A Modified Quaternionic Weierstrass Method

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Abstract

In this paper we focus on the study of monic polynomials whose coefficients are quaternions located on the left-hand side of the powers, by addressing three fundamental polynomial problems: factor, evaluate and deflate. An algorithm combining a deflaction procedure with a Weierstrass-like quaternionic method is presented. Several examples illustrate the proposed approach.

1 Introduction

In 1941, Niven [16] proved that any polynomial whose coefficients are quaternions located only on the left-hand side of the powers (one-sided polynomials) must have at least one quaternionic root. Since then, there has been a growing interest in studying the problem of characterizing and computing the zeros of these special polynomials, from the theoretical as well as from the applications point of view [7, 17, 18]. In particular, in the last decade several authors proposed algorithms for computing the zeros of one-sided polynomials. Most of these root-finding methods rely on the connection between the zeros of a quaternionic polynomial and the zeros of a certain real polynomial, usually with multiple zeros, and as such, they face the usual difficulties associated with the computation of multiple zeros or clusters of zeros [2, 13, 20, 21]. One of the few exceptions is the work [4], where a quaternionic version of the well-known Weierstrass iterative root-finding method [23], relying on quaternionic arithmetic is proposed.

Real or complex polynomials have received a lot of attention over the years and classical problems such as factor, evaluate and deflate are well studied. In this paper we consider quaternionic versions of these classical problems. In particular, we revisit the factor problem by recalling a recently proposed quaternionic Weierstrass method [4] and take a fresh look to the problem of evaluating quaternionic polynomials [3]. The main result of the paper is a deflation algorithm to be used together with the Weierstrass method which allows to reestablished its quadratic convergence without requiring higher precision.

2 Preliminary Results

We introduce the basic definitions and results needed in the sequel; we refer to [12, 15, 24] for recalling the main aspects of the quaternion algebra.

Let $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \ \ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

This non-commutative product generates the algebra of real quaternions $\mathbb{H}$. For a quaternion $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3, \ x_i \in \mathbb{R}$, we can define the real part of x, $\mathrm{Re}(x) := x_0$, the vector part of x, $\underline{x} := \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ and the conjugate of x, $\bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$. The norm of x is given by $|x| := \sqrt{x\bar{x}} = \sqrt{\bar{x}x}$. It immediately follows that each non-zero $x \in \mathbb{H}$ has an inverse given by $x^{-1} = \frac{\bar{x}}{|x|^2}$ and therefore $\mathbb{H}$ is a non-commutative division ring or a skew field.

Two quaternions q and q' are called *similar*, $q \sim q'$, if $\mathrm{Re}\, q = \mathrm{Re}\, q'$ and $|q| = |q'|$. Similarity is an equivalence relation in $\mathbb{H}$, partitioning $\mathbb{H}$ in the so-called *similarity class* of q, which we denote by $[q]$. The similarity class of a non-real quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ can be identified with the three-dimensional sphere in the hyperplane $\{(x_0, x, y, z) \in \mathbb{R}^4 : x_0 = q_0\}$, with center $(q_0, 0, 0, 0)$ and radius $\sqrt{q_1^2 + q_2^2 + q_3^2}$.

We consider now polynomials P in one formal variable x of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \ \ a_n \neq 0, \tag{1}$$

i.e. polynomials whose coefficients $a_k \in \mathbb{H}$ are located only on the left-hand side of the powers. The set of polynomials of the form (1), with the addition and multiplication defined as in the commutative case, is a ring, referred to as the ring of (left) one-sided polynomials and usually denoted by $\mathbb{H}[x]$.

The evaluation of P at q is defined as $P(q) = a_n q^n + a_{n-1} q^{n-1} + \cdots + a_1 q + a_0$. Moreover, a quaternion q is a zero of P, if $P(q) = 0$. A zero q is called an isolated zero of P, if $[q]$ contains no other zeros of P. A zero q is called a *spherical zero* of P, if q is not an isolated zero; $[q]$ is referred to as a sphere of zeros.

