On ω-identities over finite aperiodic semigroups with commuting idempotents

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Abstract

The word problems for ω-terms over the pseudovariety of aperiodic semigroups with commuting idempotents and over the pseudovariety generated by finite aperiodic inverse semigroups are studied. The two problems have different solutions, thus showing that the two pseudovarieties are distinct, a result proven by Higgins and Margolis.

1 Introduction

A pseudovariety of semigroups is a class of finite semigroups closed under taking homomorphic images of subsemigroups and finitary direct products. In this paper, we will be interested in two subpseudovarieties of A, the pseudovariety of all finite aperiodic semigroups. One of them is $\mathsf{A} \cap \mathsf{ECom}$, the pseudovariety of all finite aperiodic semigroups in which idempotents commute. Recall that, by Ash's Theorem [3], the pseudovariety ECom of idempotent commuting semigroups is generated by the class of finite inverse semigroups. The other pseudovariety we will study is denoted AInv and is the pseudovariety generated by finite aperiodic inverse semigroups. This is a proper subpseudovariety of $\mathsf{A} \cap \mathsf{ECom}$, as shown by Higgins and Margolis [6].

The objective of this paper is to investigate ω-identities over the pseudovarieties $\mathsf{A} \cap \mathsf{ECom}$ and AInv. That is, given two terms obtained from the letters of an alphabet A using the operations of multiplication and ω-power, we want to examine whether these terms coincide over all A-generated elements of $\mathsf{A} \cap \mathsf{ECom}$ (resp. AInv). Similar study was already performed

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for the pseudovariety A [7, 8] as well as for some of its most important subpseudovarieties such as J of $\mathcal{J}$-trivial semigroups [1], R of $\mathcal{R}$-trivial semigroups [2] and LSl of local semilattices [5].

If V is one of $\mathsf{A} \cap \mathsf{ECom}$ or AInv, then we describe a procedure that decides weather a given ω-identity is valid over V. These problems reduce to consider only rank 1 ω-terms. We associate to each of these ω-terms α a reversible aperiodic automaton $\mathcal{A}(\alpha)$ and an inverse aperiodic automaton $\mathcal{B}(\alpha)$. The automata $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ characterize completely the ω-term α over $\mathsf{A} \cap \mathsf{ECom}$ and AInv respectively, thus providing a tool to test rank 1 ω-identities and completing the solution of the ω-word problems over these pseudovarieties.

This article is intended to be an extended summary, without proofs, of a future paper with all the details. We leave the technical details of the results to the full paper [4] and focus on a clear description of the algorithms.

2 ω-identities

An ω-term is a formal expression obtained from the letters of an alphabet A using two operations: the binary, associative, concatenation and the unary ω-power. Any ω-term α can be given a natural interpretation on a finite semigroup S as a mapping $\alpha_S : S^A \to S$, as follows: each letter a of A is interpreted as the mapping sending each element of S^A to its image on a, the concatenation is viewed as the semigroup multiplication, while the ω-power is interpreted as the unary operation which sends each element s of S to its unique idempotent power s^ω.

An ω-*identity* is a formal equality $\alpha = \beta$ between ω-terms α and β. The ω-word problem for a pseudovariety V consists in deciding, for any given ω-identity $\alpha = \beta$, whether $\mathsf{V} \models \alpha = \beta$, that is, whether α and β have the same interpretation over every semigroup of V. Let Σ be the following set of ω-identities

$$\begin{aligned}
&(a^n)^\omega = (a^\omega)^\omega = a^\omega a^\omega = a^\omega, \quad (n \in \mathbb{N})\\
&(ab)^\omega a = a(ba)^\omega,\\
&aa^\omega = a^\omega = a^\omega a,\\
&a^\omega b^\omega = b^\omega a^\omega.
\end{aligned}$$

It is easy to verify that $\mathsf{A} \cap \mathsf{ECom}$ satisfies these ω-identities. The following lemma identifies some ω-identities that will be important to simplify the word problems under study.

Lemma 2.1. *The pseudovariety* $\mathsf{A} \cap \mathsf{ECom}$ *verifies the* ω*-identities* $(a^\omega b)^\omega = a^\omega b^\omega = (ab^\omega)^\omega$.

