ISSN 1061-9208, Russian Journal of Mathematical Physics, 2023, Vol. 30, No. 3, pp. 360–374. © Pleiades Publishing, Ltd., 2023.

Whitney–Sullivan Constructions for Transitive Lie Algebroids–Smooth Case

A. S. Mishchenko[*,**,1] **and J. R. Oliveira**[***,2]

[*]*Department of Mathematics, Lomonosov Moscow State University, Moscow, 119991 GSP-1, Moscow, Russia*
[**]*Moscow Center for Fundamental and Applied Mathematics, Russia*
[***]*Department of Mathematics, Minho University, Campus Gualtar, 4710-057 Braga, Portugal*
[1]*E-mail:* asmish-prof@yandex.ru
[2]*E-mail:* jmo@math.uminho.pt

Received July 20, 2023; Revised July 20, 2023; Accepted July 20, 2023

Abstract — Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. A piecewise smooth form on $\mathcal{A}$ is a family $\omega = (\omega_\Delta)_{\Delta\in K}$ such that ω_Δ is a smooth form on the Lie algebroid restriction of $\mathcal{A}$ to Δ, satisfying the compatibility condition concerning the restrictions of ω_Δ to the faces of Δ, that is, if Δ' is a face of Δ, the restriction of the form ω_Δ to the simplex Δ' coincides with the form $\omega_{\Delta'}$. The set $\Omega^*(\mathcal{A}; K)$ of all piecewise smooth forms on $\mathcal{A}$ is a cochain algebra. There exists a natural morphism

$$\Omega^*(\mathcal{A}; M) \to \Omega^*(\mathcal{A}; K)$$

of cochain algebras given by restriction of a smooth form defined on $\mathcal{A}$ to a smooth form defined on the Lie algebroid restriction of $\mathcal{A}$ to the simplex Δ, for all simplices Δ of K. In this paper, we prove that, for triangulated compact manifolds, the cohomology of this construction is isomorphic to the Lie algebroid cohomology of $\mathcal{A}$, in which that isomorphism is induced by the restriction mapping.

DOI 10.1134/S106192082303007X

1. INTRODUCTION

D. Sullivan considered in [1] (1977) a new model for the underlying cochain complex of classical cohomologies with rational coefficients for arbitrary simplicial spaces which gives an isomorphism with classical rational cohomologies (see also D. Sullivan [2, Theorem D]) (1975). This new model is determined by the de Rham complex of all rational polynomial forms defined on the simplicial complex triangulating the space. H. Whitney also presented in [3] (1957) other cell-like constructions of cochain complexes which induce isomorphisms in cohomology with classical cohomologies. One of those constructions presented in [1] and [3] states that the de Rham cohomology of a smooth manifold, smoothly triangulated by a simplicial complex, is isomorphic to piecewise smooth cohomology of the simplicial complex. This isomorphism is induced by restriction of smooth forms to all simplices. That construction have led us to conjecture that, given a transitive Lie algebroid on a triangulated compact smooth manifold, the morphism given by restriction, which takes smooth forms on the Lie algebroid into piecewise smooth forms on the same Lie algebroid, still remains an isomorphism in cohomology.

The aim of the present paper is to prove that conjecture. For this purpose, we use the structure which commences by fixing a smooth triangulation of the base of a transitive Lie algebroid by a simplicial complex and taking the restriction of the Lie algebroid to all simplices of the triangulation. Since the Lie algebroid is transitive, the restriction of the Lie algebroid to each simplex always exists. When this structure is given, we define the notion of piecewise smooth form in a similar way to piecewise forms on a simplicial complex. The set of all piecewise smooth forms defined on a transitive Lie algebroid over a triangulated base is naturally equipped with a differential, yielding a commutative differential graded algebra. Its cohomology is, by definition, the piecewise Lie algebroid cohomology of the Lie algebroid. Each smooth form defined on the Lie algebroid gives a piecewise smooth form defined by taking the restriction of the form to each simplex. This correspondence is a natural mapping from the usual algebra of the smooth forms on the Lie algebroid to the algebra of the piecewise smooth forms on the same Lie algebroid. Based on three crucial results, namely the triviality of a transitive Lie algebroid over a contractible smooth manifold (Mackenzie, [4, Theorem 7.3.18], 2005), the Künneth theorem for Lie algebroids (Kubarski, [5, Section 6], 2002) and the de Rham-Sullivan theorem for smooth manifolds ([1, Theorem 7.1]), we show that mapping induces an isomorphism in cohomology.

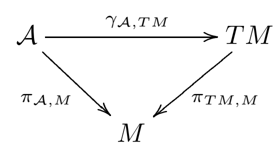
The paper consists of three sections. In the first section, we shall summarize some definitions and properties on restrictions and cohomology of transitive Lie algebroids defined over smooth manifolds. Description of definitions and results on Lie algebroids can be found in A. Cannas Silva, A. Weinstein, [6] (1999), Kubarski, Mishchenko, [7], (2003), Mackenzie, [4] and Mackenzie, Higgins, [8], (1990). In the second section, we introduce the notion of piecewise smooth form on a transitive Lie algebroid defined on a triangulated manifold and we define its piecewise Lie algebroid cohomology. We still prove that the Mayer-Vietoris sequence and the Künneth theorem remain true in this context. The goal of the third section is to prove that the restriction mapping mentioned in the previous paragraph induces an isomorphism in cohomology.

Given a smooth manifold M and an integer $p \geqslant 0$, it is well known that the subset of M consisting of all points $x \in M$ such that the pairing (M, x) is locally diffeomorphic to a sectors of index p is a submanifold of M, called the boundary of index p of M. Throughout this paper, we shall work on manifolds which are smooth, finite-dimensional and possibly with boundaries of different indices.

2. PRELIMINARIES ON RESTRICTIONS AND COHOMOLOGY OF LIE ALGEBROIDS

Let M be a smooth manifold, possibly with boundaries of different indices, TM the tangent bundle to M and $\Gamma(TM)$ the Lie algebra of the vector fields on M.

A Lie algebroid on M is a vector bundle on M whose total space is denoted by $\mathcal{A}$. So we have a total space $\mathcal{A}$ and a projection $\pi_{\mathcal{A},M} : \mathcal{A}\xrightarrow{M}$. For brevity, we will denote the bundle itself and its total space with one letter, say $\mathcal{A}$, if this does not lead to misunderstandings. The bundle $\mathcal{A}$ is equipped with a vector bundle morphism $\gamma_{\mathcal{A},TM} : \mathcal{A} \longrightarrow TM$, called anchor of $\mathcal{A}$, in which the diagram



is commutative, and a structure of real Lie algebra on the vector space $\Gamma(\mathcal{A})$ of the sections of $\mathcal{A}$ (denoted by the Lie bracket $\{\cdot,\cdot\}_{\mathcal{A}}$ or simply $\{\cdot,\cdot\}$) such that the induced mapping $\gamma_{\Gamma(\mathcal{A}),\Gamma(TM)} : \Gamma(\mathcal{A}) \longrightarrow \Gamma(TM)$ is a Lie algebra homomorphism and the action of the algebra $\mathcal{C}^\infty(M)$ of the smooth real functions on $\Gamma(\mathcal{A})$ satisfies the natural Leibniz condition:

$$\{\xi, f\eta\} = f\{\xi,\eta\} + (\gamma_\Gamma(\xi)(f))\eta$$

for each ξ, $\eta \in \Gamma(\mathcal{A})$ and $f \in \mathcal{C}^\infty(M)$. The Lie algebroid $\mathcal{A}$ is called transitive if the anchor γ is fiberwise surjective. If $\mathcal{B}$ is another Lie algebroid on a smooth manifold N and $\delta : \mathcal{B} \longrightarrow TN$ its anchor, a morphism of Lie algebroids from $\mathcal{B}$ to $\mathcal{A}$ consists of a pair of mappings (ψ, φ) in which $\psi : \mathcal{B} \longrightarrow \mathcal{A}$ and $\varphi : N \longrightarrow M$, such that (ψ, φ) is a vector bundle morphism satisfying the equality $\delta \circ \psi = T(\varphi) \circ \gamma$, in which $T(\varphi) : TN \longrightarrow TM$ denotes the tangent mapping of φ, and preserving the Lie bracket condition for ψ-decompositions (for details, see [4, Section 4.3]). We list now three examples of Lie algebroids used in this work.

Example 1. (Lie algebras) Any real finite dimensional Lie algebra $\mathfrak{g}$ over a one-point space $M = \{*\}$ with anchor equal to zero is a Lie algebroid on M. Any Lie algebra morphism between two Lie algebras is a Lie algebroid morphism for this structure of Lie algebroid.

Example 2. (Tangent Lie algebroids) If M is a smooth manifold then TM is a Lie algebroid on M. The anchor mapping is the identity mapping of TM and the Lie bracket is the usual Lie bracket of vector fields ($\{\xi,\eta\} \equiv [\xi,\eta]$). This Lie algebroid is called the tangent Lie algebroid of M. The anchor mapping $\gamma : \mathcal{A} \longrightarrow TM$ is a Lie algebroid morphism.

Example 3. (Trivial Lie algebroids)

Let $\mathfrak{g}$ be a real finite-dimensional Lie algebra and M a smooth manifold. Consider the trivial vector bundle $M \times \mathfrak{g}$ of base M. The fibre product $TM \oplus (M \times \mathfrak{g})$ is a Lie algebroid on M in which the anchor mapping is the projection $\gamma : TM \oplus (M \times \mathfrak{g}) \longrightarrow TM$ on TM and the Lie bracket on $\Gamma(TM \oplus (M \times \mathfrak{g}))$ is defined by

$$\{(X,u),(Y,v)\} = ([X,Y]_{TM}, X(v) - Y(u) - [u,v])$$

for $X, Y \in \Gamma(TM)$ and $u, v : M \longrightarrow \mathfrak{g}$ smooth mappings. This Lie algebroid is called trivial Lie algebroid on M with fibre $\mathfrak{g}$. The Lie algebroid product (compare with the paragraph preceding the proposition 1.11

of [8]) $TM \times \mathfrak{g}$, which is defined over $M \simeq M \times N$, is isomorphic to the trivial Lie algebroid $TM \oplus (M \times \mathfrak{g})$ defined over M. We will identify both Lie algebroids.