We define now two important polynomials: the conjugate of P, denoted by $\bar{P}$ and defined by conjugating the coefficients of P and the characteristic polynomial of a quaternion q, defined as

$$\Psi_q(x) := (x - q)(x - \overline{q}) = x^2 - 2\,\mathrm{Re}(q)\, x + |q|^2. \tag{2}$$

Properties concerning these two polynomials, can be seen in e.g. [6].

We end this section by recalling some results concerning the zero-structure and the factorization of polynomials in $\mathbb{H}[x]$. For the proofs and other details we refer to [1, 9, 15, 16]. Besides Niven's Fundamental Theorem of Algebra, stating that any non-constant polynomial in $\mathbb{H}[x]$ always has a zero in $\mathbb{H}$, the following results are essential for the sequel.

Theorem 1. *Consider a polynomial $P \in \mathbb{H}[x]$ of the form (1).*

1. *A quaternion q is a zero of P if and only if there exists $Q \in \mathbb{H}[x]$ such that*

$$P(x) = Q(x)(x - q).$$

2. *A non-real zero q is a spherical zero of P if and only if there exists a polynomial $Q \in \mathbb{H}[x]$ such that*

$$P(x) = Q(x)\Psi_q(x),$$

where Ψ_q is the characteristic polynomial of q given by (2).

3 Polynomial Problems Over $\mathbb{H}$

3.1 The Factor Problem

As an immediate consequence of the Fundamental Theorem of Algebra for quaternions and Theorem 1 we conclude that it is always possible, as in the classical case, to write a quaternionic polynomial P as a product of linear factors, i.e. there exist $x_1, \ldots, x_n \in \mathbb{H}$, such that

$$P(x) = (x - x_n)(x - x_{n-1}) \cdots (x - x_1). \tag{3}$$

The quaternions $x_1, \ldots, x_n$ in the factorization (3) are called factor-terms of P and the n-uple $(x_1, \ldots, x_n)$ is called a chain of P.

The link between these factor-terms and the corresponding zeros is not straightforward. In fact if $(x_1,\dots,x_n)$ is a chain of a polynomial P then any zero of P is similar to some factor-term x_k in the chain and reciprocally every factor-term x_k is similar to some zero of P [15, 21]. The next results clarify the link between zeros and factor-terms (we mostly follow [5] and references therein).

Theorem 2 (Zeros from factors)**.** *Consider a chain $(x_1,\dots,x_n)$ of a polynomial P. If the similarity classes $[x_k]$ are distinct, then P has exactly n zeros ζ_k which are related to the factor-terms x_k as follows:*

$$\zeta_k = \overline{\mathcal{P}}_k(x_k)\, x_k\, \left(\overline{\mathcal{P}}_k(x_k)\right)^{-1};\ k=1,\dots,n, \tag{4}$$

where

$$\mathcal{P}_1(x) := 1 \qquad \textit{and} \qquad \mathcal{P}_k(x) := (x-x_{k-1})\dots(x-x_1),\ k=2,\dots,n \tag{5}$$

Theorem 3 (Factors from zeros)**.** *If $\zeta_1,\dots,\zeta_n$ are quaternions such that the similarity classes $[\zeta_k]$ are distinct, then there is a unique polynomial P of degree n with zeros $\zeta_1,\dots,\zeta_n$, which can be constructed from the chain $(x_1,\dots,x_n)$, where*

$$x_k = \mathcal{P}_k(\zeta_k)\,\zeta_k\,\left(\mathcal{P}_k(\zeta_k)\right)^{-1};\ k=1,\dots,n \tag{6}$$

and $\mathcal{P}_k$ is the polynomial (5).

