By definition,

$$
\begin{aligned}
D_{\varphi,3} &= \{\vec{q} \in A^3 : \vec{q} \in D_{x_1+x_2} \cap D_{x_3} \text{ and } (x_1 + x_2)^*(\vec{q}) ⫰ {x_3}^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : \vec{q} \in D_{x_1} \cap D_{x_2} \cap D_{x_3} \text{ and } {x_1}^*(\vec{q}) ⫰ {x_2}^*(\vec{q}) \text{ and } (x_1 + x_2)^*(\vec{q}) ⫰ {x_3}^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : {x_1}^*(\vec{q}) ⫰ {x_2}^*(\vec{q}) \text{ and } (x_1 + x_2)^*(\vec{q}) ⫰ {x_3}^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : q_1 ⫰ q_2 \text{ and } (q_1 + q_2) ⫰ q_3\}
\end{aligned}
$$

and

$$
\begin{aligned}
D_{\psi,3} &= \{\vec{q} \in A^3 : \vec{q} \in D_{x_1} \cap D_{x_2+x_3} \text{ and } {x_1}^*(\vec{q}) ⫰ (x_2 + x_3)^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : \vec{q} \in D_{x_1} \cap D_{x_2} \cap D_{x_3} \text{ and } {x_2}^*(\vec{q}) ⫰ {x_3}^*(\vec{q}) \text{ and } {x_1}^*(\vec{q}) ⫰ (x_2 + x_3)^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : {x_2}^*(\vec{q}) ⫰ {x_3}^*(\vec{q}) \text{ and } {x_1}^*(\vec{q}) ⫰ (x_2 + x_3)^*(\vec{q})\} \\
&= \{\vec{q} \in A^3 : q_2 ⫰ q_3 \text{ and } q_1 ⫰ (q_2 + q_3)\}
\end{aligned}
$$

We have that $q_1 ⫰ q_2$, $q_2 ⫰ q_3$ and $q_1 ⫰ (q_2 + q_3)$. Consequently, due to the fact that $q_2 ⫰ q_2$ and $q_2 ⫰ q_3$, we have $q_2 ⫰ (q_2 + q_3)$. Therefore, any two of q_1, q_2 and $q_2 + q_3$ are commeasurable, which implies that the algebra of the polynomials in q_1, q_2 and $q_2 + q_3$ is a commutative algebra over the field of real numbers.

Additionally, we also know $(q_1 + q_2) ⫰ q_3$. Analogously to what we have previously done, since $q_2 ⫰ q_2$ and $q_2 ⫰ q_1$, $q_2 ⫰ (q_1 + q_2)$. Thus, any two of $q_1 + q_2$, q_2 and q_3 are commeasurable and, consequently, the algebra of the polynomials in $q_1 + q_2$, q_2 and q_3 is a commutative algebra over the field of real numbers. Then, we have:

$$
\begin{aligned}
\varphi^*(\vec{q}) &\stackrel{\text{(i)}}{=} (q_1 + q_2) + q_3 \\
&\stackrel{\text{(ii)}}{=} [(q_1 + q_2) + q_3] + \mathbf{0} \\
&\stackrel{\text{(iii)}}{=} [(q_1 + q_2) + q_3] + (q_2 - q_2) \\
&\stackrel{\text{(iv)}}{=} (q_1 + q_2) + [q_3 + (q_2 - q_2)] \\
&\stackrel{\text{(v)}}{=} (q_1 + q_2) + [(q_2 - q_2) + q_3] \\
&\stackrel{\text{(vi)}}{=} (q_1 + q_2) + [(-q_2 + q_2) + q_3] \\
&\stackrel{\text{(vii)}}{=} (q_1 + q_2) + [-q_2 + (q_2 + q_3)] \\
&\stackrel{\text{(viii)}}{=} [(q_1 + q_2) - q_2] + (q_2 + q_3) \\
&\stackrel{\text{(ix)}}{=} [q_1 + (q_2 - q_2)] + (q_2 + q_3) \\
&\stackrel{\text{(x)}}{=} [q_1 + \mathbf{0}] + (q_2 + q_3) \\
&\stackrel{\text{(xi)}}{=} q_1 + (q_2 + q_3) \\
&\stackrel{\text{(xii)}}{=} \psi^*(\vec{q})
\end{aligned}
$$

(i) Definition of $\varphi^*(\vec{q})$

(ii) Since $q_1 + q_2$, q_2 and q_3 are all commeasurable, then there exists the identity element for the operation $+$, which is represented by **0**

(iii) $q_1 + q_2$, q_2 and q_3 have a symmetric element. $-q_2$ is the symmetric element of q_2

(iv) Since $q_1 + q_2$, q_2 and q_3 are all commeasurable, then we can apply the associative rule

(v) Commutativity (of the algebra of the polynomials in $q_1 + q_2$, q_2 and q_3)

(vi) Commutativity (of the algebra of the polynomials in $q_1 + q_2$, q_2 and q_3)

(vii) Associativity (of the algebra of the polynomials in $q_1 + q_2$, q_2 and q_3)

(viii) Associativity (of the algebra of the polynomials in q_1, q_2 and $q_2 + q_3$)

(ix) Associativity (of the algebra of the polynomials in q_1, q_2 and $q_2 + q_3$)

(x) $q_2 - q_2 = \mathbf{0}$

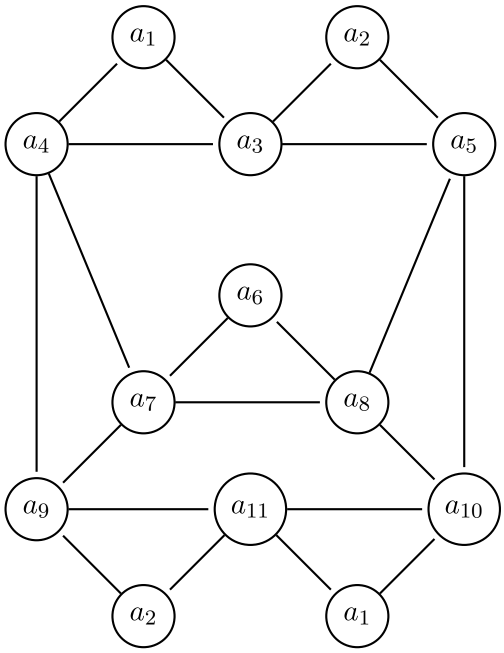
(xi) $\mathbf{0}$ is the identity for the operation $+$

(xii) Definition of $\psi^*(\vec{q})$

$\square$

Example 3. Let us consider $\varphi = (x_1 + x_2) + (x_3 + x_4)$ and $\psi = (x_1 + x_4) + (x_2 + x_3)$ two polynomials in 4 variables. The identity $\varphi = \psi$ does not weakly (and, consequently, strongly) hold in all partial algebras.

Proof. We want to show that there exists a partial algebra such that for all $\vec{q} = (q_1, q_2, q_3, q_4) \in D_{\varphi,4} \cap D_{\psi,4}$ and $\varphi^*(\vec{q}) \downarrow_{\circ} \psi^*(\vec{q})$, $\varphi^*(\vec{q}) \neq \psi^*(\vec{q})$, that is, for all $\vec{q} \in D_{\varphi,4} \cap D_{\psi,4}$ and $\varphi^*(\vec{q}) \downarrow_{\circ} \psi^*(\vec{q})$, $((x_1 + x_2) + (x_3 + x_4))^*(\vec{q}) \neq ((x_1 + x_4) + (x_2 + x_3))^*(\vec{q})$, which simplifies to $(q_1 + q_2) + (q_3 + q_4) \neq (q_1 + q_4) + (q_2 + q_3)$. Let us denote the following graph $\mathcal{G}$, which satisfies the condition C (note that the vertices a_1 and a_2 appear twice in the graph):



Now, let us consider four functions f_1, f_2, f_3 and f_4 such that $dom_{f_1} = \{a_1, a_3, a_4\}$, $dom_{f_2} = \{a_1, a_3, a_4\}$, $dom_{f_3} = \{a_2, a_3, a_5\}$ and $dom_{f_4} = \{a_2, a_9, a_{11}\}$. We are going to define them as follows:

$$f_1 : dom_{f_1} \to \mathbb{R} \quad f_2 : dom_{f_2} \to \mathbb{R} \quad f_3 : dom_{f_3} \to \mathbb{R} \quad f_4 : dom_{f_4} \to \mathbb{R}$$

$$f_1(a_1) = 1 \quad f_2(a_1) = 0 \quad f_3(a_2) = 1 \quad f_4(a_2) = 0$$

$$f_1(a_3) = 0 \quad f_2(a_3) = 1 \quad f_3(a_3) = 0 \quad f_4(a_9) = 0$$

$$f_1(a_4) = 0 \quad f_2(a_4) = 0 \quad f_3(a_5) = 0 \quad f_4(a_{11}) = 1$$

Observation: All the functions f_1, f_2, f_3 and f_4 belong to the class of functions F associated with the graph $\mathcal{G}$ as they yield real numbers and their domain is a set of three vertices of $\mathcal{G}$ any two of which are connected. So, henceforth, when introducing new functions and assuming they belong to F, it is implied that their domain is one of the triangles of the graph $\mathcal{G}$ and they produce real values.

We are going to consider the previous four functions as the representatives of the observables q_1, q_2, q_3 and q_4, i.e., $q_1 = [f_1]_E$, $q_2 = [f_2]_E$, $q_3 = [f_3]_E$ and $q_4 = [f_4]_E$.

Then, we have:

- $q_1 + q_2 = [f_1 + f_2]_E$ and we obtain

$$f_1 + f_2 : dom_{f_1} \longrightarrow \mathbb{R}$$

$$(f_1 + f_2)(a_1) = 1 + 0 = 1$$

$$(f_1 + f_2)(a_3) = 0 + 1 = 1$$

$$(f_1 + f_2)(a_4) = 0 + 0 = 0$$

- q_3+q_4 is not equal to $[f_3 + f_4]_E$ because f_3 and f_4 have different domains and, therefore, they are not commeasurable and the operation sum is not defined. So, we just need to consider, for instance, a function g_3 such that $dom_{g_3} = dom_{f_4}$ and $g_3 \in [f_3]_E$, that is, $g_3 \in F$ and $E(f_3, g_3)$. Let $dom_{g_3} = dom_{f_4}$ and $g_3(a_2) = f_3(a_2) = 1$ and $g_3(a_9) = g_3(a_{11}) = f_3(a_3) = f_3(a_5) = 0$. Then, dom_{g_3} and dom_{f_3} have one element in common and, by defining g_3 in this way, it satisfies the second statement of the definition of the relation E (3.3.3). So, $q_3 + q_4 = [g_3 + f_4]_E$ and we obtain

$$g_3 + f_4 : dom_{g_3} \longrightarrow \mathbb{R}$$

$$(g_3 + f_4)(a_2) = 1 + 0 = 1$$

$$(g_3 + f_4)(a_9) = 0 + 0 = 0$$

$$(g_3 + f_4)(a_{11}) = 0 + 1 = 1$$

- In a similar manner to the previous case, the domains of f_1 and f_4 are not the same and, therefore, it does not make sense to define $q_1 + q_4$ as $[f_1 + f_4]_E$. So, let us choose two functions g_1 and g_4 such that $g_1 \in [f_1]_E$, $g_4 \in [f_4]_E$ and $dom_{g_1} = dom_{g_4}$. We need to guarantee that $E(f_1, g_1)$ and $E(f_4, g_4)$ hold. To address this, we are going to consider $dom_{g_1} = \{a_1, a_{11}, a_{10}\}$, $g_1(a_1) = 1$ and $g_1(a_{11}) = g_1(a_{10}) = 0$ as it is aligned with the second statement of the definition of the relation E (3.3.3). In an analogous way, let us consider $dom_{g_4} = \{a_1, a_{11}, a_{10}\}$, $g_4(a_{11}) = 1$ and $g_1(a_1) = g_1(a_{10}) = 0$.

  (It should be noted that we could have chosen, for instance, a function $g_4 \in [f_4]_E$ such that $dom_{g_4} = dom_{f_1}$. However, in this case, we are following the counterexample provided by the authors of the article [9] and it is known that finding these counterexamples is not straightforward). So, $q_1 + q_4 = [g_1 + g_4]_E$ and we obtain

$$g_1 + g_4 : dom_{g_1} \longrightarrow \mathbb{R}$$
$$(g_1 + g_4)(a_1) = 1 + 0 = 1$$
$$(g_1 + g_4)(a_{10}) = 0 + 0 = 0$$
$$(g_1 + g_4)(a_{11}) = 0 + 1 = 1$$

- Once again, since dom_{f_2} and dom_{f_3} are not the same, it does not make sense to define $q_2 + q_3$ as $[f_2 + f_3]_E$. So, we are going to find a function g_2 such that $dom_{g_2} = dom_{f_3}$ and $E(f_2, g_2)$ holds. Let us consider $dom_{g_2} = dom_{f_3}$, $g_2(a_3) = f_2(a_3)$ and $g_2(a_2) = g_2(a_5) = f_2(a_1) = f_2(a_4) = 0$. Then $E(f_2, g_2)$ holds, as we are under the second statement of the definition of the relation E (3.3.3). So, $q_2 + q_3 = [g_2 + f_3]_E$ and we obtain

$$g_2 + f_3 : dom_{g_2} \longrightarrow \mathbb{R}$$
$$(g_2 + f_3)(a_2) = 0 + 1 = 1$$
$$(g_2 + f_3)(a_3) = 1 + 0 = 1$$
$$(g_2 + f_3)(a_5) = 0 + 0 = 0$$

- We need the compatibility of $q_1 + q_2$ and $q_3 + q_4$. Therefore, the previous representatives of these classes, $f_1 + f_2$ and $g_3 + f_4$, respectively, don't work for these cases. We want to find $h_1 \in q_1 + q_2$ and $h_2 \in q_3 + q_4$ such that $dom_{h_1} = dom_{h_2}$ and $E(h_1, f_1 + f_2)$ and $E(h_2, g_3 + f_4)$ hold. Let us consider $dom_{h_i} = \{a_4, a_7, a_9\}$ and h_i defined as follows, for $i \in \{1, 2\}$:

$$
\begin{array}{ll}
h_1 : dom_{h_1} \to \mathbb{R} & h_2 : dom_{h_2} \to \mathbb{R} \\
h_1(a_4) = 0 & h_2(a_4) = 1 \\
h_1(a_7) = 1 & h_2(a_7) = 1 \\
h_1(a_9) = 1 & h_2(a_9) = 0
\end{array}
$$

$h_1 \in q_1 + q_2$ because $h_1 \in F$ and $E(h_1, f_1 + f_2)$ holds (it satisfies the second statement of the definition of the relation E (3.3.3)) and $h_2 \in q_3 + q_4$ because $h_2 \in F$ and $E(h_2, g_3 + f_4)$ holds (for the same reason).

So, $(q_1 + q_2) + (q_3 + q_4) = [h_1 + h_2]_E$ and we obtain:

$$
\begin{aligned}
&h_1 + h_2 : \{a_4, a_7, a_9\} \longrightarrow \mathbb{R} \\
&(h_1 + h_2)(a_4) = 0 + 1 = 1 \\
&(h_1 + h_2)(a_7) = 1 + 1 = 2 \\
&(h_1 + h_2)(a_9) = 1 + 0 = 1
\end{aligned}
$$

- Now, we need the compatibility of $q_1 + q_4$ and $q_2 + q_3$. Once again, the representatives of these classes, $g_1 + g_4$ and $g_2 + f_3$, respectively, don't work for these cases. So, we are going to find two functions $h_3 \in q_1 + q_4$, $h_4 \in q_2 + q_3$ such that $dom_{h_3} = dom_{h_4}$, $h_3, h_4 \in F$ and $E(h_3, g_1 + g_4)$ and $E(h_4, g_2 + f_3)$ hold. Let us consider $dom_{h_3} = dom_{h_4} = \{a_5, a_8, a_{10}\}$ and h_i defined as follows, for $i \in \{3, 4\}$:

$$
\begin{array}{ll}
h_3 : dom_{h_3} \to \mathbb{R} & h_4 : dom_{h_4} \to \mathbb{R} \\
h_3(a_5) = 1 & h_4(a_5) = 0 \\
h_3(a_8) = 1 & h_4(a_8) = 1 \\
h_3(a_{10}) = 0 & h_4(a_{10}) = 1
\end{array}
$$

$h_3 \in q_1 + q_4$ because $h_3 \in F$ and $E(h_3, g_1 + g_4)$ holds (it satisfies the second statement of the definition of the relation E (3.3.3)) and $h_4 \in q_2 + q_3$ because $h_4 \in F$ and $E(h_4, g_2 + f_3)$ holds (for the same reason).

So, $(q_1 + q_4) + (q_2 + q_3) = [h_3 + h_4]_E$ and we obtain:

$$h_3 + h_4 : \{a_5, a_8, a_{10}\} \longrightarrow \mathbb{R}$$
$$(h_3 + h_4)(a_5) = 1 + 0 = 1$$
$$(h_3 + h_4)(a_8) = 1 + 1 = 2$$
$$(h_3 + h_4)(a_{10}) = 0 + 1 = 1$$

The final aim is to compare both the results of $(q_1+q_2)+(q_3+q_4)$ and $(q_1+q_4)+(q_2+q_3)$. Since, currently, they have different domains, we are going to try to find two functions $k_1 \in (q_1+q_2)+(q_3+q_4)$ and $k_2 \in (q_1+q_4)+(q_2+q_3)$ such that $dom_{k_1} = dom_{k_2}$ and they yield different values (to prove, in fact, that this identity does not hold in this partial algebra). Let us consider $dom_{k_1} = dom_{k_2} = \{a_6, a_7, a_8\}$ and k_i defined as follows, for $i \in \{1, 2\}$:

$$\begin{array}{ll} k_1 : dom_{k_1} \to \mathbb{R} & k_2 : dom_{k_2} \to \mathbb{R} \\ k_1(a_6) = 1 & k_2(a_6) = 1 \\ k_1(a_7) = 2 & k_2(a_7) = 1 \\ k_1(a_8) = 1 & k_2(a_8) = 2 \end{array}$$

$k_1 \in (q_1 + q_2) + (q_3 + q_4)$ because $k_1 \in F$ and $E(k_1, h_1 + h_2)$ holds (it satisfies the second statement of the definition of the relation E (3.3.3)) and $k_2 \in (q_1 + q_4) + (q_2 + q_3)$ because $k_2 \in F$ and $E(k_2, h_3 + h_4)$ holds (for the same reason).

So, the observables $(q_1 + q_2) + (q_3 + q_4)$ and $(q_1 + q_4) + (q_2 + q_3)$ are commeasurable, as they have the same domain, but they are different because they are represented by different functions.

Therefore, the identity $\varphi = \psi$ does not hold in all partial algebras, as it does not hold in this one. $\square$

Chapter 4

Partial Boolean algebras

In this chapter, we will introduce partial Boolean algebras independently of partial algebras, along with associated definitions and propositions, and induced by partial algebras. Boolean polynomials within the context of partial Boolean algebras, including their domain and associated mappings, will be discussed. It will also be proven, along with other results, that every Boolean algebra is a partial Boolean algebra.