The *rank* of an ω-term α is the maximum number $\mathrm{rank}(\alpha)$ of nested ω-powers in it. For instance, the expression $ab\big(a(ba)^\omega b\big)^\omega b^4\big((a^\omega)^\omega(a^3)^\omega\big)^\omega$ represents an ω-term α on the alphabet $\{a, b\}$ such that $\mathrm{rank}(\alpha) = 3$ and it is not difficult to verify that every finite semigroup satisfies $\alpha = ab(ab)^\omega b^4 a^\omega$.

Using Lemma 2.1 above, one can show the following result.

Proposition 2.2. *Let* α *be an* ω*-term such that* $\mathrm{rank}(\alpha) \geq 1$*. It is possible to compute a rank 1* ω*-term* β *such that* $\mathsf{A} \cap \mathsf{ECom} \models \alpha = \beta$.

The next lemma is a corollary of a well-known property concerning pseudoidentities over finite nilpotent semigroups.

Lemma 2.3. *Let α and β be ω-terms of rank at most 1 and let V be one of $\mathsf{A} \cap \mathsf{ECom}$ or AInv. If $\mathsf{V} \models \alpha = \beta$, then either α and β are the same finite word or they both are rank 1 ω-terms.*

So, for the study of the ω-word problem over $\mathsf{A} \cap \mathsf{ECom}$ and over AInv it remains to consider ω-identities involving ω-terms of rank 1.

3 The automaton of an ω-term

Recall that a finite automaton is a quintuple $\mathcal{A} = (Q, A, E, I, F)$ where Q is a finite set of *states*, A is an alphabet, $E \subseteq Q \times A \times Q$ is the set of *transitions*, $I \subseteq Q$ is the set of *initial* states and $F \subseteq Q$ is the set of *final* states. A transition (p, a, q) is also denoted $p \xrightarrow{a} q$ and the letter a is called its *label*. An automaton in which each letter induces a partial bijection (possibly empty) of the set of states is called *reversible*. This means that the transition semigroup of such an automaton belongs to ECom. We let $\widetilde{\mathcal{A}}$ denote the automaton $\widetilde{\mathcal{A}} = (Q, \widetilde{A}, \widetilde{E}, I, F)$ where $\widetilde{A}$ is the alphabet $A \cup A^{-1}$ and $\widetilde{E} = E \cup \{(q, a^{-1}, p) : (p, a, q) \in E\}$. The automaton $\mathcal{A}$ is said to be *isomorphic* to an automaton $\mathcal{A}' = (Q', A, E', I', F')$ if the states of $\mathcal{A}$ can be renamed so that it is identical to $\mathcal{A}'$.

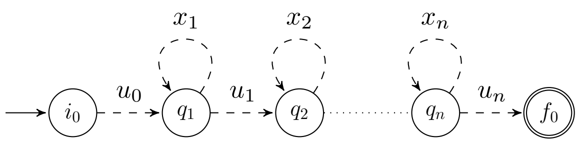
A non-empty *path* in $\mathcal{A}$ is a finite sequence

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots q_{n-1} \xrightarrow{a_n} q_n$$

of consecutive transitions of $\mathcal{A}$. The word $a_1 a_2 \cdots a_n$ is called the *label* of the path and the states q_0 and q_n are named, respectively, the *starting* and *ending* states of the path. We admit an empty path at each state and use the notation $p \overset{u}{\dashrightarrow} q$ to indicate a path from p to q labeled by a word u. A path is called a *cycle* if the starting and ending states are the same. A *simple path* is a path with no repeated states and a *simple cycle* is a cycle with no repeated states other than the starting and ending state.

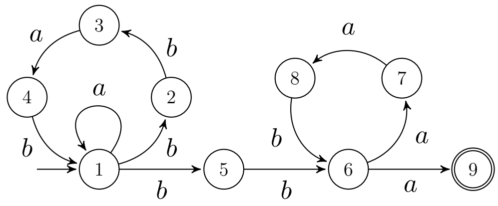
Let $\alpha = u_0 x_1^\omega u_1 x_2^\omega \cdots x_n^\omega u_n$ be a generic rank 1 ω-term over an alphabet A, where $u_i \in A^*$ and $x_j \in A^+$. In view of the ω-identities Σ, we assume that the x_j are primitive words, that is, they cannot be written in the form u^n with $n > 1$. We assume further that for any $1 \leq i < j \leq n$, the factor $x_i^\omega u_i x_{i+1}^\omega \cdots u_{j-1} x_j^\omega$ of α is not of the form $x_i^\omega x_{i+1}^\omega \cdots x_{j-1}^\omega x_i^\omega$.