Following the idea of the Weierstrass method in its sequential version [23], a quaternion version was proposed in [4] where it was also shown how to obtain sequences converging, at a quadratic rate, to the factor terms $x_1,\dots,x_n$ (cf. (3)) of a given polynomial P and with some additional little effort, how these sequences can be used to estimate the zeros $\zeta_1,\dots,\zeta_n$ of P.

Theorem 4. *Let P be a polynomial of degree n in $\mathbb{H}[x]$ with simple[1] zeros and consider, for $i=1,\dots,n$, the iterative schemes*

$$z_i^{(k+1)} = z_i^{(k)} - \left(\mathcal{L}_i^{(k)} P \mathcal{R}_i^{(k)}\right)(z_i^{(k)})\,\left(\Psi_i^{(k)}(z_i^{(k)})\right)^{-1},\ k=0,1,2,\dots \tag{7}$$

where

$$\mathcal{L}_i^{(k)}(x) := \prod_{j=i+1}^{n}(x-\overline{z_j^{(k)}}), \qquad \mathcal{R}_i^{(k)}(x) := \prod_{j=1}^{i-1}(x-\overline{z_j^{(k+1)}}) \tag{8}$$

and

$$\Psi_i^{(k)}(x) := \prod_{j=1}^{i-1}\Psi_{z_j^{(k+1)}}(x)\prod_{j=i+1}^{n}\Psi_{z_j^{(k)}}(x), \tag{9}$$

with Ψ_q denoting the characteristic polynomial of q. If the initial approximations $z_i^{(0)}$ are sufficiently close to the factor terms x_i in a factorization of P in the form (3), then the sequences $\{z_i^{(k)}\}$ converge quadratically to x_i. Moreover, the sequences $\{\zeta_i^{(k)}\}$ defined by

$$\zeta_i^{(k+1)} := \overline{\mathcal{R}_i^{(k)}}(z_i^{(k+1)})\,z_i^{(k+1)}\left(\overline{\mathcal{R}_i^{(k)}}(z_i^{(k+1)})\right)^{-1};\ k=0,1,2,\dots, \tag{10}$$

converge quadratically to the roots of P.

Remark 1. We recall that the evaluation map at a quaternion q is not an algebra homomorphism (see e.g. [15]). This means that if $P(x)=L(x)R(x)$ and $R(q)\neq 0$, then $P(q)=(LR)(q)=L(\tilde{q})\,R(q)$, where $\tilde{q}=R(q)\,q\,(R(q))^{-1}\in[q]$.

Remark 2. The proof of Theorem 4 was obtained under the assumption that all the zeros of P are isolated. Several numerical experiments show that the quaternionic Weierstrass method (7)-(9) also works for spherical roots. However, this requires to higher the precision. For example, in [4] the numerical computations have been carried out with the Mathematica system and with the precision increased to 512 significant digits. If this is not performed, then the method produces approximations to the spherical roots with much less precision than those approximating the isolated roots. As expected, for the case of the spherical root, we obtain convergence to two distinct members of the sphere of zeros.

[1] A polynomial P of degree n has only simple roots if it has n distinct isolated roots.

3.2 The Evaluating Problem

In [3] the problem of evaluating a polynomial, i.e., given a polynomial P, find its value at a given argument α was addressed. Several polynomial evaluation schemes were considered, which depend on the particular form of the polynomial. The paper mainly focus on two algorithms: the Horner's rule for quaternions and a generalization to $\mathbb{H}$ of the well-known algorithm of Goertzel [8]. The Goertzel's algorithm to obtain $P(\alpha)$ is based on the following result (see [15] for more general Euclidean division results).

Theorem 5. *Consider a quaternionic one-sided polynomial P of degree $n \geq 2$ and the characteristic polynomial Ψ_α of a quaternion α. Then there exists a unique polynomial Q of degree $n-2$ such that*

$$P(x) = Q(x)\Psi_\alpha(x) + c_1 x + c_0, \tag{11}$$

with $c_1, c_0 \in \mathbb{H}$.