4.1 Partial Boolean algebra and its properties

Definition 4.1.1. *A partial Boolean algebra* $\mathcal{B} = (B, \downarrow\!\!\circ, \vee, \neg, \mathbf{1}, \mathbf{0})$ *is defined by a nonempty set* B*, a binary relation* $\downarrow\!\!\circ$ *on* B*, called compatibility or commeasurability, a partial binary function on* B*,* $\vee$*, a unary function,* $\neg$*, and two elements of* B*,* **0** *and* **1***, with the following properties:*

1. *The relation* $\downarrow\!\!\circ$ *is reflexive and symmetric*
2. *For all* $q \in B$*,* $q \downarrow\!\!\circ \mathbf{0}$ *and* $q \downarrow\!\!\circ \mathbf{1}$
3. *The partial binary function* $\vee$ *is defined exactly for those pairs* $(q_1, q_2) \in B \times B$ *for which* $q_1 \downarrow\!\!\circ q_2$
4. *If any two of* q_1*,* q_2 *and* q_3 *are commeasurable (*q_1*,* q_2*,* $q_3 \in B$*), then* $(q_1 \vee q_2) \downarrow\!\!\circ q_3$ *and* $(\neg q_1) \downarrow\!\!\circ q_2$
5. *If any two of* q_1*,* q_2 *and* q_3 *are commeasurable (*q_1*,* q_2*,* $q_3 \in B$*), then the algebra of the Boolean polynomials in* q_1*,* q_2 *and* q_3 *(defined in the observation below) is a Boolean algebra*

Observation: Let us assume $\mathcal{B} = (B, \downarrow\!\!\circ, \vee, \neg, \mathbf{1}, \mathbf{0})$ defined as in the previous definition. If any two of q_1, q_2 and q_3 are commeasurable, then the algebra of the Boolean polynomials in q_1, q_2 and q_3 is the structure $\mathcal{B}' = (B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$, where:

- $B' \subseteq B$ is inductively defined:

    1. $q_1, q_2, q_3 \in B'$
    2. $\mathbf{0}, \mathbf{1} \in B'$
    3. If $x, y \in B'$ and $x \mathbin{\lfloor_\circ} y$, then $x \vee y \in B'$ and $x \wedge' y \in B'$
    4. If $x \in B'$, then $\neg x \in B'$

- The operations $\vee'$, $\neg'$ and $\wedge'$ are defined as:

  $\vee' = \vee|_{B' \times B'}$
  $\neg' = \neg|_{B'}$
  $\wedge' = B' \times B' \to B'$ such that for all $a, b \in B'$, with $a \mathbin{\lfloor_\circ} b$, $a \wedge' b = \neg'(\neg' a \vee' \neg' b)$

Now, we will verify that the previous structure $(B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$ constitutes an algebraic structure, that is, the operations $\vee'$, $\wedge'$ and $\neg'$ are total functions and that B' is closed under these operations.

Lemma 4.1.2. *Any two elements in B' are compatible.*

Proof. Let us suppose that $x \in B'$, $q_1 \mathbin{\lfloor_\circ} q_2$, $q_1 \mathbin{\lfloor_\circ} q_3$ and $q_2 \mathbin{\lfloor_\circ} q_3$. Let us consider $P(x)$ the property: for all $z \in B'$, $x \mathbin{\lfloor_\circ} z$. The proof follows by induction on $x \in B'$.

1. We want to show $P(q_1)$, that is, for all $z \in B'$, $q_1 \mathbin{\lfloor_\circ} z$. Let $z \in B'$ and let us consider, now, the following property: for all $y \in B'$, $Q(y)$ iff $y \mathbin{\lfloor_\circ} q_1$.

    (i) $Q(q_1)$ iff $y \mathbin{\lfloor_\circ} y$. Since $\lfloor_\circ$ is reflexive, then $q_1 \mathbin{\lfloor_\circ} q_1$ holds.

    (ii) $Q(q_2)$ iff $q_1 \mathbin{\lfloor_\circ} q_2$, which is true based on the hypothesis.

    (iii) $Q(q_3)$ iff $q_1 \mathbin{\lfloor_\circ} q_3$, which is true based on the hypothesis.

    (iv) $Q(\mathbf{1})$ iff $\mathbf{1} \mathbin{\lfloor_\circ} q_1$ and $Q(\mathbf{0})$ iff $\mathbf{0} \mathbin{\lfloor_\circ} q_1$. Since $q_1 \in B'$, $\mathbf{1}$ and $\mathbf{0}$ are commeasurable with all the elements in B and $B' \subseteq B$, then $\mathbf{1} \mathbin{\lfloor_\circ} q_1$ and $\mathbf{0} \mathbin{\lfloor_\circ} q_1$.

    (v) Let us suppose $Q(x), Q(y)$ and $x \mathbin{\lfloor_\circ} y$, for $x, y \in B'$. We want to show $Q(x \vee y)$ and $Q(x \wedge' y)$, that is, $(x \vee y) \mathbin{\lfloor_\circ} q_1$ and $(x \wedge' y) \mathbin{\lfloor_\circ} q_1$, respectively. Since $x \mathbin{\lfloor_\circ} q_1$, $y \mathbin{\lfloor_\circ} q_1$ and $x \mathbin{\lfloor_\circ} y$, then we have that any two of x, y, q_1 are commeasurable. On one hand, by definition 4.1.1, $(x \vee y) \mathbin{\lfloor_\circ} q_1$. So, $Q(x \vee y)$. On the other hand, we also have that $\neg x$, $\neg y$ and q_1 are all commeasurable. Therefore, by definition 4.1.1, $(\neg(\neg x \vee \neg y)) \mathbin{\lfloor_\circ} q_1$. Since $\neg'$ and $\vee'$ are defined such that their domain is restricted to B' and $B' \times B'$, respectively, we have, by definition of restriction of a function, that the value of $\neg'(\neg' x \vee' \neg' y)$ is the same of $\neg(\neg x \vee \neg y)$. So, $(\neg'(\neg' x \vee' \neg' y)) ⫰ q_1$, that is, $(x \wedge' y) ⫰ q_1$, and we conclude $Q(x \wedge' y)$.

(vi) Let us suppose $Q(x)$, for $x \in B'$. We want to show $Q(\neg x)$, that is, $(\neg x) ⫰ q_1$. Since, by $Q(x)$, $x ⫰ q_1$ then, by definition 4.1.1, $(\neg x) ⫰ q_1$.

The proofs of $P(q_2)$ and $P(q_3)$ are analogous.

2. We want to show $P(\mathbf{1})$ and $P(\mathbf{0})$, that is, for all $z \in B'$, $\mathbf{1} ⫰ z$ and $\mathbf{0} ⫰ z$, respectively. Since **1** and **0** are commeasurable with all the elements in B and $B' \subseteq B$, then $\mathbf{1} ⫰ z$ and $\mathbf{0} ⫰ z$.

3. Let us suppose $P(x), P(y)$ and $x ⫰ y$, for $x, y \in B'$. We want to show $P(x \vee y)$ and $P(x \wedge' y)$, that is, for all $z \in B'$, $(x \vee y) ⫰ z$ and $(x \wedge' y) ⫰ z$, respectively. Let $z \in B'$. By the hypothesis $P(x)$ and $P(y)$ we have, respectively, that $x ⫰ z$ and $y ⫰ z$, and by the fact that $x ⫰ y$, any two of x, y, z are commeasurable. So, by definition 4.1.1, $(x \vee y) ⫰ z$, i.e. $Q(x \vee z)$. To prove $Q(x \wedge' y)$, we just need to observe that from the hypothesis we also have that $\neg x$, $\neg y$, z are all commeasurable and, by definition 4.1.1, $(\neg(\neg x \vee \neg y)) ⫰ z$. Once again, by definition of restriction of a function, the value of $\neg'(\neg' x \vee' \neg' y)$ is the same of $\neg(\neg x \vee \neg y)$. So, $(x \wedge' y) ⫰ z$, that is, $Q(x \wedge' y)$.

4. Let us suppose $P(x)$, for $x \in B'$. We want to show $P(\neg x)$, that is, for all $z \in B'$, $x ⫰ z$. Let $z \in B'$. By the hypothesis $P(x)$, $x ⫰ z$. So, by definition 4.1.1, $(\neg x) ⫰ z$.

So $P(x)$, for all $x \in B'$. □

Proposition 4.1.3. *The operations $\vee'$, $\wedge'$ and $\neg'$ are total functions and B' is closed under these operations.*

Proof. Let us consider $x, y \in B'$. We want to show that $x \vee' y \in B'$, $x \wedge' y \in B'$, and $\neg' x \in B'$. By lemma 4.1.2, $x ⫰ y$. Consequently, the elements $x \vee y, x \wedge' y, \neg x \in B'$ (due to the statements 3 and 4 of the definition of B'). It remains to prove that $x \vee' y, \neg' x \in B'$. By definition of restriction of a function, the value of $x \vee' y$ is $x \vee y$ and the value of $\neg' x$ is $\neg x$. So, $x \vee' y, \neg' x \in B'$. □

Proposition 4.1.4. *We can generalize the statement 5 of the definition of partial Boolean algebra (definition 4.1.1) to any number of observables, that is, if any two of $q_1, \cdots, q_n$ are commeasurable, $n \in \mathbb{N}$, then the algebra of the Boolean polynomials in $q_1, \cdots, q_n$ is a Boolean algebra.*

4.2 Partial Boolean algebra induced by a partial algebra

Definition 4.2.1. *Let $\mathcal{A} = (A, ⫰, +, \cdot, \circ, \mathbf{1})$ be a partial algebra. The structure induced by $\mathcal{A}$, $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$, is defined as follows:*

1. *$B = \{a \in A : a \cdot a = a\}$, that is, the elements of B are the idempotent elements of A*
2. *$a ⫰ b$ iff $a ⫰ b$ in $\mathcal{A}$*
3. *$a \vee b = (a + b) - a \cdot b$*
4. *$\neg a = \mathbf{1} - a$*
5. *$\mathbf{1}$ is the $\mathbf{1}$ in $\mathcal{A}$*
6. *$\mathbf{0} = 0 \circ \mathbf{1}$, where 0 is the real number*

Observation: *$a - b$ is to be interpreted as $a + (-1) \circ b$, for all $a, b \in A$.*

Theorem 4.2.2. *The structure $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$, induced by the partial algebra $\mathcal{A} = (A, ⫰, +, \cdot, \circ, \mathbf{1})$, is a partial Boolean algebra.*

Proof. Let $\mathcal{A} = (A, ⫰, +, \cdot, \circ, \mathbf{1})$ be a partial algebra and $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$ be the induced structure. We check the conditions 1. to 5. of the definition 4.1.1.

1. By definition 4.2.1, the relation $⫰$ in $\mathcal{B}$ is the same as in $\mathcal{A}$. So, $⫰$ is reflexive and symmetric.
2. Since $⫰$ is the same as in $\mathcal{A}$, then $q ⫰ \mathbf{1}$. It remains to show that $q ⫰ \mathbf{0}$.

   Let $q \in B$. By definition 4.2.1, $q ⫰ \mathbf{0}$ if $q ⫰ (0 \circ \mathbf{1})$. But $q ⫰ (0 \circ \mathbf{1})$ if, by definition 3.1.1, $q ⫰ \mathbf{1}$, which we already know it is always true. So, $q ⫰ \mathbf{0}$.
3. Let $q_1, q_2 \in B$. We want to prove that $\vee$ is exactly defined when $q_1 ⫰ q_2$. By definition 4.2.1, $q_1 \vee q_2 = (q_1 + q_2) - q_1 \cdot q_2$ and, by definition 3.1.1, the partial binary functions $+$ and $\cdot$ are defined exactly when $q_1 ⫰ q_2$.
4. Let $q_1, q_2, q_3 \in B$ such that $q_i ⫰ q_j$, for all $i, j \in \{1, 2, 3\}$. Then, by definition 3.1.1, $(q_1 + q_2) ⫰ q_3$ and $(q_1 \cdot q_2) ⫰ q_3$. Applying, again, definition 3.1.1, we obtain $[(-1) \circ (q_1 \cdot q_2)] ⫰ q_3$, which is the same as $[-(q_1 \cdot q_2)] ⫰ q_3$. Once $(q_1 + q_2) ⫰ q_3$, $[-(q_1 \cdot q_2)] ⫰ q_3$ and $(q_1 + q_2) ⫰ [-(q_1 \cdot q_2)]$ (see the observation below), then, by definition 3.1.1, $[(q_1 + q_2) - (q_1 \cdot q_2)] ⫰ q_3$. So, by definition 4.2.1, $(q_1 \vee q_2) ⫰ q_3$.

It remains to show that $(\neg q_1) \mathrel{\underset{\circ}{\rfloor}} q_2$. Once $q_1 \mathrel{\underset{\circ}{\rfloor}} q_2$ then, by definition 3.1.1, $(-q_1) \mathrel{\underset{\circ}{\rfloor}} q_2$. So, we have that $-q_1$, q_2 and **1** are all compatible and, by definition 3.1.1, $(\mathbf{1} - q_1) \mathrel{\underset{\circ}{\rfloor}} q_2$, which is equal, by definition 4.2.1, to $(\neg q_1) \mathrel{\underset{\circ}{\rfloor}} q_2$.

Observation: Since q_1, q_2 and $q_1 + q_2$ are all commeasurable, then $(q_1 + q_2) \mathrel{\underset{\circ}{\rfloor}} (q_1 \cdot q_2)$. From here, follows that $(q_1 + q_2) \mathrel{\underset{\circ}{\rfloor}} [-(q_1 \cdot q_2)]$.

5. Let us consider that any two of q_1, q_2 and q_3 are commeasurable. We want to prove that the algebra of the polynomials in q_1, q_2, q_3, that is, $\mathcal{B}' = (B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$, is a Boolean algebra, where $\mathcal{B}'$ is defined as in the observation of the previous section (4.1).

   Observations:

   - We demonstrated above that any two elements in B' are commeasurable, the operations $\vee'$, $\wedge'$ and $\neg'$ are total functions and B' is closed under these operations.
   - It is also important to note that the algebra of the polynomials in q_1, q_2 and q_3, i.e., $(B', +', \cdot', \circ', \mathbf{1})$, is a commutative algebra over the field of real numbers, where $+' : +|_{B' \times B'}$, $\cdot' : \cdot|_{B' \times B'}$ and $\circ' : \circ|_{\mathbb{R} \times B'}$. This fact will be utilized in this proof.
   - We will denote $-1 \circ' a$ as $-'a$, for all $a \in B'$.
   - Throughout this demonstration, we will state that for all $a \in B'$, $-'a$ is the symmetric of a. This happens because $-a' +' a = (-1 \circ' a) +' (1 \circ' a) = (-1 + 1) \circ' a = 0 \circ' a = \mathbf{0}$, where $0, 1$ and -1 are real numbers, $+$ is defined in the field of real numbers and $\mathbf{0} \in B'$ (this is a result mentioned in the preliminaries).
   - $\mathbf{0}$ is the identity for the operation $+$ because for all $a \in B'$, $\mathbf{0} +' a = (0 \circ' a) +' (1 \circ' a) = (0 + 1) \circ' a = a$.

   Let us consider the structure $(B', \wedge', \vee')$. We need to show that it is a lattice, that is, for all $a, b, c \in B'$, we have:

   (i) Idempotency for $\vee'$ and $\wedge'$, that is, $a \vee' a = a \wedge' a = a$

   $a \vee' a \overset{(a)}{=} a +' a -' a \cdot' a \overset{(b)}{=} a +' a -' a \overset{(c)}{=} a$

   (a) Definition of $\vee'$

   (b) a is an idempotent element

   (c) For all $a \in B'$, there exists $-a \in B'$ such that $a +' (-'a) = \mathbf{0}$, $\mathbf{0} \in B'$, which is the identity for the $+$ operation

$$
\begin{aligned}
a \wedge' a &\overset{\text{(a)}}{=} \neg'(\neg' a \vee' \neg' a) \\
&\overset{\text{(b)}}{=} \neg'(\neg' a +' \neg' a -' \neg' a \cdot' \neg' a) \\
&\overset{\text{(c)}}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' a) -' (\mathbf{1} -' a) \cdot' (\mathbf{1} -' a)) \\
&\overset{\text{(d)}}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' a) -' (\mathbf{1} \cdot' \mathbf{1} -' \mathbf{1} \cdot' a -' a \cdot \mathbf{1} +' a \cdot' a)) \\
&\overset{\text{(e)}}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' a) -' (\mathbf{1} -' a)) \\
&\overset{\text{(f)}}{=} \neg'(\mathbf{1} -' a +' \mathbf{1} -' a -' \mathbf{1} +' a) \\
&\overset{\text{(g)}}{=} \mathbf{1} -' (\mathbf{1} -' a) \\
&\overset{\text{(h)}}{=} a
\end{aligned}
$$

(a) Definition of $\wedge'$

(b) Definition of $\vee'$

(c) Definition of $\neg'$

(d) Distributivity of $\cdot'$ with respect to $+'$

(e) **1** is the identity for the operation $\cdot'$; **1** and a are idempotent elements; all the elements in B' have a symmetric one

(f) Distributivity of scalar multiplication with respect to vector addition; associativity

(g) All the elements in B' have a symmetric one; **0** is the identity of the operation $+'$

(h) Distributivity of scalar multiplication with respect to vector addition; all the elements in B' have a symmetric one; associativity; **0** is the identity of the operation $+'$

(ii) Commutativity for $\vee'$ and $\wedge'$, that is, $a \vee' b = b \vee' a$ and $a \wedge' b = b \wedge' a$

$$a \vee' b \overset{\text{(a)}}{=} (a +' b) -' a \cdot b \overset{\text{(b)}}{=} (b +' a) -' b \cdot a \overset{\text{(c)}}{=} b \vee' a$$

(a) Definition of $\vee'$

(b) Commutativity of the operations $+'$ and $\cdot'$

(c) Definition of $\vee'$

It is analogous for $\wedge'$.

(iii) Associativity for $\vee'$ and $\wedge'$, that is, $a \vee' (b \vee' c) = (a \vee' b) \vee' c$ and $a \wedge' (b \wedge' c) = (a \wedge' b) \wedge' c$

$$
\begin{aligned}
a \vee' (b \vee' c) &\overset{\text{(a)}}{=} (a +' (b \vee' c)) -' (a \cdot' (b \vee' c)) \\
&\overset{\text{(b)}}{=} a +' (b +' c -' (b \cdot' c)) -' (a \cdot' (b +' c -' (b \cdot' c))) \\
&\overset{\text{(c)}}{=} a +' (b +' c -' (b \cdot' c)) -' ((a \cdot' b) + (a \cdot' c) -' (a \cdot' (b \cdot' c))) \\
&\overset{\text{(d)}}{=} a +' (b +' c -' (b \cdot' c)) -' a \cdot' b - a \cdot' c +' (a \cdot' (b \cdot' c)) \\
&\overset{\text{(e)}}{=} a +' b -' (a \cdot' b) +' c -' (a +' b +' (a \cdot' b)) \cdot' c \\
&\overset{\text{(f)}}{=} (a \vee' b) +' c -' (a \vee' b) \cdot' c \\
&\overset{\text{(g)}}{=} (a \vee' b) \vee' c
\end{aligned}
$$

(a) Definition of $\vee'$

(b) Definition of $\vee'$

(c) Distributivity of $\cdot'$ with respect to $+'$

(d) Distributivity of scalar multiplication with respect to vector addition

(e) Commutativity; Associativity; Distributivity of scalar multiplication with respect to vector addition

(f) Definition of $\vee'$

(g) Definition of $\vee'$

It is analogous for $\wedge'$.

