Fixed Points for Cubic Coquaternionic Maps

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Abstract

This paper deals with the dynamics of a special two-parameter family of coquaternionic cubic maps. By making use of recent results for the zeros of one-sided coquaternionic polynomials, we analytically determine the fixed points of these maps. Some numerical examples illustrating the theory are also presented. The results obtained show an unexpected richness for the dynamics of cubic coquaternionic maps when compared to the already studied dynamics of quadratic maps.

1 Introduction

The four dimensional algebra of coquaternions, also known in the literature as split quaternions, was introduced by Sir James Cockle (1819-1895) at about the same time that Sir William Hamilton (1805-1865) discovered the algebra of quaternions.

Although never as popular as their famous "cousins" quaternions, coquaternions have recently been attracting the attention of mathematicians and physicists who recognize the potential of applications of these hypercomplex numbers numbers [1, 2, 3, 9, 10, 11, 12, 13, 14, 15, 16].

In a previous study [4] the authors discussed the dynamics of the family of coquaternionic quadratic maps of the form $x^2+\mathrm{c}$ and, more recently, they considered quadratic maps of the form $x^2+\mathrm{b}x$ [7]. The present paper can be seen as continuation of the studies initiated in [4] and [7] and is a first step to deal with the – naturally much more interesting, but also much more demanding – problem of the dynamics of cubic coquaternionic maps.

The remaining of the paper is organized as follows: Section 2 contains a brief revision of the main definitions and results on the algebra of coquaternions and on unilateral coquaternionic polynomials; Section 3 contains the main results of the paper. Finally, Section 4 contains carefully chosen examples illustrating some of the conclusions contained in Section 3.

2 Preliminary results

In this section, we present a summary of the results on coquaternions and coquaternionic polynomials which are essential to the understanding of the rest of the paper. For more details, we refer the reader to [4, 5, 6, 7].

2.1 The algebra $\mathbb{H}_{\rm coq}$

Let $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathsf{i}^2 = -1,\ \mathsf{j}^2 = \mathsf{k}^2 = 1,\ \ \mathsf{ij} = -\mathsf{ji} = \mathsf{k}.$$

This non-commutative product generates the algebra of real coquaternions, which we will denote by $\mathbb{H}_{\rm coq}$. We will identify the space $\mathbb{R}^4$ with $\mathbb{H}_{\rm coq}$ by associating the element $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ with the coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. Given $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \in \mathbb{H}_{\rm coq}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} = q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k}$; the number q_0 is called the *real part* of q and is denoted by $\operatorname{re}\mathsf{q}$ and the *vector part* of q, denoted by $\operatorname{vec}\mathsf{q}$, is $\operatorname{vec}\mathsf{q} = q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. We will identify the set of coquaternions whose vector part is zero with the set $\mathbb{R}$ of real numbers. We call *determinant* of q and denote by $\det\mathsf{q}$ the quantity given by $\det\mathsf{q} = \mathsf{q}\,\overline{\mathsf{q}} = q_0^2 + q_1^2 - q_2^2 - q_3^2$. Not all non-zero coquaternions are invertible. It can be shown that a coquaternion q is invertible (also referred to as nonsingular) if and only if $\det\mathsf{q} \neq 0$. In that case, we have, $\mathsf{q}^{-1} = \frac{\overline{\mathsf{q}}}{\det\mathsf{q}}$.

We identify three particularly important subspaces of dimension two of $\mathbb{H}_{\rm coq}$, usually called the *canonical planes* or *cycle planes*. The first is simply the complex plane $\mathbb{C}$; the second, which we denote by $\mathbb{P}$ and whose elements are usually called *perplex numbers* is given by $\mathbb{P} = \operatorname{span}_{\mathbb{R}}(1, \mathsf{j})$ and corresponds to the classical *Minkowski plane*; the third, denoted by $\mathbb{D}$, is the subspace of the so-called *dual numbers*, $\mathbb{D} = \operatorname{span}_{\mathbb{R}}(1, \mathsf{i} + \mathsf{j})$ and can be identified with the classical *Laguerre plane*.

A concept which will play an important role in this paper is the concept of *quasi-similarity* of coquaternions. We say that two elements $\mathsf{p}, \mathsf{q} \in \mathbb{H}_{\rm coq}$ are *quasi-similar* if and only if they satisfy $\operatorname{re}\mathsf{p} = \operatorname{re}\mathsf{q}$ and $\det\mathsf{p} = \det\mathsf{q}$ (or, equivalently, if $\operatorname{re}\mathsf{p} = \operatorname{re}\mathsf{q}$ and $\det(\operatorname{vec}\mathsf{p}) = \det(\operatorname{vec}\mathsf{q})$). This is easily seen to be an equivalence relation in $\mathbb{H}_{\rm coq}$; the class of an element $\mathsf{q} \in \mathbb{H}_{\rm coq}$ with respect to this relation is denoted by $[\![\mathsf{q}]\!]$ and referred to as the *quasi-similarity class* of q. Observe that the quasi-similarity class of a coquaternion q is given by

$$[\![\mathsf{q}]\!] = \left\{x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = \det(\operatorname{vec}\mathsf{q})\right\}$$

which we can identify with a hyperboloid in the hyperplane $x_0 = q_0$: a hyperboloid of two sheets if $\det(\operatorname{vec}\mathsf{q}) > 0$, a hyperboloid of one sheet if $\det(\operatorname{vec}\mathsf{q}) < 0$ and a degenerate hyperboloid, i.e. a cone, if $\det(\operatorname{vec}\mathsf{q}) = 0$.

2.2 Unilateral coquaternionic polynomials

We now summarize some results on the zeros of coquaternionic polynomials [6]. We deal only with monic unilateral left polynomials, i.e. polynomials of the form

$$P(x) = x^n + \mathsf{a}_{n-1}x^{n-1} + \cdots + \mathsf{a}_1 x + \mathsf{a}_0,\ \ \mathsf{a}_i \in \mathbb{H}_{\rm coq}, \tag{1}$$

with addition and multiplication of such polynomials defined as in the commutative case,where the variable is allowed to commute with the coefficients.

Given a quasi-similarity class $[\![\mathsf{q}]\!] = [\![q_0 + \operatorname{vec}\mathsf{q}]\!]$, the *characteristic polynomial* of $[\![\mathsf{q}]\!]$, denoted by $\Psi_{[\![\mathsf{q}]\!]}$, is the polynomial given by

$$\Psi_{[\![\mathsf{q}]\!]}(x) = x^2 - 2q_0\,x + \det\mathsf{q}.$$

This is a second degree monic polynomial with real coefficients with discriminant $\Delta = -4\det(\operatorname{vec}\mathsf{q})$. Hence, $\Psi_{[\![\mathsf{q}]\!]}$ has two complex conjugate roots, if $\det(\operatorname{vec}\mathsf{q}) > 0$, and is a polynomial of the form $(x - r_1)(x - r_2)$, with $r_1, r_2 \in \mathbb{R}$, if $\det(\operatorname{vec}\mathsf{q}) \le 0$. Reciprocally, any second degree monic polynomial $S(x)$ with real coefficients is the characteristic polynomial of a uniquely defined quasi-similarity class; if $S(x)$ is irreducible with two complex conjugate roots α and $\overline{\alpha}$, then $S = \Psi_{[\![\alpha]\!]}$; if S has real roots r_1 and r_2 (with, eventually, $r_1 = r_2$), then $S = \Psi_{[\![\mathsf{q}]\!]}$ with $\mathsf{q} = \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2}\mathsf{j}$.

