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Mathematica Tools for Coquaternions

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Information

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Abstract

Coquaternions form a four dimensional real algebra generalizing complex numbers and were introduced by James Cockle at about the same time that Hamilton discovered the famous algebra of quaternions. Although not as popular as quaternions, in recent years one can observe an emerging interest among mathematicians and physicists on the study of these numbers. In this work we revisit a Mathematica package for implementing the algebra of coquaternions – `Coquaternions` – and discuss a set of Mathematica functions – `CoqPolynomial` – to deal with coquaternionic polynomials. These two sets of functions provide the basic tools necessary for manipulating coquaternions and unilateral coquaternionic polynomials, reflecting, in its present form, the recent interests of the authors in the area.

1 Introduction

Quaternions, introduced in 1843 by the Irish mathematician William Rowan Hamilton (1805-1865) as a generalization of complex numbers [15], have become a powerful tool for solving problems in almost all applied sciences [17], unlocking also new approaches in many branches of applied mathematics. This increasing interest in using quaternions has motivated the emergence of several software packages to perform computations in the algebra of the real quaternions (see, for example, [5, 9, 10, 21]), or more generally, in Clifford Algebras (see [1, 2] and the references therein for details).

In 1849, the English mathematician James Cockle, introduced another *system of quadruple algebra* [4] whose elements he designated by the name *coquaternions*. Although coquaternions, also known in the literature as split quaternions, are not as popular as the Hamilton's quaternions, one can say that recently they have also become an active research area [3, 6, 7, 14, 16, 18, 19].

Recently [12], we have introduced a Mathematica add-on application – `Coquaternions` – whose main purpose is to define arithmetic rules for coquaternions and have described a collection of functions – `CoqPolynomial` – for dealing with some polynomial problems over the algebra of coquaternions. As far as we know, these are the only computational tools specially designed to work with coquaternions.

Table 1: Basic operations on coquaternions

Function	Description
`Abs[q]`	extends the `Abs` function to coquaternion objects
`ComplexMatrixToCoquaternion[m]`	gives the coquaternion corresponding to `m` $\in \mathcal{M}_2(\mathbb{C})$
`ComplexToCoquaternion[{a,b}]`	gives the coquaternion $\mathsf{a} + \mathsf{bj}$, with $\mathsf{a}, \mathsf{b} \in \mathbb{C}$
`Conjugate[q]`	extends the `Conjugate` function to coquaternion objects
`Coquaternion[q0,q1,q2,q3]`	represents the coquaternion $\mathsf{q}_0 + \mathsf{q}_1\mathsf{i} + \mathsf{q}_2\mathsf{j} + \mathsf{q}_3\mathsf{k}$
`CoquaternionQ[q]`	gives `True` if q is a coquaternion and `False` otherwise
`CoquaternionToComplex[q]`	gives a list $\{q_0 + \mathrm{i}q_1, q_2 + \mathrm{i}q_3\}$ for the complex form of q
`CoquaternionToComplexMatrix[q]`	gives the complex 2×2 representation matrix of q
`CoquaternionToMatrix[q]`	gives the real 2×2 representation matrix of q
`CoquaternionTo4DMatrixL[q]`	gives the real 4×4 left representation matrix of q
`CoquaternionTo4DMatrixR[q]`	gives the real 4×4 right representation matrix of q
`Det[q]`	extends the `Det` function to coquaternion objects
`MatrixToCoquaternion[m]`	gives the coquaternion corresponding to `m` $\in \mathcal{M}_2(\mathbb{R})$
`Norm[q]`	extends the `Norm` function to coquaternion objects
`Power[q,n]`	extends the `Power` function to coquaternion objects for integer exponents
`Re[q]`	extends the `Re` function to coquaternion objects
`ToCoquaternion[a]`	gives de coquaternion form of a when a is a real or complex number
`Tr[q]`	extends the `Tr` function to coquaternion objects
`Vec[q]`	gives the vector part of q

In this paper we discuss a new set of functions for dealing with unilateral coquaternionic polynomials, which, in particular, contains a function implementing a recently proposed algorithm to determine and classify the zeros of such polynomials [13]. A function for computing the nth roots of a coquaternion, accordingly to [11], is also added to `CoqPolynomial`. Several examples have been designed to show clearly the use and performance of the computational tools presented in this work.

2 The Package `Coquaternions` revisited

In this section we recall some basic definitions and results on coquaternions and we revisit the Mathematica package `Coquaternions`. We mainly follow the presentation of [12], where more details can be found.