(iv) Absorption, that is, $a \wedge' (a \vee' b) = a \vee' (a \wedge' b) = a$

$$
\begin{aligned}
a \vee' (a \wedge' b) &\overset{\text{(a)}}{=} a \vee' (\neg'(\neg' a \vee' \neg' b)) \\
&\overset{\text{(b)}}{=} a \vee' (\neg'(\neg' a + \neg' b - (\neg' a \cdot' \neg' b))) \\
&\overset{\text{(c)}}{=} a \vee' (\neg'((\mathbf{1} -' a) +' (\mathbf{1} -' b) -' ((\mathbf{1} -' a) \cdot' (\mathbf{1} -' b)))) \\
&\overset{\text{(d)}}{=} a \vee' (\neg'((\mathbf{1} -' a) + (\mathbf{1} -' b) -' (\mathbf{1} -' b -' a +' (a \cdot' b)))) \\
&\overset{\text{(e)}}{=} a \vee' (\neg'(\mathbf{1} -' a + \mathbf{1} -' b -' \mathbf{1} +' b +' a -' (a \cdot' b))) \\
&\overset{\text{(f)}}{=} a \vee' (\neg'(\mathbf{1} -' (a \cdot' b))) \\
&\overset{\text{(g)}}{=} a \vee' (\mathbf{1} -' \mathbf{1} +' (a \cdot' b)) \\
&\overset{\text{(h)}}{=} a +' (a \cdot' b) -' (a \cdot' (a \cdot' b)) \\
&\overset{\text{(i)}}{=} a +' (a \cdot' b) -' (a \cdot' b) \\
&\overset{\text{(j)}}{=} a
\end{aligned}
$$

(a) Definition of $\wedge'$

(b) Definition of $\vee'$

(c) Definition of $\neg'$

(d) Distributivity of $\cdot'$ with respect to $+'$; **1** is the identity element for $\cdot'$

(e) Associativity; Distributivity of scalar multiplication with respect to vector addition

(f) $-'a$ is the symmetric of a; $-'\mathbf{1}$ is the symmetric of **1**; **0** is the identity for $+'$

(g) Definition of $\neg'$; Distributivity of scalar multiplication with respect to $+'$

(h) $-'\mathbf{1}$ is the symmetric of **1**; **0** is the identity for $+'$; Definition of $\vee'$

(i) Associativity; a is an idempotent element

(j) $-'(a \cdot' b)$ is the symmetric of $a \cdot' b$; **0** is the identity for $+'$

It is analogous to $a \wedge' (a \vee' b)$.

So, since in $(B', \wedge', \vee')$ we have the idempotency, commutativity, associativity and absorption laws, $(B', \wedge', \vee')$ is a lattice.

Now, we need to show that the lattice $(B', \wedge', \vee')$ is distributive, that is, for all $a, b, c \in B'$, $a \wedge' (b \vee' c) = (a \wedge' b) \vee' (a \wedge' c)$.

$a \wedge' (b \vee' c) \overset{(a)}{=}$

$\overset{(a)}{=} \neg'(\neg' a \vee' \neg'(b \vee' c))$

$\overset{(b)}{=} \neg'(\neg' a \vee' \neg'(b +' c -' b \cdot' c))$

$\overset{(c)}{=} \neg'(\neg' a \vee' (\mathbf{1} -' b -' c + b \cdot' c))$

$\overset{(d)}{=} \neg'(\neg' a +' (\mathbf{1} -' b' -' c +' b \cdot' c) -' \neg' a \cdot' (\mathbf{1} -' b -' c -' +' b \cdot' c))$

$\overset{(e)}{=} \neg'(\mathbf{1} -' a +' \mathbf{1} -' b -' c +' b \cdot' c -' (\mathbf{1} -' a) \cdot' (\mathbf{1} -' b -' c +' b \cdot' c))$

$\overset{(f)}{=} \neg(\mathbf{1} -' a +' \mathbf{1} -' b -' c +' b \cdot' c -' (\mathbf{1} -' b -' c +' b \cdot' c -' a +' a \cdot' b +'$

$a \cdot' c -' a \cdot' (b \cdot' c)))$

$\overset{(g)}{=} \mathbf{1} -' \mathbf{1} +' a \cdot' b +' a \cdot' c -' a \cdot' (b \cdot' c)$

$\overset{(h)}{=} a \cdot' b +' a \cdot' c -' (a \cdot' a) \cdot' (b \cdot' c)$

$\overset{(i)}{=} a \cdot' b +' a \cdot' c -' (a \cdot' b) \cdot' (a \cdot' c)$

$\overset{(j)}{=} (a \cdot' b) \vee' (a \cdot' c)$

$\overset{(k)}{=} (a \wedge' b) \vee' (a \wedge' c)$

(a) Definition of $\wedge'$

(b) Definition of $\vee'$

(c) Definition of $\neg'$

(d) Definition of $\vee'$

(e) Definition of $\neg'$

(f) Distributivity of $\cdot'$ with respect to $+'$; **1** is the identity for $\cdot'$

(g) Definition of $\neg'$; $-'a$ is the symmetric of a; $-'b$ is the symmetric of b; $-'c$ is the symmetric of c; $-'(b \cdot' c)$ is the symmetric of $b \cdot' c$; **0** is the identity for $+'$

(h) $-'\mathbf{1}$ is the symmetric element of **1**; **0** is the identity for $+'$; a is an idempotent element; Associativity

(i) Associativity; Commutativity

(j) Definition of $\vee'$

(k) $a \wedge' b = \neg'(\neg' a \vee' \neg' b)$

$$= \neg'(\neg' a + \neg' b -' (\neg' a) \cdot' (\neg' b))$$

$$= \neg'(\mathbf{1} -' a +' \mathbf{1} -' b -' ((\mathbf{1} -' a) \cdot' (\mathbf{1} -' b)))$$

$$= \neg'(\mathbf{1} -' a +' \mathbf{1} -' b -' \mathbf{1} +' b +' a -' a \cdot' b)$$

$$= \mathbf{1} -' \mathbf{1} +' a \cdot' b$$

$$= a \cdot' b$$

It is analogous to $a \wedge' c = a \cdot' c$

Finally, we need to demonstrate that $a \wedge' \mathbf{0} = \mathbf{0}$, $a \vee' \mathbf{1} = \mathbf{1}$, $a \wedge' \neg' a = \mathbf{0}$ and $a \vee' \neg a = \mathbf{1}$.

$a \vee' \mathbf{1} \overset{\text{(a)}}{=} a +' \mathbf{1} - a \cdot' \mathbf{1} \overset{\text{(b)}}{=} a +' \mathbf{1} -' a \overset{\text{(c)}}{=} \mathbf{1}$

(a) Definition of $\vee'$

(b) **1** is the identity for the operation $\cdot'$

(c) $-'a$ is the symmetric element of a; **0** is the identity for the operation $+$

$$
\begin{aligned}
a \wedge' \mathbf{0} &\overset{(a)}{=} \neg'(\neg' a \vee' \neg'\mathbf{0}) \\
&\overset{(b)}{=} \neg'(\neg' a +' \neg'\mathbf{0} -' \neg' a \cdot' \neg'\mathbf{0}) \\
&\overset{(c)}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' \mathbf{0}) -' ((\mathbf{1} -' a) \cdot' (\mathbf{1} -' \mathbf{0}))) \\
&\overset{(d)}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' \mathbf{0}) -' (\mathbf{1} -' \mathbf{0} -' a +' \mathbf{0})) \\
&\overset{(e)}{=} \neg'((\mathbf{1} -' a) +' \mathbf{1} -' (\mathbf{1} -' a)) \\
&\overset{(f)}{=} \neg'(\mathbf{1} -' a +' \mathbf{1} -' \mathbf{1} +' a) \\
&\overset{(g)}{=} \mathbf{1} -' \mathbf{1} \\
&\overset{(h)}{=} \mathbf{0}
\end{aligned}
$$

(a) Definition of $\wedge'$

(b) Definition of $\vee'$

(c) Definition of $\neg'$

(d) Distributivity of $\cdot'$ with respect to $+'$; **1** is the identity for the operation $\cdot'$

(e) **0** is the identity for the operation $+$

(f) Associativity; Distributivity of scalar multiplication with respect to $+'$

(g) $-'a$ is the symmetric of a; $-'\mathbf{1}$ is the symmetric of **1**; **0** is the identity for the operation $+$; Definition of $\neg'$

(h) $-'\mathbf{1}$ is the symmetric of **1**

$$
\begin{aligned}
a \vee' \neg' a &\overset{(a)}{=} a +' \neg' a -' (a \cdot' \neg' a) \\
&\overset{(b)}{=} a +' (\mathbf{1} -' a) -' (a \cdot' (\mathbf{1} -' a)) \\
&\overset{(c)}{=} \mathbf{1} -' (a \cdot' \mathbf{1} -' a \cdot' a) \\
&\overset{(d)}{=} \mathbf{1} -' (a -' a) \\
&\overset{(e)}{=} \mathbf{1}
\end{aligned}
$$

(a) Definition of $\vee'$

(b) Definition of $\neg'$

(c) Associativity; Commutativity; **0** is the identity for the operation $+$

(d) **1** is the identity for the operation $\cdot'$; a is an idempotent element

(e) $-'a$ is the symmetric of a; **0** is the identity for the operation $+$

$$
\begin{aligned}
a \wedge' \neg' a &\overset{\text{(a)}}{=} \neg'(\neg' a \vee' \neg'(\neg' a)) \\
&\overset{\text{(b)}}{=} \neg'(\neg' a +' \neg'(\neg' a) +' (\neg' a \cdot' \neg'(\neg' a))) \\
&\overset{\text{(c)}}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' (\mathbf{1} -' a)) +' (\mathbf{1} -' a) \cdot' (\mathbf{1} -' (\mathbf{1} -' a))) \\
&\overset{\text{(d)}}{=} \neg'((\mathbf{1} -' a) +' (\mathbf{1} -' \mathbf{1} +' a) +' ((\mathbf{1} -' a) \cdot' (\mathbf{1} -' \mathbf{1} +' a))) \\
&\overset{\text{(e)}}{=} \neg'(\mathbf{1} +' ((\mathbf{1} -' a) \cdot' a')) \\
&\overset{\text{(f)}}{=} \neg'(\mathbf{1} +' a' -' a) \\
&\overset{\text{(g)}}{=} \mathbf{1} -' \mathbf{1} \\
&\overset{\text{(h)}}{=} \mathbf{0}
\end{aligned}
$$

(a) Definition of $\wedge'$

(b) Definition of $\vee'$

(c) Definition of $\neg'$

(d) Distributivity of scalar multiplication with respect to $+'$

(e) $-'\mathbf{1}$ is the symmetric of $\mathbf{1}$; Associativity; $\mathbf{0}$ is the identity for the operation $+'$

(f) Distributivity of $\cdot'$ with respect to $+'$; $\mathbf{1}$ is the identity for the operation $\cdot'$; a is an idempotent element

(g) $-'a$ is the symmetric of a; $\mathbf{0}$ is the identity for the operation $+'$

(h) $-'\mathbf{1}$ is the symmetric of $\mathbf{1}$

So, since $(B', \wedge', \vee')$ is a distributive lattice and for all $a \in B'$, $a \wedge' \mathbf{0} = \mathbf{0}$, $a \vee' \mathbf{1} = \mathbf{1}$, $a \wedge' \neg' a = \mathbf{0}$ and $a \vee' \neg' a = \mathbf{1}$, we conclude that $(B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$ is a Boolean algebra.

Therefore, we proved that $\mathcal{B}$, induced by the partial algebra $\mathcal{A}$, is a partial Boolean algebra. $\square$

4.3 Boolean polynomials in the context of a partial Boolean algebra

Definition 4.3.1. *Let $n \in \mathbb{N}$. Let Q_n, the set of Boolean polynomials, be defined as:*

(i) $1 \in Q_n$

(ii) $0 \in Q_n$

(iii) $x_i \in Q_n$*, for all* $1 \leq i \leq n$

(iv) If $\varphi \in Q_n$*, then* $\neg\varphi \in Q_n$

(v) If $\varphi, \psi \in Q_n$*, then* $\varphi \vee \psi \in Q_n$

The set of all Boolean polynomials is $\bigcup_{n\in\mathbb{N}} Q_n$.

Observations:

- From the previous definition is immediate that the formulas in n variables are also Boolean polynomials, more precisely, $\Sigma_n \subseteq Q_n$.
- Both lower case letters of the Greek alphabet φ, ψ, χ and α, β, γ (formulas of Σ_n) will be used to denote Boolean polynomials.

In the context of a partial Boolean algebra on the set B, every polynomial $\varphi \in Q_n$ determines a map $\varphi^* : Dom_{\varphi,n} \to B$, with $Dom_{\varphi,n}$ being a subset of B^n, according to the following definition.

Definition 4.3.2. *Let* $\mathcal{B} = (B, \downarrow\!\!\circ, \vee, \neg, \mathbf{1}, \mathbf{0})$ *be a partial Boolean algebra and* Q_n *be the set of Boolean polynomials previously defined.*

We define recursively on a polynomial $\varphi \in Q_n$ *the set* $Dom_{\varphi,n} \subseteq B^n$ *and the map* $\varphi^* : Dom_{\varphi,n} \to B$*, as follows:*

1. *If* $\varphi = 1$*, then* $Dom_{\varphi,n} = B^n$ *and* $\varphi^*(\vec{q}) = \mathbf{1}$
2. *If* $\varphi = 0$*, then* $Dom_{\varphi,n} = B^n$ *and* $\varphi^*(\vec{q}) = \mathbf{0}$
3. *If* $\varphi = x_i$*, then* $Dom_{\varphi,n} = B^n$ *and* $\varphi^*(\vec{q}) = q_i$
4. *If* $\varphi = \neg\psi$*, then* $Dom_{\varphi,n} = Dom_{\psi,n}$ *and* $\varphi^*(\vec{q}) = \neg\psi^*(\vec{q})$

5. *If* $\varphi = \psi \vee \chi$*, then* $Dom_{\varphi,n} = \{\vec{q} \in B^n : \vec{q} \in Dom_{\psi,n} \cap Dom_{\chi,n}$ *and* $\psi^*(\vec{q}) \downarrow \chi^*(\vec{q})\}$ *and* $\varphi^*(\vec{q}) = \psi^*(\vec{q}) \vee \chi^*(\vec{q})$

$Dom_{\varphi,n}$ *and* $\varphi^*(\vec{q})$ *are, respectively, the domain and the map associated to the polynomial* φ *relative to* B.

Next, we show that the definition 4.3.2 is coherent with the definition 3.2.2.

Theorem 4.3.3. *Let* $\mathcal{A} = (A, \downarrow, +, \cdot, \circ, \mathbf{1})$ *be a partial algebra,* $\mathcal{B} = (B, \downarrow, \vee, \neg, \mathbf{1}, \mathbf{0})$ *be the induced partial Boolean algebra and* p_n *be the following function:*

$$p_n : Q_n \to P_n$$

$$p_n(1) = 1$$

$$p_n(0) = 0 \circ 1, \text{ where 0 is the real number}$$

$$p_n(x_i) = x_i$$

$$p_n(\varphi \vee \psi) = (p_n(\varphi) + p_n(\psi)) - p_n(\varphi) \cdot p_n(\psi)$$

$$p_n(\neg\varphi) = 1 - p_n(\varphi)$$

Then, for all $\varphi \in Q_n$*:*

1. $Dom_{\varphi,n} = D_{p_n(\varphi),n}\big|_{B^n}$*, where* $D_{p_n(\varphi),n}\big|_{B^n}$ *denotes the restriction of the domain* $D_{p_n(\varphi),n}$ *to* B^n*,* $B^n \subseteq A^n$

2. *For all* $\vec{q} \in Dom_{\varphi,n}, \varphi^*(\vec{q}) = p_n(\varphi)^*(\vec{q})$

Proof. Let

$$P(\varphi) = \begin{cases} 1.\ Dom_{\varphi,n} = D_{p_n(\varphi),n}\big|_{B^n} \\ 2.\ \text{for all } \vec{q} \in Dom_{\varphi,n}, \varphi^*(\vec{q}) = p_n(\varphi)^*(\vec{q}) \end{cases}$$

The proof follows by induction on φ.

- $\varphi = 1$:

  1. $Dom_{\varphi,n} \overset{\text{(i)}}{=} B^n \overset{\text{(ii)}}{=} D_{p_n(\varphi),n}\big|_{B^n}$

     (i) Definition of $Dom_{1,n}$ in the partial Boolean algebra

     (ii) $D_{p_n(\varphi),n}\big|_{B^n} = D_{1,n}\big|_{B^n} = A^n\big|_{B^n} = B^n$

  2. $\varphi^*(\vec{q}) \overset{\text{(i)}}{=} \mathbf{1} \overset{\text{(ii)}}{=} p_n(\varphi)^*(\vec{q})$

(i) Definition of $1^*(\vec{q})$ in the partial Boolean algebra

(ii) $p_n(\varphi)^*(\vec{q}) = 1^*(\vec{q}) = \mathbf{1}$

Observation: $\mathbf{1}$ is in B because, by definition, $\mathbf{1}$ is an identity element of the product in A. In particular, $\mathbf{1} = \mathbf{1} \cdot \mathbf{1}$, that is, $\mathbf{1}$ is an idempotent element of A.

- $\varphi = 0$:

1. $D_{p_n(\varphi),n}\big|_{B^n} \overset{\text{(i)}}{=} D_{0\circ 1,n}\big|_{B^n} \overset{\text{(ii)}}{=} B^n \overset{\text{(iii)}}{=} Dom_{\varphi,n}$

   (i) Definition of $p_n(0)$

   (ii) $D_{0\circ 1,n}|_{B^n} = D_{1,n}|_{B^n} = A^n|_{B^n} = B^n$

   (iii) Definition of $Dom_{0,n}$ in the partial Boolean algebra

2. $p_n(\varphi)^*(\vec{q}) \overset{\text{(i)}}{=} (0 \circ 1)^*(\vec{q}) \overset{\text{(ii)}}{=} 0 \circ \mathbf{1} \overset{\text{(iii)}}{=} \mathbf{0} \overset{\text{(iv)}}{=} \varphi^*(\vec{q})$

   (i) Definition of $p_n(0)$

   (ii) $(0 \circ 1)^*(\vec{q}) = 0 \circ 1^*(\vec{q}) = 0 \circ \mathbf{1}$

   (iii) Definition of $\mathbf{0}$

   (iv) Definition of $0^*(\vec{q})$ in the partial Boolean algebra

Observations:

- Since $\mathbf{0}$ and $\mathbf{1}$ are commeasurable, the algebra of the polynomials in $\mathbf{0}$ and $\mathbf{1}$ is a commutative algebra over the field of real numbers.
- $\mathbf{0}$ is in B because $(0 \circ \mathbf{1}) \cdot (0 \circ \mathbf{1}) = (0 \times 0) \circ (\mathbf{1} \cdot \mathbf{1}) = 0 \circ \mathbf{1}$.

- $\varphi = x_i$:

1. $Dom_{\varphi,n} \overset{\text{(i)}}{=} B^n \overset{\text{(ii)}}{=} D_{p_n(\varphi),n}\big|_{B^n}$

   (i) Definition of $Dom_{x_i,n}$ in the partial Boolean algebra

   (ii) $D_{p_n(\varphi),n}\big|_{B^n} = D_{x_i,n}|_{B^n} = A^n|_{B^n} = B^n \ (1 \le i \le n)$

2. $\varphi^*(\vec{q}) \overset{\text{(i)}}{=} q_i \overset{\text{(ii)}}{=} p_n(\varphi)^*(\vec{q})$

   (i) Definition of ${x_i}^*(\vec{q})$ in the partial Boolean algebra

   (ii) $p_n(\varphi)^*(\vec{q}) = x_i^*(\vec{q}) = q_i$

Observation: $q_i = q_i \cdot q_i$ because, by definition, $\vec{q} = (q_1, \cdots, q_n) \in B^n$. So, $q_i \in B$ and the elements in B are the idempotent of A.

- $\varphi = \neg\psi$:

Suppose $P(\psi)$. We want to show $P(\neg\psi)$.