We say that $\mathsf{z} \in \mathbb{H}_{\rm coq}$ is a *zero* of the polynomial P if $P(\mathsf{z}) = 0$ and we denote by $Z(P)$ the *zero set* of P, i.e. the set of all the zeros of P.

Given a polynomial P of the form (1), its *conjugate polynomial* is the polynomial defined by $\overline{P}(x) = x^n + \bar{\mathsf{a}}_{n-1}x^{n-1} + \cdots + \bar{\mathsf{a}}_1 x + \bar{\mathsf{a}}_0$ and its *companion polynomial* is the polynomial given by $\mathcal{C}_P(x) = P(x)\overline{P}(x)$.

The following theorem contains an important result relating the characteristic polynomials of the quasi-similarity classes of zeros of a given polynomial P and the companion polynomial of P [6].

Theorem 1. *Let P be a polynomial of the form (1). If $\mathsf{z} \in \mathbb{H}_{\text{coq}}$ is a zero of P, then $\Psi_{[\![\mathsf{z}]\!]}$ is a divisor of $\mathcal{C}_P$.*

It can be shown easily that $\mathcal{C}_P$ is a polynomial of degree $2n$ with real coefficients and, as such, considered as a polynomial in $\mathbb{C}$, has $2n$ roots. If these roots are $\alpha_1, \overline{\alpha}_1, \ldots, \alpha_m, \overline{\alpha}_m \in \mathbb{C} \setminus \mathbb{R}$ and $r_1, r_2, \ldots, r_\ell \in \mathbb{R}$, where $\ell = 2(n-m)$, $(0 \leq m \leq n)$, then it is easy to conclude that the characteristic polynomials which divide $\mathcal{C}_P$ are the ones associated with the following quasi-similarity classes:

$$[\![\alpha_k]\!];\ k = 1, \ldots, m, \tag{2a}$$

$$[\![\mathsf{r}_{ij}]\!];\ i = 1, \ldots, \ell-1,\ j = i+1, \ldots, \ell, \tag{2b}$$

with

$$\mathsf{r}_{ij} = \frac{r_i + r_j}{2} + \frac{r_i - r_j}{2}\mathsf{j}. \tag{2c}$$

We thus have the following result concerning the zero set of P.

Theorem 2. *Let P be a polynomial of the form (1). Then:*

$$Z(P) \subseteq \bigcup_k [\![\alpha_k]\!] \bigcup_{i,j} [\![\mathsf{r}_{ij}]\!],$$

where $[\![\alpha_k]\!]$ and $[\![\mathsf{r}_{ij}]\!]$ are the quasi-similarity classes defined by (2).

We call the classes given by (2) the *admissible classes* of the polynomial P.

The results given in the following theorem show how to find the set of zeros of P belonging to a given admissible class [6].

Theorem 3. *Let $P(x)$ be a polynomial of the form (1) and let $[\![\mathsf{q}]\!] = [\![q_0 + \operatorname{vec}\mathsf{q}]\!]$ be a given admissible class of $P(x)$. Also, let $\mathsf{A} + \mathsf{B}x$, with $\mathsf{B} = B_0 + B_1\mathsf{i} + B_2\mathsf{j} + B_3\mathsf{k}$, be the remainder of the division of $P(x)$ by the characteristic polynomial of $[\![\mathsf{q}]\!]$.*

1. *If $\det \mathsf{B} \neq 0$, then $[\![\mathsf{q}]\!]$ contains only one zero of P, given by $\mathsf{z} = -\mathsf{B}^{-1}\mathsf{A}$.*
2. *If $\mathsf{A} = \mathsf{B} = 0$, then $[\![\mathsf{q}]\!] \subseteq Z(P)$.*
3. *If $\mathsf{B} \neq 0, \det \mathsf{B} = 0$, $\det(\operatorname{vec}\mathsf{q}) \leq 0$ and $\mathsf{A} = -\gamma_0 \mathsf{B}$ with*

$$\gamma_0 = q_0 \pm \sqrt{-\det(\operatorname{vec}\mathsf{q})} \tag{3}$$

   then the zeros of P in $[\![\mathsf{q}]\!]$ form the following line in the hyperplane $x_0 = q_0$,

$$\mathcal{L} = \left\{ q_0 + \alpha\mathsf{i} + (k_2\alpha + k_1(q_0 - \gamma_0))\,\mathsf{j} + (k_2(q_0 - \gamma_0) - k_1\alpha +)\,\mathsf{k} : \alpha \in \mathbb{R} \right\}, \tag{4a}$$

   with k_1 and k_2 given by

$$k_1 = -\frac{B_0B_2 + B_1B_3}{B_0^2 + B_1^2} \quad \text{and} \quad k_2 = \frac{B_1B_2 - B_0B_3}{B_0^2 + B_1^2}. \tag{4b}$$

4. *If $\mathsf{B} \neq 0, \det \mathsf{B} = 0$ and $\mathsf{A} = -\mathsf{B}(\gamma_0 + \gamma_1\mathsf{i})$ $(\gamma_1 \neq 0)$, then the class $[\![\mathsf{q}]\!]$ contains only one zero of P, given by*

$$\mathsf{z} = q_0 + (\beta + \gamma_1)\mathsf{i} + (k_2\beta + k_1(q_0 - \gamma_0))\,\mathsf{j} + (-k_1\beta + k_2(q_0 - \gamma_0))\,\mathsf{k},$$

   where

$$\beta = \frac{\det(\operatorname{vec}\mathsf{q}) + (q_0 - \gamma_0)^2 - \gamma_1^2}{2\gamma_1}$$

   and k_1 and k_2 are given by (4b).

5. *If none of the above conditions holds, then there are no zeros of P in $[\![\mathsf{q}]\!]$.*

In cases (1) and (4), we say that the zero z is an *isolated zero* of P; in case (2), we say that the class $[\![\mathsf{q}]\!]$ (or any of its elements) is a *hyperboloidal zero* of P and in case (3) we call the line $\mathcal{L}$ (or any of its elements) a *linear zero* of P.