Restrictions of Lie algebroids to submanifolds of the base are given by inverse image of Lie algebroids through the inclusion of submanifolds. In the following paragraph, we summarize the image inverse of Lie algebroids.

Let $\mathcal{A}$ be a transitive Lie algebroid on a smooth manifold M and $\gamma : \mathcal{A} \longrightarrow TM$ its anchor. Let N be a smooth submanifold and $\varphi : N \longrightarrow M$ and the smooth inclusion.

We recall that the Lie algebroid inverse image of $\mathcal{A}$ by φ, denoted by $\varphi^{!!}\mathcal{A}$, is defined as follows. Consider the differential $T\varphi : TN \longrightarrow TM$ and the composition of the following two diagrams:

$$\begin{array}{ccc} (T\varphi)^*\mathcal{A} & \xrightarrow{\widehat{T\varphi}} & \mathcal{A} \\ {\scriptstyle\widehat{\gamma}}\downarrow & & \downarrow{\scriptstyle\gamma} \\ TN & \xrightarrow{T\varphi} & TM \\ {\scriptstyle\pi_{TN,N}}\downarrow & & \downarrow{\scriptstyle\pi_{TM,M}} \\ N & \xrightarrow{\varphi} & M \end{array}$$

the inverse image (of $T\varphi$) (or pullback of cospan $T\varphi$ and γ) (or fiber-product of spaces TN and $\mathcal{A}$) with the differential diagram. The total space of the bundle $\varphi^{!!}\mathcal{A}$ on the manifold N is defined as isomorphic to

$$\varphi^{!!}\mathcal{A} \overset{\approx}{\dashrightarrow} (T\varphi)^*\mathcal{A}$$

(for further details, see [4] or [8]).

We notice that the vector bundle $\widehat{\gamma} : (T\varphi)^*\mathcal{A} \longrightarrow TN$ exists because γ is surjective. The smooth mapping $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathcal{A}$ defined by $\varphi^{!!}(X, a) = a$ will be called the canonical mapping induced by φ and the pair of mappings $(\varphi^{!!}, \varphi)$ the canonical Lie algebroid morphism induced by φ.

In order to define a commutator Lie bracket on the set $\Gamma(\varphi^{!!}\mathcal{A}, N)$ of the sections of $\varphi^{!!}\mathcal{A}$, consider the Atiyah exact sequence of vector bundles (which are Lie algebroids, see [7] or [4]):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{i_{L,\mathcal{A}}} & \mathcal{A} & \xrightarrow{\gamma_{\mathcal{A},TM}} & TM & \longrightarrow & 0 \\ & & & {\scriptstyle\pi_{L,M}}\searrow & \downarrow{\scriptstyle\pi_{\mathcal{A},M}} & \swarrow{\scriptstyle\pi_{TM,M}} & & & \\ & & & & M & & & & \end{array}$$

Consider the change of the base M for the submanifold N:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_N & \xrightarrow{i_{L,\mathcal{A}}} & \mathcal{A}_N & \xrightarrow{\gamma_{\mathcal{A},TM}} & (TM)_N & \longrightarrow & 0 \\ & & & {\scriptstyle\pi_{L,M}}\searrow & \downarrow{\scriptstyle\pi_{\mathcal{A},M}} & \swarrow{\scriptstyle\pi_{TM,M}} & & & \\ & & & & N & & & & \end{array}$$

The bundle TN is the subbundle of $(TM)_N$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_N & \xrightarrow{i} & \mathcal{A}_N & \xrightarrow{\gamma} & (TM)_N & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & L_N & \longrightarrow & \varphi^{!!}\mathcal{A} = \gamma^{-1}(TN) & \xrightarrow{\gamma} & TN & \longrightarrow & 0 \end{array}$$

On the level of total spaces, there is a sequence of inclusions:

$$\begin{array}{ccccc} \varphi^{!!}\mathcal{A} & \hookrightarrow & \mathcal{A}_N & \hookrightarrow & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ N & = & N & \hookrightarrow & M \end{array}$$

Given two sections $\sigma_1, \sigma_2 \in \Gamma(\varphi^{!!}\mathcal{A}, N)$ consider extensions of the sections σ_1, σ_2 to sections $\widetilde{\sigma_1}, \widetilde{\sigma_2} \in \Gamma(\mathcal{A}, M)$. Then, by definition,

$$\{\sigma_1, \sigma_2\}_{\varphi^{!!}\mathcal{A}} = \{\widetilde{\sigma_1}, \widetilde{\sigma_2}\}_{|N}$$

we obtain the Lie bracket on the space $\Gamma(\varphi^{!!}\mathcal{A}, N)$. The definition of the Lie bracket $\{\bullet, \bullet\}_{\varphi^{!!}\mathcal{A}}$ does not depend on the choice of the extension $(\widetilde{\sigma_1}, \widetilde{\sigma_2})$ of the sections (σ_1, σ_2).

Definition 1. Keeping these hypotheses and notation, the Lie algebroid $\varphi^{!!}\mathcal{A}$ constructed as introduced above is called the Lie algebroid restriction of $\mathcal{A}$ to the submanifold N and denoted by $\mathcal{A}^{!!}_N$.

We note that restrictions of transitive Lie algebroids enjoy the transitivity property and that $(TM)^{!!}_N = TN$.

Let $\mathcal{A}_N$ denote the restriction of the underlying vector bundle of $\mathcal{A}$ to the submanifold N. In the case in which N is an open subset U of M, the Lie algebroid restriction $\mathcal{A}^{!!}_U$ constructed according to the dentition 1 is naturally isomorphic to the Lie algebroid $\mathcal{A}_U$ according to [4, Proposition 3.3.2]. In the general case, the canonical mapping $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathcal{A}$ can be restricted to the injective N-morphism of vector bundles $\varphi^{!!} : \varphi^{!!}\mathcal{A} \longrightarrow \mathcal{A}_N$. Consequently, the vector bundle Im $\varphi^{!!}$ is a transitive Lie algebroid, which is identified to the Lie algebroid $\mathcal{A}^{!!}_N$. In the proof of the proposition 4, the canonical mapping $\varphi^{!!} : \mathcal{A}^{!!}_N \longrightarrow \mathcal{A}$ will be denoted by $(\varphi^{\mathcal{A}}_{M,N})^{!!}$ in order to distinguish the canonical mappings corresponding to the restrictions of two Lie algebroids defined over the same manifold.

We introduce now the cochain complex of the smooth forms on a Lie algebroid and its cohomology. Let M be a smooth manifold and $\mathcal{A}$ a transitive Lie algebroid on M. Let $M \times \mathrm{R}$ denote the trivial vector bundle of base M and fibre R. A smooth form of degree p on $\mathcal{A}$ is a section of the exterior algebra bundle $\left(\bigwedge^p \mathcal{A}^*\right) \otimes \left(M \times \mathrm{R}\right)$. The set of all smooth forms of degree p on $\mathcal{A}$ will be denoted by $\Omega^p(\mathcal{A}; M)$. For $p = 0$, we have that $\Omega^0(\mathcal{A}; M) = C^\infty(M)$. The set $\Omega^p(\mathcal{A}; M)$ is a $C^\infty(M)$-module for each $p \geqslant 0$. Since the dimension of the fibres of $\mathcal{A}$ is finite, the module $\Omega^p(\mathcal{A}; M)$ is also the module of the sections of the alternated vector bundle $\mathrm{Alt}^p(\mathcal{A}; M \times \mathrm{R})$. Hence, the exterior product of alternated multilinear mappings induces an exterior product in

$$\Omega^*(\mathcal{A}; M) = \bigoplus_{p \geqslant 0} \Omega^p(\mathcal{A}; M)$$

making $\Omega^*(\mathcal{A}; M)$ into a skew-commutative graded algebra, in which the constant mapping $M \longrightarrow \mathrm{R}$, $x \mapsto 1$, is the unit.

For the definition of exterior derivative on $\Omega^*(\mathcal{A}; M)$, we first consider the algebra $\Omega^0(\mathcal{A}; M) = C^\infty(M)$. Let $f \in C^\infty(M)$ be a smooth mapping. The exterior derivative of f, denoted by $d^0_{\mathcal{A}} f$, is the smooth form belonging to $\Omega^1(\mathcal{A}; M)$ such that, on each $X \in \Gamma(\mathcal{A})$, the equality

$$d^0_{\mathcal{A}} f(X) = (\gamma \circ X) \cdot f$$

holds. Now, for each $p \geqslant 1$ we define

$$d^p_{\mathcal{A}} : \Omega^p(\mathcal{A}; M) \longrightarrow \Omega^{p+1}(\mathcal{A}; M)$$

$$d^p_{\mathcal{A}} \omega(X_1, X_2, \cdots, X_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} (\gamma \circ X_j) \cdot (\omega(X_1, \cdots, \widehat{X_j}, \cdots, X_{p+1}))$$
$$+ \sum_{i<k} (-1)^{i+k} \omega(\{X_i, X_k\}, X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_k}, \cdots, X_{p+1})$$

for $\omega \in \Omega^p(\mathcal{A}; M)$ and $X_1, X_2, \cdots, X_{p+1} \in \Gamma(\mathcal{A})$. The family of differential operators $d^*_{\mathcal{A}} = (d^p_{\mathcal{A}})_{p \geqslant 0}$ satisfies the properties

1. For each $p \geqslant 0$, $d^p_{\mathcal{A}}$ is linear;

2. For each $p \geqslant 0$, $d^{p+1}_{\mathcal{A}} \circ d^p_{\mathcal{A}} = 0$;

3. For each $\xi \in \Omega^p(\mathcal{A}; M)$ and $\eta \in \Omega^q(\mathcal{A}; M)$,

$$d^{p+q}_{\mathcal{A}}(\xi \wedge \eta) = d^p_{\mathcal{A}}(\xi) \wedge \eta + (-1)^p \xi \wedge (d^q_{\mathcal{A}} \eta).$$

Hence, the family $d^*_{\mathcal{A}} = (d^p_{\mathcal{A}})_{p\geqslant 0}$ defines, on the algebra $\Omega^*(\mathcal{A};M)$, a structure of differential graded algebra and so $\Omega^*(\mathcal{A};M)$ is a skew commutative cochain algebra defined over R.