1. $D_{p_n(\neg\psi),n}\big|_{B^n} \overset{\text{(i)}}{=} D_{1-p_n(\psi),n}\big|_{B^n}$

$\overset{\text{(ii)}}{=} \{\vec{q} \in B^n : \vec{q} \in D_{1,n}|_{B^n} \cap D_{p_n(\psi),n}\big|_{B^n}$ and $1^*(\vec{q}) \downarrow p_n(\psi)^*(\vec{q})\}$

$\overset{\text{(iii)}}{=} \{\vec{q} \in B^n : \vec{q} \in B^n \cap D_{p_n(\psi),n}\big|_{B^n}$ and $\mathbf{1} \downarrow p_n(\psi)^*(\vec{q})\}$

$\overset{\text{(iv)}}{=} \{\vec{q} \in B^n : \vec{q} \in D_{p_n(\psi),n}\big|_{B^n}\}$

$\overset{\text{(v)}}{=} \{\vec{q} \in B^n : \vec{q} \in Dom_{\psi,n}\}$

$\overset{\text{(vi)}}{=} Dom_{\psi,n}$

(i) Definition of $p_n(\neg\psi)$

(ii) Definition of $D_{1-p_n(\psi),n}$ in the partial algebra

(iii) $D_{1,n}|_{B^n} = B^n$

$1^*(\vec{q}) = \mathbf{1}$

(iv) Once $D_{p_n(\psi),n}\big|_{B^n} \subseteq B^n$, then $D_{p_n(\psi),n}\big|_{B^n} \cap B^n = D_{p_n(\psi),n}\big|_{B^n}$

By definition, it is always true that $\mathbf{1} \downarrow p_n(\psi)^*(\vec{q})$

(v) Induction hypothesis $P(\psi)$

(vi) Definition of $Dom_{\psi,n}$

2. $(p_n(\neg\psi))^*(\vec{q}) \overset{\text{(i)}}{=} (1 - p_n(\psi))^*(\vec{q})$

$\overset{\text{(ii)}}{=} 1^*(\vec{q}) - p_n(\psi)^*(\vec{q})$

$\overset{\text{(iii)}}{=} \mathbf{1} - \psi^*(\vec{q})$

$\overset{\text{(iv)}}{=} \neg\psi^*(\vec{q})$

$\overset{\text{(v)}}{=} (\neg\psi)^*(\vec{q})$

(i) Definition of $p_n(\neg\psi)$

(ii) Definition of $(1 - p_n(\psi))^*(\vec{q})$ in the partial algebra

(iii) $1^*(\vec{q}) = \mathbf{1}$

Induction hypothesis $P(\psi)$

(iv) Definition of the connective $\neg$

(v) Definition of $(\neg\psi)^*(\vec{q})$ in the partial Boolean algebra

Observations:

- Since **1** and $\psi^*(\vec{q})$ are commeasurable, the algebra of the polynomials in **1** and $\psi^*(\vec{q})$ is a commutative algebra over the field of real numbers.
- $\neg\psi^*(\vec{q}) = \mathbf{1} - \psi^*(\vec{q})$ is an idempotent element of A. For the sake of simplification of notation, let us consider $\psi^*(\vec{q}) = a$. We want to show that $(\mathbf{1} - a) \cdot (\mathbf{1} - a) = \mathbf{1} - a$

$$(\mathbf{1} - a) \cdot (\mathbf{1} - a) \overset{\text{(a)}}{=} \mathbf{1} \cdot \mathbf{1} - \mathbf{1} \cdot a - a \cdot \mathbf{1} + a \cdot a \overset{\text{(b)}}{=} \mathbf{1} - a - a + a \overset{\text{(c)}}{=} \mathbf{1} - a$$

(a) Distributivity of $\cdot$ with respect to $+$

(b) **1** is the identity of $\mathcal{B}$

Since $a \in B$, then $a \cdot a = a$

(c) $-a$ is the symmetric element of a and **0** is the identity for the operation $+$

- $\varphi = \psi \vee \chi$:

Suppose $P(\psi)$ and $P(\chi)$. We want to show $P(\psi \vee \chi)$.

1. $D_{p_n(\psi\vee\chi),n}\big|_{B^n} \overset{\text{(i)}}{=}$

$\overset{\text{(i)}}{=} D_{(p_n(\psi)+p_n(\chi))-p_n(\psi)\cdot p_n(\chi),n}\big|_{B^n}$

$\overset{\text{(ii)}}{=} \{\vec{q} \in B^n : \vec{q} \in D_{p_n(\psi),n}\big|_{B^n} \cap D_{p_n(\chi),n}\big|_{B^n}$ and $p_n(\psi)^*(\vec{q}) \between p_n(\chi)^*(\vec{q})\}$

$\overset{\text{(iii)}}{=} \{\vec{q} \in B^n : \vec{q} \in Dom_{\psi,n} \cap Dom_{\chi,n}$ and $\psi^*(\vec{q}) \between \chi^*(\vec{q})\}$

$\overset{\text{(iv)}}{=} Dom_{\psi\vee\chi,n}$

(i) Definition of $p_n(\psi \vee \chi)$

(ii) $D_{(p_n(\psi)+p_n(\chi))-p_n(\psi)\cdot p_n(\chi),n} =$

$= \{\vec{q} \in B^n : \vec{q} \in D_{p_n(\psi)+p_n(\chi),n}\big|_{B^n} \cap D_{p_n(\psi)\cdot p_n(\chi),n}\big|_{B^n}$ and

$(p_n(\psi) + p_n(\chi))^*(\vec{q}) \between (p_n(\psi) \cdot p_n(\chi))^*(\vec{q})\}$

$= \{\vec{q} \in B^n : \vec{q} \in D_{p_n(\psi),n}\big|_{B^n} \cap D_{p_n(\chi),n}\big|_{B^n}$ and $p_n(\psi)^*(\vec{q}) \between p_n(\chi)^*(\vec{q})$

and $(p_n(\psi) + p_n(\chi))^*(\vec{q}) \between (p_n(\psi) \cdot p_n(\chi))^*(\vec{q})\}$

$\overset{\text{(a)}}{=} \{\vec{q} \in B^n : \vec{q} \in D_{p_n(\psi),n}\big|_{B^n} \cap D_{p_n(\chi),n}\big|_{B^n}$ and $p_n(\psi)^*(\vec{q}) \between p_n(\chi)^*(\vec{q})\}$

(a) We can omit the compatibility $(p_n(\psi) + p_n(\chi))^*(\vec{q}) \between (p_n(\psi) \cdot p_n(\chi))^*(\vec{q})$ due to the fact that from the compatibility of $p_n(\psi)^*(\vec{q})$ and $p_n(\chi)^*(\vec{q})$ we can obtain the omitted one. So, once $p_n(\psi)^*(\vec{q})$ and $p_n(\chi)^*(\vec{q})$ are compatible, then $p_n(\psi)^*(\vec{q}) + p_n(\chi)^*(\vec{q})$ and $p_n(\psi)^*(\vec{q})$ are compatible (as well as $p_n(\psi)^*(\vec{q}) + p_n(\chi)^*(\vec{q})$ and $p_n(\chi)^*(\vec{q})$). Since any two of $p_n(\psi)^*(\vec{q}) + p_n(\chi)^*(\vec{q})$, $p_n(\psi)^*(\vec{q})$ and $p_n(\chi)^*(\vec{q})$ are compatible, then $p_n(\psi)^*(\vec{q}) \cdot p_n(\chi)^*(\vec{q})$ and $p_n(\psi)^*(\vec{q}) +$

$p_n(\chi)^*(\vec{q})$ are compatible. By definition of p_n^*, $p_n(\psi)^*(\vec{q}) + p_n(\chi)^*(\vec{q}) = (p_n(\psi) + p_n(\chi))^*(\vec{q})$ and $p_n(\psi)^*(\vec{q}) \cdot p_n(\chi)^*(\vec{q}) = (p_n(\psi) \cdot p_n(\chi))^*(\vec{q})$

(iii) Induction hypothesis $P(\psi)$ and $P(\chi)$

(iv) Definition of $Dom_{\psi \vee \chi, n}$

2. $$\begin{aligned}(p_n(\psi \vee \chi))^*(\vec{q}) &\overset{\text{(i)}}{=} [p_n(\psi) + p_n(\chi) - (p_n(\psi) \cdot p_n(\chi))]^*(\vec{q}) \\ &\overset{\text{(ii)}}{=} p_n(\psi)^*(\vec{q}) + p_n(\chi)^*(\vec{q}) - p_n(\psi)^*(\vec{q}) \cdot p_n(\chi)^*(\vec{q}) \\ &\overset{\text{(iii)}}{=} \psi^*(\vec{q}) + \chi^*(\vec{q}) - \psi^*(\vec{q}) \cdot \chi^*(\vec{q}) \\ &\overset{\text{(iv)}}{=} \psi^*(\vec{q}) \vee \chi^*(\vec{q}) \\ &\overset{\text{(v)}}{=} (\psi \vee \chi)^*(\vec{q})\end{aligned}$$

(i) Definition of $p_n(\psi \vee \chi)$

(ii) Definition of $(p_n(\psi) + p_n(\chi) - p_n(\psi) \cdot p_n(\chi))^*(\vec{q})$ in the partial algebra

(iii) Induction hypothesis $P(\psi)$ and $P(\chi)$

(iv) Definition of the connective $\vee$

(v) Definition of $(\psi \vee \chi)^*(\vec{q})$ in the partial Boolean algebra

Observations:

- Since $\psi^*(\vec{q})$ and $\chi^*(\vec{q})$ are commeasurable, the algebra of the polynomials in $\psi^*(\vec{q})$ and $\chi^*(\vec{q})$ is a commutative algebra over the field of real numbers.
- For the sake of simplification of notation, let us consider $\psi^*(\vec{q}) = a$ and $\chi^*(\vec{q}) = b$. We want to show that $((a+b) - a \cdot b) \cdot ((a+b) - a \cdot b) = (a+b) - a \cdot b$

  $$((a+b) - a \cdot b) \cdot ((a+b) - a \cdot b) \overset{\text{(a)}}{=}$$

  $$\overset{\text{(a)}}{=} a \cdot a + a \cdot b - a \cdot (a \cdot b) + b \cdot a + b \cdot b - b \cdot (a \cdot b) - (a \cdot b) \cdot a - (a \cdot b) \cdot b + (a \cdot b) \cdot (a \cdot b)$$

  $$\overset{\text{(b)}}{=} a + a \cdot b - a \cdot b + a \cdot b + b - a \cdot b - a \cdot b - a \cdot b + a \cdot b$$

  $$\overset{\text{(c)}}{=} (a+b) - a \cdot b$$

  (a) Distributivity of $\cdot$ with respect to $+$

  (b) Associativity; Commutativity; a and b are idempotent elements

  (c) There exists a symmetric element for a, b and any other element obtained from the operation involving a and b, and a **0** element, which is the identity for the operation $+$

Then, for all $\varphi \in Q_n$, $P(\varphi)$. □

From the lemma that follows, we can conclude that every Boolean algebra $\mathcal{B}$ is a partial Boolean algebra and that the value of α determined by a valuation in $\mathcal{B}$ is the value of α^* on a tuple determined by the valuation.

Lemma 4.3.4. *Let $\mathcal{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0})$ be a Boolean algebra and let us consider the following structure $\mathcal{B}_p = (B, B^2, \vee, \neg, \mathbf{1}, \mathbf{0})$. Then,*

(i) $\mathcal{B}_p$ is a partial Boolean algebra

(ii) Let $\alpha \in \Sigma_n$. Then,

(a) $Dom_{\alpha,n} = B^n$;

(b) For all $\vec{q} \in B^n$ and for all Boolean valuation v such that $v(x_i) = q_i$, $i \in \{1, \cdots, n\}$, $\alpha^*(\vec{q}) = \overline{\alpha}(v)$

Proof. Let $\mathcal{B} = (B, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0})$ be a Boolean algebra and $\mathcal{B}_p = (B, B^2, \vee, \neg, \mathbf{1}, \mathbf{0})$.

(i) We want to prove that $\mathcal{B}_p$ is a partial Boolean algebra, that is:

1. B^2 is reflexive and symmetric;

   Since the compatibility relation is $B^2 = \{(q_i, q_j) : q_i, q_j \in B\}$, then all the elements in B are commeasurable. In particular, we have that $(q_i, q_i), (q_i, q_j), (q_j, q_i) \in B^2$, for all $q_i, q_j \in B$, $q_i \neq q_j$, that is, B^2 is reflexive and symmetric.

2. For all $q \in B$, $(q, \mathbf{0}) \in B^2$ and $(q, \mathbf{1}) \in B^2$;

   Let $q \in B$. Since $\mathbf{1} \in B$ and $\mathbf{0} \in B$ and all the elements in B are compatible, then $(q, \mathbf{1}) \in B^2$ and $(q, \mathbf{0}) \in B^2$.

3. $\vee$ is exactly defined for those pairs $(q_1, q_2) \in B \times B$ such that $(q_1, q_2) \in B^2$;

   Let $(q_1, q_2) \in B \times B$. Then, the pair (q_1, q_2) is in the relation B^2.

4. If any two of $q_1, q_2, q_3 \in B$ are commeasurable, then $(q_1 \vee q_2, q_3) \in B^2$ and $(\neg q_1, q_2) \in B^2$.

   Let $q_1, q_2, q_3 \in B$ such that $(q_i, q_j) \in B^2$, for all $i, j \in \{1, 2, 3\}$. Since $q_1 \vee q_2 \in B$, $\neg q_1 \in B$ and all the elements in B are compatible, then $(q_1 \vee q_2, q_3) \in B^2$ and $(\neg q_1, q_2) \in B^2$.

5. If any two of $q_1, q_2, q_3 \in B$ are commeasurable, then the algebra of the Boolean polynomials in q_1, q_2 and q_3 form a Boolean algebra.

   This proof is similar to the one provided in the theorem 4.2.2 in the statement 5.

Then, $\mathcal{B}_p$ is a partial Boolean algebra.

(ii) Let $\alpha \in \Sigma_n$, $\vec{q} \in B^n$ and v be a Boolean valuation such that $v(x_i) = q_i$, for $i \in \{1, \cdots, n\}$. Let

$$P(\alpha) = \begin{cases} \text{(a) } Dom_{\alpha,n} = B^n \\ \text{(b) } \alpha^*(\vec{q}) = \overline{\alpha}(v) \end{cases}$$

The proof follows by induction on α.

- $\alpha = x_i$:

  (a) $Dom_{x_i,n} \overset{\text{(i)}}{=} B^n$

   (i) Definition of $Dom_{x_i,n}$ in the partial Boolean algebra

  (b) ${x_i}^*(\vec{q}) \overset{\text{(i)}}{=} q_i \overset{\text{(ii)}}{=} v(x_i) \overset{\text{(iii)}}{=} \overline{x_i}(v)$

   (i) Definition of ${x_i}^*(\vec{q})$ in the partial Boolean algebra

   (ii) By hypothesis

   (iii) Definition of value in a Boolean algebra

- $\alpha = \neg\varphi$:

  Suppose $P(\alpha)$. We want to show $P(\neg\alpha)$.

  (a) $Dom_{\neg\alpha,n} \overset{\text{(i)}}{=} Dom_{\alpha,n} \overset{\text{(ii)}}{=} B^n$

   (i) Definition of $Dom_{\neg\alpha,n}$ in the partial Boolean algebra

   (ii) By induction hypothesis $P(\alpha)$

  (b) $(\neg\alpha)^*(\vec{q}) \overset{\text{(i)}}{=} \neg\alpha^*(\vec{q}) \overset{\text{(ii)}}{=} \neg\overline{\alpha}(v) \overset{\text{(iii)}}{=} \overline{\neg\alpha}(v)$

   (i) Definition of $(\neg\alpha)^*(\vec{q})$ in the partial Boolean algebra

   (ii) Induction hypothesis $P(\alpha)$

   (iii) Definition of value in a Boolean algebra

- $\alpha = \beta \vee \gamma$:

  Suppose $P(\beta)$ and $P(\gamma)$. We want to show $P(\beta \vee \gamma)$.

(a) $Dom_{\beta\vee\gamma,n} \overset{\text{(i)}}{=} \{\vec{q} \in B^n : \vec{q} \in Dom_{\beta,n} \cap Dom_{\gamma,n} \text{ and } (\beta^*(\vec{q}), \gamma^*(\vec{q})) \in B^2\}$

$\overset{\text{(ii)}}{=} \{\vec{q} \in B^n : \vec{q} \in Dom_{\beta,n} \cap Dom_{\gamma,n}\}$

$\overset{\text{(iii)}}{=} Dom_{\beta,n} \cap Dom_{\gamma,n}$

$\overset{\text{(iv)}}{=} B^n \cap B^n$

$\overset{\text{(v)}}{=} B^n$

(i) Definition of $Dom_{\beta\vee\gamma,n}$ in the partial Boolean algebra

(ii) Since $\beta^*(\vec{q}), \gamma^*(\vec{q}) \in B$, then it is always true that $(\beta^*(\vec{q}), \gamma^*(\vec{q})) \in B^2$

(iii) Simplification of notation

(iv) Induction hypothesis $P(\beta)$ and $P(\gamma)$

(v) $A \cap A = A$, for all set A

(b) $(\beta \vee \gamma)^*(\vec{q}) \overset{\text{(i)}}{=} \beta^*(\vec{q}) \vee \gamma^*(\vec{q}) \overset{\text{(ii)}}{=} \overline{\beta}(v) \vee \overline{\gamma}(v) \overset{\text{(iii)}}{=} \overline{\beta \vee \gamma}(v)$

(i) Definition of $(\beta \vee \gamma)^*(\vec{q})$ in the partial Boolean algebra

(ii) Induction hypothesis $P(\beta)$ and $P(\gamma)$

(iii) Definition of value in a Boolean algebra

Then, for all $\alpha \in \Sigma_n$, $P(\alpha)$. □

Definition 4.3.5. *Let $\mathcal{B}$ be a partial Boolean algebra and $\alpha \in \Sigma_n$. We say that α holds in $\mathcal{B}$ if for all $\vec{q} \in Dom_{\alpha,n}$, $\alpha^*(\vec{q}) = \mathbf{1}$.*

In the next section, we will study the formulas holding in all partial Boolean algebras.

Chapter 5

Partial classical propositional logic

In this chapter, we will explore the Q-validity of a formula, introduce the axiomatic system of [9] along with new definitions and theorems. Furthermore, the chapter will cover the soundness and completeness theorems, accompanied by their respective proofs. Each introduced concept will be illustrated through examples provided in their respective sections.

5.1 Q-validity

In this section, the concept of formula we will use corresponds to the one provided in the preliminaries. In addition to introducing the concept of Q-validity, we will present a result that compares Q-validity with C-validity, exploring these concepts in both quantum and classical propositional logic.