3 Fixed Points for Cubic Coquaternionic Maps

We consider the following two-parameter family of cubic coquaternionic maps

$$f_{\mathsf{a},\mathsf{b}}(x) = x^3 - (\mathsf{a}+\mathsf{b})x^2 + \mathsf{ab}x + x, \quad x \in \mathbb{H}_{\rm coq} \tag{5}$$

with $\mathsf{a},\mathsf{b} \in \mathbb{H}_{\rm coq}$ and seek to determine its fixed points, i.e. the points $x \in \mathbb{H}_{\rm coq}$ such that $f_{\mathsf{a},\mathsf{b}}(x) = x$.

The reason for choosing this rather peculiar family of maps has to do with the fact that its fixed points turn out to be the zeros of the simple factorized polynomial

$$P(x) = f_{\mathsf{a},\mathsf{b}}(x) - x = (x-\mathsf{a})x(x-\mathsf{b}). \tag{6}$$

Since $x(x-\mathsf{b})$ is a right factor of $P(x)$, its zeros are always zeros of $P(x)$; this means that the fixed points of the quadratic $g_{\mathsf{b}}(x) = x^2 - (\mathsf{b}+1)x$ – which is a kind of family studied in detail by the authors in [7] – are always fixed points of $f_{\mathsf{a},\mathsf{b}}$. This will enable us to highlight the richness of the dynamics of cubic coquaternionic maps when compared to the dynamics of quadratic coquaternionic maps.

This paper deals only with the case where the parameter b is a non-real perplex number, i.e. is of the form

$$\mathsf{b} = b_0 + b_2\mathsf{j}, \; b_2 \neq 0. \tag{7}$$

Other type of values for the parameter b will be object of future studies. We note that this choice of b is the one for which the quadratic $g_{\mathsf{b}}(x)$ shows the most interesting dynamics; see [7] for details.

To determine the fixed points of the cubic map (5), or, in other words, to find the zeros of the polynomial P given by (6), we proceed as follows: we first compute the roots of the companion polynomial $\mathcal{C}_P$ of P and identify all the admissible classes; then, to discuss the zeros in each class $[\![\mathsf{q}]\!] = [\![q_0 + \operatorname{vec}\mathsf{q}]\!]$ we determine the remainder of the division of P by the polynomial $x^2 - 2q_0x + \det\mathsf{q}$ and make use of Theorem 3.

This remainder is easily seen to be the polynomial $\mathsf{B}x + \mathsf{A}$, where

$$\begin{aligned} \mathsf{A} &= K_2(K_1 - \mathsf{a} - \mathsf{b}) \\ \mathsf{B} &= K_1(K_1 - \mathsf{a} - \mathsf{b}) + \mathsf{ab} - K_2 \end{aligned} \tag{8}$$

with $K_1 = 2q_0$ and $K_2 = \det\mathsf{q}$.

In what follows, for the sake of simplicity, we refer to a quadratic or cubic with i isolated fixed points, ℓ lines of fixed points and h hyperboloids of fixed points as a (i,ℓ,h) map.

3.1 Case $\mathsf{a} \in \mathbb{P}$

We start by discussing with detail the case where the parameter a is a (non-real) perplex number

$$\mathsf{a} = a_0 + a_2\mathsf{j}, \; a_2 \neq 0. \tag{9}$$

In this case, the zeros of the companion polynomial $\mathcal{C}_P$ of P are

$$r_1 = r_2 = 0, \; r_3 = a_0 - a_2, \; r_4 = a_0 + a_2, \; r_5 = b_0 - b_2, \; r_6 = b_0 + b_2,$$

and so there are, at most, 11 admissible classes

$$\mathscr{C}_1 = [\![0]\!], \quad \mathscr{C}_2 = [\![\tfrac{a_0-a_2}{2} + \tfrac{a_0-a_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_3 = [\![\tfrac{a_0+a_2}{2} + \tfrac{a_0+a_2}{2}\mathsf{j}]\!]$$

$$\mathscr{C}_4 = [\![\tfrac{b_0-b_2}{2} + \tfrac{b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_5 = [\![\tfrac{b_0+b_2}{2} + \tfrac{b_0+b_2}{2}\mathsf{j}]\!]$$

$$\mathscr{C}_6 = [\![a_0 + a_2\mathsf{j}]\!] = [\![a]\!], \quad \mathscr{C}_7 = [\![\tfrac{a_0-a_2+b_0-b_2}{2} + \tfrac{a_0-a_2-b_0+b_2}{2}\mathsf{j}]\!],$$

$$\mathscr{C}_8 = [\![\tfrac{a_0-a_2+b_0+b_2}{2} + \tfrac{a_0-a_2-b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_9 = [\![\tfrac{a_0+a_2+b_0-b_2}{2} + \tfrac{a_0+a_2-b_0+b_2}{2}\mathsf{j}]\!],$$

$$\mathscr{C}_{10} = [\![\tfrac{a_0+a_2+b_0+b_2}{2} + \tfrac{a_0+a_2-b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_{11} = [\![b_0 + b_2\mathsf{j}]\!] = [\![b]\!]$$

It is important to remark that, for a of the form (9) and b of the form (7), one has $\mathsf{ab} = \mathsf{ba}$ and so the polynomials $x(x-\mathsf{a})(x-\mathsf{b})$ and $x(x-\mathsf{b})(x-\mathsf{a})$ coincide. This means, in particular, that the results concerning the classes $\mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_9$ and $\mathscr{C}_{11}$ may be obtained easily from the ones for the classes $\mathscr{C}_2, \mathscr{C}_3$, $\mathscr{C}_8$ and $\mathscr{C}_6$, respectively, by simply swapping a with b; this also implies that, in addition to 0 and b, we always have a as a zero of P.

We first consider the case where the classes are all distinct i.e. we assume that none of the following conditions holds:

$$\begin{aligned}
&\mathsf{C}_{1,1}: a_2 = a_0, \quad \mathsf{C}_{1,2}: a_2 = -a_0, \quad \mathsf{C}_{1,3}: b_2 = b_0, \quad \mathsf{C}_{1,4}: b_2 = -b_0,\\
&\mathsf{C}_{1,5}: b_2 = a_0 + a_2 - b_0, \quad \mathsf{C}_{1,6}: b_2 = -a_0 - a_2 + b_0,\\
&\mathsf{C}_{1,7}: b_2 = a_0 - a_2 - b_0, \quad \mathsf{C}_{1,8}: b_2 = -a_0 + a_2.
\end{aligned} \tag{10}$$

Zeros in class $\mathscr{C}_1$
This corresponds to $K_1 = K_2 = 0$, and so we have

$$\begin{aligned}
&\mathsf{A} = 0\\
&\mathsf{B} = a_0 b_0 + a_2 b_2 + (a_2 b_0 + a_0 b_2)\mathsf{j}\\
&\det \mathsf{B} = (a_0^2 - a_2^2)(b_0^2 - b_2^2).
\end{aligned} \tag{11}$$

Conditions (10) guarantee that $\det \mathsf{B} \neq 0$ and hence the class contains only the zero $\mathsf{z} = 0$.