Observe that the form (11) corresponds to the division of P by Ψ_α and can be presented in a compact form by the use of an expanded synthetic division as follows.

Quaternionic Goertzel's algorithm

INPUT: Coefficients a_k of P and α
OUTPUT: Coefficients c_k of Q and $p = P(\alpha)$

- Obtain $r = 2\,\mathrm{Re}(\alpha)$; $s = |\alpha|^2$;
- Let $c_{n+1} = 0$; $c_n = a_n$;
- **for** $k = n-1 : -1 : 1$ **do**

  $c_k = a_k + r\, c_{k+1} - s\, c_{k+2}$;

- **end for**
- Compute $c_0 = a_0 - s\, c_2$;
- Compute $p = c_1 \alpha + c_0$;

The algorithm produces not only the value $P(\alpha)$, but also the coefficients of the quotient polynomial Q, i.e.

$$p_n(x) = (c_n x^{n-2} + c_{n-1} x^{n-3} + \cdots + c_3 x + c_2)\Psi_\alpha(x) + c_1 x + c_0. \tag{12}$$

We point out that when P is a polynomial with real (or complex) coefficients, this quaternionic algorithm coincides with the classical Goertzel's algorithm for computing $P(\alpha)$, $\alpha \in \mathbb{C}$. This algorithm has advantages, from the complexity of the algorithm point of view, over the well-known Horner's method [5, 14]. The Goertzel?s algorithm for computing $P(\alpha)$ is componentwise backward stable both in the complex case [22] and in the quaternionic case [5].

3.3 The Deflate Problem

Given a real polynomial P and one of its roots α, deflating a polynomial means to find a polynomial Q such that

$$P(x) = Q(x)(x - \alpha);$$

being the additional roots of P exactly the roots of Q. In the case $\alpha \in \mathbb{C}$, one can deflate by a quadratic factor

$$P(x) = Q(x)(x - \alpha)(x - \bar{\alpha}).$$

In both cases, the polynomial Q can be obtained by synthetic long division through Goertzel's algorithm (see Sect. 3.2).

In this section we propose a method to deflate a given polynomial so that the deflating polynomial has no spherical zeros.

The motivation for considering this problem comes from the fact that the proof of the quadratic convergence to the factor-terms (or equivalently to the roots) of the Weierstrass method, relies on the assumption that all the roots are simple (cf. Theorem 4). Several numerical experiments show that it is possible to reach quadratic convergent to spherical roots at the expense of increasing the accuracy of calculations, something that is not accessible to all numerical systems.

We are looking for a factorization of a quaternionic polynomial P of degree n, of the form

$$P(x) = Q(x)S(x) = S(x)Q(x) \tag{13}$$

such that all the zeros of Q are isolated and all the zeros of S are spherical. Assuming that P has the spherical roots $\zeta_1, \ldots, \zeta_k$, $(2k \leq n)$ and recalling Theorem 1, we can write S as

$$S(x) = \Psi_{\zeta_1}(x) \ldots \Psi_{\zeta_k}(x), \tag{14}$$

where $\Psi_{\zeta_i}(x)$ is the characteristic polynomial of ζ_i. Since P can be written as

$$P(x) = P_1(x) + P_{\mathbf{i}}(x)\mathbf{i} + P_{\mathbf{j}}(x)\mathbf{j} + P_{\mathbf{k}}(x)\mathbf{k}, \tag{15}$$

where $P_1, P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$ are real polynomials, it is immediate to conclude that Ψ_ζ is a divisor of P if and only if it is a common divisor of $P_1, P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$.

The starting point to obtain the factorization (13) is to compute the set Z of all the $2k$ common complex zeros of $P_1, P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$, i.e. to construct

$$Z = \{\zeta_1, \bar{\zeta}_1, \ldots, \zeta_k, \bar{\zeta}_k\}.$$

The polynomial Q in (13) is obtained by dividing the polynomial P successively by Ψ_{ζ_i}, $i = 1, \ldots, k$.