Definition 5.1.1. *A formula is Q-valid if it holds in all partial Boolean algebras.*

Theorem 5.1.2. *Every Q-valid formula is a C-valid formula.*

Proof. Let $\alpha \in \Sigma_n$ be a Q-valid formula. To show that α is a C-valid formula, by theorem 2.3.7, is to prove that $\mathcal{B} \models \alpha$, for all Boolean algebras $\mathcal{B}$. Let $\mathcal{B} = (B, \vee, \wedge, \neg, \mathbf{1}, \mathbf{0})$ be a Boolean algebra and v a valuation in $\mathcal{B}$. We want to conclude that $\overline{\alpha}(v) = \mathbf{1}$. By lemma 4.3.4, $\mathcal{B}_p = (B, B^2, \vee, \neg, \mathbf{1}, \mathbf{0})$ is a partial Boolean algebra. Let $\vec{q} \in Dom_{\alpha,n}$ such that $q_i = v(x_i)$. Then, by lemma 4.3.4, $\alpha^*(\vec{q}) = \overline{\alpha}(v)$. Since α is Q-valid, $\alpha^*(\vec{q}) = \mathbf{1}$. Therefore, $\overline{\alpha}(v) = \mathbf{1}$. □

Theorem 5.1.3. *Let α be a formula in n variables whose only subformulas in x_i alone are x_i or $\neg x_i$, for $i \in \{1, \cdots, n\}$, and such that for all i,j ($1 \leq i < j \leq n$) there exists a subformula $\alpha_{i,j}$ in x_i and x_j alone. Then, α is Q-valid if it is C-valid.*

Proof. Let $\mathcal{B} = (B, \between, \vee, \neg, \mathbf{1}, \mathbf{0})$ be a partial Boolean algebra and α the Boolean polynomial in n variables. Let us also consider $\vec{q} \in Dom_{\alpha,n}$ and a subpolynomial $\alpha_{i,j}$ such that no subpolynomial of

$\alpha_{i,j}$ has occurrences of both x_i and x_j (see the Observation below). We want to show that $\alpha^*(\vec{q}) = \mathbf{1}$. Given the construction of $\alpha_{i,j}$, it can only be in one of the following formats: $x_i \vee x_j$, $\neg x_i \vee x_j$, $x_i \vee \neg x_j$ or $\neg x_i \vee \neg x_j$. By definition of $Dom_{\_,n}$, we obtain that $q_i \between q_j$, for all $1 \leq i < j \leq n$ and by proposition 4.1.4, the algebra of the Boolean polynomials in $q_1, \cdots, q_n$ is a Boolean algebra. Let $\mathcal{B}'$ be the Boolean algebra of the polynomials in $q_1, \cdots, q_n$ and α^{**} the function associated to $\mathcal{B}'$. Note that $\alpha^*(\vec{q}) = \alpha^{**}(\vec{q})$. Given that α is C-valid and $\mathcal{B}'$ is a Boolean algebra, $\alpha^{**}(\vec{q}) = \mathbf{1}$. Then, $\alpha^*(\vec{q}) = \mathbf{1}$. □

Observation: Such $\alpha_{i,j}$ exists. We begin with $\alpha_{i,j}$ as stated in the theorem. If this $\alpha_{i,j}$ still does not satisfy the additional condition required, it is because we can choose a subpolynomial $\alpha'_{i,j}$ where both variables x_i and x_j occur. Then, we select $\alpha'_{i,j}$ and repeat the process.

Theorem 5.1.4. *A formula in one or two variables is Q-valid if it is C-valid.*

Proof. We are going to consider two scenarios, the first one α being a formula in one variable and the second one in two variables.

- Let us consider a partial Boolean algebra $\mathcal{B} = (B, \between, \vee, \neg, \mathbf{1}, \mathbf{0})$ and a formula $\alpha \in \Sigma_1$ such that α is C-valid. We want to show that $\alpha^*(\vec{q}) = \mathbf{1}$. By the proposition 4.1.4, the algebra of the Boolean polynomials in q_1 is a Boolean algebra. Let $\mathcal{B}'$ be the Boolean algebra of the polynomials in q_1 and α^{**} the function associated to $\mathcal{B}'$. Note that $\alpha^*(\vec{q}) = \alpha^{**}(\vec{q})$. Given that α is C-valid and $\mathcal{B}'$ is a Boolean algebra, $\alpha^{**}(\vec{q}) = \mathbf{1}$. Then, $\alpha^*(\vec{q}) = \mathbf{1}$.

- Let us consider a partial Boolean algebra $\mathcal{B} = (B, \between, \vee, \neg, \mathbf{1}, \mathbf{0})$ and a formula $\alpha \in \Sigma_2$ such that α is C-valid. We want to show that $\alpha^*(\vec{q}) = \mathbf{1}$. Let us consider the property $P(\alpha)$ iff if $\vec{q} = (q_1, q_2) \in Dom_{\alpha,2}$, then $q_1 \between q_2$. The proof of this property follows by induction on α.

  - $P(x_i)$, for $i \in \{1, 2\}$, iff $\vec{q} \in Dom_{x_i,2}$ implies $q_1 \between q_2$. By definition, $Dom_{x_i,2} = B^2$. So, $q_1 \between q_2$.
  - $P(\neg\alpha)$ iff $\vec{q} \in Dom_{\neg\alpha,2}$ implies $q_1 \between q_2$. Let us suppose $P(\alpha)$ and that $\vec{q} \in Dom_{\neg\alpha,2}$. By definition, $Dom_{\neg\alpha,2} = Dom_{\alpha,2}$ and, consequently, $\vec{q} \in Dom_{\alpha,2}$. By induction hypothesis $P(\alpha)$, $q_1 \between q_2$.
  - $P(\alpha \vee \beta)$ iff $\vec{q} \in Dom_{\alpha\vee\beta,2}$ implies $q_1 \between q_2$. Let us suppose $P(\alpha)$, $P(\beta)$ and that $\vec{q} \in Dom_{\alpha\vee\beta,2}$. By definition, $Dom_{\alpha\vee\beta,2} = \{\vec{q} \in B^2 : \vec{q} \in Dom_{\alpha,2} \cap Dom_{\beta,2}$ and $\alpha^*(\vec{q}) \between \beta^*(\vec{q})\}$. Since $\vec{q} \in Dom_{\alpha,2}$ then, by induction hypothesis $P(\alpha)$, $q_1 \between q_2$.

So, in all these cases, for all $\vec{q}$ in the domain of a formula $\alpha \in \Sigma_2$, $q_1 \downarrow\!\!\circ\, q_2$. By proposition 4.1.4, the algebra of the Boolean polynomials in q_1 and q_2 is a Boolean algebra. Let $\mathcal{B}'$ be the Boolean algebra of the polynomials in q_1 and q_2 and α^{**} the function associated to $\mathcal{B}'$. Note that $\alpha^*(\vec{q}) = \alpha^{**}(\vec{q})$. Given that α is C-valid and $\mathcal{B}'$ is a Boolean algebra, $\alpha^{**}(\vec{q}) = \mathbf{1}$. Then, $\alpha^*(\vec{q}) = \mathbf{1}$.

□

Now, we will give some examples of Q-valid formulas and not Q-valid formulas, which were taken from the article [9].

Example 1. The formula $\alpha = ((x_1 \vee x_2) \wedge x_3) \leftrightarrow [(x_1 \wedge x_3) \vee (x_2 \wedge x_3)]$ is a Q-valid formula.

Proof. Since the subformulas in x_1 alone is x_1, in x_2 alone is x_2 and in x_3 alone is x_3, and for all $1 \leq i < j \leq 3$ there exists $\alpha_{i,j}$, where $\alpha_{1,2} = x_1 \vee x_2$, $\alpha_{1,3} = x_1 \wedge x_3$ and $\alpha_{2,3} = x_2 \wedge x_3$, and given that α is C-valid (it is the distributive law), then by theorem 5.1.3, α is Q-valid. □

Example 2. The formula $\alpha = [(x_1 \vee x_2) \vee x_3] \leftrightarrow [x_1 \vee (x_2 \vee x_3)]$ is Q-valid.

Proof. Firstly, we can't apply theorem 5.1.3 because this formula does not satisfy all the required hypothesis, specifically there does not exist $\alpha_{1,3}$, that is, a subformula of α involving only the variables x_1 and x_3. However, this does not mean that it is not Q-valid.
Let us consider a partial Boolean algebra $\mathcal{B} = (B, \downarrow\!\!\circ, \vee, \neg, \mathbf{1}, \mathbf{0})$ and let $\vec{q} = (q_1, q_2, q_3) \in Dom_{\alpha,3}$. By definition,

$$\begin{aligned} Dom_{\alpha,3} = \{\vec{q} \in B^3 : \vec{q} \in Dom_{(x_1 \vee x_2) \vee x_3,3} \cap Dom_{x_1 \vee (x_2 \vee x_3),3} \text{ and } \\ ((x_1 \vee x_2) \vee x_3)^*(\vec{q}) \downarrow\!\!\circ\, (x_1 \vee (x_2 \vee x_3))^*(\vec{q})\} \\ = \{\vec{q} \in B^3 : \vec{q} \in Dom_{x_1,3} \cap Dom_{x_2,3} \cap Dom_{x_3,3} \text{ and } \\ ((x_1 \vee x_2) \vee x_3)^*(\vec{q}) \downarrow\!\!\circ\, (x_1 \vee (x_2 \vee x_3))^*(\vec{q}) \text{ and } (x_1 \vee x_2)^*(\vec{q}) \downarrow\!\!\circ\, {x_3}^*(\vec{q}) \text{ and } \\ {x_1}^*(\vec{q}) \downarrow\!\!\circ\, (x_2 \vee x_3)^*(\vec{q}) \text{ and } {x_1}^*(\vec{q}) \downarrow\!\!\circ\, {x_2}^*(\vec{q}) \text{ and } {x_2}^*(\vec{q}) \downarrow\!\!\circ\, {x_3}^*(\vec{q})\} \\ = \{\vec{q} \in B^3 : ((q_1 \vee q_2) \vee q_3) \downarrow\!\!\circ\, (q_1 \vee (q_2 \vee q_3)) \text{ and } (q_1 \vee q_2) \downarrow\!\!\circ\, q_3 \text{ and } q_1 \downarrow\!\!\circ\, (q_2 \vee q_3) \\ \text{and } q_1 \downarrow\!\!\circ\, q_2 \text{ and } q_2 \downarrow\!\!\circ\, q_3\} \end{aligned}$$

We have that $q_1 \downarrow\!\!\circ\, q_2$, $q_2 \downarrow\!\!\circ\, q_3$ and $q_1 \downarrow\!\!\circ\, (q_2 \vee q_3)$. Consequently, due to the fact that $q_2 \downarrow\!\!\circ\, q_2$ and $q_2 \downarrow\!\!\circ\, q_3$, we have $q_2 \downarrow\!\!\circ\, (q_2 \vee q_3)$. Therefore, any two of q_1, q_2 and $q_2 \vee q_3$ are commeasurable, which implies that the algebra of the Boolean polynomials in q_1, q_2 and $q_2 \vee q_3$ is a Boolean algebra.

Additionally, we also know $q_1 \vee q_2 \text{⫰} q_3$. Analogously to what we have previously done, since $q_2 \text{⫰} q_2$ and $q_2 \text{⫰} q_1$, $q_2 \text{⫰} (q_1 \vee q_2)$. Thus, any two of $q_1 \vee q_2$, q_2 and q_3 are commeasurable and, consequently, the algebra of the Boolean polynomials in $q_1 \vee q_1$, q_2 and q_3 is a Boolean algebra. Then, on one hand,

$$
\begin{aligned}
&(q_1 \vee q_2) \vee (q_2 \vee q_3) = \\
&\overset{\text{(i)}}{=} q_1 \vee [q_2 \vee (q_2 \vee q_3)] \\
&\overset{\text{(ii)}}{=} q_1 \vee [(q_2 \vee q_2) \vee q_3] \\
&\overset{\text{(iii)}}{=} q_1 \vee (q_2 \vee q_3)
\end{aligned}
$$

(i) Associativity (of the algebra of the Boolean polynomials in q_1, q_2 and $q_2 \vee q_3$)

(ii) Associativity (of the algebra of the Boolean polynomials in $q_1 \vee q_2$, q_2 and q_3)

(iii) Idempotency (of the algebra of the Boolean polynomials in $q_1 \vee q_2$, q_2 and q_3)

On the other hand,

$$
\begin{aligned}
&(q_1 \vee q_2) \vee (q_2 \vee q_3) = \\
&\overset{\text{(i)}}{=} [(q_1 \vee q_2) \vee q_2] \vee q_3 \\
&\overset{\text{(ii)}}{=} [q_1 \vee (q_2 \vee q_2)] \vee q_3 \\
&\overset{\text{(iii)}}{=} (q_1 \vee q_2) \vee q_3
\end{aligned}
$$

(i) Associativity (of the algebra of the Boolean polynomials in $q_1 \vee q_2$, q_2 and q_3)

(ii) Associativity (of the algebra of the Boolean polynomials in q_1, q_2 and $q_2 \vee q_3$)

(iii) Idempotency (of the algebra of the Boolean polynomials in q_1, q_2 and $q_2 \vee q_3$)

Since $q_1 \vee (q_2 \vee q_3) = (q_1 \vee q_2) \vee (q_2 \vee q_3) = (q_1 \vee q_2) \vee q_3$, we conclude that $q_1 \vee (q_2 \vee q_3) = (q_1 \vee q_2) \vee q_3$ and, consequently, the formulas $(x_1 \vee x_2) \vee x_3$ and $x_1 \vee (x_2 \vee x_3)$ have the same value, establishing that the formula α is Q-valid. $\square$

Example 3. The formula $[(x_1 \leftrightarrow x_2) \leftrightarrow (x_3 \leftrightarrow x_4)] \leftrightarrow [(x_1 \leftrightarrow x_4) \leftrightarrow (x_2 \leftrightarrow x_3)]$ is C-valid but it is not Q-valid. The proof is by considering the same algebra and the same observables as in the example of the identity that does not hold in all partial algebras (3.4) for the corresponding formula but substituting $\leftrightarrow$ for $+$.

5.2 Axiomatic system

Let $\Sigma^{\downarrow_\circ}$ be the set of formulas $\Sigma \cup \{\downarrow_\circ(\alpha_1, \cdots, \alpha_m) : \alpha_1, \cdots, \alpha_m \in \Sigma,$ for $m \in \mathbb{N}\}$. $\Sigma_n^{\downarrow_\circ}$ will be the subset of $\Sigma^{\downarrow_\circ}$ defined as $\Sigma_n \cup \{\downarrow_\circ(\alpha_1, \cdots, \alpha_m) : \alpha_1, \cdots, \alpha_m \in \Sigma_n,$ for $m \in \mathbb{N}\}$.

Observation: We will assume that $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined just with the connectives mentioned above, i.e., $\alpha_1 \wedge \alpha_2$ is an abbreviation of $\neg(\neg\alpha_1 \vee \neg\alpha_2)$; $\alpha_1 \rightarrow \alpha_2$ is an abbreviation of $\neg\alpha_1 \vee \alpha_2$ and $\alpha_1 \leftrightarrow \alpha_2$ is an abbreviation of $\neg(\neg(\neg\alpha_1 \vee \alpha_2) \vee \neg(\neg\alpha_2 \vee \alpha_1))$.

Definition 5.2.1. *Let Φ be a subset of $\Sigma_n^{\downarrow_\circ}$. A sequence $\gamma_1, \cdots, \gamma_k$ of formulas of $\Sigma_n^{\downarrow_\circ}$ is Φ-admissible if the following condition is satisfied:*

For all $i \in \{1, \cdots, k\}$, γ_i is either of the type $\downarrow_\circ(\alpha_1, \alpha_1)$, where α_1 is a subformula of a formula $\alpha \in \Phi$ or of the type $\downarrow_\circ(\alpha_1, \alpha_2)$, where $\alpha_1 \vee \alpha_2$ is a subformula of a formula $\alpha \in \Phi$ (we will call both of these subformulas "axioms extracted from α"); or there exist indices $i_1, \cdots, i_p$ such that $1 \leq i_k < i$ and γ_i follows from $\gamma_{i_1}, \cdots, \gamma_{i_p}$ by one of the rules below (rules of inference):

R₁: $\dfrac{\downarrow_\circ(\alpha_1, \cdots, \alpha_m)}{\downarrow_\circ(\alpha_i, \alpha_j)}$ where $1 \leq i, j \leq m$

R₂: $\dfrac{\downarrow_\circ(\alpha_1, \alpha_1) \quad \downarrow_\circ(\alpha_1, \alpha_2) \quad \cdots \quad \downarrow_\circ(\alpha_i, \alpha_j) \quad \cdots \quad \downarrow_\circ(\alpha_m, \alpha_m)}{\downarrow_\circ(\alpha_1, \cdots, \alpha_m)}$

(There are m^2 premisses of the type $\downarrow_\circ(\alpha_i, \alpha_j)$, where $1 \leq i, j \leq m$)

R₃: $\dfrac{\downarrow_\circ(\alpha_1, \alpha_2) \quad \alpha_2 \leftrightarrow \alpha_3}{\downarrow_\circ(\alpha_1, \alpha_3)}$

R₄: $\dfrac{\downarrow_\circ(\neg\alpha_1, \alpha_2)}{\downarrow_\circ(\alpha_1, \alpha_2)}$

R₅: $\dfrac{\downarrow_\circ(\alpha_1, \alpha_2, \alpha_3)}{\downarrow_\circ(\alpha_1 \vee \alpha_2, \alpha_3)}$

S₁: $\dfrac{\downarrow_\circ(\alpha_1, \cdots, \alpha_n)}{\beta(\alpha_1, \cdots, \alpha_n)}$ (where $\beta(x_1, \cdots, x_n)$ is a C-valid formula)

S₂: $\dfrac{\alpha_1 \quad \alpha_1 \rightarrow \alpha_2}{\alpha_2}$

5.2.1 Q-proof of a formula

Definition 5.2.2. *A sequence $\gamma_1, \cdots, \gamma_k$ of formulas of $\Sigma_n^{⫰}$ is a Q-proof of a formula $\alpha \in \Sigma_n$ if it is $\{\alpha\}$-admissible and there exists $i \in \{1, \cdots, k\}$ such that $\alpha = \gamma_i$.*

Observations: As we said before, the connectives $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined with the connectives $\neg$ and $\vee$. It is important to note that if we have a formula α with a subformula of the type $\alpha_1 \wedge \alpha_2$, $\alpha_1 \rightarrow \alpha_2$ or $\alpha_1 \leftrightarrow \alpha_2$, we can extract the axiom $⫰(\alpha_1, \alpha_2)$, just like with the case of $\alpha_1 \vee \alpha_2$, as we show below:

- $\alpha_1 \wedge \alpha_2 = \neg(\neg\alpha_1 \vee \neg\alpha_2)$. From here, we can extract a few axioms but the one needed is $⫰(\neg\alpha_1, \neg\alpha_2)$. We want to show that we can obtain $⫰(\alpha_1, \alpha_2)$ from $⫰(\neg\alpha_1, \neg\alpha_2)$:

$$
\begin{array}{c}
\text{Hypothesis} \\ \hline
⫰(\neg\alpha_1, \neg\alpha_2) \\ \hline
⫰(\neg\alpha_2, \neg\alpha_1) \\ \hline
⫰(\alpha_2, \neg\alpha_1) \\ \hline
⫰(\neg\alpha_1, \alpha_2) \\ \hline
⫰(\alpha_1, \alpha_2)
\end{array}
\begin{array}{l}
\\ (R_1) \\ (R_4) \\ (R_1) \\ (R_4)
\end{array}
$$

- $\alpha_1 \rightarrow \alpha_2 = \neg\alpha_1 \vee \alpha_2$. From here, the most relevant axiom is $⫰(\neg\alpha_1, \alpha_2)$. Let us show that we can obtain $⫰(\alpha_1, \alpha_2)$ from $⫰(\neg\alpha_1, \alpha_2)$:

$$
\begin{array}{c}
\text{Hypothesis} \\ \hline
⫰(\neg\alpha_1, \alpha_2) \\ \hline
⫰(\alpha_1, \alpha_2)
\end{array}
\begin{array}{l}
\\ (R_4)
\end{array}
$$

- $\alpha_1 \leftrightarrow \alpha_2 = \neg(\neg(\neg\alpha_1 \vee \alpha_2) \vee \neg(\neg\alpha_2 \vee \alpha_1))$. It is useful to see $\alpha_1 \leftrightarrow \alpha_2$ as $(\alpha_1 \rightarrow \alpha_2) \wedge (\alpha_2 \rightarrow \alpha_1)$, because $\alpha_1 \rightarrow \alpha_2$ is a subformula of $(\alpha_1 \rightarrow \alpha_2) \wedge (\alpha_2 \rightarrow \alpha_1)$ and we know that from $\alpha_1 \rightarrow \alpha_2$ we can extract $⫰(\alpha_1, \alpha_2)$.