Zeros in class $\mathscr{C}_2$
This corresponds to $K_1 = a_0 - a_2$ and $K_2 = 0$, leading to

$$\begin{aligned}
&\mathsf{A} = 0\\
&\mathsf{B} = a_2(a_2 - a_0 + b_2 + b_0) + a_2(a_2 - a_0 + b_2 + b_0)\mathsf{j}\\
&\det \mathsf{B} = 0
\end{aligned} \tag{12}$$

We thus have $\mathsf{B} \neq 0$ and, since we are dealing with a class $[\![\mathsf{q}]\!]$ with q of the form $\mathsf{q} = q_0 + q_0\mathsf{j}$, we have $\det(\operatorname{vec}\mathsf{q}) = -q_0^2 < 0$ and the condition $\gamma_0 \mathsf{B} = -\mathsf{A}$ is trivially satisfied for $\gamma_0 = q_0 - \sqrt{-\det(\operatorname{vec}\mathsf{q})} = 0$; hence we are in case (3) of Theorem 3; moreover, we have, with the notations of that theorem, $B_0 = B_2$ and $B_1 = B_3 = 0$ and so we get $k_1 = 1$ and $k_2 = 0$, leading us to conclude that the class contains the following line of zeros

$$\mathcal{L} = \left\{ \tfrac{a_0 - a_2}{2} + \alpha\mathsf{i} - \tfrac{a_0 - a_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \tag{13}$$

Zeros in class $\mathscr{C}_3$
The study of the zeros in this class is very similar to the study of the previous class. In this case, we obtain

$$\begin{aligned}
&\mathsf{A} = 0\\
&\mathsf{B} = a_2(a_0 + a_2 - b_0 + b_2) - (a_2(a_0 + a_2 - b_0 + b_2))\mathsf{j}\\
&\det \mathsf{B} = 0
\end{aligned} \tag{14}$$

and it will follow that the zeros in the class will form the following line

$$\mathcal{L} = \left\{ \tfrac{a_0 + a_2}{2} + \alpha\mathsf{i} + \tfrac{a_0 + a_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \tag{15}$$

Zeros in class $\mathscr{C}_4$
The expression for the zeros in this class is obtained by simply replacing a_0 and a_2 with b_0 and b_2, respectively, in the expression of the zeros of class $\mathscr{C}_2$; we thus conclude that the zeros form the line

$$\mathcal{L} = \left\{ \tfrac{b_0 - b_2}{2} + \alpha\mathsf{i} - \tfrac{b_0 - b_2}{2} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}.$$

Zeros in class $\mathscr{C}_5$
The zeros in this class form the line

$$\mathcal{L} = \left\{ \tfrac{b_0+b_2}{2} + \alpha\mathsf{i} - \tfrac{b_0+b_2}{2} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}.$$

Zeros in class $\mathscr{C}_6$
This corresponds to $K_1 = 2a_0$ and $K_2 = a_0^2 - a_2^2$, from which we obtain

$$\begin{aligned} \mathsf{A} &= (a_0^2 - a_2^2)\big(a_0 - b_0 - (a_2 + b_2)\mathsf{j}\big) \\ \mathsf{B} &= a_0(a_0 - b_0) + a_2(a_2 + b_2) + (a_2 b_0 - a_0(2a_2 + b_2))\,\mathsf{j} \\ \det \mathsf{B} &= (a_0^2 - a_2^2)(a_0 - a_2 - b_0 - b_2)(a_0 + a_2 - b_0 + b_2). \end{aligned} \tag{16}$$

Conditions (10) guarantee that $\det \mathsf{B} \neq 0$, leading us to conclude that this class contains only the isolated zero $\mathsf{z} = \mathsf{a}$.

Zeros in class $\mathscr{C}_7$
Here, we have $K_1 = a_0 - a_2 + b_0 - b_2$ and $K_2 = (a_0 - a_2)(b_0 - b_2)$ and we get

$$\begin{aligned} \mathsf{A} &= (a_0 - a_2)(b_0 - b_2)(a_2 + b_2) + (a_0 - a_2)(b_0 - b_2)(a_2 + b_2)\mathsf{j} \\ \mathsf{B} &= a_2(a_2 - a_0) + b_2(2a_2 - b_0 + b_2) + (a_2(a_2 - a_0) + b_2(2a_2 - b_0 + b_2))\,\mathsf{j} \\ \det \mathsf{B} &= 0 \end{aligned} \tag{17}$$

We now show that, when none of the condition (10) holds, this class contains no zeros of P. First, we note that we cannot have $A = B = 0$, since

$$a_2 + b_2 = 0 \wedge 2a_2 - b_0 + b_2 = 0 \Rightarrow b_2 = b_0.$$

Hence we are not in case 1 of Theorem 3. Since $A = A_0 + A_0\mathsf{j}$ and $B = B_0 + B\mathsf{j}$, we cannot be in case (4); it remains to verify that we cannot be in case (3). For $\gamma_0 = q_0 + \sqrt{-\det(\operatorname{vec}\mathsf{q})} = a_0 - a_2$, we have

$$\begin{aligned} \mathsf{B}\gamma_0 = -\mathsf{A} &\Rightarrow a_2(a_2 - a_0) + b_2(2a_2 - b_0 + b_2) = -(b_0 - b_2)(a_2 + b_2) \\ &\Rightarrow a_2(a_2 - a_0 + b_0 + b_2) = 0 \end{aligned}$$

which contradicts the hypothesis that $\mathsf{C}_{1,8}$ does not hold. The case $\gamma_0 = q_0 - \sqrt{-\det(\operatorname{vec}\mathsf{q})} = b_0 - b_2$ is analogous.

Zeros in class $\mathscr{C}_8$
In this case, $K_1 = a_0 - a_2 + b_0 + b_2$ and $K_2 = (a_0 - a_2)(b_0 + b_2)$ and we get

$$\begin{aligned} \mathsf{A} &= (a_0 - a_2)(a_2 - b_2)(b_0 + b_2) + (a_0 - a_2)(a_2 + b_2)(b_0 + b_2)\mathsf{j} \\ \mathsf{B} &= a_2(a_2 - a_0) + b_2(b_0 + b_2) + (a_2(a_2 - a_0) - b_2(b_0 + b_2))\mathsf{j} \\ \det \mathsf{B} &= -4a_2 b_2(a_0 - a_2)(b_0 + b_2). \end{aligned} \tag{18}$$

Since none of the conditions (10) is satisfied, we have $\det \mathsf{B} \neq 0$, and we may conclude that there is only one zero in this class, given by

$$\mathsf{z} = -\tfrac{\overline{\mathsf{B}}}{\det \mathsf{B}}\mathsf{A} = \tfrac{a_0 - a_2 + b_0 + b_2}{2} + \tfrac{a_2 - a_0 + b_0 + b_2}{2}\mathsf{j}.$$