Definition 2. The Lie algebroid cohomology of $\mathcal{A}$ is the cohomology space of the cochain algebra $\Omega^*(\mathcal{A};M)$ equipped with the structures defined above. This cohomology space will be denoted by $H^*(\mathcal{A};M)$.

Let $\mathcal{B}$ be another Lie algebroid defined on a smooth manifold N and $\lambda = (\psi,\varphi)$ a morphism of Lie algebroids defined by the smooth mappings $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ and $\varphi : M \longrightarrow N$. Let $\omega \in \Omega^p(\mathcal{B};N)$ be a smooth form on $\mathcal{B}$ of degree p. The inverse image of ω by the morphism λ, denoted by $\lambda^*\omega$, is the form defined by

$$\left(\lambda^*\omega\right)_x(v_1, v_2,\cdots, v_p) = \omega_{\varphi(x)}(\psi(v_1),\psi(v_2),\cdots,\psi(v_p))$$

$x \in M$ and $v_1,\, v_2,\, \ldots\, v_p \in \mathcal{A}_x$. Thus, for each $p \geqslant 0$, there is a mapping

$$\lambda^{*p} : \Omega^p(\mathcal{B};N) \longrightarrow \Omega^p(\mathcal{A};M)$$

$$\omega \longrightarrow \lambda^*\omega.$$

The family $\lambda^* = (\lambda^{*p})_{p\geqslant 0}$ is a morphism of cochain algebras. In particular, if $\varphi : N \hookrightarrow M$ is a submanifold and $\lambda = (\varphi^{!!},\varphi)$ the canonical Lie algebroid morphism, in which $\varphi^{!!} : \mathcal{A}^{!!}_N \longrightarrow \mathcal{A}$ is defined by $\varphi^{!!}(X,a) = a$, we can use the induced morphism $\lambda^* : \Omega^*(\mathcal{A};M) \longrightarrow \Omega^*(\mathcal{A}^{!!}_N;N)$ for the definition of restriction of smooth forms. We summarize it in our next definition.

Definition 3. For each form $\omega \in \Omega^*(\mathcal{A};M)$, the form $\lambda^*(\omega) \in \Omega^*(\mathcal{A}^{!!}_N;N)$ is called the form restriction of ω to N and denoted by $\omega^{!!}_N$ or simply by $\omega_{/_N}$, if there is no danger of confusion with the restriction of ω to N in the context of the restriction of the underlying vector bundle of $\mathcal{A}$ to N. In the subsequent sections, the morphism λ^* will be often denoted by $\varphi^{\mathcal{A}}_{M,N}$.

If U is an open subset of M and $\varphi : U \hookrightarrow M$ the inclusion mapping then, for each $p \geqslant 0$, the spaces $\Omega^p(\mathcal{A}_U;U)$ and $\Omega^p(\mathcal{A}^{!!}_U;U)$ are isomorphic. In the general case, if $\varphi : N \hookrightarrow M$ is a submanifold, we have that the spaces $\Omega^p(\mathcal{A}^{!!}_N;N)$ and $\Omega^p(\mathrm{Im}\ \varphi^{!!};N)$ are isomorphic. Let us notice now a proposition concerning extensions of smooth forms.

Proposition 1. *Let M be a smooth manifold and $\varphi : N \hookrightarrow M$ a submanifold such that N is a closed subset in M in the topological sense. Let $\mathcal{A}$ be a transitive Lie algebroid on M and consider the canonical Lie algebroid morphism $\lambda = (\varphi^{!!},\varphi)$ in which $\varphi^{!!} : \mathcal{A}^{!!}_N \longrightarrow \mathcal{A}$ is defined by $\varphi^{!!}(X,a) = a$. Then, the morphism of cochain algebras*

$$\lambda^* : \Omega^*(\mathcal{A};M) \longrightarrow \Omega^*(\mathcal{A}^{!!}_N;N)$$

is surjective.

Proof. Let $\gamma : \mathcal{A} \longrightarrow TM$ be the anchor of $\mathcal{A}$ and $\widetilde{\omega} \in \Omega^p(\mathcal{A}^{!!}_N;N)$ a smooth form. We define the smooth form $\widehat{\omega} \in \Omega^p(\mathrm{Im}\ \varphi^{!!};N)$ by

$$\widehat{\omega}(\xi_1,\ldots,\xi_p) = \widetilde{\omega}((\gamma\circ\xi_1,\xi_1),\ldots,(\gamma\circ\xi_1,\xi_p))$$

Since N is closed, the form $\widehat{\omega}$ extends to a smooth form $\omega \in \Omega^*(\mathcal{A};M)$. We have that ω satisfies the equality $\lambda^*(\omega) = \widetilde{\omega}$.

We finalize this section by taking a theorem into consideration which will be used in the proof of the proposition 7. Let $\mathcal{A}$ be a transitive Lie algebroid on a contractible smooth manifold M and $\mathfrak{g} = \ker\gamma$, in which γ denotes the anchor of $\mathcal{A}$. Then, $\mathcal{A}$ is isomorphic to the trivial Lie algebroid $TM \times \mathfrak{g}$.

This theorem is a direct consequence of [4, Theorem 7.3.18], and its proof follows the theory of non-Abelian Lie algebra extensions. Alternatively, a proof of this theorem, based in integrability of Lie algebroids, is given in [10, Corollary 5.6].

3. PIECEWISE SMOOTH FORMS AND COHOMOLOGY

For each simplicial complex K, its polytope will be denoted by $|K|$. In our context, simplex will always mean closed simplex. A smooth manifold M is said to be smoothly triangulated by a simplicial complex K if there exists a homeomorphism $\lambda : |K| \longrightarrow M$ such that, for each simplex $\Delta \in K$, the mapping $\lambda_{/\Delta} : \Delta \longrightarrow M$ is a smooth embedding in all points including the boundary. As usually is done, we shall not make no notational distinction between the manifold and the complex triangulating it. In what follows, all simplicial complexes considered are geometric and finite. Each simplex can be represented as the convex body generated by its vertices and, if its vertices are the points a_0, a_1, …, a_p, we eventually denote this simplex by $[a_0, a_1, \ldots, a_p]$. We shall write $\Delta' \prec \Delta$, if Δ' is a face of the simplex Δ.

The notation $\varphi : \Delta' \hookrightarrow \Delta$, in which φ is the inclusion mapping, will be also used when Δ' is a face of Δ. The open star of the simplex Δ in a simplicial complex K, denoted by $\mathrm{St}(\Delta)$, is the union of the interiors of all (closed) simplices of K having Δ as a face. The closed star of the simplex Δ in K is the union of all (closed) simplices of K having Δ as a face. The star $\mathrm{St}(\Delta)$ is an open subset in $|K|$ for the weak topology (which is the same as the topology of subspace induced by the topology of the ambient space since the simplicial complex is finite).

Several variants of piecewise cohomology can be contemplated. What we consider in the present work is the cohomology obtained from the cochain algebra of the piecewise smooth forms on a transitive Lie algebroid which is defined over a smooth manifold triangulated by simplicial complex. It is in this context that our theorem 1 arises.

Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let Δ be a simplex of K. Since $\mathcal{A}$ is transitive, the Lie algebroid restriction $\mathcal{A}^{!!}_{\Delta}$ on Δ is well defined. Suppose that Δ' is another simplex of K such that $\varphi_{\Delta,\Delta'} : \Delta' \hookrightarrow \Delta$ is a face of Δ. By transitivity of restrictions, we have that $\mathcal{A}^{!!}_{\Delta'} \simeq (\mathcal{A}^{!!}_{\Delta})^{!!}_{\Delta'}$. Consequently, the cochain algebras $\Omega^*(\mathcal{A}^{!!}_{\Delta'};\Delta')$ and $\Omega^*((\mathcal{A}^{!!}_{\Delta})^{!!}_{\Delta'};\Delta')$ are isomorphic. The morphism of cochain algebras induced by the inclusion $\varphi_{\Delta,\Delta'} : \Delta' \hookrightarrow \Delta$ is denoted by

$$\varphi^{\mathcal{A}^{!!}_{\Delta}}_{\Delta,\Delta'} : \Omega^*(\mathcal{A}^{!!}_{\Delta};\Delta) \longrightarrow \Omega^*(\mathcal{A}^{!!}_{\Delta'};\Delta')$$

and, for each smooth form $\omega_\Delta \in \Omega^p(\mathcal{A}^{!!}_{\Delta};\Delta)$, the smooth form $\varphi^{\mathcal{A}^{!!}_{\Delta}}_{\Delta,\Delta'}(\omega_\Delta)$ is denoted by $(\omega_\Delta)^{!!}_{\Delta'}$ or $(\omega_\Delta)_{/_{\Delta'}}$ (compare with the definition 3). Keeping these hypotheses and notations, we give below the definition of piecewise smooth form. The idea of this definition is based both in the Whitney's book [3] and in the Sullivan's paper [1]. Morgan and Griffiths has also presented in [9] the notion of piecewise smooth form on an ambient space made up of a set of manifolds with transverse intersections.