So, any kind of occurrences of formulas of this type, we will extract this axiom trivially.

Let us consider some examples of Q-proofs to elucidate the definitions 5.2.1 and 5.2.2. It is important to mention that these Q-proofs will be presented in a tree format to enhance comprehension.

Example 1. We want to construct a Q-proof of $\alpha = (x_1 \vee \neg x_1) \vee x_2$. From α, we extract the axioms: $⫰(x_1, \neg x_1)$, $⫰(x_1 \vee \neg x_1, x_2)$ (as well as the reflexive ones, that is, $⫰(x_1, x_1)$, $⫰(\neg x_1, \neg x_1)$, $⫰(x_1 \vee \neg x_1, x_1 \vee \neg x_1)$, $⫰(x_2, x_2)$, $⫰(\alpha, \alpha)$). Then,

$$
\dfrac{\dfrac{\dfrac{\text{Axiom}}{⫰(x_1, x_1)}\,(R_2)}{\dfrac{⫰(x_1)}{x_1 \vee \neg x_1}\,(S_1)} \qquad \dfrac{\dfrac{\text{Axiom}}{⫰(x_1 \vee \neg x_1, x_2)}}{\beta(x_1 \vee \neg x_1, x_2)}\,(S_1)}{\alpha}\,(S_2)
$$

is a Q-proof of α.

Observations:

$\beta(x_1, x_2) = x_1 \to (x_1 \vee x_2)$ is a classical tautology. So,

$$
\beta(x_1 \vee \neg x_1, x_2) = (x_1 \vee \neg x_1) \to ((x_1 \vee \neg x_1) \vee x_2)
$$

Example 2. We want to construct a Q-proof of $\alpha = (x_1 \vee x_2) \vee \neg x_1$. We extract the axioms $⫰(x_1, x_2)$ and $⫰(x_1 \vee x_2, \neg x_1)$ (as well as the reflexive ones) from α. Then,

$$
\dfrac{\dfrac{\text{Axiom}}{⫰(x_1, x_2)}}{\beta(x_1, x_2)}\,(S_1)
$$

is a Q-proof of α.

Observations:

$\beta(x_1, x_2) = (x_1 \vee x_2) \vee \neg x_1 = \alpha$ is a classical tautology.

Example 3. We want to construct a Q-proof of $\alpha = ((x_2 \vee x_2) \vee x_1) \vee \neg x_2$. The axioms extracted from α are: $⫰(x_2 \vee x_2, x_1)$, $⫰((x_2 \vee x_2) \vee x_1), \neg x_2)$ (as well as the reflexive ones). Then,

$$
\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom}}{⫰(x_2 \vee x_2, x_1)}}{⫰(x_1, x_2 \vee x_2)}\,(R_1) \qquad \dfrac{\dfrac{\dfrac{\text{Axiom}}{⫰(x_2, x_2)}}{⫰(x_2)}\,(R_2)}{\gamma(x_2)}\,(S_1)}{⫰(x_1, x_2)}\,(R_3)}{\beta(x_1, x_2)}\,(S_1)
$$

is a Q-proof of α.

Observations:

$\beta(x_1, x_2) = \alpha$ and $\gamma(x_2) = (x_2 \vee x_2) \leftrightarrow x_2$ are C-valid formulas.

Example 4. We want to construct a Q-proof of $\alpha = ((x_1 \vee x_2) \wedge x_3) \leftrightarrow ((x_1 \wedge x_3) \vee (x_2 \wedge x_3))$. The axioms extracted from α are: $⫰(x_1 \vee x_2, x_3)$, $⫰(x_1 \wedge x_3, x_2 \wedge x_3)$, $⫰(x_1, x_2)$, $⫰(x_1, x_3)$, $⫰(x_2, x_3)$, $⫰((x_1 \vee x_2) \wedge x_3, (x_1 \wedge x_3) \vee (x_2 \wedge x_3))$ (as well as the reflexive ones). Then,

$$\dfrac{\dfrac{\text{Axiom}}{\cdots} \quad \dfrac{\text{Axiom}}{⫰(x_2, x_3)} \quad \dfrac{\text{Axiom}}{⫰(x_1, x_3)} \quad \dfrac{\text{Axiom}}{⫰(x_1, x_2)}}{\dfrac{⫰(x_1, x_2, x_3)}{\beta(x_1, x_2, x_3)}\ (S_1)}\ (R_2)$$

is a Q-proof of α.

Observations:

In $\cdots$ are the formulas of the type $⫰(x_i, x_i), 1 \leq i \leq 3$ (reflexivity), which are axioms, and the formulas of the type $⫰(x_j, x_i), 1 \leq i < j \leq 3, x_i \neq x_j$ (symmetry), which can be obtained by the rule R_1 from $⫰(x_i, x_j)$, that we already know to be axioms.

$\beta(x_1, x_2, x_3) = \alpha$ is a C-valid formula (it is the distributive law).

Proposition 5.2.3. *If α is a C-valid formula in n variables from which we can extract the axioms $⫰(x_i, x_j)$, for all $1 \leq i < j \leq n$, then there exists a Q-proof of α.*

Proof. Let $\alpha = \beta(x_1, \cdots, x_n)$ be a C-valid formula such that $⫰(x_i, x_j)$ are axioms, for, at least, $1 \leq i < j \leq n$. Then,

$$\dfrac{\dfrac{\text{Axiom}}{⫰(x_1, x_1)} \quad \dfrac{\text{Axiom}}{\cdots ⫰(x_i, x_j) \cdots} \quad \dfrac{\dfrac{\text{Axiom}}{\cdots ⫰(x_i, x_j) \cdots}}{⫰(x_j, x_i)}\ (R_1) \quad \dfrac{\text{Axiom}}{⫰(x_{n-1}, x_n)} \quad \dfrac{\text{Axiom}}{⫰(x_n, x_n)}}{\dfrac{⫰(x_1, \cdots, x_n)}{\beta(x_1, \cdots, x_n)}\ (S_1)}\ (R_2)$$

is a Q-proof of α. □

5.3 Soundness of the axiomatic system

Lemma 5.3.1. *Let $\alpha \in \Sigma_n$ and $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$ be a partial Boolean algebra. Let us also consider $\vec{q} \in Dom_{\alpha,n}$. Then,*

for all $i \in \mathbb{N}$, for all $\gamma \in \Sigma_n^{⫰}$, if γ is the i-th element of a Q-proof of α, then $P'(\gamma)$, where $P'(\gamma)$ is defined as:

If γ is a formula of Σ_n, then $\vec{q}$ is in the domain of the Boolean polynomial γ and $\gamma^*(\vec{q}) = \mathbf{1}$; If γ is a formula of the type $⫰(\alpha_1, \cdots, \alpha_k)$, then $\vec{q}$ is in the domain of the Boolean polynomials $\alpha_1, \cdots, \alpha_k$ and the elements ${\alpha_m}^*(\vec{q})$ are all in relation $⫰$, for $m \in \{1, \cdots, k\}$.

Proof. Let $\alpha \in \Sigma_n$ and $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$ be a partial Boolean algebra. Let $P(i)$ be defined as:

for all $\gamma \in \Sigma_n^{⫰}$, if γ is the i-th element of a Q-proof of α, then $P'(\gamma)$. We are going to prove $P(i)$ by induction on i, for all $i \in \mathbb{N}$.

- $P(1)$ iff for all $\gamma \in \Sigma_n^{⫰}$, if γ is the first element of a Q-proof of α, then $P'(\gamma)$.

  - Case $\gamma = ⫰(\alpha_1, \alpha_1)$:

    Then, α_1 is a subformula of α and due to the recursive way the domain of the Boolean polynomials are defined, we have that $\vec{q}$ is in the domain of α_1. Since $⫰$ is reflexive, ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_1}^*(\vec{q})$.

  - Case $\gamma = ⫰(\alpha_1, \alpha_2)$, $\alpha_1 \neq \alpha_2$:

    Then, $\alpha_1 \vee \alpha_2$ is a subformula of α. By definition of $Dom_{\alpha_1 \vee \alpha_2, n}$, $\vec{q} \in Dom_{\alpha_1, n} \cap Dom_{\alpha_2, n}$ and ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_2}^*(\vec{q})$.

- Let us assume $P(j)$, for all $j < k$. We want to show $P(k)$, i.e., for all $\gamma \in \Sigma_n^{⫰}$, if γ is the k-th element of a Q-proof of α, then $P'(\gamma)$.

  - **R₁**: Case $\gamma = ⫰(\alpha_i, \alpha_j)$, where $i, j \in \{1, \cdots, m\}$, is the k-th element of a Q-proof of α:

    Then, $\gamma_1 = ⫰(\alpha_1, \cdots, \alpha_m)$ is the $k - t$-th element of the Q-proof of α, for some $t \in \mathbb{N}$, and, by induction hypothesis, $P'(\gamma_1)$. Since γ_1 is a formula of the type $⫰(\alpha_1, \cdots, \alpha_m)$, we have that $\vec{q}$ is in the domain of the Boolean polynomials $\alpha_1, \cdots, \alpha_m$ and the elements ${\alpha_l}^*(\vec{q})$ are all in relation $⫰$, for all $l \in \{1, \cdots, m\}$, that is, for all $i, j \in \{1, \cdots, m\}$, ${\alpha_i}^*(\vec{q}) ⫰ {\alpha_j}^*(\vec{q})$.

  - **R₂**: Case $\gamma = (\alpha_1, \cdots, \alpha_m)$ is the k-th element of a Q-proof of α:

    Then, $\gamma_1 = ⫰(\alpha_1, \alpha_1), \gamma_2 = ⫰(\alpha_1, \alpha_2), \cdots, \gamma_p = ⫰(\alpha_i, \alpha_j), \cdots, \gamma_{m^2} = ⫰(\alpha_m, \alpha_m), 1 < p < m^2$, are the $k - t_1, k - t_2, \cdots, k - t_{m^2}$, for some $t_1, t_2, \cdots, t_{m^2} \in \mathbb{N}$, elements of the Q-proof of α. By induction hypothesis applied to each $\gamma_r, r \in \{1, \cdots, m^2\}$, and due to the fact that $\gamma_r \in \Sigma_n^{⫰} \setminus \Sigma_n$, $\vec{q}$ is in the domain of the Boolean polynomials $\alpha_1, \cdots, \alpha_m$ and ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_1}^*(\vec{q})$, ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_2}^*(\vec{q})$, $\cdots$, ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_m}^*(\vec{q}), \cdots$, ${\alpha_i}^*(\vec{q}) ⫰ {\alpha_j}^*(\vec{q}), \cdots, {\alpha_m}^*(\vec{q}) ⫰ {\alpha_m}^*(\vec{q})$, i.e., the elements ${\alpha_l}^*(\vec{q})$ are all in relation $⫰$, for all $l \in \{1, \cdots, m\}$.

  - **R₃**: Case $\gamma = ⫰(\alpha_1, \alpha_3)$ is the k-th element of a Q-proof of α:

    Then, $\gamma_1 = ⫰(\alpha_1, \alpha_2)$, $\gamma_2 = \alpha_2 \leftrightarrow \alpha_3$ are the $k - t_1$-th, $k - t_2$-th elements of the Q-proof of α, for some $t_1, t_2 \in \mathbb{N}$. By induction hypothesis applied to γ_1 and once $\gamma_1 \in \Sigma_n^{⫰} \setminus \Sigma_n$, we have that $\vec{q}$ is in the domain of the Boolean polynomials α_1, α_2 and ${\alpha_1}^*(\vec{q}) ⫰ {\alpha_2}^*(\vec{q})$. By induction hypothesis applied to γ_2 and once $\alpha_2 \in \Sigma_n$, $\vec{q}$ is in the domain of the Boolean polynomial $\theta = \alpha_2 \leftrightarrow \alpha_3$ and $\theta^*(\vec{q}) = \mathbf{1}$. $\theta^*(\vec{q}) = \mathbf{1}$ means, because of the observation below, that $\vec{q} \in Dom_{\alpha_2,n} \cap Dom_{\alpha_3,n}$, ${\alpha_2}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q}) = {\alpha_3}^*(\vec{q})$.

Since ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_2}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q}) = {\alpha_3}^*(\vec{q})$, follows that ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$.

So, once $\vec{q} \in Dom_{\alpha_1,n} \cap Dom_{\alpha_3,n}$ and ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$, we conclude $P'(\gamma)$.

Observation:

$\theta^*(\vec{q}) = (\alpha_2 \leftrightarrow \alpha_3)^*(\vec{q})$ is equivalent to $(\neg(\neg(\neg\alpha_2 \vee \alpha_3) \vee \neg(\neg\alpha_3 \vee \alpha_2)))^*(\vec{q})$. Applying multiple times the definition of $\_^*$ in a partial Boolean algebra, we get that $\theta^*(\vec{q}) = \neg(\neg(\neg{\alpha_2}^*(\vec{q}) \vee {\alpha_3}^*(\vec{q})) \vee \neg(\neg{\alpha_3}^*(\vec{q}) \vee {\alpha_2}^*(\vec{q})))$, i.e., $\theta^*(\vec{q}) = {\alpha_2}^*(\vec{q}) \leftrightarrow {\alpha_3}^*(\vec{q})$.

- **R₄**: Case $\gamma = \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}}(\alpha_1, \alpha_2)$ is the k-th element of a Q-proof of α:

  Then, $\gamma_1 = \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}}(\neg\alpha_1, \alpha_2)$ is the $k-t$-th element of the Q-proof of α, for some $t \in \mathbb{N}$, and, by induction hypothesis applied to γ_1 and due to the fact that $\gamma_1 \in \Sigma_n^{\rotatebox[origin=c]{-90}{$\multimap$}} \setminus \Sigma_n$, $\vec{q}$ is in the domain of the Boolean polynomials $\neg\alpha_1$ and α_2 and $(\neg\alpha_1)^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_2}^*(\vec{q})$.

  By definition, $Dom_{\neg\alpha_1,n} = Dom_{\alpha_1,n}$. So, $\vec{q} \in Dom_{\alpha_1,n}$. Since $(\neg\alpha_1)^*(\vec{q}) = \neg{\alpha_1}^*(\vec{q})$ and $(\neg\alpha_1)^*(\vec{q})$ and ${\alpha_2}^*(\vec{q})$ are commeasurable then, by theorem 4.2.2, $\neg\neg{\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_2}^*(\vec{q})$. By the observation below, $\neg\neg{\alpha_1}^*(\vec{q}) = {\alpha_1}^*(\vec{q})$. So, ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_2}^*(\vec{q})$ and we conclude $P'(\gamma)$.

  Observation:

  Since $\neg{\alpha_1}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q})$ are commeasurable, then the Boolean polynomials in $\neg{\alpha_1}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q})$ form a Boolean algebra and, by lemma 2.3.2, we have the property $\neg\neg{\alpha_1}^*(\vec{q}) = {\alpha_1}^*(\vec{q})$.

- **R₅**: Case $\gamma = \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}}(\alpha_1 \vee \alpha_2, \alpha_3)$ is the k-th element of a Q-proof of α:

  Then, $\gamma_1 = \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}}(\alpha_1, \alpha_2, \alpha_3)$ is the $k-t$-th element of the Q-proof of α, for some $t \in \mathbb{N}$, and, by induction hypothesis applied to γ_1 and since $\gamma \in \Sigma_n^{\rotatebox[origin=c]{-90}{$\multimap$}} \setminus \Sigma_n$, $\vec{q}$ is in the domain of the Boolean polynomials α_1, α_2 and α_3 and ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_2}^*(\vec{q})$, ${\alpha_1}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$ (as well as the symmetric elements and the reflexive ones).

  Since any two of the three previous elements are commeasurable then, by theorem 4.2.2, ${\alpha_1}^*(\vec{q}) \vee {\alpha_2}^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$, which is, by definition, $(\alpha_1 \vee \alpha_2)^*(\vec{q}) \mathrel{\rotatebox[origin=c]{-90}{$\multimap$}} {\alpha_3}^*(\vec{q})$. It remains to show that $\vec{q}$ is in the domain of the Boolean polynomial $\alpha_1 \vee \alpha_2$. By definition, $\vec{q} \in Dom_{\alpha_1 \vee \alpha_2,n}$ if, in particular, $\vec{q} \in Dom_{\alpha_1,n} \cap Dom_{\alpha_2,n}$. So, $P'(\gamma)$.

- $\mathbf{S_1}$: Case $\gamma = \beta(\alpha_1, \cdots, \alpha_n)$ (where $\beta(x_1, \cdots, x_n)$ is C-valid) is the k-th element of a Q-proof of α:

  Then, $\gamma_1 = \rfloor_{\circ}(\alpha_1, \cdots, \alpha_n)$ is the $k - t$-th element of the Q-proof of α, for some $t \in \mathbb{N}$. By induction hypothesis applied to γ_1 and since $\alpha \in \Sigma_n^{\rfloor_{\circ}} \setminus \Sigma_n$, $\vec{q}$ is in the domain of the Boolean polynomials $\alpha_1, \cdots, \alpha_n$ and ${\alpha_i}^*(\vec{q}) \rfloor_{\circ} {\alpha_j}^*(\vec{q})$, for all $i, j \in \{1, \cdots, n\}$. We want to prove that $\vec{q} \in Dom_{\gamma,n}$ and that $\gamma^*(\vec{q}) = \mathbf{1}$.

  By definition 4.3.2, since $\vec{q} \in Dom_{\alpha_1,n} \cap \cdots \cap Dom_{\alpha_n,n}$ and ${\alpha_i}^*(\vec{q}) \rfloor_{\circ} {\alpha_j}^*(\vec{q})$, for all $i, j \in \{1, \cdots, n\}$, $\vec{q} \in Dom_{\gamma,n} = B^n$. It remains to show that $\gamma^*(\vec{q}) = \mathbf{1}$. Given that any pair among ${\alpha_1}^*(\vec{q}), \cdots, {\alpha_n}^*(\vec{q})$ are commeasurable, the algebra of the Boolean polynomials in ${\alpha_1}^*(\vec{q}), \cdots, {\alpha_n}^*(\vec{q})$ is a Boolean algebra. As $\beta(x_1, \cdots, x_n)$ is a C-valid formula then, by the principle of substitution for tautologies (2.3.8), $\beta(\alpha_1, \cdots, \alpha_n) = \gamma$ is also a C-valid formula, that is, for all Boolean algebras $\mathcal{B}_1$ and for all valuation v, $\overline{\gamma}(v) = \mathbf{1}$. Once more, due to the commeasurability of all the elements of B, we have that $\mathcal{B} = (B, B^2, \vee, \neg, \mathbf{1}, \mathbf{0})$. Let us consider the Boolean algebra $\mathcal{B}_1 = (B, \wedge, \vee, \neg, \mathbf{1}, \mathbf{0})$ such that for all valuation v in B, $v(x_i) = {\alpha_i}^*(\vec{q})$, for all $i \in \{1, \cdots, n\}$. Then, by lemma 4.3.4, $\gamma^*(\vec{q}) = \overline{\gamma}(v) = \mathbf{1}$.