Zeros in class $\mathscr{C}_9$
This class contains only the zero

$$\mathsf{z} = \tfrac{a_0 + a_2 + b_0 - b_2}{2} + \tfrac{a_0 + a_2 - b_0 + b_2}{2}\mathsf{j}.$$

Zeros in class $\mathscr{C}_{10}$
The study of this case is trivial adaptation of the study conducted for the class $\mathscr{C}_7$; in this case, we obtain

$$\begin{aligned} \mathsf{A} &= -(a_0 + a_2)(b_0 + b_2)(a_2 + b_2) + (a_0 + a_2)(b_0 + b_2)(a_2 + b_2)\mathsf{j} \\ \mathsf{B} &= a_2(a_2 + a_0) + b_2(2a_2 + b_0 + b_2) - (a_2(a_2 + a_0) + b_2(2a_2 + b_0 + b_2))\,\mathsf{j} \\ \det \mathsf{B} &= 0 \end{aligned} \tag{19}$$

and we can conclude that there are no zeros of P in this class.
Zeros in class $\mathscr{C}_{11}$
This class contains only the zero $\mathsf{z} = \mathsf{b}$.

From the previous discussion, it is clear that, if none of conditions $\mathsf{C}_{1,1} - \mathsf{C}_{1,8}$ given in (10) holds, then the cubic (5) has the following five isolated fixed points

$$\begin{array}{ll} \mathsf{z}_1 = 0, \quad \mathsf{z}_2 = \mathsf{a}, \quad \mathsf{z}_3 = \mathsf{b} & \\ \mathsf{z}_4 = \frac{a_0 - a_2 + b_0 + b_2}{2} + \frac{a_2 - a_0 + b_0 + b_2}{2}\mathsf{j} & \mathsf{z}_5 = \frac{a_0 + a_2 + b_0 - b_2}{2} + \frac{a_0 + a_2 + b_2 - b_0}{2}\mathsf{j} \end{array} \tag{20}$$

and the following four lines of fixed points

$$\begin{array}{l} \mathcal{L}_1 = \left\{ \frac{a_0 - a_2}{2} + \alpha\mathsf{i} - \frac{a_0 - a_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \\ \mathcal{L}_2 = \left\{ \frac{a_0 + a_2}{2} + \alpha\mathsf{i} + \frac{a_0 + a_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \\ \mathcal{L}_3 = \left\{ \frac{b_0 - b_2}{2} + \alpha\mathsf{i} - \frac{b_0 - b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \\ \mathcal{L}_4 = \left\{ \frac{b_0 + b_2}{2} + \alpha\mathsf{i} + \frac{b_0 + b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\} \end{array} \tag{21}$$

i.e. is (5,4,0) map.

Naturally, when some of the conditions (10) hold, the situation is different. Consider, for example, the case where condition $\mathsf{C}_{1,1} : a_0 = a_2$ is satisfied, but none of the other conditions $\mathsf{C}_{1,2} - \mathsf{C}_{1,8}$ holds. In this case, we have $\mathscr{C}_2 \equiv \mathscr{C}_1$, $\mathscr{C}_6 \equiv \mathscr{C}_3$, $\mathscr{C}_7 \equiv \mathscr{C}_4$ and $\mathscr{C}_8 \equiv \mathscr{C}_5$, i.e., there are only 7 admissible classes. Let us see what modifications occur in the zeros contained in these classes due to the fact that $a_2 = a_0$.

Zeros in class $\mathscr{C}_1$ when $a_2 = a_0$
In this case, we have $\mathsf{A} = 0$, $\mathsf{B} = a_0(b_0 + b_2) + a_0(b_0 + b_2)\mathsf{j}$ and $\det \mathsf{B} = 0$; see (11). Since $\mathsf{B} \neq 0$ and $\exists \gamma_0 = 0$ such that $\mathsf{B}\gamma_0 = -\mathsf{A}$, we are in case 3.2 (i) of Therorem 3 and we conclude that the zeros of P in this class form the line

$$\mathcal{L} = \{\alpha\mathsf{i} + \alpha\mathsf{k} : k \in \mathbb{R}\}.$$

Zeros in class $\mathscr{C}_3$ when $a_2 = a_0$
It is simple to verify that the zeros in this class will still form the line (15) which, for $a_0 = a_2$, takes the simpler form

$$\mathcal{L} = \{a_0 + \alpha\mathsf{i} + a_0\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\}.$$

Zeros in class $\mathscr{C}_7$ when $a_2 = a_0$
In this case, we have $\mathsf{A} = 0$, $\mathsf{B} = b_2(2a_2 - b_0 + b_2) + b_2(2a_2 - b_0 + b_2)\mathsf{j}$ and $\det \mathsf{B} = 0$; see (17). Since condition $\mathsf{C}_{1,5}$ does not hold, we conclude that $\mathsf{B} \neq 0$ and hence it is simple to see that the zeros in this class form the line

$$\mathcal{L} = \left\{ \frac{b_0 - b_2}{2} + \alpha\mathsf{i} - \frac{b_0 - b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}.$$

Zeros in class $\mathscr{C}_8$ when $a_2 = a_0$
In this case, $\mathsf{A} = 0$, $\mathsf{B} = b_2(b_0 + b_2) - b_2(b_0 + b_2)\mathsf{j}$ and $\det \mathsf{B} = 0$; see (18). Since condition $\mathsf{C}_{1,4}$ does not hold, we have $\mathsf{B} \neq 0$ and we may conclude that the zeros in this class form the following line

$$\mathcal{L} = \left\{ \frac{b_0 + b_2}{2} + \alpha\mathsf{i} + \frac{b_0 + b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}.$$

Zeros in class $\mathscr{C}_{10}$ when $a_2 = a_0$
The situation concerning the zeros in class $\mathscr{C}_{10}$ previously described does not change when $a_2 = a_0$, i.e. the class has no zeros.

In summary, we conclude that when $a_2 = a_0$ is the unique of conditions (10) holding, the cubic is a $(2, 4, 0)$ map.

A similar study was conducted for all the different situations that occur due to the fulfillment of one or more of the conditions (10) and the corresponding results are summarized in Table 1.

A little explanation of how the table must be read is due. When, for example, we write $\mathsf{a} = a_0 + a_0\mathsf{j}$ and $\mathsf{b} = b_0 + b_2\mathsf{j}$, this must be interpreted as meaning that a satisfies condition $\mathsf{C}_{1,1}$, but b_2 does not have any of the special forms given by conditions $\mathsf{C}_{1,3} - \mathsf{C}_{1,8}$. For completeness, the first line of the table gives the situation when none of the conditions (10) is verified. Finally, we point out that we do not include in the table the cases which can be obtained by simply replacing the roles of a and b.