Let $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. The algebra of real coquaternions, which we denote by $\mathbb{H}_{\rm coq}$, is generated by the product given according to the following rules

$$\mathsf{i}^2 = -1,\ \mathsf{j}^2 = \mathsf{k}^2 = 1,\ \ \mathsf{ij} = -\mathsf{ji} = \mathsf{k}.$$

Given $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \in \mathbb{H}_{\rm coq}$, its *conjugate* $\overline{\mathsf{q}}$ is defined as $\overline{\mathsf{q}} = q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k}$; the number q_0 is called the *real part* of q, denoted by $\operatorname{Re}\mathsf{q}$, and the *vector part* of q, denoted by $\underline{\mathsf{q}}$, is given by $\underline{\mathsf{q}} = q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. We call *trace* of q, denoted by $\operatorname{tr}\mathsf{q}$, the quantity given by $\operatorname{tr}\mathsf{q} = \mathsf{q} + \overline{\mathsf{q}} = 2\operatorname{Re}\mathsf{q}$ and call *determinant* of q, denoted by $\det\mathsf{q}$, the quantity given by $\det\mathsf{q} = \mathsf{q}\,\overline{\mathsf{q}} = q_0^2 + q_1^2 - q_2^2 - q_3^2$.

Unlike $\mathbb{C}$ and $\mathbb{H}$, $\mathbb{H}_{\rm coq}$ is not a division algebra. In fact, a coquaternion q is invertible if and only if $\det\mathsf{q} \neq 0$. In that case, we have $\mathsf{q}^{-1} = \frac{\overline{\mathsf{q}}}{\det\mathsf{q}}$. Finally, we endow $\mathbb{H}_{\rm coq}$ with the *semi-norm* $\|\mathsf{q}\| = \sqrt{|\det\mathsf{q}|}$ and call q a *unit coquaternion* if $\|\mathsf{q}\| = 1$.

In the `Coquaternions` package, a coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ is an object of the form `Coquaternion[q0,q1,q2,q3]`, whose entries are numeric quantities or symbols (in such case the package assumes that all symbols represent real numbers).

The package adds rules to the functions `Plus`, `Minus`, `Times`, `Power` and `NonCommutativeMultiply` to make it easy to perform basic arithmetic operations and also extends some standard functions to coquaternion objects. A summary of the functions included in the package is given in Table 1.

3 Coquaternionic Polynomials

In this section we describe a new collection of functions added to the Mathematica tool `CoqPolynomial` for dealing with some polynomial problems in $\mathbb{H}_{\rm coq}$. The first part of the section follows mainly [12].

Let $\mathbb{H}_{\rm coq}[x]$ denote the set of polynomials of the form

$$P(x) = \mathsf{c}_n x^n + \mathsf{c}_{n-1} x^{n-1} + \cdots + \mathsf{c}_1 x + \mathsf{c}_0,\ \mathsf{c}_i \in \mathbb{H}_{\rm coq}, \tag{1}$$

i.e., the set of polynomials whose coefficients are only located on the left of the variable, with the addition and multiplication of such polynomials defined as in the commutative case, where the variable is assumed to commute with the coefficients. This is a ring, referred to as the ring of (left) *one-sided* or *unilateral* polynomials in $\mathbb{H}_{\rm coq}$, or simply the ring of *coquaternionic polynomials*. We usually omit the reference to the variable and write simply P when referring to an element $P(x) \in \mathbb{H}_{\rm coq}[x]$, since all the polynomials considered in this work are in the indeterminate x.

As usual, if $\mathsf{c}_n \neq 0$, we say that the *degree* of the polynomial P is n and refer to c_n as the leading coefficient of the polynomial. When $\mathsf{c}_n = 1$, we say that P is *monic*. If the coefficients c_i in (1) are real, then we say that P is a *real polynomial*. Given a polynomial P of the form (1), its *conjugate polynomial* is the polynomial defined by

$$\overline{P}(x) = \overline{\mathsf{c}}_n x^n + \overline{\mathsf{c}}_{n-1} x^{n-1} + \cdots + \overline{\mathsf{c}}_1 x + \overline{\mathsf{c}}_0.$$

For a coquaternionic polynomial P, the *evaluation map* at a given element $\mathsf{q} \in \mathbb{H}_{\rm coq}$ is defined by

$$P(\mathsf{q}) = \mathsf{c}_n \mathsf{q}^n + \mathsf{c}_{n-1} \mathsf{q}^{n-1} + \cdots + \mathsf{c}_1 \mathsf{q} + \mathsf{c}_0.$$

If $P(\mathsf{q}) = 0$, then we say that q is a *zero* (or a *root*) of P.