Definition 4. Let p be a natural number $\geqslant 0$. A piecewise smooth form of degree p on $\mathcal{A}$ is a family $\omega = (\omega_\Delta)_{\Delta\in K}$ such that the subsequent conditions are satisfied.

1. For each $\Delta \in K$, $\omega_\Delta \in \Omega^p(\mathcal{A}^{!!}_{\Delta};\Delta)$ is a smooth form of degree p on $\mathcal{A}^{!!}_{\Delta}$.

2. For each Δ, $\Delta' \in K$, if $\varphi_{\Delta,\Delta'} : \Delta' \hookrightarrow \Delta$ is a face of Δ,

$$\varphi^{\mathcal{A}^{!!}_{\Delta}}_{\Delta,\Delta'}(\omega_\Delta) = \omega_{\Delta'}.$$

By the paragraph preceding Proposition 1, the Lie algebroid $\mathcal{A}^{!!}_{\Delta'}$ can be identified with the Lie algebroid $\mathrm{Im}\,(\varphi_{\Delta,\Delta'})^{!!}$ and so, for each $x \in \Delta'$, the fibre $(\mathcal{A}^{!!}_{\Delta'})_x$ is a vector subspace of the fibre $(\mathcal{A}^{!!}_{\Delta})_x$. Moreover, the cochain algebras $\Omega^*(\mathcal{A}^{!!}_{\Delta'};\Delta')$ and $\Omega^*(\mathrm{Im}\,(\varphi_{\Delta,\Delta'})^{!!};\Delta')$ are identified. Hence, the second condition of the definition 4 can be stated in the following form: for each point $x \in \Delta'$ and vectors $u_1, \ldots, u_p \in (A^{!!}_{\Delta'})_x$

$$\omega_{\Delta'}(x)(u_1,\cdots,u_p) = \omega_\Delta(x)(u_1,\cdots,u_p).$$

Thus, a piecewise smooth form on $\mathcal{A}$ is a collection of smooth forms, each one defined on the Lie algebroid restriction of $\mathcal{A}$ to a simplex of K, which are compatible under restriction to faces. The set of all piecewise smooth forms of degree p on $\mathcal{A}$ will be denoted by $\Omega^p(\mathcal{A};K)$. When we need to emphasize the manifold which we work on, we write $\Omega^p_{\mathrm{ps}}(\mathcal{A};M)$ instead of $\Omega^p(\mathcal{A};K)$. We have then

$$\Omega^p(\mathcal{A};K) = \{(\omega_\Delta)_{\Delta\in K} : \omega_\Delta \in \Omega^p(\mathcal{A}^{!!}_{\Delta};\Delta),\quad \Delta' \prec \Delta \implies (\omega_\Delta)_{/_{\Delta'}} = \omega_{\Delta'}\}.$$

When $p = 0$, a piecewise smooth form of degree zero on the Lie algebroid $\mathcal{A}$ is a family $(\varphi_\Delta)_{\Delta\in K} \in \prod_{\Delta\in K} C^\infty(\Delta)$ such that $\varphi_\Delta : \Delta \longrightarrow \mathrm{R}$ is smooth function and the equality $\varphi_{\Delta'} = \varphi_{\Delta_{/_{\Delta'}}}$ holds for each face Δ' of Δ. The compatibility condition of restrictions to faces gives a mapping $\varphi : |K| \longrightarrow \mathrm{R}$ which is piecewise smooth. Obviously, $\Omega^0(\mathcal{A};K)$ has a structure of an unitary associative algebra over R (with respect to pointwise multiplication).

Since the restrictions of smooth forms are compatible with sums and products, various operations on $\Omega^p(\mathcal{A};K)$ can be defined by the corresponding operations on $\Omega^p(\mathcal{A}^{!!}_{\Delta};\Delta)$, for each simplex Δ of K. The set $\Omega^p(\mathcal{A};K)$, equipped with these operations, becomes a real vector subspace of $\prod_{\Delta\in K}\Omega^p(\mathcal{A}_\Delta;\Delta)$ and, additionally, is a module over the algebra $\Omega^0(\mathcal{A};K)$. Moreover, the direct sum

$$\Omega^*(\mathcal{A};K)=\bigoplus_{p\geqslant 0}\Omega^p(\mathcal{A};K)$$

equipped with the exterior product defined by the corresponding exterior product on each algebra $\Omega^*(\mathcal{A}_\Delta;\Delta)=\bigoplus_{p\geqslant 0}\Omega^p(\mathcal{A}_\Delta;\Delta)$, is a skew-commutative graded algebra defined over R.

In order to obtain a complex of cochains, especially important is the analogue to the exterior derivative. This operator is also obtained by the corresponding exterior derivative on $\Omega^p(\mathcal{A}^{!!}_\Delta;\Delta)$, for each simplex Δ of K. Such as in the case of smooth forms on a Lie algebroid, the space $\Omega^*(\mathcal{A};K)$, with the operations and differentiation above, becomes a cochain algebra, which is defined over R with the structure of differential graded algebra

$$d_{\mathcal{A},K}:\Omega^p(\mathcal{A};K)\longrightarrow\Omega^{p+1}(\mathcal{A};K).$$

Keeping these hypotheses and notation, we give the definition of piecewise Lie algebroid cohomology.

Definition 5. The piecewise Lie algebroid cohomology of $\mathcal{A}$ is the cohomology space of the cochain algebra $(\Omega^*(\mathcal{A};K),d_{\mathcal{A},K})$. Its cohomology, $H(\Omega^*(\mathcal{A};K),d_{\mathcal{A},K})$, will be denoted by $H^*(\mathcal{A};K)$.

We shall formulate now the main problem of this paper. Let M be a compact smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let $\omega\in\Omega^p(\mathcal{A};M)$ be a smooth form on $\mathcal{A}$ of degree p. For each simplex Δ of K, let $\varphi_{M,\Delta}:\Delta\longrightarrow M$ be the inclusion mapping. We can restrict the form ω to the smooth form $\omega_{/\Delta}=\varphi^{\mathcal{A}}_{M,\Delta}(\omega)\in\Omega^*(\mathcal{A}^{!!}_\Delta;\Delta)$. It is obvious that the family $\omega=(\omega_{/\Delta})_{\Delta\in K}$ is a piecewise smooth form on $\mathcal{A}$. Hence, we have a linear mapping

$$\Psi^p_{M,K}:\Omega^p(\mathcal{A};M)\longrightarrow\Omega^p(\mathcal{A};K)$$

defined by

$$\omega\longrightarrow(\omega_{/\Delta})_{\Delta\in K}.$$

Since the exterior derivative $d_{\mathcal{A}}$ commutes with the restrictions to any submanifold of M, the family $\Psi_{M,K}=(\Psi^p_{M,K})_{p\geqslant 0}$ defines a morphism of cochain algebras from the cochain algebra $(\Omega^*(\mathcal{A};M),d_{\mathcal{A},M})$ to the cochain algebra $(\Omega^*(\mathcal{A};K),d_{\mathcal{A},K})$. This mapping $\Psi_{M,K}$ will be called the *ps*–restriction mapping. We claim that the *ps*–restriction mapping $\Psi_{M,K}$ induces an isomorphism in cohomology. This is our main result whose the statement is indexed in the next section as Theorem 1.

There are some facts that we will need for the proof of this theorem such as the Mayer–Vietoris sequence and the Künneth theorem, both in the smooth and piecewise smooth contexts, the triviality of Lie algebroids over contractible manifolds and the de Rham–Sullivan theorem for cell manifolds. It should be remarked that, for the statement of the Mayer–Vietoris sequence in the piecewise smooth context, we are going to deal with a complex of piecewise smooth forms which may not be defined over a set of closed simplices of a simplicial complex but over a set of submanifolds obtained by the intersection of closed simplices with unions of open stars in the polytope of the simplicial complex. We specify now the generalization of this piecewise smooth setting, by providing some definitions and notations. Once these ideas are established, we shall then turn towards to the statement of the Mayer–Vietoris sequence in this setting.

Generalization of the piecewise setting. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K. Assume that $\mathrm{s}_1,\ldots,\mathrm{s}_e$ are simplices of K and consider the open subsets

$$U_1=\mathrm{St}(\mathrm{s}_1),\ldots,U_e=\mathrm{St}(\mathrm{s}_e).$$

Denote the open subset $U_1\cup\cdots\cup U_e$ by U. Consider the set $\underline{K}^U$ consisting of all submanifolds $\Delta\cap U$ such that Δ is a simplex of K in which s_j is a face of Δ for some $j\in\{1,\ldots,e\}$, that is,

$$\underline{K}^U=\big\{U\cap\Delta:\Delta\in K,\ \mathrm{s}_j\prec\Delta,\ j\in\{1,\ldots,e\}\big\}.$$

From now on, the submanifold $\Delta \cap U$ will be denoted by Δ_U. We have that U is equal to the union of all submanifolds $\Delta_U \in \underline{K}^U$. Suppose that $\mathcal{A}$ is a transitive Lie algebroid on U. A piecewise smooth form of degree p on $\mathcal{A}$ is a family

$$\omega = (\omega_{\Delta_U})_{\Delta_U \in \underline{K}^U} \in \prod_{\Delta_U \in \underline{K}^U} \Omega^p(\mathcal{A}^{!!}_{\Delta_U}; \Delta_U)$$

such that, if Δ and Δ' are simplices of K, with $\mathrm{s}_j \prec \Delta' \prec \Delta$ for some $j \in \{1, \ldots, e\}$, and $\varphi_{\Delta_U, \Delta'_U} : \Delta'_U \hookrightarrow \Delta_U$ the inclusion mapping, one has

$$\varphi^{\mathcal{A}^{!!}_{\Delta_U}}_{\Delta_U, \Delta'_U}(\omega_{\Delta_U}) = \omega_{\Delta'_U}$$

or simply $(\omega_{\Delta_U})_{/\Delta'_U} = \omega_{\Delta'_U}$.