Table 1: Case $\mathsf{a} = a_0 + a_2\mathsf{j}$ and $\mathsf{b} = b_0 + b_2\mathsf{j}$

a	b	type of map
$a_0 + a_2\mathsf{j}$	$b_0 + b_2\mathsf{j}$	$(5,4,0)$
$a_0 + a_2\mathsf{j}$	$b_0 + (a_0 + a_2 - b_0)\mathsf{j}$	$(3,3,0)$
$a_0 + a_2\mathsf{j}$	$b_0 - (a_0 + a_2 - b_0)\mathsf{j}$	$(3,4,1)$
$a_0 + a_2\mathsf{j}$	$b_0 + (-a_0 + a_2 + b_0)\mathsf{j}$	$(3,3,0)$
$a_0 + a_2\mathsf{j}$	$a_0 + a_2\mathsf{j}$	$(2,2,0)$
$a_0 + a_2\mathsf{j}$	$a_0 - a_2\mathsf{j}$	$(3,0,3)$
$a_0 \pm a_0\mathsf{j}$	$b_0 + b_2\mathsf{j}$	$(2,4,0)$
$a_0 + a_0\mathsf{j}$	$b_0 + b_0\mathsf{j}$	$(0,3,0)$
$a_0 + a_0\mathsf{j}$	$b_0 - b_0\mathsf{j}$	$(1,2,1)$
$a_0 + a_0\mathsf{j}$	$a_0 + a_0\mathsf{j}$	$(0,2,0)$
$a_0 + a_0\mathsf{j}$	$a_0 - a_0\mathsf{j}$	$(1,0,2)$
$a_0 + a_0\mathsf{j}$	$b_0 + (2a_0 - b_0)\mathsf{j}$	$(1,3,0)$
$a_0 + a_0\mathsf{j}$	$b_0 - (2a_0 - b_0)\mathsf{j}$	$(1,3,1)$
$a_0 - a_0\mathsf{j}$	$b_0 - b_0\mathsf{j}$	$(0,3,0)$
$a_0 - a_0\mathsf{j}$	$b_0 \pm (2a_0 - b_0)\mathsf{j}$	$(1,3,0)$
$a_0 - a_0\mathsf{j}$	$a_0 - a_0\mathsf{j}$	$(0,2,0)$

3.2 Case $\mathsf{a} \in \mathbb{R}$

When $\mathsf{a} = a_0 \in \mathbb{R}, a_0 \neq 0$, the zeros of the companion polynomial $\mathcal{C}_P$ of P are

$$r_1 = r_2 = 0,\ r_3 = r_4 = a_0,\ r_5 = b_0 - b_2,\ r_6 = b_0 + b_2$$

and so there are, at most, 8 admissible classes

$$\mathscr{C}_1 = [\![0]\!], \quad \mathscr{C}_2 = [\![\tfrac{a_0}{2} + \tfrac{a_0}{2}\mathsf{j}]\!],$$
$$\mathscr{C}_3 = [\![\tfrac{b_0-b_2}{2} + \tfrac{b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_4 = [\![\tfrac{b_0+b_2}{2} + \tfrac{b_0+b_2}{2}\mathsf{j}]\!],$$
$$\mathscr{C}_5 = [\![a_0]\!] = [\![a]\!], \quad \mathscr{C}_6 = [\![\tfrac{a_0+b_0-b_2}{2} + \tfrac{a_0-b_0+b_2}{2}\mathsf{j}]\!],$$
$$\mathscr{C}_7 = [\![\tfrac{a_0+b_0+b_2}{2} + \tfrac{a_0-b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_8 = [\![b_0 + b_2\mathsf{j}]\!] = [\![\mathsf{b}]\!]$$

Corresponding to conditions (10), in this case we have to consider the following conditions

$$\mathsf{C}_{2,1} : b_2 = b_0, \quad \mathsf{C}_{2,2} : b_2 = -b_0, \quad \mathsf{C}_{2,3} : b_2 = b_0 - a_0, \quad \mathsf{C}_{2,4} : b_2 = a_0 - b_0 \tag{22}$$

When none of these conditions holds, the cubic has the isolated fixed points $\mathsf{z}_1 = 0$, $\mathsf{z}_2 = \mathsf{a}$ and $\mathsf{z}_3 = \mathsf{b}$, four lines of fixed points

$$\mathcal{L}_1 = \left\{ \tfrac{b_0-b_2}{2} + \alpha\mathsf{i} - \tfrac{b_0-b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}$$

$$\mathcal{L}_2 = \left\{ \tfrac{b_0+b_2}{2} + \alpha\mathsf{i} + \tfrac{b_0+b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}$$

$$\mathcal{L}_3 = \left\{ \tfrac{b_0-b_2+a_0}{2} + \alpha\mathsf{i} - \tfrac{b_0-b_2-a_0}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}$$

$$\mathcal{L}_4 = \left\{ \tfrac{b_0+b_2+a_0}{2} + \alpha\mathsf{i} + \tfrac{b_0+b_2-a_0}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\}$$

Table 2: Case $\mathsf{a} = a_0$ and $\mathsf{b} = b_0 + b_2\mathsf{j}$

b	type of map
$b_0 + b_2\mathsf{j}$	$(3, 4, 1)$
$b_0 \pm b_0\mathsf{j}$	$(1, 3, 1)$
$b_0 \pm (a_0 - b_0)\mathsf{j}$	$(1, 3, 1)$
$\frac{a_0}{2} \pm \frac{a_0}{2}\mathsf{j}$	$(0, 2, 1)$

and also the hyperboloid of fixed points $\mathcal{H} = [\![\frac{a_0}{2} + \frac{a_0}{2}\mathsf{j}]\!]$, i.e. is a $(3, 4, 1)$ map.

Table 2 gives a full description of the types of maps that are obtained when one or more of the conditions (22) are satisfied.

We observe that in this case, we always have a hyperboloid of fixed points coexisting with lines of fixed points, a situation that never occurs with the quadratic map.