We associate with α the following A-labeled automaton $\mathcal{A}_0(\alpha) = (Q_0, A, E_0, \{i_0\}, \{f_0\})$



where $s_i \overset{u_i}{\dashrightarrow} s_{i+1}$ represents a simple path if $u_i \neq 1$, $s_i = s_{i+1}$ if $u_i = 1$, and $q_j \circlearrowleft x_j$ is a simple cycle. Moreover, when $q_j = q_k$ with $j \neq k$, we require that the only common state of the cycles $q_j \circlearrowleft x_j$ and $q_k \circlearrowleft x_k$ is q_j.

Example 3.1. *Consider the rank 1 ω-term $\alpha = a^\omega(bbab)^\omega bb(aab)^\omega a$. The automaton $\mathcal{A}_0(\alpha)$ associated with α is*



The automata of the form $\mathcal{A}_0(\alpha)$ will be fundamental to solve the ω-word problem over $\mathsf{A} \cap \mathsf{ECom}$, while the automata $\widetilde{\mathcal{A}}_0(\alpha)$ will play the same role in the case of the pseudovariety AInv. The automata $\widetilde{\mathcal{A}}_0(\alpha)$ will be represented simply by $\mathcal{B}_0(\alpha)$.

4 Reversible aperiodic quotient of an automaton

Let $\mathcal{A}_0$ be a (finite) automaton. We describe an effective procedure for building a quotient of $\mathcal{A}_0$ which is a reversible aperiodic automaton, denoted $\mathcal{A}_0^{\mathrm{ra}}$. Its transition semigroup is therefore aperiodic with commuting idempotents. In particular, if $\mathcal{A}_0$ is either the automaton $\mathcal{A}_0(\alpha)$ or the automaton $\mathcal{B}_0(\alpha)$, defined in Section 3 above for a rank 1 ω-term α, then the reversible aperiodic quotient $\mathcal{A}_0^{\mathrm{ra}}$ will be denoted respectively by $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$. The automata $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ characterize completely the values of α over the pseudovarieties $\mathsf{A} \cap \mathsf{ECom}$ and AInv, respectively.

The automaton $\mathcal{A}_0^{\mathrm{ra}}$ is built by a recursive process. We begin by transforming $\mathcal{A}_0$ into a reversible quotient $\mathcal{A}_1$ using a certain "reversibility procedure". The "reversibility procedure" consists in the identification of the (eventual) states of $\mathcal{A}_0$ that prevent the automaton from being reversible. The automaton $\mathcal{A}_1$ is therefore a quotient of $\mathcal{A}_0$ and its transition semigroup has commuting idempotents. It is, however, not aperiodic in general. If $\mathcal{A}_1$ is aperiodic, then $\mathcal{A}^{\mathrm{ra}}$ is taken as $\mathcal{A}_1$. Otherwise, we apply to $\mathcal{A}_1$ a certain "aperiodicity rule". This identifies (a part of) the states that make the automaton not aperiodic, thus defining a quotient $\mathcal{A}_2$ of $\mathcal{A}_1$ and, so, of $\mathcal{A}_0$. The reversibility property may fail in $\mathcal{A}_2$, but this automaton is closer to aperiodicity. The alternating application of the "reversibility procedure" with the "aperiodicity rule" will produce a chain $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k, \ldots$, of quotients of the original automaton $\mathcal{A}_0$. This chain is necessarily finite and, so, after a finite number of steps the reversible aperiodic automaton $\mathcal{A}_0^{\mathrm{ra}}$ will be obtained. In case $\mathcal{A}_0$ is of the form $\mathcal{A}_0(\alpha)$, we will write $\mathcal{A}_k(\alpha)$ for $\mathcal{A}_k$.

Reversibility procedure. *For a given automaton $\mathcal{A}$, the reversibility procedure is to apply the two following transformation rules as long as possible:*

$(R.1)$ *Identify two states q_1 and q_2 for which there are transitions of the form $q_1 \xleftarrow{a} p \xrightarrow{a} q_2$;*

$(R.2)$ *Identify two states q_1 and q_2 for which there are transitions of the form $q_1 \xrightarrow{a} p \xleftarrow{a} q_2$.*

For instance, if α is the ω-term $a^\omega(bbab)^\omega bb(aab)^\omega a$ of Example 3.1, then $\mathcal{A}_1(\alpha)$ is the automaton of Figure 1.