4 A Modified Weierstrass Method

In this section we combine the deflation technique described in Sect. 3.3 with the Weierstrass method of Sect. 3.1.

Consider the representation (15) of the monic polynomial P. It is clear that the polynomial P_1 has degree n and the other polynomials $P_{\mathbf{i}}$, $P_{\mathbf{j}}$, $P_{\mathbf{k}}$ have lower degree.

Denoting by P_4, P_3, P_2, P_1 a non-descending sorting of the polynomials P_1, $P_{\mathbf{i}}$, $P_{\mathbf{j}}$, $P_{\mathbf{k}}$ by their degree, we start the process by constructing a list ℓ_C of the complex zeros with positive imaginary part of the real polynomial P_4 and selecting first the one that has the smallest module (see remark at the end of the section), say ζ.

Using the Goertzel's algorithm (in the complex case) to evaluate $P_i(\zeta)$, $i = 1, 2, 3$, we compute simultaneously the (real) coefficients of Q_i such that

$$P_i(x) = Q_i(x)\Psi_\zeta(x) + c_1^i x + c_0^i \tag{16}$$

as in (11)-(12). If ζ is a common zero to all the four polynomials, then we replace the polynomials P_i by the polynomials Q_i (applying in that case the Goertzel's algorithm also to polynomial P_4) and we jump to the next element of the list ℓ_C.

At the end of the procedure, the polynomial $Q(x)$ in (13) is the polynomial

$$Q(x) = P_1(x) + P_{\mathbf{i}}(x)\mathbf{i} + P_{\mathbf{j}}(x)\mathbf{j} + P_{\mathbf{k}}(x)\mathbf{k}, \tag{17}$$

where $P_1, P_{\mathbf{i}}, P_{\mathbf{j}}, P_{\mathbf{k}}$ are the updated polynomials P_1, P_2, P_3, P_4, after a convenient reordering, that comes out of the process.

Once all the spherical roots of P are identified and the deflating polynomial Q is constructed, the isolated zeros of P are computed by applying the Weierstrass method proposed in [4] to Q. The overall process can be summarized as follows:

Modified Weierstrass Algorithm

INPUT: Polynomial P and a tolerance `tol`.
OUTPUT: Lists ℓ_S and ℓ_I with the spherical and isolated roots of P

- Compute the polynomials P_i, $i = 1, 2, 3, 4$ resulting from sorting the real polynomials in (15) in decreasing order of their degree

- Construct the list ℓ_C of the complex roots with positive imaginary part of P_4 sorted in increasing order of the absolute value of its elements

- **for** each $\zeta_l \in \ell_C$ **do**

  - **for** i=3:-1:1 **do**

    Compute the coefficients c_k^i of P_i as in (16) % Division of P_i by Ψ_{ζ_l}

    if $|c_1^i \zeta_l + c_0^i|$ >`tol` **then**

    break % jump to the next ζ_l; ζ_l is not a root of P_i

    end if

    end for

  - Add ζ_l to the list ℓ_S

  - Compute the coefficients c_k^4 of P_4 as in (16)

  - Update P_i % $P_i(x) = Q_i(x)$

- **end for**

- Construct the polynomial Q from (17)

- Apply Weierstrass method to Q to obtain ℓ_I

Remark 3. In practice, deflation should be used with care, since the use of floating point arithmetic leads to non-exact coefficients in Q. The modified Weierstrass method computes the coefficient of Q in the order from highest power down to the constant term (forward deflation). This turns out to be stable if the roots of smallest absolute value are computed first ([19]). Example 2 illustrates this situation.

5 Numerical Experiments

In this section we illustrate how the deflation technique introduced in the previous section can be combined together with the Weierstrass method to produce accurate approximations to the zeros of a polynomial.