The set $\Omega^*_{ps}(\mathcal{A}; U)$ of all piecewise smooth forms on $\mathcal{A}$ is a graded real vector space. A wedge product and a differential can be defined on $\Omega^*_{ps}(\mathcal{A}; U)$ by the corresponding operations on each cochain algebra $\Omega^*(\mathcal{A}^{!!}_{\Delta_U}; \Delta_U)$, giving to $\Omega^*_{ps}(\mathcal{A}; U)$ a structure of cochain algebra defined over R. The cohomology space of this cochain algebra is denoted by $H^*_{ps}(\mathcal{A}; U)$. As done before, we can define a restriction mapping

$$\Psi : \Omega^*(\mathcal{A}; U) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$$

by

$$\omega \longrightarrow (\omega_{/\Delta_U})_{\Delta_U \in \underline{K}^U}.$$

This mapping Ψ is a morphism of cochain algebras.

Next, we are concerned with the Mayer–Vietoris sequence under these hypotheses. As in the smooth case, the Mayer–Vietoris sequence will be the long sequence induced from the canonical short exact sequence corresponding to two open subsets with union equal to the space and mappings given by restriction and difference of forms. We begin by stating this short exact sequence in this context.

Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let $\mathrm{s}_1, \ldots, \mathrm{s}_e$ be simplices of K and consider the open subset $U_j = \mathrm{St}(\mathrm{s}_j)$ for each $j \in \{1, \ldots, e\}$. For $l \in \{1, \ldots, e\}$ fixed, consider the open subsets U and V, in which $U = U_1 \cup \cdots \cup U_l$ and $V = U_{l+1} \cup \cdots \cup U_e$, and assume that $M = U \cup V$. Consider the following sets of manifolds:

1. $\underline{K}^U$ is the set of all submanifolds $U \cap \Delta$ such that $\Delta \in K$ and s_j is a face of Δ for some $j \in \{1, \ldots, l\}$;
2. $\underline{K}^V$ is the set of all submanifolds $V \cap \Delta$ such that $\Delta \in K$ and s_i is a face of Δ for some $i \in \{l+1, \ldots, e\}$;
3. $\underline{K}^{U\cap V}$ is the set of all submanifolds $U \cap V \cap \Delta$ such that $\Delta \in K$ and s_j and s_i are faces of Δ for some $j \in \{1, \ldots, l\}$ and for some $i \in \{l+1, \ldots, e\}$.

As done above, the manifolds $U \cap \Delta$, $V \cap \Delta$ and $U \cap V \cap \Delta$ are denoted by Δ_U, Δ_U and $\Delta_{U\cap V}$, respectively. Consider the following two mappings δ and π:

$$\delta : \Omega^p(\mathcal{A}; K) \longrightarrow \Omega^p_{ps}(\mathcal{A}_U; U) \times \Omega^p_{ps}(\mathcal{A}_V; V),$$

$$\pi : \Omega^p_{ps}(\mathcal{A}_U; U) \times \Omega^p_{ps}(\mathcal{A}_V; V) \longrightarrow \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V)$$

defined by

$$\delta\Big((\omega_\Delta)_{\Delta\in K}\Big) = \Big((\omega_{\Delta_{/\Delta_U}})_{\Delta_U \in \underline{K}^U}, (\omega_{\Delta_{/\Delta_V}})_{\Delta_V \in \underline{K}^V}\Big),$$

$$\pi\Big((\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V}\Big) = \Big((\eta_{\Delta_V})_{/\Delta_{U\cap V}} - (\xi_{\Delta_U})_{/\Delta_{U\cap V}}\Big)_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}}.$$

Keeping these hypotheses and notation, we present our next result.

Proposition 2. *The sequence*

$$\{0\} \longrightarrow \Omega^p(\mathcal{A}; K) \xrightarrow{\ \delta\ } \Omega^p_{ps}(\mathcal{A}_U; U) \oplus \Omega^p_{ps}(\mathcal{A}_V; V) \xrightarrow{\ \pi\ } \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V) \longrightarrow \{0\}$$

is a short exact sequence.

Proof. Let Δ be a simplex of K and v the barycenter of this simplex. Since $M = U \cup V$, the set made up of the stars $\{U_j : j \in \{1, \dots, e\}\}$ is an open covering of M. Then, there exists an index $j \in \{1, \dots, e\}$ and a simplex Δ' of K such that s_j is a face of Δ' and v belongs to the interior of Δ'. Since v also belongs to the interior of Δ then, by uniqueness, $\Delta' = \Delta$. Hence, the simplex s_j is a face of the simplex Δ. Therefore, for each simplex Δ of K, there exists an index $j \in \{1, \dots, e\}$ such that s_j is a face Δ. With this property, we shall check now that the mapping δ is injective and that $\mathrm{Im}\,\delta = \mathrm{Ker}\,\pi$. Let

$$\omega = (\omega_\Delta)_{\Delta \in K} \in \Omega^p(\mathcal{A}; K)$$

be a piecewise smooth form such that $\delta(\omega) = (0,0)$. For each simplex $\Delta \in K$, we can take an index $j \in \{1, \dots, e\}$ such that s_j is a face of Δ. Without loss of generality, we may assume that $j \in \{1, \dots, l\}$. By hypothesis, $\delta(\omega) = 0$ and this means that $\omega_{\Delta_{/\Delta_U}} = 0$. Applying the same argument to all faces of Δ, we obtain that $\omega_\Delta = 0$. Consequently, we have that $\omega = 0$ and so δ is injective. Since the forms

$$\omega_{\Delta_{/\Delta_U}} \quad \text{and} \quad \omega_{\Delta_{/\Delta_V}}$$

have the same restriction to $\Delta_{U\cap V}$, we conclude that $\mathrm{Im}\,\delta \subset \mathrm{Ker}\,\pi$. For checking the reciprocal inclusion, if

$$\pi\Big((\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V}\Big) = (0,0)$$

then

$$(\xi_{\Delta_U})_{/\Delta_{U\cap V}} = (\eta_{\Delta_V})_{/\Delta_{U\cap V}}$$

and this equality allows to define the form ω_Δ on each simplex $\Delta \in K$ by

$$\omega_\Delta(x) = \begin{cases} \xi_{\Delta_U}(x) & \text{if } \ x \in \Delta_U \\ \\ \eta_{\Delta_V}(x) & \text{if } \ x \in \Delta_V \end{cases}$$

The sets $\Delta_U = \Delta \cap U$ and $\Delta_V = \Delta \cap V$ are open in Δ with union equal to Δ. Hence, the form ω_Δ is smooth. We will check now that the form

$$(\omega_\Delta)_{\Delta \in K}$$

is piecewise. If Δ' is a face of Δ, in view of fact that

$$\Big(\Delta_U \cap \Delta'\Big) \cup \Big(\Delta_V \cap \Delta'\Big) = \Delta'$$

the restriction of the form ω_Δ to Δ' is equal to

$$\Big(\omega_\Delta\Big)_{\Delta'} = \begin{cases} \Big(\xi_{\Delta_U}\Big)_{\Delta_U \cap \Delta'} & \text{for } \ x \in \Delta_U \cap \Delta' \\ \\ \Big(\eta_{\Delta_V}\Big)_{\Delta_V \cap \Delta'} & \text{for } \ x \in \Delta_V \cap \Delta' \end{cases}$$

On the other hand,

$$\Delta_U \cap \Delta' = \Delta \cap U \cap \Delta' = \Delta' \cap U = \Delta'_U.$$

Therefore,

$$\Big(\xi_{\Delta_U}\Big)_{\Delta_U \cap \Delta'} = \xi_{\Delta'_U} \quad \text{and} \quad \Big(\eta_{\Delta_V}\Big)_{\Delta_V \cap \Delta'} = \eta_{\Delta'_V}.$$

Hence, $(\omega_\Delta)_{/\Delta'} = \omega_{\Delta'}$ and so $(\omega_\Delta)_{\Delta \in K}$ is piecewise smooth. By construction, we also have that

$$\delta((\omega_\Delta)_{\Delta \in K}) = \Big((\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V}\Big)$$

which shows that $\Big((\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}, (\eta_{\Delta_V})_{\Delta_V \in \underline{K}^V}\Big)$ belongs to the image of δ. Hence, we have that $\mathrm{Ker}\,\pi \subset \mathrm{Im}\,\delta$.