3.3 Case $\mathsf{a} \in \mathbb{C}$

We now discuss the case where $\mathsf{a} = a_0 + a_1\mathsf{i}$, with $a_1 \neq 0$. In this case, the zeros of the companion polynomial $\mathcal{C}_P$ of P are

$$r_1 = r_2 = 0,\ r_3 = a_0 + a_1\mathsf{i},\ r_4 = a_0 - a_1\mathsf{i},\ r_5 = b_0 - b_2,\ r_6 = b_0 + b_2,$$

and so there are, at most, 5 admissible classes

$$\begin{aligned}
&\mathscr{C}_1 = [\![0]\!], \quad \mathscr{C}_2 = [\![a_0 + a_1\mathsf{i}]\!] = [\![\mathsf{a}]\!],\\
&\mathscr{C}_3 = [\![\tfrac{b_0-b_2}{2} + \tfrac{b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_4 = [\![\tfrac{b_0+b_2}{2} + \tfrac{b_0+b_2}{2}\mathsf{j}]\!]\\
&\mathscr{C}_5 = [\![b_0 + b_2\mathsf{j}]\!] = [\![b]\!]
\end{aligned}$$

In this case we have to consider the conditions

$$\mathsf{C}_{3,1} : b_2 = b_0, \quad \mathsf{C}_{3,2} : b_2 = -b_0, \quad \mathsf{C}_{3,3} : b_2^2 = (a_0 - b_0)^2 + a_1^2 \tag{23}$$

When none of these conditions holds, the cubic has the isolated fixed points

$$\begin{aligned}
&\mathsf{z}_1 = 0, \quad \mathsf{z}_2 = \mathsf{b},\\
&\mathsf{z}_3 = a_0 + \tfrac{a_1\left((a_0-b_0)^2+a_1^2+b_2^2\right)}{(a_0-b_0)^2+a_1^2-b_2^2}\mathsf{i} + \tfrac{2a_1^2 b_2}{(a_0-b_0)^2+a_1^2-b_2^2}\mathsf{j} + \tfrac{2a_1(-a_0+b_0)b_2}{(a_0-b_0)^2+a_1^2-b_2^2}\mathsf{k}
\end{aligned} \tag{24}$$

and the two lines of fixed points

$$\begin{aligned}
\mathcal{L}_1 &= \left\{\tfrac{b_0+b_2}{2} + \alpha\mathsf{i} + \tfrac{b_0+b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\}\\
\mathcal{L}_2 &= \left\{\tfrac{b_0-b_2}{2} + \alpha\mathsf{i} - \tfrac{b_0-b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\right\}.
\end{aligned} \tag{25}$$

Table 3 gives a full description of the types of maps that are obtained when one or more of the conditions (23) are satisfied. Since conditions (23) do not impose any restriction on a, the table only contains the special forms of b that influence the type of map we obtain.

3.4 Case $\mathsf{a} \in \mathbb{D}$

Finally, we consider the case where a is a dual number $\mathsf{a} = a_0 + \mathsf{i} + \mathsf{j}$. In this case the zeros of the companion polynomial $\mathcal{C}_P$ of P are

$$r_1 = r_2 = 0,\ r_3 = r_4 = a_0,\ r_5 = b_0 - b_2,\ r_6 = b_0 + b_2,$$

Table 3: Case $\mathsf{a} = a_0 + a_1\mathsf{i}$ and $\mathsf{b} = b_0 + b_2\mathsf{j}$

b	type of map
$b_0 + b_2\mathsf{j}$	$(3,2,0)$
$b_0 \pm b_0\mathsf{j}$	$(1,2,0)$
$b_0 \pm \sqrt{a_1^2 + (a_0 - b_0)^2}\mathsf{j}$	$(2,2,0)$
$\frac{a_0^2+a_1^2}{2a_0} \pm \frac{a_0^2+a_1^2}{2a_0}\mathsf{j}$ $(a_0 \neq 0)$	$(0,2,0)$

and so there are, at most, 6 admissible classes

$$\mathscr{C}_1 = [\![0]\!], \quad \mathscr{C}_2 = [\![\tfrac{a_0}{2} + \tfrac{a_0}{2}\mathsf{j}]\!], \quad \mathscr{C}_3 = [\![a_0]\!] = [\![\mathsf{a}]\!],$$

$$\mathscr{C}_4 = [\![\tfrac{b_0-b_2}{2} + \tfrac{b_0-b_2}{2}\mathsf{j}]\!], \quad \mathscr{C}_5 = [\![\tfrac{b_0+b_2}{2} + \tfrac{b_0+b_2}{2}\mathsf{j}]\!]$$

$$\mathscr{C}_6 = [\![b_0 + b_2\mathsf{j}]\!] = [\![\mathsf{b}]\!]$$

In this case, we must consider the conditions

$$\begin{aligned}
&\mathsf{C}_{4,1} : a_0 = 0, \quad \mathsf{C}_{4,2} : b_2 = b_0, \quad \mathsf{C}_{4,3} : b_2 = -b_0,\\
&\mathsf{C}_{4,4} : b_2 = a_0 - b_0, \quad \mathsf{C}_{4,5} : b_2 = -a_0 + b_0,\\
&\mathsf{C}_{4,6} : (a_0 - b_0)^2 = b_2(2 + b_2)
\end{aligned} \tag{26}$$

When none of the above conditions holds, the fixed points of the cubic are

$$\mathsf{z}_1 = 0, \qquad \mathsf{z}_2 = \mathsf{b},$$

$$\mathsf{z}_3 = \tfrac{b_0+b_2+a_0}{2} + \tfrac{b_0-b_2-a_0}{2}\mathsf{i} + \tfrac{b_0+b_2-a_0}{2}\mathsf{j} - \tfrac{b_0-b_2-a_0}{2}\mathsf{k},$$

$$\mathsf{z}_4 = \tfrac{b_0-b_2+a_0}{2} - \tfrac{b_0+b_2-a_0}{2}\mathsf{i} - \tfrac{b_0-b_2-a_0}{2}\mathsf{j} - \tfrac{b_0+b_2-a_0}{2}\mathsf{k},$$

$$\mathsf{z}_5 = a_0 + \tfrac{(a_0-b_0)^2+b_2^2}{(a_0-b_0)^2-b_2(2+b_2)}\mathsf{i} + \tfrac{(a_0-b_0)^2-b_2^2}{(a_0-b_0)^2-b_2(2+b_2)}\mathsf{j} + \tfrac{2(-a_0+b_0)b_2}{(a_0-b_0)^2-b_2(2+b_2)}\mathsf{k}$$

and

$$\mathcal{L}_1 = \left\{ \tfrac{b_0+b_2}{2} + \alpha\mathsf{i} + \tfrac{b_0+b_2}{2}\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\},$$

$$\mathcal{L}_2 = \left\{ \tfrac{b_0-b_2}{2} + \alpha\mathsf{i} - \tfrac{b_0-b_2}{2}\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R} \right\},$$

$$\mathcal{L}_3 = \left\{ \tfrac{a_0}{2} + \alpha\mathsf{i} + \tfrac{a_0b_2(a_0-b_0)+\alpha\left((a_0-b_0)^2-b_2^2\right)}{(a_0-b_0)^2+b_2^2}\mathsf{j} + \tfrac{a_0\left((a_0-b_0)^2-b_2^2\right)+4b_2\alpha(b_0-a_0)}{2\left((a_0-b_0)^2+b_2^2\right)}\mathsf{k} : \alpha \in \mathbb{R} \right\},$$

i.e. we have a $(5,3,0)$ map.

In Table 4, one can see the types of maps occurring when one or more of the conditions (26) are satisfied.