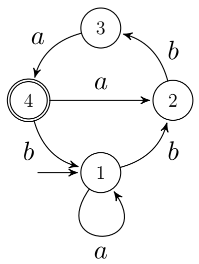


Figure 1: The automaton $\mathcal{A}_1(a^\omega(bbab)^\omega bb(aab)^\omega a)$

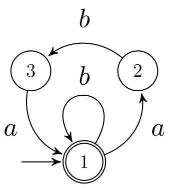
Notice that in this example the automaton $\mathcal{A}_1(\alpha)$ is aperiodic, so that $\mathcal{A}(\alpha)$ is $\mathcal{A}_1(\alpha)$ and its transition semigroup $S(\alpha)$ belongs to the pseudovariety $\mathsf{A} \cap \mathsf{ECom}$. The letters define the partial bijections $\bar{a} = \left(\begin{smallmatrix}1&2&3&4\\1&-&4&2\end{smallmatrix}\right)$ and $\bar{b} = \left(\begin{smallmatrix}1&2&3&4\\2&3&-&1\end{smallmatrix}\right)$ and, in $S(\alpha)$, one has

$$\alpha_{S(\alpha)}(\bar{a},\bar{b}) = \left(\begin{smallmatrix}1&2&3&4\\4&-&-&-\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}1&2&3&4\\-&-&-&-\end{smallmatrix}\right) = \bar{a}^\omega\bar{b}^\omega.$$

Hence, the semigroup $S(\alpha)$ and therefore the pseudovariety $\mathsf{A} \cap \mathsf{ECom}$ does not satisfy the ω-identity $\alpha = a^\omega b^\omega$. On the contrary, we shall see in Example 5.5 below that AInv verifies this ω-identity thus showing that $\mathsf{AInv} \neq \mathsf{A} \cap \mathsf{ECom}$.

We now present an ω-term whose corresponding reversible automaton is not aperiodic.

Example 4.1. *Consider the rank 1 ω-term $\beta = (aba)^\omega b^\omega$. In this case, the automaton $\mathcal{A}_1(\beta)$ coincides with $\mathcal{A}_0(\beta)$ since this one is the following reversible automaton*



This automaton is not aperiodic since, in $S_1(\beta)$, $(\bar{a}\bar{b})^\omega = (\bar{a}\bar{b})^2 = \left(\begin{smallmatrix}1&2&3\\1&-&3\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}1&2&3\\3&-&1\end{smallmatrix}\right) = (\bar{a}\bar{b})^{\omega+1}$.

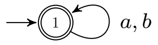
We introduce a third reduction rule that will serve to achieve aperiodicity. Let $\mathcal{A}$ be an automaton in which every letter induces a partial map on the state set. Notice that the aperiodicity of $\mathcal{A}$ can be tested as follows: if s is the number of states of $\mathcal{A}$, then it suffices to verify whether there is a cycle

$$q_1 \overset{u}{\dashrightarrow} q_2 \overset{u}{\dashrightarrow} q_3 \cdots q_n \overset{u}{\dashrightarrow} q_{n+1} = q_1 \tag{4.1}$$

for an integer $n \in \{2, \ldots, s\}$, pairwise distinct states $q_1, \ldots, q_n$ and a primitive word $u \in A^+$ such that $|u| \leq |S(\mathcal{A})|$. The automaton is aperiodic if and only if such a cycle does not exist.

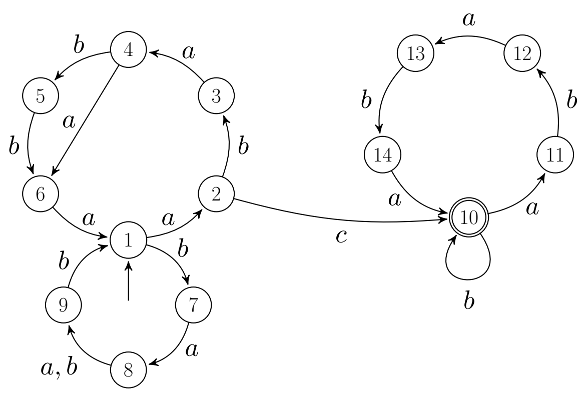
Aperiodicity rule. *The third transformation rule is the following:*