We shall check now that the mapping π is surjective. Since the set $\{U, V\}$ is an open covering of M, we can fix two smooth mappings $\varphi, \psi : M \longrightarrow [0,1]$ such that $\mathrm{supp}\,\varphi \subset U$, $\mathrm{supp}\,\psi \subset V$ and $\varphi(x) + \psi(x) = 1$ for each x in M. Let

$$(\gamma_{\Delta_{U\cap V}})_{\Delta_{U\cap V} \in \underline{K}^{U\cap V}} \in \Omega^p_{ps}(\mathcal{A}_{U\cap V}; U \cap V)$$

be a piecewise smooth form on $\mathcal{A}_{U\cap V}$. We shall define a differential form

$$(\xi_{\Delta_U})_{\Delta_U\in \underline{K}^U}\in \Omega^p_{ps}(\mathcal{A}_U;U)$$

as follows. For each $\Delta_U\in \underline{K}^U$, define

$$\xi_{\Delta_U}(x)=\begin{cases}-\psi(x)\ \gamma_{\Delta_{U\cap V}}(x) & \text{if } x\in \Delta\cap U\cap V\\ \\ 0_x\in (\mathcal{A}^{!!}_{\Delta_U})_x & \text{if } x\in \Delta\cap U\cap (M\setminus \operatorname{supp}\psi).\end{cases}$$

The sets $\Delta_U\cap V$ and $\Delta_U\cap(M\setminus\operatorname{supp}\psi)$ are open in Δ_U with union equal to Δ_U. Obviously, the restrictions of ξ_{Δ_U} to $\Delta_U\cap V$ and to $\Delta_U\cap(M\setminus\operatorname{supp}\psi)$ are smooth. Therefore, we conclude that $\xi_{\Delta_U}\in\Omega^p(\mathcal{A}^{!!}_{\Delta_U};\Delta_U)$. In order to obtain a piecewise smooth form belonging to $\Omega^p_{ps}(\mathcal{A}_U;U)$, it remains to check that $(\xi_{\Delta_U})_{\Delta_U\in\underline{K}^U}$ is compatible with restrictions to faces. Let Δ and Δ' be two simplices of K such that $s_j\prec\Delta'\prec\Delta$ for some $j\in\{1,\dots,e\}$. Then, $\Delta'_U\cap V\subset\Delta_U\cap V$ and, since γ is piecewise smooth, we have

$$\gamma_{\Delta'_{U\cap V}}(x)=(\gamma_{\Delta_{U\cap V}})_{/\Delta'_{U\cap V}}(x)$$

for each $x\in\Delta'_{U\cap V}$. Hence, if $x\in\Delta'_{U\cap V}$,

$$\xi_{\Delta'_U}(x)=-\psi(x)\ \gamma_{\Delta'_{U\cap V}}(x)=-\psi(x)(\gamma_{\Delta_{U\cap V}})_{/\Delta'_{U\cap V}}(x)=(\xi_{\Delta_U})_{/\Delta'_U}(x).$$

If $x\in\Delta_U\cap(M\setminus\operatorname{supp}\psi)$ we have that $\xi_{\Delta_U}(x)=(\xi_{\Delta'_U})_{\Delta_U}(x)=0$. Hence, the differential form $(\xi_{\Delta_U})_{\Delta_U\in\underline{K}^U}$ is a piecewise smooth form on $\mathcal{A}_U$. Analogously, we define a piecewise smooth form $(\eta_{\Delta_V})_{\Delta_V\in\underline{K}^V}\in\Omega^p_{ps}(\mathcal{A}_V;V)$ by

$$\eta_{\Delta_V}(x)=\begin{cases}-\varphi(x)\ \gamma_{\Delta_{U\cap V}}(x) & \text{if } x\in\Delta_V\cap U\\ \\ 0_x\in(\mathcal{A}^{!!}_{\Delta_V})_x & \text{if } x\in\Delta_V\cap(M\setminus\operatorname{supp}\varphi)\end{cases}$$

and we have that, for each $x\in\Delta_{U\cap V}\in\underline{K}^{U\cap V}$,

$$(\eta_{\Delta_V})_{/\Delta_{U\cap V}}(x)-(\xi_{\Delta_U})_{/\Delta_{U\cap V}}(x)=\gamma_{\Delta_{U\cap V}}(x)$$

that is,

$$\pi\big((\xi_{\Delta_U})_{\Delta_U\in\underline{K}^U},(\eta_{\Delta_V})_{\Delta_V\in\underline{K}^V}\big)=(\gamma_{\Delta_{U\cap V}})_{\Delta_{U\cap V}\in\underline{K}^{U\cap V}}.$$

Hence, the result is proved.

The Mayer–Vietoris sequence in the piecewise context is the long sequence of cohomology corresponding to the short sequence shown in the statement of the previous proposition. If we put that short sequence together with the short exact sequence presented in the third section of [5] (smooth case), we obtain a commutative diagram of short exact sequences, in which the vertical mappings are the restriction mappings Ψ defined above. This is the statement of our next proposition.

Proposition 3. *Keeping the same hypotheses and notation as above, the diagram*

$$\begin{array}{ccccccccc}\{0\} & \longrightarrow & \Omega^p(\mathcal{A};M) & \xrightarrow{\alpha} & \Omega^p(\mathcal{A}_U;U)\oplus\Omega^p(\mathcal{A}_V;V) & \xrightarrow{\beta} & \Omega^p(\mathcal{A}_{U\cap V};U\cap V) & \longrightarrow & \{0\}\\ & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \\ \{0\} & \longrightarrow & \Omega^p(\mathcal{A};K) & \xrightarrow{\delta} & \Omega^p_{ps}(\mathcal{A}_U;U)\oplus\Omega^p_{ps}(\mathcal{A}_V;V) & \xrightarrow{\pi} & \Omega^p_{ps}(\mathcal{A}_{U\cap V};U\cap V) & \longrightarrow & \{0\}\end{array}$$

is commutative.

We finalize this section with a proposition concerning the Künneth isomorphism in a particular case of piecewise setting. Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and s a simplex of K. Denote by U the open subset St(s). Assume that $\mathfrak{g}$ is a real Lie algebra and consider the trivial Lie algebroid $\mathcal{A}=TU\times\mathfrak{g}$.

Proposition 4. *Keeping the same hypotheses and notation as above, one has*

$$H^*_{ps}(\mathcal{A};U)\simeq H^*_{ps}(U)\otimes(\mathfrak{g}).$$

Proof. As done above, we denote the submanifold $U\cap\Delta$ by Δ_U and $\underline{K}^U$ the set of all submanifolds Δ_U for each simplex $\Delta\in K$ such that s is a face of Δ. Consider the Lie algebroids morphisms

$$\gamma_{\Delta_U}: T\Delta_U\times\mathfrak{g}\longrightarrow T\Delta_U$$

and

$$\pi_{\Delta_U}: T\Delta_U\times\mathfrak{g}\longrightarrow \mathfrak{g}$$

given by the projections on the first and second factors, respectively. The remaining proof will be split into three parts.

Part 1. We will check now that

$$\Omega^*_{ps}(U)\otimes\Omega^*(\mathfrak{g})\simeq\Omega^*_{ps}(\mathcal{A};U).$$

Let $\xi=(\xi_{\Delta_U})_{\Delta_U\in\underline{K}^U}$ be a form belonging to $\Omega^*_{ps}(U)$ and η belonging to $\Omega^*(\mathfrak{g})$. Assume that Δ' and Δ are two simplices of K such that $s\prec\Delta'$ and $\varphi:\Delta'\hookrightarrow\Delta$ the inclusion mapping. Denote by $\varphi_{\Delta,\Delta'}$ the inclusion mapping $U\cap\Delta'\hookrightarrow U\cap\Delta$. Consider the canonical mappings

$$(\varphi^{T\Delta_U\times\mathfrak{g}}_{\Delta,\Delta'})^{!!}:(T\Delta_U\times\mathfrak{g})^{!!}_{\Delta'_U}\longrightarrow T\Delta_U\times\mathfrak{g}$$

and

$$(\varphi^{T\Delta_U}_{\Delta,\Delta'})^{!!}:(T\Delta_U)^{!!}_{\Delta'_U}\longrightarrow T\Delta_U$$

(see the paragraph soon after the definition 1). We have that

$$\gamma_{\Delta_U}\circ(\varphi^{T\Delta_U\times\mathfrak{g}}_{\Delta,\Delta'})^{!!}=(\varphi^{T\Delta_U}_{\Delta,\Delta'})^{!!}\circ\gamma_{\Delta'_U}\quad \textit{and}\quad \pi_{\Delta_U}\circ(\varphi^{T\Delta_U\times\mathfrak{g}}_{\Delta,\Delta'})^{!!}=\pi_{\Delta'_U}$$

so the equalities

$$(\gamma^*_{\Delta_U}\xi_{\Delta_U})_{/\Delta'_U}=\gamma^*_{\Delta'_U}\xi_{\Delta'_U}\quad \textit{and}\quad (\pi^*_{\Delta_U}\eta)_{/\Delta'_U}=\pi^*_{\Delta'_U}\eta$$

also hold. These equalities show that the differential form

$$(\gamma^*_{\Delta_U}\xi_{\Delta_U}\wedge\pi^*_{\Delta_U}\eta)_{\Delta_U\in\underline{K}^U}$$

belongs to $\Omega^*_{ps}(\mathcal{A};U)$. Hence, we can consider a mapping

$$k_{ps}:\Omega^*_{ps}(U)\otimes\Omega^*(\mathfrak{g})\longrightarrow\Omega^*_{ps}(\mathcal{A};U)$$

such that

$$k_{ps}(\xi\otimes\eta)=(\gamma^*_{\Delta_U}\xi_{\Delta_U}\wedge\pi^*_{\Delta_U}\eta)_{\Delta_U\in\underline{K}^U}$$

in which $\xi=(\xi_{\Delta_U})_{\Delta_U\in\underline{K}^U}\in\Omega^*_{ps}(U)$ and $\eta\in\Omega^*(\mathfrak{g})$. This mapping is well defined. Now, we shall see that the mapping k_{ps} is an isomorphism of differential graded algebras. Obviously, the mapping k_{ps} is a morphism of graded algebras. For each $\Delta\in K$ such that $s\prec\Delta$, let

$$k_{\Delta_U}:\Omega^*(\Delta_U)\otimes\Omega(\mathfrak{g})\longrightarrow\Omega^*(T\Delta_U\times\mathfrak{g};\Delta_U)$$

be the K$\ddot{u}$nneth isomorphism described in the sixth section of [5]. We have that,

$$(k_{ps}(\xi\otimes\eta))_{\Delta_U}=\gamma^*_{\Delta_U}\xi_{\Delta_U}\wedge\pi^*_{\Delta_U}\eta=k_{\Delta_U}(\xi_{\Delta_U}\otimes\eta).$$