In all the computation, we have used the Matlab system with double floating point arithmetic. For details on the Weierstrass method, in particular, the stopping criteria and the choice of the initial approximations we refer the readers to [4].

Our first example was borrowed from the aforementioned paper, where the results were obtained by the use of the Mathematica system with the precision extended to 512 significant digits. Without using this strategy, it is clear that the approximations to the spherical roots cannot reach the same accuracy as the one exhibited by isolated roots.

Example 1. The polynomial

$$P(x) = x^4 + (-1 + \mathbf{i})x^3 + (2 - \mathbf{i} + \mathbf{j} + \mathbf{k})x^2 + (-1 + \mathbf{i})x + 1 - \mathbf{i} + \mathbf{j} + \mathbf{k},$$

has, apart from the isolated zeros $-\mathbf{i} + \mathbf{k}$ and $1 - \mathbf{j}$, a whole sphere of zeros, $[\,\mathbf{i}\,]$.

Table 1: Weierstrass Method for Example 1

Roots	Type	Error
$1-\mathbf{j}$	Isolated	3×10^{-16}
$-\mathbf{i}+\mathbf{k}$	Isolated	2×10^{-15}
$[\,\mathbf{i}\,]$	Spherical	8×10^{-9}

Table 2: Modified Weierstrass Method for Example 1

Roots	Method	Type	Error
$1-\mathbf{j}$	Weierstrass	Isolated	7×10^{-17}
$-\mathbf{i}+\mathbf{k}$	Weierstrass	Isolated	2×10^{-18}
$[\,\mathbf{i}\,]$	Deflation	Spherical	0

Starting with the initial guess $z^{(0)} = (1,-2,0.5\mathbf{i},1+\mathbf{i})$, we obtained, after 14 iterations, the results presented in Table 1.

Observe that P can be written in the form

$$P(x) = P_1(x) + P_2(x)\mathbf{i} + P_3(x)\mathbf{j} + P_4(x)\mathbf{k},$$

where

$$P_1(x) = x^4 - x^3 + 2x^2 - x + 1,\ \ P_2(x) = x^3 - x^2 + x - 1,\ \ P_3(x) = P_4(x) = x^2 + 1.$$

In this case, Matlab finds exactly the roots $\pm\mathbf{i}$ of P_4 and the use of Goerstzel's algorithm shows that these roots are also roots of P_2 and P_1 and that the division of P by $\Psi_{\mathbf{i}}$ produces the polynomial

$$Q(x) = -x^3 + 2x - 1 + (-x+1)\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Applying now the Weierstrass algorithm to Q with the initial approximation $z^{(0)} = (1, 1+\mathbf{i})$, we obtain, after 11 iterations, the results presented in Table 2.

Example 2. The polynomial

$$\begin{aligned} P(x) = x^6 &+ (-1+\mathbf{i})x^5 + (6-\mathbf{i}+\mathbf{j}+\mathbf{k})x^4 + (-5+5\mathbf{i})x^3 \\ &+ (9-5\mathbf{i}+5\mathbf{j}+5\mathbf{k})x^2 + (-4+4\mathbf{i})x + 4 - 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} \end{aligned}$$

has the same zeros as the polynomial P in Example 1 and the additional spherical zero $[\,2\mathbf{i}\,]$.

The results of the Weierstrass and Modified Weierstrass methods are presented in Table 3 and 4, respectively, where the advantage of the modified method is already evident.

Example 3. In the last example we apply both methods to the 8th degree polynomial

$$\begin{aligned} P(x) = x^8 &+ (-7+\mathbf{i})x^7 + (37-7\mathbf{i}+\mathbf{j}+\mathbf{k})x^6 \\ &+ (-66+36\mathbf{i}-6\mathbf{j}-6\mathbf{k})x^5 + (189-60\mathbf{i}+30\mathbf{j}+30\mathbf{k})x^4 \\ &+ (-183+159\mathbf{i}-30\mathbf{j}-30\mathbf{k})x^3 + (253-153\mathbf{i}+129\mathbf{j}+129\mathbf{k})x^2 \\ &+ (-124+124\mathbf{i}-24\mathbf{j}-24\mathbf{k})x + 100 - 100\mathbf{i} + 100\mathbf{j} + 100\mathbf{k}. \end{aligned}$$

This polynomial shares all its roots with the polynomial P of Example 2 and has also the extra spherical root $[3+4\mathbf{i}]$.