4 Examples

We now present some examples illustrating the theoretical results obtained in the previous section. In the determination of the zeros of the polynomials considered in the examples, we made use of a set of Mathematica functions – the package `CoqPolynomial` – which were specially designed to deal with coquaternionic polynomials [8].

Table 4: Case $\mathsf{a} = a_0 + \mathsf{i} + \mathsf{j}$ and $\mathsf{b} = b_0 + b_2\mathsf{j}$

a	b	type of map
$a_0 + \mathsf{i} + \mathsf{j}$	$b_0 + b_2\mathsf{j}$	$(5, 3, 0)$
$a_0 + \mathsf{i} + \mathsf{j}$	$b_0 \pm b_0\mathsf{j}$	$(2, 3, 0)$
$a_0 + \mathsf{i} + \mathsf{j}$	$b_0 \pm (a_0 - b_0)\mathsf{j}$	$(3, 2, 0)$
$a_0 + \mathsf{i} + \mathsf{j}$	$b_0 + (-1 \pm \sqrt{1 + (a_0 - b_0)^2})\mathsf{j}$	$(4, 3, 0)$
$a_0 + \mathsf{i} + \mathsf{j}$	$\frac{a_0}{2} \pm \frac{a_0}{2}\mathsf{j}$	$(1, 2, 0)$
$a_0 + \mathsf{i} + \mathsf{j}$ $(a_0 \neq -1)$	$\frac{a_0^2}{2(1+a_0)} + \frac{a_0^2}{2(1+a_0)}\mathsf{j}$	(1,3,0)
$a_0 + \mathsf{i} + \mathsf{j}$ $(a_0 \neq 1)$	$\frac{a_0^2}{2(1-a_0)} - \frac{a_0^2}{2(1-a_0)}\mathsf{j}$	(1,3,0)
$\mathsf{i} + \mathsf{j}$	$b_0 + b_2\mathsf{j}$	$(1, 3, 0)$
$\mathsf{i} + \mathsf{j}$	$b_0 \pm b_0\mathsf{j}$	$(0, 2, 0)$
$\mathsf{i} + \mathsf{j}$	$b_0 + (-1 \pm \sqrt{1 + b_0^2})\mathsf{j}$	$(1, 3, 0)$

Example 1. For the choice of parameters $\mathsf{a} = 12 - 4\mathsf{j}$ and $\mathsf{b} = 10 + 8\mathsf{j}$, the cubic map $f_{\mathsf{a},\mathsf{b}}$ has the following five isolated fixed points:

$$\mathsf{z}_1 = 0, \quad \mathsf{z}_2 = 12 - 4\mathsf{j}, \quad \mathsf{z}_3 = 10 + 8\mathsf{j}, \quad \mathsf{z}_4 = 17 + \mathsf{j}, \quad \mathsf{z}_5 = 5 + 3\mathsf{j}$$

and the following four lines of fixed points:

$$\mathcal{L}_1 = \{8 + \alpha\mathsf{i} - 8\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \qquad \mathcal{L}_2 = \{4 + \alpha\mathsf{i} + 4\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\},$$

$$\mathcal{L}_3 = \{1 + \alpha\mathsf{i} - \mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \qquad \mathcal{L}_4 = \{9 + \alpha\mathsf{i} + 9\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\}.$$

The results of [7] show that the quadratic g_{b} has z_1, z_3, $\mathcal{L}_3$ and $\mathcal{L}_4$ as fixed points and we would like to remark how the simple introduction of a new linear factor made us move form a $(2, 2, 0)$ map to a $(5, 4, 0)$ map.

Example 2. For the choice of parameters $\mathsf{a} = 11 + \mathsf{j}$ and $\mathsf{b} = 7 - 5\mathsf{j}$, the cubic map $f_{\mathsf{a},\mathsf{b}}$ has the three isolated fixed points

$$\mathsf{z}'_1 = 0, \quad \mathsf{z}'_2 = 6 - 4\mathsf{j}, \quad \mathsf{z}'_3 = 12,$$

the four lines of fixed points

$$\mathcal{L}'_1 = \{5 + \alpha\mathsf{i} - 5\mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \qquad \mathcal{L}'_2 = \{1 + \alpha\mathsf{i} + \mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\},$$
$$\mathcal{L}'_3 = \{7 + \alpha\mathsf{i} - 5\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \qquad \mathcal{L}'_4 = \{11 + \alpha\mathsf{i} + \mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}$$

and the hyperboloid of fixed points

$$\mathcal{H} = [\![6 + 6\mathsf{j}]\!].$$

Thus we have a $(3, 4, 1)$ map, as predicted by the theory, since we are in the case where b has the special form $\mathsf{b} = b_0 + (-a_0 - a_2 + b_0)\mathsf{j}$; see Table 1.

We would like to observe that all the fixed points listed in (20) and (21) are still fixed points of the cubic, but $\mathsf{z}_2 = \mathsf{a}$ and $\mathsf{z}_3 = \mathsf{b}$ are no longer isolated: $\mathsf{z}_2 \in \mathcal{L}'_4$ and $\mathsf{z}_3 \in \mathcal{L}'_3$. But, more interesting is the fact that the lines $\mathcal{L}_2$ and $\mathcal{L}_3$ given in (21) are both contained in the hyperboloid $\mathcal{H}$.

Example 3. In this example we consider the parameters $\mathsf{a} = -4 + \mathsf{i} + \mathsf{j}$ and $\mathsf{b} = -4 - 2\mathsf{j}$. The corresponding cubic map $f_{\mathsf{a},\mathsf{b}}$ has the four isolated fixed points

$$\mathsf{z}_1 = 0, \quad \mathsf{z}_2 = -4 - 2\mathsf{j}, \quad \mathsf{z}_3 = -5 + \mathsf{i} - \mathsf{j} - \mathsf{k}, \quad \mathsf{z}_4 = -3 + \mathsf{i} - \mathsf{j} + \mathsf{k},$$

and the three lines of fixed points

$$\begin{aligned}
\mathcal{L}_1 &= \{-3 + \alpha\mathsf{i} - 3\mathsf{j} - \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \\
\mathcal{L}_2 &= \{-1 + \alpha\mathsf{i} + \mathsf{j} + \alpha\mathsf{k} : \alpha \in \mathbb{R}\}, \\
\mathcal{L}_3 &= \{-2 + \alpha\mathsf{i} - \alpha\mathsf{j} + 2\mathsf{k} : \alpha \in \mathbb{R}\}.
\end{aligned}$$

Note that we are in the case where b has the special form $\mathsf{b} = b_0 + (-1 - \sqrt{1 + (a_0 - b_0)^2})\mathsf{j}$ and that the type of map obtained is as predicted by the theory; see Table 4.

This is an interesting example in the sense that neither a nor any of the elements in its class are fixed points of the map.

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