Therefore, if $\omega=\sum\xi\otimes\eta\in\Omega^*_{ps}(U)\otimes\Omega^*(\mathfrak{g})$ and $k_{ps}(\omega)=0$, then $(k_{ps}(\omega))_{\Delta_U}=0$ and so

$$0=\big(k_{ps}\big(\sum\xi\otimes\eta\big)\big)_{\Delta_U}=k_{\Delta_U}\big(\sum(\xi_{\Delta_U}\otimes\eta)\big).$$

Hence $\omega=\sum(\xi_{\Delta_U}\otimes\eta)=0$ and, with this, we have checked that k is injective. Take now $\lambda=(\lambda_{\Delta_U})_{\Delta_U\in\underline{K}^U}\in\Omega^*_{ps}(\mathcal{A};U)$. We want to find $\omega\in\Omega^*_{ps}(U)\otimes\Omega^*(\mathfrak{g})$ such that $k_{ps}(\omega)=\lambda$. Since k_{Δ_U} is surjective, we can consider smooth forms $\xi_{j_{\Delta_U}}\in\Omega^*(\Delta_U)$ and $\eta\in\Omega^*(\mathfrak{g})$ such that

$$k_{\Delta_U}\big(\sum_j(\xi_{j_{\Delta_U}}\otimes\eta)\big)=\lambda_{\Delta_U}.$$

Take then the form $\omega_{\Delta_U} = \sum_j(\xi_{j_{\Delta_U}} \otimes \eta)$. If Δ' and Δ are simplices of K with $s \prec \Delta' \prec \Delta$, we have the equalities

$$k_{\Delta'_U}\big(\sum_j(\xi_{j_{\Delta_U}})_{\Delta'_U} \otimes \eta\big) = \sum_j k_{\Delta'_U}((\xi_{j_{\Delta_U}})_{\Delta'_U} \otimes \eta\big) = \sum_j \big(\gamma^*_{\Delta'_U}(\xi_{j_{\Delta_U}})_{\Delta'_U} \wedge \pi^*_{\Delta'}\eta\big) = (*)$$

and

$$\begin{aligned} k_{\Delta'_U}\big(\sum_j(\xi_{j_{\Delta'_U}} \otimes \eta)\big) &= \lambda_{\Delta'_U} = (\lambda_{\Delta_U})_{/\Delta'_U} = \big(k_{\Delta_U}\big(\sum_j(\xi_{j_{\Delta_U}} \otimes \eta)\big)_{/\Delta'_U} \\ &= \big(\sum_j(\gamma^*_{\Delta_U}(\xi_{j_{\Delta_U}}) \wedge \pi^*_\Delta\eta)\big)_{/\Delta'_U} = \sum_j(\gamma^*_{\Delta_U}(\xi_{j_\Delta}) \wedge \pi^*_\Delta\eta)\big)_{/\Delta'_U} \\ &= \sum_j(\gamma^*_{\Delta_U}((\xi_{j_{\Delta_U}})_{/\Delta'_U})) \wedge \pi^*_{\Delta'}\eta)) = \sum_j \big(\gamma^*_{\Delta'_U}(\xi_{j_{\Delta_U}})_{\Delta'_U} \wedge \pi^*_{\Delta'}\eta\big) = (*). \end{aligned}$$

Hence,

$$k_{\Delta'_U}\big(\sum_j(\xi_{j_{\Delta_U}})_{\Delta'_U} \otimes \eta\big) = k_{\Delta'_U}\big(\sum_j(\xi_{j_{\Delta'_U}} \otimes \eta)\big)$$

and, since $k_{\Delta'_U}$ is bijective, $\sum_j(\xi_{j_{\Delta_U}})_{\Delta'_U} \otimes \eta = \sum_j(\xi_{j_{\Delta'_U}} \otimes \eta)$. Therefore, we can conclude that $(\xi_{j_{\Delta_U}})_{/\Delta'_U} = \xi_{j_{\Delta'_U}}$. Then, the form $\omega = (\omega_{\Delta_U})_{\Delta_U \in \underline{K}^U}$, in which $\omega_{\Delta_U} = \sum_j(\xi_{j_{\Delta_U}} \otimes \eta)$, belongs to $\Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})$. Obviously, $k_{ps}(\omega) = \lambda$ and then it is checked that k_{ps} is an isomorphism of graded algebras.

Part 2. Next, we will check that k_{ps} commutes with the differential. For each $\Delta \in K$ such that s is a face of Δ, the differentials on the complexes $\Omega^*_{ps}(\mathcal{A}; U)$, $\Omega^*_{ps}(U)$ and $\Omega^*(\Delta_U)$ will be denoted by $d^{\mathcal{A}}_{ps}$, d^U_{ps} and d^{Δ_U} respectively. Let $\xi = (\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}$ be a form belonging to $\Omega^*_{ps}(U)$ and η belonging to $\Omega^*(\mathfrak{g})$. We have

$$\begin{aligned} (d^{\mathcal{A}}_{ps} \circ k_{ps})(\xi \otimes \eta) &= d^{\mathcal{A}}_{ps}((\gamma^*_{\Delta_U}\xi_{\Delta_U} \wedge \pi^*_{\Delta_U}\eta)_{\Delta_U \in \underline{K}^U}) \\ = d^{\mathcal{A}}_{ps}\big((\gamma^*_{\Delta_U}\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U}\big) \wedge \pi^*\eta &+ (-1)^{deg\xi}(\gamma^*_{\Delta_U}\xi_{\Delta_U})_{\Delta_U \in \underline{K}^U} \wedge d^{\mathcal{A}}_{ps}(\pi^*\eta) \\ = \big(\gamma^*_{\Delta_U}(d^{\Delta_U}(\xi_{\Delta_U}))\big)_{\Delta_U \in \underline{K}^U} &\wedge \pi^*\eta + (-1)^{deg\omega}\gamma^*\xi \wedge \pi^*(d_{\mathfrak{g}}\eta) = \\ = k_{ps}((d^U_{ps}\xi) \otimes \eta) &+ (-1)^{deg\xi}k_{ps}(\xi \otimes d_{\mathfrak{g}}\eta) = k_{ps} \circ \delta(\xi \otimes \eta) \end{aligned}$$

This proves that k_{ps} is an isomorphism of differential graded algebras.

Part 3. The isomorphism k_{ps} above induces an isomorphism in cohomology. By applying the Künneth theorem, we obtain

$$H^*_{ps}(\mathcal{A}; U) \simeq H^*(\Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g})) \simeq H^*_{ps}(U) \otimes H^*(\mathfrak{g})$$

and the result is proved.

4. MAIN THEOREM

Whitney in [3] and Sullivan in [1] have proved that cohomologies obtained by using the cell structure of a space are isomorphic to the singular cohomology of the polytope (for further details, see [3, Theorem IV-29A and Theorem VII-12A] and [1, Theorem 7.1]). Therefore, those piecewise cohomologies also are isomorphic to the Rham cohomology, if the space is a cell smooth manifold. Based both in their work and in the work presented by Mackenzie in [4] as well by Kubarski in [5], we have claimed in the previous section that the Lie algebroid cohomology and the piecewise Lie algebroid cohomology of a transitive Lie algebroid over a triangulated compact manifold are isomorphic. In the present section, we build the parts of the proof of this assertion for all Lie algebroids under these hypotheses.

Let M be a smooth manifold, smoothly triangulated by a finite simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Consider the ps–restriction mapping

$$\Psi_{M,K} : \Omega^*(\mathcal{A}; M) \longrightarrow \Omega^*(\mathcal{A}; K),$$

$$\omega \longrightarrow (\omega_{/\Delta})_{\Delta \in K}.$$

Theorem 1. *The mapping $\Psi_{M,K}$ induces an isomorphism in cohomology.*

The proof of Theorem 1 involves, beyond theorems already mentioned before, the Steenrod five lemma applied to the commutative diagram shown in the Proposition 3 of the previous section. In fact, we shall be able to apply the Steenrod five lemma to that diagram as long as we know that the mapping Ψ is a quasi-isomorphism for trivial Lie algebroids over stars. Therefore, our first step is to show that the theorem 1 holds for these trivial Lie algebroids.

Proposition 5. *Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and* s *a simplex of K. Denote the open subset* $\mathrm{St}(\mathrm{s})$ *by U and the submanifold $U \cap \Delta$ by Δ_U. Let $\underline{K}^U$ be the set of all submanifolds Δ_U such that $\Delta \in K$ and* s *is a face of Δ. Assume that $\mathfrak{g}$ is a real Lie algebra and consider the trivial Lie algebroid $\mathcal{A} = TU \times \mathfrak{g}$. Then, the ps–restriction mapping*

$$\Psi : \Omega^*(\mathcal{A}; U) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U),$$

$$\omega \longrightarrow (\omega_{/\Delta_U})_{\Delta_U \in \underline{K}^U}$$

induces an isomorphism in cohomology.