Table 3: Weierstrass Method for Example 2

Roots	Type	Error
$1-\mathbf{j}$	Isolated	3×10^{-12}
$-\mathbf{i}+\mathbf{k}$	Isolated	1×10^{-14}
$[\,\mathbf{i}\,]$	Spherical	5×10^{-9}
$[\,2\mathbf{i}\,]$	Spherical	8×10^{-10}

Table 4: Modified Weierstrass Method for Example 2

Roots	Method	Type	Error
$1-\mathbf{j}$	Weierstrass	Isolated	7×10^{-16}
$-\mathbf{i}+\mathbf{k}$	Weierstrass	Isolated	8×10^{-16}
$[\,\mathbf{i}\,]$	Deflation	Spherical	3×10^{-16}
$[\,2\mathbf{i}\,]$	Deflation	Spherical	5×10^{-16}

It is visible in Table 5 that the difficulties of the Weierstrass in dealing with spherical roots increases with the number of spheres. Moreover these problems are also deteriorating the quality of the approximations for the isolated roots. On the other hand, the Modified Weierstrass method produces very good approximations for both isolated and spherical roots: see Table 6.

6 Conclusions

We have proposed an algorithm to compute simultaneously all the roots of an one-sided quaternionic polynomial. This algorithm combines a deflation technique, based on the Goertzel's method with the quaternionic Weierstrass method. Several examples specially designed to include spherical roots show the substantial increase of precision of the corresponding approximations.

Given a polynomial P of the form (1), the Goertzel's method produces accurate approximations to the value of $P(\alpha)$, as far as the condition number for the evaluation of the polynomial P at α, i.e.

$$\text{cond } (P,\alpha) := \frac{\sum_{k=0}^{n} |a_k||\alpha|^k}{|\sum_{k=0}^{n} a_k\alpha^k|}$$

Table 5: Weierstrass Method for Example 3

Roots	Type	Error
$1-\mathbf{j}$	Isolated	9×10^{-6}
$-\mathbf{i}+\mathbf{k}$	Isolated	1×10^{-5}
$[\,\mathbf{i}\,]$	Spherical	1×10^{-5}
$[\,2\mathbf{i}\,]$	Spherical	2×10^{-6}
$[\,3+4\mathbf{i}\,]$	Spherical	9×10^{-7}

Table 6: Modified Weierstrass Method for Example 3

Roots	Method	Type	Error
$1 - \mathbf{j}$	Weierstrass	Isolated	1×10^{-14}
$-\mathbf{i} + \mathbf{k}$	Weierstrass	Isolated	9×10^{-15}
$[\,\mathbf{i}\,]$	Deflation	Spherical	3×10^{-16}
$[\,2\mathbf{i}\,]$	Deflation	Spherical	3×10^{-16}
$[\,3 + 4\mathbf{i}\,]$	Deflation	Spherical	2×10^{-15}

is not large.

For large values of the condition number, the method (in the classical and quaternionic cases) suffers from instability, producing a computed value with few exact digits, which in turn can lead the modified Weierstrass method to failure in the identification of ζ as a spherical root of P. In the classical case, several techniques to overcome such difficulties are known ([10, 11, 22]); we believe they are worth considering in our future work.