- $\mathbf{S_2}$: Case $\gamma = \alpha_2$ is the k-th element of a Q-proof of α:

  Then, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_1 \to \alpha_2$ are the $k - t_1$-th, $k - t_2$-th elements of the Q-proof of α, for some $t_1, t_2 \in \mathbb{N}$. By induction hypothesis applied to γ_1, $\vec{q}$ is in the domain of the Boolean polynomial α_1 and ${\alpha_1}^*(\vec{q}) = \mathbf{1}$. By induction hypothesis applied to γ_2, $\vec{q}$ is in the domain of the Boolean polynomial θ and $\theta^*(\vec{q}) = \mathbf{1}$, where $\theta = \alpha_1 \to \alpha_2$. Once θ is equivalent to $\neg\alpha_1 \vee \alpha_2$ then, by definition, $Dom_{\neg\alpha_1 \vee \alpha_2,n} = \{\vec{q} \in B^n : \vec{q} \in Dom_{\neg\alpha_1,n} \cap Dom_{\alpha_2,n}$ and $\neg{\alpha_1}^*(\vec{q}) \rfloor_{\circ} {\alpha_2}^*(\vec{q})\} = \{\vec{q} \in B^n : \vec{q} \in Dom_{\alpha_1,n} \cap Dom_{\alpha_2,n}$ and ${\alpha_1}^*(\vec{q}) \rfloor_{\circ} {\alpha_2}^*(\vec{q})\}$.

  We have that ${\alpha_1}^*(\vec{q}) = \mathbf{1}$. So, $\theta^*(\vec{q}) = (\alpha_1 \to \alpha_2)^*(\vec{q}) = \mathbf{1}$ implies, by the observation below, that ${\alpha_2}^*(\vec{q}) = \mathbf{1}$.

  Since $\vec{q} \in Dom_{\alpha_2,n}$ and ${\alpha_2}^*(\vec{q}) = \mathbf{1}$, we conclude $P'(\gamma)$.

  Observations:

  - Since ${\alpha_1}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q})$ are commeasurable, then the Boolean polynomials in ${\alpha_1}^*(\vec{q})$ and ${\alpha_2}^*(\vec{q})$ form a Boolean algebra and since ${\alpha_1}^*(\vec{q}) = \mathbf{1}$, we have by definition 4.3.2, by lemma 2.3.2 and by definition 2.3.1, the following equalities:

$(\alpha_1 \to \alpha_2)^*(\vec{q}) = (\neg\alpha_1 \vee \alpha_2)^*(\vec{q}) = \neg{\alpha_1}^*(\vec{q}) \vee {\alpha_2}^*(\vec{q}) = \neg\mathbf{1} \vee {\alpha_2}^*(\vec{q}) = \mathbf{0} \vee {\alpha_2}^*(\vec{q}) = {\alpha_2}^*(\vec{q})$.

- The rule S_2 does not preserve Q-validity, as remarked in [9], i.e., there are Q-valid formulas α_1 and $\alpha_1 \to \alpha_2$ such that α_2 is not Q-valid.

□

Soundness Theorem. *If there is a Q-proof of $\alpha \in \Sigma_n$ then, for all partial Boolean algebra $\mathcal{B}$, α holds in $\mathcal{B}$.*

Proof. Suppose that there is a Q-proof of $\alpha \in \Sigma_n$, say $S = \gamma_1, \cdots, \gamma_m$. Let $\mathcal{B}$ be a partial Boolean algebra and $\vec{q} \in Dom_{\alpha,n}$. Since S is a Q-proof of α, then $\alpha = \gamma_i$, for some $i \in \{1, \cdots, m\}$. By lemma 5.3.1, we have $P'(\gamma)$. Since $\alpha \in \Sigma_n$, we conclude that $\alpha^*(\vec{q}) = \mathbf{1}$. □

5.4 Completeness of the axiomatic system

In order to demonstrate the completeness theorem, it will be necessary to introduce a few lemmas as well as new concepts.

Definition 5.4.1. *Let $\alpha \in \Sigma_n$. A formula γ of $\Sigma_n^{⫰}$ is called "α-provable" if there exists an $\{\alpha\}$-admissible sequence $\gamma_1, \cdots, \gamma_k$ such that $\gamma = \gamma_i$, for some $i \in \{1, \cdots, k\}$.*

Observation: From now on, we will assume that α is a fixed formula in exactly n variables.

Definition 5.4.2. *Let us define Ω_α as the subset of formulas of Σ_n such that $⫰(\beta, \beta)$ is α-provable, that is, $\Omega_\alpha = \{\beta \in \Sigma_n : ⫰(\beta,\beta)$ is $\alpha -$ provable$\}$. Formulas of Ω_α contain no other variables than $x_1, \cdots, x_n$.*[1]

Lemma 5.4.3. *The formulas $x_1, \cdots, x_n$ and α are formulas of Ω.*

Proof. Let us consider the set Ω previously defined. We want to show that $x_1, \cdots, x_n$ and α are formulas of Ω, that is, that $x_1, \cdots, x_n, \alpha \in \Sigma_n$ (which is trivially true) and $⫰(x_i, x_i)$ and $⫰(\alpha, \alpha)$ are α-provable, for all $1 \leq i \leq n$. Since α is a formula in exactly n variables, then x_i is a subformula of α, for all $i \in \{1, \cdots, n\}$. So, we extract the axioms $⫰(x_i, x_i)$. Similarly, since α is a subformula of itself, then we also extract the axiom $⫰(\alpha, \alpha)$. So, $⫰(x_i, x_i)$ and $⫰(\alpha, \alpha)$ are α-provable and, consequently, $x_i, \alpha \in \Omega$, for all $i \in \{1, \cdots, n\}$. □

[1] Whenever there is no ambiguity, we will write Ω instead of Ω_α.

Definition 5.4.4. *Given $\alpha_1, \alpha_2 \in \Sigma_n$, we say "$\alpha_1$ is α-provable equivalent to α_2" (notation: $\alpha_1 \leftrightarrow_\alpha \alpha_2$) when $\alpha_1 \leftrightarrow \alpha_2$ is α-provable.*

Lemma 5.4.5. *The relation $\leftrightarrow_\alpha$ is an equivalence relation on the set Ω.*

Proof. We want to show that $\leftrightarrow_\alpha$ is an equivalence relation on Ω.

- $\leftrightarrow_\alpha$ is reflexive, that is, for all $\alpha_1 \in \Omega$, $\alpha_1 \leftrightarrow_\alpha \alpha_1$. Since $\alpha_1 \in \Omega$, then $⫰(\alpha_1, \alpha_1)$ is α-provable. So, we have

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)}}{⫰(\alpha_1)}\,(R_2)}{\alpha_1 \leftrightarrow \alpha_1}\,(S_1)$$

  and, consequently, $\alpha_1 \leftrightarrow_\alpha \alpha_1$.

- $\leftrightarrow_\alpha$ is symmetric, that is, for all $\alpha_1, \alpha_2 \in \Omega$, if $\alpha_1 \leftrightarrow_\alpha \alpha_2$, then $\alpha_2 \leftrightarrow_\alpha \alpha_1$. Since $\alpha_1, \alpha_2 \in \Omega$, we have that $⫰(\alpha_1, \alpha_1), ⫰(\alpha_2, \alpha_2)$ are α-provable. So, the tree

$$\dfrac{\dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2} \quad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)} \quad \dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2}}{⫰(\alpha_1,\alpha_2)}\,(R_3)}{(\alpha_1 \leftrightarrow \alpha_2) \to (\alpha_2 \leftrightarrow \alpha_1)}\,(S_1)}{\alpha_2 \leftrightarrow \alpha_1}\,(S_2)$$

  proves that $\alpha_2 \leftrightarrow_\alpha \alpha_1$.

- $\leftrightarrow_\alpha$ is transitive, that is, for all $\alpha_1, \alpha_2, \alpha_3 \in \Omega$, if $\alpha_1 \leftrightarrow_\alpha \alpha_2$ and $\alpha_2 \leftrightarrow_\alpha \alpha_3$, then $\alpha_1 \leftrightarrow_\alpha \alpha_3$. Since $\alpha_1, \alpha_2, \alpha_3 \in \Omega$, we have that $⫰(\alpha_1, \alpha_1), ⫰(\alpha_2, \alpha_2), ⫰(\alpha_3, \alpha_3)$ are α-provable.

$$\dfrac{\dfrac{\alpha\text{-provable}}{\alpha_2 \leftrightarrow \alpha_3} \quad \dfrac{\dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2} \quad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_i,\alpha_i)} \quad \dfrac{P_3}{⫰(\alpha_1,\alpha_2)} \quad \dfrac{P_2}{⫰(\alpha_2,\alpha_3)} \quad \dfrac{P_1}{⫰(\alpha_1,\alpha_3)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_j,\alpha_i)}}{⫰(\alpha_1,\alpha_2,\alpha_3)}\,(R_2)}{(\alpha_1 \leftrightarrow \alpha_2) \to [(\alpha_2 \leftrightarrow \alpha_3) \to (\alpha_1 \leftrightarrow \alpha_3)]}\,(S_1)}{(\alpha_2 \leftrightarrow \alpha_3) \to (\alpha_1 \leftrightarrow \alpha_3)}\,(S_2)}{\alpha_1 \leftrightarrow \alpha_3}\,(S_2)$$

  where $1 \leq i < j \leq 3$.

  P_1 is the subtree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)} \quad \dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2}}{⫰(\alpha_1,\alpha_2)}\,(R_3) \quad \dfrac{\alpha\text{-provable}}{\alpha_2 \leftrightarrow \alpha_3}}{⫰(\alpha_1,\alpha_3)}\,(R_3)$$

P_2 is the subtree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\alpha_2,\alpha_2)} \qquad \dfrac{\alpha\text{-provable}}{\alpha_2 \leftrightarrow \alpha_3}}{\text{⫰}(\alpha_2,\alpha_3)}\;(R_3)$$

P_3 is the subtree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\alpha_1,\alpha_1)} \qquad \dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2}}{\text{⫰}(\alpha_1,\alpha_2)}\;(R_3)$$

So, since $\leftrightarrow_\alpha$ is reflexive, symmetric and transitive, we conclude that it is an equivalence relation on Ω. ☐

Lemma 5.4.6. *Let $\beta \in \Omega$. Then, the formula $\text{⫰}(\beta, \beta \leftrightarrow \beta)$ is α-provable.*

Proof. Let us consider $\beta \in \Omega$. Then, $\text{⫰}(\beta,\beta)$ is α-provable. To show that $\text{⫰}(\beta, \beta \leftrightarrow \beta)$ is α-provable, we just need to take into account the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\cdots} \quad \dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)} \quad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\;(R_2)}{\beta \leftrightarrow \neg\neg\beta}\;(S_1)}{\text{⫰}(\beta,\neg\neg\beta)}\;(R_3)}{\text{⫰}(\neg\neg\beta,\beta)}\;(R_1)}{\text{⫰}(\neg\beta,\beta)}\;(R_4)}{\text{⫰}(\beta,\neg\beta)}\;(R_1)}{\text{⫰}(\beta,\beta,\neg\beta)}\;(R_2)}{\text{⫰}(\beta\vee\neg\beta,\beta)}\;(R_5)}{\text{⫰}(\beta,\beta\vee\neg\beta)}\;(R_1) \qquad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\;(R_2)}{(\beta\vee\neg\beta)\leftrightarrow(\beta\leftrightarrow\beta)}\;(S_1)}{\text{⫰}(\beta,\beta\leftrightarrow\beta)}\;(R_3)$$

In $\cdots$ appear the following formulas (which are obviously α-provable): $\text{⫰}(\neg\beta,\neg\beta)$ and $\text{⫰}(\neg\beta,\beta)$. In fact, there are in total 9 formulas of $\Sigma_n^{\text{⫰}}$ but since two of the formulas in $\text{⫰}(\beta,\beta,\neg\beta)$ are the same, we are going to have some of them repeated. ☐

Definition 5.4.7. *Given a formula $\alpha \in \Sigma_n$, the structure associated with it is $\mathcal{B} = (B, \text{⫰}, \vee, \neg, \mathbf{1}, \mathbf{0})$, with $B = \Omega/\leftrightarrow_\alpha$ [2], where:*

1. *for all $[\alpha_1], [\alpha_2] \in B$, $[\alpha_1]\text{⫰}[\alpha_2]$ iff $\text{⫰}(\alpha_1,\alpha_2)$ is α-provable*

[2] Whenever there is no ambiguity, we will write $[\beta]$ instead of $[\beta]_{\leftrightarrow_\alpha}$, for all elements $[\beta]$ of B.

2. *for all* $[\alpha_1], [\alpha_2] \in B$ *such that* $[\alpha_1]⫰[\alpha_2]$, $[\alpha_1] \vee [\alpha_2]$ *is the class* $[\alpha_1 \vee \alpha_2]$

3. *for all* $[\alpha_1] \in B$, $\neg[\alpha_1]$ *is the class* $[\neg\alpha_1]$

4. **1** *is the class of* α*-provable formulas, that is,* $\mathbf{1} = \{\beta \in \Omega : \beta \text{ is } \alpha\text{-provable}\}$

5. **0** *is the class* $\neg\mathbf{1}$*, that is,* $\mathbf{0} = [\neg\beta]$*, for* β α*-provable*

Lemma 5.4.8. *Let* $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$ *be defined as in the previous structure. The relation* $⫰$ *and the operations* $\vee$ *and* $\neg$ *are well defined on* B.

Proof. Let us consider $\mathcal{B} = (B, ⫰, \vee, \neg, \mathbf{1}, \mathbf{0})$. To show that $⫰$, $\neg$ and $\vee$ are well defined on B, we need to demonstrate:

- If $\alpha_1 \leftrightarrow_\alpha \alpha_2$, $\beta_1 \leftrightarrow_\alpha \beta_2$ and $⫰(\alpha_1, \beta_1)$ is α-provable, then $⫰(\alpha_2, \beta_2)$ is α-provable. Let us suppose that $\alpha_1 \leftrightarrow_\alpha \alpha_2$, $\beta_1 \leftrightarrow_\alpha \beta_2$ and $⫰(\alpha_1, \beta_1)$ is α-provable. The following tree demonstrates the desired proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\beta_1)}}{⫰(\beta_1,\alpha_1)}\,(R_1) \quad \dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2}}{⫰(\beta_1,\alpha_2)}\,(R_3)}{⫰(\alpha_2,\beta_1)}\,(R_1) \qquad \dfrac{\alpha\text{-provable}}{\beta_1 \leftrightarrow \beta_2}}{⫰(\alpha_2,\beta_2)}\,(R_3)$$

- If $\alpha_1 \leftrightarrow_\alpha \alpha_2$, then $\neg\alpha_1 \leftrightarrow_\alpha \neg\alpha_2$. Let us suppose that $\alpha_1 \leftrightarrow_\alpha \alpha_2$. Then, we just need to consider the following tree to prove that $\neg\alpha_1 \leftrightarrow_\alpha \neg\alpha_2$:

$$\dfrac{\dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2} \qquad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)} \quad \dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2}}{⫰(\alpha_1,\alpha_2)}\,(R_3)}{(\alpha_1 \leftrightarrow \alpha_2) \to (\neg\alpha_1 \leftrightarrow \neg\alpha_2)}\,(S_1)}{\neg\alpha_1 \leftrightarrow \neg\alpha_2}\,(S_2)$$

- If $\alpha_1 \leftrightarrow_\alpha \alpha_2$, $\beta_1 \leftrightarrow_\alpha \beta_2$, $⫰(\alpha_1, \beta_1)$ is α-provable and $⫰(\alpha_2, \beta_2)$ is α-provable, then $\alpha_1 \vee \beta_1 \leftrightarrow_\alpha \alpha_2 \vee \beta_2$. Let us suppose that $\alpha_1 \leftrightarrow_\alpha \alpha_2$, $\beta_1 \leftrightarrow_\alpha \beta_2$, $⫰(\alpha_1, \beta_1)$ is α-provable and $⫰(\alpha_2, \beta_2)$ is α-provable. We just need to consider the following tree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{\alpha_1 \leftrightarrow \alpha_2} \qquad \dfrac{\dfrac{\alpha\text{-provable}}{\beta_1 \leftrightarrow \beta_2} \qquad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_i,\alpha_i)} \quad \dfrac{\alpha\text{-provable}}{⫰(\beta_i,\beta_i)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\beta_1)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_2,\beta_2)} \quad P_1 \quad P_2 \quad P_3 \quad P_4}{⫰(\beta_1,\beta_2,\alpha_1,\alpha_2)}\,(R_2)}{(\beta_1 \leftrightarrow \beta_2) \to [(\alpha_1 \leftrightarrow \alpha_2) \to (\alpha_1 \vee \beta_1 \leftrightarrow \alpha_2 \vee \beta_2)]}\,(S_1)}{(\alpha_1 \leftrightarrow \alpha_2) \to (\alpha_1 \vee \beta_1 \leftrightarrow \alpha_2 \vee \beta_2)}\,(S_2)}{\alpha_1 \vee \beta_1 \leftrightarrow \alpha_2 \vee \beta_2}\,(S_2)$$

  where:

P_1 :

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\alpha_1,\alpha_1)} \qquad \dfrac{\alpha\text{-provable}}{\alpha_1\leftrightarrow\alpha_2}}{\text{⫰}(\alpha_1,\alpha_2)}\,(R_3)$$

P_2 :

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta_1,\beta_1)} \qquad \dfrac{\alpha\text{-provable}}{\beta_1\leftrightarrow\beta_2}}{\text{⫰}(\beta_1,\beta_2)}\,(R_3)$$

P_3 :

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\alpha_1,\beta_1)} \qquad \dfrac{\alpha\text{-provable}}{\beta_1\leftrightarrow\beta_2}}{\text{⫰}(\alpha_1,\beta_2)}\,(R_3)$$

P_4 :

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\alpha_1,\beta_1)}}{\text{⫰}(\beta_1,\alpha_1)}\,(R_1) \qquad \dfrac{\alpha\text{-provable}}{\alpha_1\leftrightarrow\alpha_2}}{\text{⫰}(\beta_1,\alpha_2)}\,(R_3)}{\text{⫰}(\alpha_2,\beta_1)}\,(R_1)$$

Observations:

- The formula $\sigma(x_1,x_2,x_3,x_4) = (x_1 \leftrightarrow x_2) \to [(x_3 \leftrightarrow x_4) \to (x_3 \vee x_1 \leftrightarrow x_4 \vee x_2)]$ is C-valid. Note that $\sigma(\beta_1,\beta_2,\alpha_1,\alpha_2)$ is the formula

$$(\beta_1 \leftrightarrow \beta_2) \to [(\alpha_1 \leftrightarrow \alpha_2) \to (\alpha_1 \vee \beta_1 \leftrightarrow \alpha_2 \vee \beta_2)]$$

- When we apply the R_2 rule in the main tree, theoretically, we should have had $4^2 = 16$ formulas, but we only presented 8. This decision was made because the remaining ones are very similar to demonstrate (the idea is exactly the same as the other ones). So, we chose to omit them.

□

Lemma 5.4.9. *The structure previously defined, that is, $\mathcal{B} = (B, \text{⫰}, \vee, \neg, \mathbf{1}, \mathbf{0})$, is a partial Boolean algebra.*

Proof. We want to prove that:

1. The relation $⫰$ is reflexive and symmetric. Let us consider $[\alpha_1] \in B$. We know that $[\alpha_1]⫰[\alpha_1]$ if $⫰(\alpha_1, \alpha_1)$ is α-provable. By definition 5.4.7, $\alpha_1 \in \Omega$ and by definition of Ω, $⫰(\alpha_1, \alpha_1)$ is α-provable.

   Now, let us consider $[\alpha_1], [\alpha_2] \in B$ such that $[\alpha_1]⫰[\alpha_2]$. Then, $⫰(\alpha_1, \alpha_2)$ is α-provable. We want to show that $⫰(\alpha_2, \alpha_1)$ is α-provable. By the R_1 rule, from $⫰(\alpha_1, \alpha_2)$ we can conclude $⫰(\alpha_2, \alpha_1)$. Consequently, $⫰(\alpha_2, \alpha_1)$ is α-provable.