$(R.3)$ *For a cycle of the form* (4.1)*, identify the states at the same position in each sub-path* $q_i \overset{u}{\dashrightarrow} q_{i+1}$ *(that is to say that any two sub-paths* $q_i \overset{u}{\dashrightarrow} q_{i+1}$ *and* $q_j \overset{u}{\dashrightarrow} q_{j+1}$ *are identified).*

Consider for instance the ω-term $\beta = (aba)^\omega b^\omega$ of Example 4.1. The automaton $\mathcal{A}_1(\beta)$ is not aperiodic since it contains the cycle $1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{a} 1 \xrightarrow{b} 1$ of the form (4.1) with $u = ab$. The automaton $\mathcal{A}_2(\beta)$ produced by the application of the aperiodicity rule to this cycle is 

. Since this is an aperiodic reversible automaton, it follows that $\mathcal{A}(\beta) = \mathcal{A}_2(\beta)$.

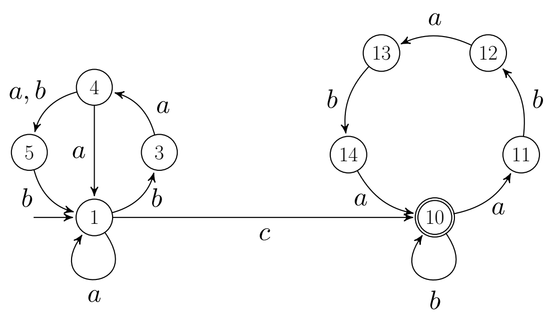
Example 4.2. *Let* $\alpha = (abab^2a)^\omega(aba^3)^\omega(bab^2)^\omega(ba^2b)^\omega ac(ababa)^\omega b^\omega$*. The automaton* $\mathcal{A}_1(\alpha)$ *is*



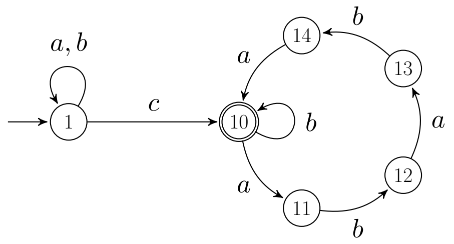
Now, to apply rule $(R.3)$ *in order to obtain an automaton* $\mathcal{A}_2(\alpha)$*, we have to choose a cycle of the form* (4.1)*. So, for the choice of the cycle*

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{a} 4 \xrightarrow{b} 5 \xrightarrow{b} 6 \xrightarrow{a} 1 \xrightarrow{b} 7 \xrightarrow{a} 8 \xrightarrow{b} 9 \xrightarrow{b} 1$$

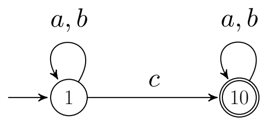
with $u = abab^2$*, we get for* $\mathcal{A}_2(\alpha)$ *the automaton*



This automaton is not reversible. So, the reversible automaton $\mathcal{A}_3(\alpha)$ *is distinct from* $\mathcal{A}_2(\alpha)$ *and is equal to*



Now, the cycle $10 \xrightarrow{b} 10 \xrightarrow{a} 11 \xrightarrow{b} 12 \xrightarrow{a} 13 \xrightarrow{b} 14 \xrightarrow{a} 10$ *in* $\mathcal{A}_3(\alpha)$ *of the form* (4.1) *with* $u = ba$ *determines for* $\mathcal{A}_4(\alpha)$ *the automaton*



This is reversible aperiodic, so $\mathcal{A}(\alpha) = \mathcal{A}_4(\alpha)$.

5 The ω-word problems over $\mathsf{A} \cap \mathsf{ECom}$ and AInv

As already mentioned in Section 4, the reversible aperiodic automaton $\mathcal{A}(\alpha)$, associated with a rank 1 ω-term α, characterizes completely the value of α over the pseudovariety $\mathsf{A} \cap \mathsf{ECom}$.

Theorem 5.1. *Let α and β be rank 1 ω-terms. Then,* $\mathsf{A} \cap \mathsf{ECom} \models \alpha = \beta$ *if and only if* $\mathcal{A}(\alpha)$ *and* $\mathcal{A}(\beta)$ *are isomorphic automata.*

Since, for any rank 1 ω-term α, the automaton $\mathcal{A}(\alpha)$ is effectively computable, the above theorem combined with Proposition 2.2 and Lemma 2.3 proves the following result.