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References

[1] Beck, B.: Sur les équations polynomiales dans les quaternions. Enseign. Math. 25, 193–201 (1979)

[2] De Leo, S., Ducati, G., Leonardi, V.: Zeros of unilateral quaternionic polynomials. Electron. J. Linear Algebra 15, 297–313 (2006)

[3] Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Evaluation schemes in the ring of quaternionic polynomials. BIT Numerical Mathematics 58(1), 51–72 (2018)

[4] Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Weierstrass method for quaternionic polynomial root-finding. Mathematical Methods in the Applied Sciences 41(1), 423–437 (2018)

[5] Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Mathematica tools for quaternionic polynomials. Lecture Notes in Comput. Sci. 10405, 394–408 (2017)

[6] Falcão, M.I., Miranda, F., Severino, R., Soares, M.J.: Polynomials over quaternions and coquaternions: A unified approach. Lecture Notes in Comput. Sci. 10405, 379–393 (2017)

[7] Farouki, R.T., Gentili, G., Giannelli, C., Sestini, A., Stoppato, C.: A comprehensive characterization of the set of polynomial curves with rational rotation-minimizing frames. Adv. Comput. Math. 43(1), 1–24 (2017)

[8] Goertzel, G.: An algorithm for the evaluation of finite trigonometric series. Amer. Math. Monthly 65, 34–35 (1958)

[9] Gordon, B., Motzkin, T.: On the zeros of polynomials over division rings I. Trans. Amer. Math. Soc. 116, 218–226 (1965)

[10] Graillat, S., Langlois, P., Louvet, N.: Algorithms for accurate, validated and fast polynomial evaluation. Japan J. Indust. Appl. Math. 26(2-3), 191–214 (2009)

[11] Graillat, S., Ménissier-Morain, V.: Accurate summation, dot product and polynomial evaluation in complex floating point arithmetic. Information and Computation 216, 57–71 (2012)

[12] Gürlebeck, K., Sprößig, W.: Quaternionic and Clifford calculus for physicists and engineers. John Wiley & Sons (1997)

[13] Janovská, D., Opfer, G.: A note on the computation of all zeros of simple quaternionic polynomials. SIAM J. Numer. Anal. 48(1), 244–256 (2010)

[14] Knuth, D.E.: Seminumerical Algorithms, The Art of Computer Programming, vol. 2. Addison-Wesley, second edn. (1981)

[15] Lam, T.Y.: A first course in noncommutative rings. Graduate Texts in Mathematics, Springer-Verlag, New York (1991)

[16] Niven, I.: Equations in quaternions. Amer. Math. Monthly 48, 654–661 (1941)

[17] Pereira, R., Rocha, P.: On the determinant of quaternionic polynomial matrices and its application to system stability. Math Methods Appl Sci. 31(1), 99–122 (2008)

[18] Pereira, R., Rocha, P., Vettori, P.: Algebraic tools for the study of quaternionic behavioral systems. Linear Algebra Appl. 400(1-3), 121–140 (2005)

[19] Press, W.H., Teukolsky, S.A., Vetterling, W.T., Flannery, B.P.: Numerical Recipes 3rd Edition: The Art of Scientific Computing. Cambridge University Press, USA, 3 edn. (2007)

[20] Serôdio, R., Pereira, E., Vitória, J.: Computing the zeros of quaternion polynomials. Comput. Math. Appl. 42(8-9), 1229–1237 (2001)

[21] Serôdio, R., Siu, L.S.: Zeros of quaternion polynomials. Appl. Math. Lett. 14(2), 237–239 (2001)

[22] Smoktunowicz, A., Wróbel, I.: On improving the accuracy of Horner's and Goertzel's algorithms. Numer. Algorithms 38(4), 243–258 (2005)

[23] Weierstrass, K.: Neuer Beweis des Satzes, dass jede ganze rationale Function einer Veränderlichen dargestellt werden kann als ein Product aus linearen Functionen derselben Veränderlichen. In: Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin, vol. II. S.1085-1101. Berlin (1891)

[24] Zhang, F.: Quaternions and matrices of quaternions. Linear Algebra Appl. 251, 21–57 (1997)