2. For all $[\alpha_1] \in B$, $[\alpha_1]⫰\mathbf{1}$ and $\alpha_1⫰\mathbf{0}$. Let us consider $[\alpha_1] \in B$. It is going to be useful to see **1** as $[\alpha_1 \leftrightarrow \alpha_1]$ and **0** as $[\neg(\alpha_1 \leftrightarrow \alpha_1)]$. To prove that, in fact, $\mathbf{1} = [\alpha_1 \leftrightarrow \alpha_1]$ and $\mathbf{0} = [\neg(\alpha_1 \leftrightarrow \alpha_1)]$ we just need to show that $\alpha_1 \leftrightarrow \alpha_1$ is α-provable. So, the following tree proves it:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1, \alpha_1)}}{⫰(\alpha_1)}\,{\scriptstyle (R_2)}}{\alpha_1 \leftrightarrow \alpha_1}\,{\scriptstyle (S_1)}$$

   Now, $[\alpha_1]⫰[\alpha_1 \leftrightarrow \alpha_1]$ iff $⫰(\alpha_1, \alpha_1 \leftrightarrow \alpha_1)$ is α-provable. By lemma 5.4.6, since $\alpha_1 \in \Omega$, then $⫰(\alpha_1, \alpha_1 \leftrightarrow \alpha_1)$ is α-provable.

   In an analogous way, one can easily prove that $[\alpha_1]⫰[\neg(\alpha_1 \leftrightarrow \alpha_1)]$, i.e., that $⫰(\alpha_1, \neg(\alpha_1 \leftrightarrow \alpha_1))$ is α-provable. The only difference is in the initial steps:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1, \alpha_1 \leftrightarrow \alpha_1)} \qquad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1, \alpha_1)}}{⫰(\alpha_1)}\,{\scriptstyle (R_2)}}{(\alpha_1 \leftrightarrow \alpha_1) \leftrightarrow \neg\neg(\alpha_1 \leftrightarrow \alpha_1)}\,{\scriptstyle (S_1)}}{⫰(\alpha_1, \neg\neg(\alpha_1 \leftrightarrow \alpha_1))}\,{\scriptstyle (R_3)}}{⫰(\neg\neg(\alpha_1 \leftrightarrow \alpha_1), \alpha_1)}\,{\scriptstyle (R_1)}}{\dfrac{⫰(\neg(\alpha_1 \leftrightarrow \alpha_1), \alpha_1)}{⫰(\alpha_1, \neg(\alpha_1 \leftrightarrow \alpha_1))}\,{\scriptstyle (R_1)}}\,{\scriptstyle (R_4)}$$

3. The function $\vee$ is defined exactly for those pairs $([\alpha_1], [\alpha_2]) \in B \times B$ such that $[\alpha_1]⫰[\alpha_2]$. By lemma 5.4.8, it is trivial that this holds.

4. If any two of $[\alpha_1], [\alpha_2], [\alpha_3] \in B$ are commeasurable, then $[\alpha_1] \vee [\alpha_2]⫰[\alpha_3]$ and $\neg[\alpha_1]⫰[\alpha_2]$. By definition, $[\alpha_1] \vee [\alpha_2] = [\alpha_1 \vee \alpha_2]$. So, we want to show that $⫰(\alpha_1 \vee \alpha_2, \alpha_3)$ is α-provable. By the hypothesis, we extract that $⫰(\alpha_1, \alpha_2)$, $⫰(\alpha_1, \alpha_3)$ and $⫰(\alpha_2, \alpha_3)$ are all α-provable and by

the fact that $\alpha_k \in \Omega$, we have that $⫰(\alpha_k, \alpha_k)$ are α-provable, for all $k \in \{1,2,3\}$. So, to prove that $⫰(\alpha_1 \vee \alpha_2, \alpha_3)$ is α-provable we just need to consider the following tree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_k,\alpha_k)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_2)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_3)} \quad \dfrac{\alpha\text{-provable}}{⫰(\alpha_2,\alpha_3)} \quad \dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_i,\alpha_j)}}{⫰(\alpha_j,\alpha_i)}\,(R_1)}{\dfrac{⫰(\alpha_1,\alpha_2,\alpha_3)}{⫰(\alpha_1\vee\alpha_2,\alpha_3)}\,(R_5)}\,(R_2)$$

where $1 \leq i < j \leq 3$.

Now, by definition, $\neg[\alpha_1] = [\neg\alpha_1]$. We want to show that $⫰(\neg\alpha_1, \alpha_2)$ is α-provable. Acknowledging some of the facts mentioned above, we just need to consider the following tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_2)}}{⫰(\alpha_2,\alpha_1)}\,(R_1) \quad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)}}{⫰(\alpha_1)}\,(R_2)}{\alpha_1 \leftrightarrow \neg\neg\alpha_1}\,(S_1)}{⫰(\alpha_2,\neg\neg\alpha_1)}\,(R_3) \quad \dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\alpha_1,\alpha_1)}}{⫰(\alpha_1)}\,(R_2)}{\neg\neg\alpha_1 \leftrightarrow \alpha_1}\,(S_1)}{\dfrac{\dfrac{⫰(\alpha_2,\neg\neg\alpha_1)}{⫰(\neg\neg\alpha_1,\alpha_2)}\,(R_1)}{⫰(\neg\alpha_1,\alpha_2)}\,(R_4)}\,(R_3)$$

5. Let us consider that any two of $[\alpha_1], [\alpha_2], [\alpha_3] \in B$ are commeasurable. We want to prove that the algebra of the polynomials in $[\alpha_1], [\alpha_2], [\alpha_3]$, that is, $\mathcal{B} = (B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$, is a Boolean algebra, where:

   - $B' \subseteq B$ is inductively defined:
     1. $[\alpha_1], [\alpha_2], [\alpha_3] \in B'$
     2. $\mathbf{0}, \mathbf{1} \in B'$
     3. If $[\beta], [\sigma] \in B'$ and $[\beta]⫰[\sigma]$, then $[\beta] \vee [\sigma] \in B'$ and $[\beta] \wedge' [\sigma] \in B'$
     4. If $[\beta] \in B'$, $[\neg\beta] \in B'$
   - The operations $\vee'$, $\neg'$ and $\wedge'$ are defined as:

     $\vee' : \vee|_{B' \times B'}$

     $\neg' : \neg|_{B'}$

     $\wedge' : B' \times B' \to B'$ such that for all $[\beta], [\sigma] \in B'$, with $[\beta]⫰[\sigma]$, $[\beta] \wedge' [\sigma] = [\beta \wedge \sigma] = [\neg(\neg\beta \vee \neg\sigma)]$

Observations: We are going to assume that any two elements in B' are compatible, the three new operations are total functions and B' is closed under these operations. The proof is similar to the ones provided in lemma 4.1.2 and proposition 4.1.3.

Since any two elements in B' are compatible, we will state some facts which are going to be useful throughout the following proofs:

- For all $[\beta] \in B'$, $⫰(\beta, \beta)$ is α-provable
- For all $[\beta], [\sigma] \in B'$, $⫰(\beta, \sigma)$ is α-provable (definition of compatibility)

Let us consider the structure $(B', \wedge', \vee')$. We want to show that it is a lattice, that is, for all $[\beta], [\sigma], [\theta] \in B'$, we have:

(a) Idempotency for $\vee'$ and $\wedge'$, that is, $[\beta] \vee' [\beta] = [\beta] = [\beta] \wedge' [\beta]$. By definition, $[\beta] \vee' [\beta] = [\beta \vee \beta]$. We want show that $(\beta \vee \beta) \leftrightarrow_\alpha \beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\beta,\beta)}}{⫰(\beta)}\,(R_2)}{(\beta \vee \beta) \leftrightarrow \beta}\,(S_1)$$

It is analogous to the operation $\wedge'$.

(b) Commutativity for $\vee'$ and $\wedge'$, that is, $[\beta] \vee' [\sigma] = [\sigma] \vee' [\beta]$ and $[\beta] \wedge' [\sigma] = [\sigma] \wedge' [\beta]$. By definition, $[\beta] \wedge' [\sigma] = [\beta \wedge \sigma]$ and $[\sigma] \wedge' [\beta] = [\sigma \wedge \beta]$. We want to show that $\beta \wedge \sigma \leftrightarrow_\alpha \sigma \wedge \beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{⫰(\beta,\sigma)}}{(\beta \wedge \sigma) \leftrightarrow (\sigma \wedge \beta)}\,(S_1)$$

It is analogous to $\vee'$

(c) Associativity for $\vee'$ and $\wedge'$, that is, $([\beta] \vee' [\sigma]) \vee' [\theta] = [\beta] \vee' ([\sigma] \vee' [\theta])$. By definition, $([\beta] \vee' [\sigma]) \vee' [\theta] = [(\beta \vee \sigma) \vee \theta]$ and $[\beta] \vee' ([\sigma] \vee' [\theta]) = [\beta \vee (\sigma \vee \theta)]$. We want to show that $(\beta \vee \sigma) \vee \theta \leftrightarrow_\alpha \beta \vee (\sigma \vee \theta)$. We just need o consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\cdots} \quad \dfrac{\alpha\text{-provable}}{⫰(\beta,\sigma)} \quad \dfrac{\alpha\text{-provable}}{⫰(\beta,\theta)} \quad \dfrac{\alpha\text{-provable}}{⫰(\sigma,\theta)}}{⫰(\beta,\sigma,\theta)}\,(R_2)}{(\beta \vee \sigma) \vee \theta \leftrightarrow \beta \vee (\sigma \vee \theta)}\,(S_1)$$

In $\cdots$ are the formulas of $\Sigma_n^{⫯}$ of the type $⫰(\beta, \beta)$, $⫰(\theta, \theta)$ and $⫰(\sigma, \sigma)$ and the formulas of the type $⫰(\sigma, \beta)$, $⫰(\theta, \beta)$ and $⫰(\theta, \sigma)$, that by R_1 rule, one obtains α-provable formulas.

It is analogous to $\wedge'$.

(d) Absorption, that is, $[\beta] \wedge' ([\beta \vee' [\sigma]]) = [\beta] = [\beta] \vee' ([\beta \wedge' [\sigma]])$. By definition, $[\beta] \wedge' ([\beta \vee' [\sigma]]) = [\beta \wedge (\beta \vee \sigma)]$. We want to show that $\beta \wedge (\beta \vee \sigma) \leftrightarrow_\alpha \beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\sigma)}}{\beta \wedge (\beta \vee \sigma) \leftrightarrow \beta}\ (S_1)$$

It is analogous to $[\beta] \vee' ([\beta \wedge' [\sigma]])$.

So, we proved that $(B', \wedge', \vee')$ is a lattice.

Now, we need to show that the lattice $(B', \wedge', \vee')$ is distributive, that is, for all $[\beta], [\sigma], [\theta] \in B'$, $[\beta] \wedge' ([\sigma] \vee' [\theta]) = ([\beta] \wedge' [\sigma]) \vee' ([\beta] \wedge' [\theta])$. By definition, $[\beta] \wedge' ([\sigma] \vee' [\theta]) = [\beta \wedge (\sigma \vee \theta)]$ and $([\beta] \wedge' [\sigma]) \vee' ([\beta] \wedge' [\theta]) = [(\beta \wedge \sigma) \vee (\beta \wedge \theta)]$. We want to show that $\beta \wedge (\sigma \vee \theta) \leftrightarrow_\alpha (\beta \wedge \sigma) \vee (\beta \wedge \theta)$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\cdots} \quad \dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\sigma)} \quad \dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\theta)} \quad \dfrac{\alpha\text{-provable}}{\text{⫰}(\sigma,\theta)}}{\text{⫰}(\beta,\sigma,\theta)}\ (R_2)}{\beta \wedge (\sigma \vee \theta) \leftrightarrow (\beta \wedge \sigma) \vee (\beta \wedge \theta)}\ (S_1)$$

In $\cdots$ are the formulas of $\Sigma_n^{\text{⫰}}$ of the type $\text{⫰}(\beta,\beta)$, $\text{⫰}(\theta,\theta)$ and $\text{⫰}(\sigma,\sigma)$ and the formulas of the type $\text{⫰}(\sigma,\beta)$, $\text{⫰}(\theta,\beta)$ and $\text{⫰}(\theta,\sigma)$, that by R_1 rule, one obtains α-provable formulas.

So, $(B', \wedge', \vee')$ is a distributive lattice.

Finally, it remains to show that for all $[\beta] \in B'$, $[\beta] \wedge' \mathbf{0} = \mathbf{0}$, $[\beta] \vee' \mathbf{1} = \mathbf{1}$, $[\beta] \wedge' \neg[\beta] = \mathbf{0}$ and $[\beta] \vee' \neg[\beta] = \mathbf{1}$.

Let us consider $[\beta] \in \mathbf{1}$.

Observations: For this part of the proof, it will be useful to see **1** as $[\beta \vee \neg\beta]$ and **0** as $[\neg(\beta \vee \neg\beta)] = [\beta \wedge \neg\beta]$ (actually, we could consider any tautology in classical logic). Let us prove that $\mathbf{1} = [\beta \vee \neg\beta]$ and $\mathbf{0} = [\beta \wedge \neg\beta]$. We want to show that $\beta \vee \neg\beta$ is α-provable. Then, we just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\ (R_2)}{\beta \vee \neg\beta}\ (S_1)$$

By definition, we have $[\beta] \wedge' \mathbf{0} = [\beta] \wedge' [\beta \wedge \neg\beta] = [\beta \wedge (\beta \wedge \neg\beta)]$. We want to show that $\beta \wedge (\beta \wedge \neg\beta) \leftrightarrow_\alpha \beta \wedge \neg\beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\,(R_2)}{\beta \wedge (\beta \wedge \neg\beta) \leftrightarrow \beta \wedge \neg\beta}\,(S_1)$$

By definition, we have $[\beta] \vee' \mathbf{1} = [\beta] \vee' [\beta \vee \neg\beta] = [\beta \vee (\beta \vee \neg\beta)]$. We want to show that $\beta \vee (\beta \vee \neg\beta) \leftrightarrow_\alpha \beta \vee \neg\beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\,(R_2)}{\beta \vee (\beta \vee \neg\beta) \leftrightarrow \beta \vee \neg\beta}\,(S_1)$$

By definition, $[\beta] \wedge' \neg[\beta] = [\beta \wedge \neg\beta]$. We want to show that $\beta \wedge \neg\beta \leftrightarrow_\alpha \beta \wedge \neg\beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\,(R_2)}{\beta \wedge \neg\beta \leftrightarrow \beta \wedge \neg\beta}\,(S_1)$$

By definition, $[\beta] \vee' \neg[\beta] = [\beta \vee \neg\beta]$. We want to show that $\beta \vee \neg\beta \leftrightarrow_\alpha \beta \vee \neg\beta$. We just need to consider the tree:

$$\dfrac{\dfrac{\dfrac{\alpha\text{-provable}}{\text{⫰}(\beta,\beta)}}{\text{⫰}(\beta)}\,(R_2)}{\beta \vee \neg\beta \leftrightarrow \beta \vee \neg\beta}\,(S_1)$$

So, since $(B', \wedge', \vee')$ is a distributive lattice and for all $[\beta], [\sigma] \in B'$, $[\beta] \wedge' \mathbf{0} = \mathbf{0}$, $[\beta] \vee' \mathbf{1} = \mathbf{1}$, $[\beta] \wedge' \neg[\beta] = \mathbf{0}$ and $[\beta] \vee' \neg[\beta] = \mathbf{1}$, we conclude that $(B', \vee', \wedge', \neg', \mathbf{1}, \mathbf{0})$ is a Boolean algebra. □

Completeness Theorem. *If a formula $\alpha \in \Sigma_n$ holds in all partial Boolean algebras, then there exists a Q-proof of α.*

Proof. Let us assume that there does not exist a Q-proof of the formula $\alpha \in \Sigma_n$. We want to construct a partial Boolean algebra $\mathcal{B}$ such that α does not hold in $\mathcal{B}$. Let us consider the partial Boolean algebra $\mathcal{B} = (B, \text{⫰}, \vee, \neg, \mathbf{1}, \mathbf{0})$ previously defined in the definition 5.4.7. Let q_i be the class of the formula $x_i \in \Omega$ and let $\beta \in \Omega$. So, $[x_i] = {x_i}^*(\vec{q}) = q_i$. Similarly, the class of β is the element $\beta^*(\vec{q})$, that is,

$[\beta] = \beta^*(\vec{q})$, which is easily proven by induction on β; we have chosen to omit it. By definition, β holds in the partial Boolean algebra $\mathcal{B}$ iff for all $\vec{q} \in Dom_{\beta,n}$, $\beta^*(\vec{q}) = \mathbf{1}$ (definition 4.3.5). Consequently, $\beta^*(\vec{q}) = \mathbf{1}$ iff $[\beta] = \mathbf{1}$ iff β is α-provable (the first equivalence is by the previous observation that $\beta^*(\vec{q}) = [\beta]$, and the second one is by definition of **1**). So, β is α-provable iff β holds in $\mathcal{B}$. In particular, α is α-provable[3] iff α holds in $\mathcal{B}$. Since α is not α-provable, α does not hold in $\mathcal{B}$. □

[3] α is α-provable if there exists a Q-proof of α.

Chapter 6

Conclusion

Having all the basic concepts clarified, our study began with an exploration of partial algebras, the foundational structures from which the compatibility relation originated. We studied polynomials within this context, their domains and their respective function, crucial for assigning values to these polynomials in the partial algebra. Subsequently, we extended our study to partial Boolean algebras, delving into Boolean polynomials, their domains and their respective function, in order to assign values to these Boolean polynomials in the partial Boolean algebra. We concluded that the set of formulas in n variables constitutes a subset of the Boolean polynomials in n variables, implying that the value of a propositional calculus formula aligns with the value of a Boolean polynomial, when it makes sense to do such a comparison, that is, when we have all the compatibilities inherent to the formula within the domain of the Boolean polynomial.

The dissertation's title, "Partial classical propositional logic", was elucidated through the study of Q-valid formulas, accompanied by illustrative examples and counterexamples. The creation of a counterexample, which is not straightforward, involved utilizing partial algebras. So, although we initially defined partial Boolean algebras independently of partial algebras, studying them became necessary. The process of proving theorems within this newly formal system proved to be complex and, occasionally, counterintuitive. Certain seemingly straightforward logical deductions required significant effort. For instance, the direct demonstration (without resorting to the theorems of soundness and completeness) that any C-valid formula in one or two variables is Q-provable was omitted, because we could not prove it in full generality.

The dissertation's beginning involved the study of orthologic and ortholattices, although these studies did not make it into the dissertation. This exploration was essential in understanding the varying semantics of different quantum logics. Initially, our plan was to study two articles, one of which was [9] and the other [8]. However, we focused on the [9] because on the other one the formal system seemed to be less intuitive, due to the lack of resemblance to the formal system of classical logic, and more complex. Additionally, this paper did not explore the use of partial algebras. An area I had hoped to explore was transitive Boolean algebras. Unfortunately, due to the complexity of the quantum logic currently being studied, I did not have the opportunity of such exploration. Diving into additional literature might have offered a deeper understanding of the different possibilities of interpretation of this logic, as seen in [8].

Looking ahead, delving deeper into this quantum logic and its related counterparts, such as transitive partial Boolean algebras, along with their connection to partially ordered orthomodular sets, would be a logical continuation. Understanding the alignment of these new concepts with the logic explored here and determining whether orthological and orthomodular quantum logics offer advantages over partial classical propositional logic would mark a promising starting point for future research. Additionally, fully understanding the recent article [1] would be interesting and it could be the next step to delve deeper into the world of quantum computing.

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