Proof. By the Künneth theorem for trivial Lie algebroids stated in [5, Corollary 6.2], we have that

$$H(\mathcal{A}; U) \simeq H_{dR}(U) \otimes H(\mathfrak{g})$$

We shall check now that Ψ induces an isomorphism in cohomology. Take the diagram

$$\begin{array}{ccc}
\Omega^*(U) \otimes \Omega^*(\mathfrak{g}) & \xrightarrow{\Phi\otimes \mathrm{Id}} & \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k_{ps}} \\
\Omega^*(\mathcal{A}; U) & \xrightarrow{\quad\Psi\quad} & \Omega^*_{ps}(\mathcal{A}; U)
\end{array}$$

in which $k_{ps} : \Omega^*_{ps}(U) \otimes \Omega^*(\mathfrak{g}) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$ is the isomorphism defined in the proof of the proposition 4, k is the Künneth isomorphism described in [5, Lemma 6.1] and Φ is the restriction mapping induced from the Rham–Sullivan theorem for cell manifolds (see the diagram preceding the proposition 8.7 of [9]). Obviously, the diagram is commutative and, by the de Rham–Sullivan theorem ([1, Theorem 7.1] or section VIII-F of [9]), the mapping Φ induces an isomorphism in cohomology. Therefore, in cohomology, we have the commutative diagram

$$\begin{array}{ccc}
H^*_{dR}(U) \otimes H^*(\mathfrak{g}) & \xrightarrow{H(\Phi\otimes \mathrm{Id})} & H^*_{ps}(U) \otimes H^*(\mathfrak{g}) \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\
H^*(\mathcal{A}; U) & \xrightarrow{\quad H(\Psi)\quad} & H^*_{ps}(\mathcal{A}; U)
\end{array}$$

Hence, $H(\Psi)$ is an isomorphism.

Next, we will show that Ψ induces an isomorphism in cohomology, not only for trivial Lie algebroids defined over open stars, but for any arbitrary transitive Lie algebroid defined over an open star. For that, we begin by stating a basic result which is a direct consequence from the functor homology.

Proposition 6. *Let M be a smooth manifold, smoothly triangulated by a simplicial complex K and* s *a simplex of K. Denote the open subset* $\mathrm{St}(\mathrm{s})$ *by U. Let $\mathcal{A}$ and $\mathcal{B}$ be two transitive Lie algebroids on M and suppose there exists an isomorphism of Lie algebroids between them. Then, the cohomology spaces $H_{ps}(\mathcal{A}; U)$ and $H_{ps}(\mathcal{B}; U)$ are isomorphic.*

Proposition 7. *Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and* s *a simplex of K. Denote the open subset* $\mathrm{St}(\mathrm{s})$ *by U and the submanifold $U \cap \Delta$ by Δ_U. Let $\underline{K}^U$ be the set of all submanifolds Δ_U such that $\Delta \in K$ and* s *is a face of Δ. Assume that $\mathcal{A}$ is a transitive Lie algebroid on U. Then, the morphism*

$$\Psi : \Omega^*(\mathcal{A}; U) \longrightarrow \Omega^*_{ps}(\mathcal{A}; U)$$

$$\omega \longrightarrow (\omega_{/\Delta_U})_{\Delta_U \in \underline{K}^U}$$

induces an isomorphism in cohomology.

Proof. Let $\gamma : \mathcal{A} \longrightarrow TU$ be the anchor of $\mathcal{A}$ and $\mathfrak{g} = \ker \gamma$. The set U is contractible. Consequently, as noted at the end of the first section, the Lie algebroid $\mathcal{A}$ is isomorphic to the trivial Lie algebroid $TU \times \mathfrak{g}$ defined over U. We achieve the result by the commutativity of the diagram

$$\begin{array}{ccc} \Omega^p(\mathcal{A};U) & \longrightarrow & \Omega^p_{ps}(\mathcal{A};U) \\ \downarrow & & \downarrow \\ \Omega^p(TU\times\mathfrak{g}) & \xrightarrow{\Psi} & \Omega^p_{ps}(TU\times\mathfrak{g}) \end{array}$$

and by applying the proposition 6 to the vertical mappings and the Proposition 5 to the mapping Ψ displayed on the diagram.

Proof of Theorem 1. We shall prove the result by induction on the number of vertices of the simplicial complex K. Suppose then that $\{v_0, \ldots, v_n\}$ is the family of all vertices of K. If K has only one vertex, the result is trivial. Suppose we have proved the result for each set $\{v_0, \ldots, v_l\}$ in which $l < n$. It is known that the family $\{\mathrm{St}(v_j) : j \in \{0, \ldots, n\}\}$ is an open covering of M. Consider the open subsets of M

$$U = \bigcup_{j=0}^{n-1} \mathrm{St}(v_j) \quad \text{and} \quad V = \mathrm{St}(v_n)$$

We have that

$$U \cap V = \big(\bigcup_{j=0}^{n-1} \mathrm{St}(v_j)\big) \cap \mathrm{St}(v_n) = \bigcup_{j=0}^{n-1} \big(\mathrm{St}(v_j) \cap \mathrm{St}(v_n)\big) = \bigcup_{j} \mathrm{St}([v_j, v_n])$$

in which the union is taken over all indexes j such that the vertices v_j and v_n generate a simplex of K (otherwise the intersection $\mathrm{St}(v_j) \cap \mathrm{St}(v_n)$ is empty) and $[v_j, v_n]$ denotes the closed simplex generated by the vertices v_j and v_n. The Proposition 3 sates that

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & \Omega^p(\mathcal{A};M) & \xrightarrow{\alpha} & \Omega^p(\mathcal{A}_U;U)\oplus\Omega^p(\mathcal{A}_V;V) & \xrightarrow{\beta} & \Omega^p(\mathcal{A}_{U\cap V};U\cap V) & \longrightarrow & \{0\} \\ & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \downarrow{\scriptstyle\Psi} & & \\ \{0\} & \longrightarrow & \Omega^p(\mathcal{A};K) & \xrightarrow{\delta} & \Omega^p_{ps}(\mathcal{A}_U;U)\oplus\Omega^p_{ps}(\mathcal{A}_V;V) & \xrightarrow{\pi} & \Omega^p_{ps}(\mathcal{A}_{U\cap V};U\cap V) & \longrightarrow & \{0\} \end{array}$$

is a commutative diagram of short exact sequences. The mapping Ψ on the right hand side of the diagram is a quasi-isomorphism by induction. The mapping Ψ in the middle of the diagram is a quasi-isomorphism by induction and by the Proposition 7. By the Steenrod five lemma, the mapping Ψ on the left hand side of the diagram is also a quasi-isomorphism. The Theorem 1 is proved.

From Theorem 1, we infer that the piecewise Lie algebroid cohomology of a compact triangulated manifold does not depend on the triangulation used, that is, for any simplicial division of the simplicial complex, the piecewise Lie algebroid cohomology spaces of both triangulated manifolds are isomorphic. This statement is the substance of our next proposition.

Corollary 1. *Let M be a smooth manifold, smoothly triangulated by a simplicial complex K, and $\mathcal{A}$ a transitive Lie algebroid on M. Let L be another simplicial complex and assume that L is a subdivision of K. Then, the piecewise Lie algebroid cohomology $H(\mathcal{A};K)$ is isomorphic to the piecewise Lie algebroid cohomology $H(\mathcal{A};L)$. Furthermore, the morphism from $\Omega^*(\mathcal{A};K)$ to $\Omega^*(A;L)$ which induces that isomorphism in cohomology is also given by restriction of forms.*

Proof. The result follows from the commutativity of the next diagram

$$\begin{array}{ccccc} & & \Omega^*(\mathcal{A};M) & & \\ & \swarrow{\scriptstyle\Psi_{M,K}} & & \searrow{\scriptstyle\Psi_{M,L}} & \\ \Omega^*(\mathcal{A};K) & & \xrightarrow{\Phi_{K,L}} & & \Omega^*(\mathcal{A};L) \end{array}$$

in which $\Phi_{K,L}$ is also given by restriction.

ACKNOWLEDGMENTS

The authors are grateful to Jesus Alvarez, James Stasheff and Nicolae Teleman for their valuable ideas concerning this work. The second author strongly appreciates the support given by Paula Smith, Lisa Santos and Fernando Miranda.

REFERENCES

[1] D. Sullivan, "Infinitesimal Computations in Topology", *Publ. Math. Inst. Hautes Études Sci.*, **47** (1977), 269–331.

[2] D. Sullivan, *Differential Forms and the Topology of Manifolds*, Manifolds - Tokyo 1973 (Proc. Internat. Conf., Tokyo), Univ. of Tokyo Press, Tokyo, 1975.

[3] H. Whitney, *Geometric Integration Theory*, **16**, Princet. Leg. Libr., Princeton Univ. Press, Princeton, 1957.

[4] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, **213**, London Math. Soc. Lecture Note Ser., Cambridge U. Press, Cambridge, 2005.

[5] J. Kubarski, "Poincaré duality for transitive unimodular invariantly oriented Lie algebroids", *Topology Appl.*, **121**:3 (2002), 333–355.

[6] A. Cannas Silva and A. Weinstein, *Geometric Models for Noncommutative Algebras*, **10**, Berkeley Math. Lect. Notes, Amer. Math. Soc., Providence, 1999.

[7] J. Kubarski and A. Mishchenko, "Lie Algebroids: Spectral Sequences and Signature", *Mat. Sb.*, **194**:7 (2003), 127–154.

[8] K. Mackenzie and P. J. Higgins, "Algebraic Constructions in the Category of Lie Algebroids", *J. Algebra*, **129**:1 (1990), 194–230.

[9] J. M. Morgan and P. A. Griffiths, *Rational Homotopy Theory and Differential Forms*, Progr. Math. vol. 16, Birkhäuser, Boston, 1981.

[10] M. Crainic and R. L. Fernandes, "Integrability of Lie brackets", *Ann. of Math.*, **157**:2 (2003), 575–620.