Corollary 5.2. *The ω-word problem over* $\mathsf{A} \cap \mathsf{ECom}$ *is decidable.*

In turn, the characterization of the value of a rank 1 ω-term α over the pseudovariety AInv is given by the automata $\mathcal{B}(\alpha)$, introduced in Section 4 as the reversible (inverse, in this case) aperiodic quotient of the automaton $\mathcal{B}_0(\alpha) := \widetilde{\mathcal{A}}_0(\alpha)$. The analogue of Theorem 5.1 for the pseudovariety AInv is therefore the following result.

Theorem 5.3. *Let α and β be rank 1 ω-terms. Then,* $\mathsf{AInv} \models \alpha = \beta$ *if and only if* $\mathcal{B}(\alpha)$ *and* $\mathcal{B}(\beta)$ *are isomorphic automata.*

This theorem together with Proposition 2.2 and Lemma 2.3 proves the following decidability result.

Corollary 5.4. *The ω-word problem over* AInv *is decidable.*

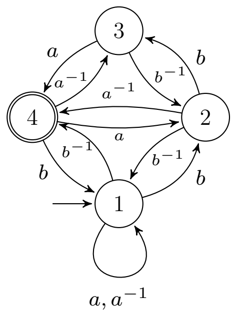
Before giving an example of application of Theorem 5.3, let us present some remarks about the automaton $\mathcal{B}(\alpha)$ and its relationship with automaton $\mathcal{A}(\alpha)$. Recall first that $\mathcal{A}(\alpha)$ and $\mathcal{B}(\alpha)$ denote, respectively, the automata $(\mathcal{A}_0(\alpha))^{\mathrm{ra}}$ and $(\mathcal{B}_0(\alpha))^{\mathrm{ra}}$. If $\mathcal{A}_0(\alpha), \mathcal{A}_1(\alpha), \ldots,$

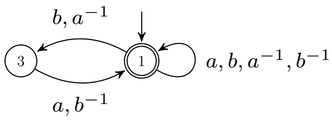
$\mathcal{A}_m(\alpha)$ is a sequence that generates automaton $\mathcal{A}(\alpha)$ using the procedure in Section 4, then it can be seen that $\mathcal{B}_0(\alpha), \mathcal{B}_1(\alpha), \ldots, \mathcal{B}_m(\alpha)$, where $\mathcal{B}_j(\alpha)$ is the automaton $\widetilde{\mathcal{A}}_j(\alpha)$ for every j, is a chain of quotients of $\mathcal{B}_0(\alpha)$ that can be obtained with the same procedure, choosing the same cycles used in the computation of $\mathcal{A}(\alpha)$ when the aperiodicity rule is applied. In particular $\mathcal{B}_m(\alpha)$ is the inverse automaton $\widetilde{\mathcal{A}}(\alpha)$. We deduce therefore that

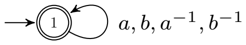
$$\mathcal{B}(\alpha) = (\widetilde{\mathcal{A}}(\alpha))^{\mathrm{ra}},$$

meaning that the computation of the automaton $\mathcal{B}(\alpha)$ can be made from the automaton $\widetilde{\mathcal{A}}(\alpha)$. Although $\mathcal{A}(\alpha)$ is always aperiodic, the automaton $\widetilde{\mathcal{A}}(\alpha)$ can be not aperiodic.

Example 5.5. *Let α be the ω-term $a^\omega(bbab)^\omega bb(aab)^\omega a$ of Example 3.1. Then $\mathcal{A}(\alpha)$ is the automaton $\mathcal{A}_1(\alpha)$ exhibited in Figure 1. The inverse automaton $\mathcal{B}_1(\alpha)$ is the automaton $\widetilde{\mathcal{A}}(\alpha)$ pictured below*



The automaton $\mathcal{B}_1(\alpha)$ is not aperiodic since it admits the cycle $1 \xrightarrow{a} 1 \xrightarrow{b^{-1}} 4 \xrightarrow{a} 2 \xrightarrow{b^{-1}} 1$ of form (4.1) *with $u = ab^{-1}$. The rule $(R.3)$ applied to this cycle identifies the states* 1, 2 *and* 4*, thus setting for $\mathcal{B}_2(\alpha)$ the automaton* 

. Hence, $\mathcal{B}(\alpha)$ is $\mathcal{B}_3(\alpha)$, the automaton 

. In view of Theorem 5.3 this means that the pseudovariety AInv *satisfies the ω-identity $\alpha = a^\omega b^\omega$.*

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