The evaluation map is not a homomorphism from the ring $\mathbb{H}_{\rm coq}[x]$ into $\mathbb{H}_{\rm coq}$; given two polynomials $L, R \in \mathbb{H}_{\rm coq}[x]$, in general, we do not have $(LR)(\mathsf{q}) = L(\mathsf{q})R(\mathsf{q})$.

Before we present some results on the structure of the sets of zeros of a coquaternionic polynomial, we first recall the concept of quasi-similarity for coquaternions; see e.g. [8, 13] and references therein.

We say that two elements $\mathsf{p}, \mathsf{q} \in \mathbb{H}_{\rm coq}$ are *quasi-similar* if and only if $\operatorname{Re}\mathsf{p} = \operatorname{Re}\mathsf{q}$ and $\det\mathsf{p} = \det\mathsf{q}$. This is an equivalence relation in $\mathbb{H}_{\rm coq}$;[1] the class of an element $\mathsf{q} \in \mathbb{H}_{\rm coq}$ with respect to this relation will be denoted by $\llbracket \mathsf{q} \rrbracket$ and referred to as the *quasi-similarity class* of q. Observe that

$$\llbracket \mathsf{q} \rrbracket = \{x_0 + x_1\mathsf{i} + x_2\mathsf{j} + x_3\mathsf{k} : x_0 = q_0 \text{ and } x_1^2 - x_2^2 - x_3^2 = \det \underline{\mathsf{q}}\}$$

can be identified with a hyperboloid in the hyperplane $\{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_0 = q_0\}$: a hyperboloid of two sheets if $\det \underline{\mathsf{q}} > 0$, a hyperboloid of one sheet if $\det \underline{\mathsf{q}} < 0$ and a degenerate hyperboloid, i.e. a cone, if $\det \underline{\mathsf{q}} = 0$.

Given a quasi-similarity class $\llbracket \mathsf{q} \rrbracket = \llbracket q_0 + \underline{\mathsf{q}} \rrbracket$, we call *characteristic polynomial* of this class and denote by $\Psi_{\llbracket \mathsf{q} \rrbracket}$, the second degree monic polynomial with real coefficients given by

$$\Psi_{\llbracket \mathsf{q} \rrbracket}(x) = x^2 - 2q_0\, x + \det \mathsf{q}.$$

It is important to observe that any second degree monic polynomial S with real coefficients is the characteristic polynomial of a (uniquely defined) quasi-similarity class: if S has two roots $\alpha, \overline{\alpha} \in \mathbb{C}\setminus\mathbb{R}$, then $S = \Psi_{\llbracket \mathsf{q} \rrbracket}$, with $\mathsf{q} = \alpha$, and if S has real roots r_1 and r_2 (with, eventually, $r_1 = r_2$), then $S = \Psi_{\llbracket \mathsf{q} \rrbracket}$, with $\mathsf{q} = \frac{r_1 + r_2}{2} + \frac{r_1 - r_2}{2}\mathsf{j}$.

We recall that in the Mathematica tool `CoqPolynomial`, introduced in [12], a coquaternionic polynomial is an object of the form `Polynomial[`c_n`,`c_{n-1}`,...,`c_1`,`c_0`]` accordingly to (1). For such objects, several rules and functions are defined.

A function `CharacteristicPolynomial` for computing the characteristic polynomial of a (representative of a given) class was recently added to `CoqPolynomial`. We illustrate its use with three simple examples.

```
In[1]:= <<CoqPolynomial`

In[2]:= q1 = ToCoquaternion[1+2I]; ChP1 = CharacteristicPolynomial[q1]
```

[1] For a discussion on how this concept relates to another well-known equivalence in $\mathbb{H}_{\rm coq}$, the similarity relation, see e.g. [8].

```
Out[2]= Polynomial[1, −2, 5]

 In[3]:= Solve[PolyForm[ChP1,x] == 0, x]
Out[3]= {{x → 1 − 2 ⅈ}, {x → 1 + 2 ⅈ}}

 In[4]:= q2 = ToCoquaternion[2]; ChP2 = CharacteristicPolynomial[q2]
Out[4]= Polynomial[1, −4, 4]

 In[5]:= Solve[PolyForm[ChP2,x] == 0, x]
Out[5]= {{x → 2}, {x → 2}}

 In[6]:= q3 = Coquaternion[2,0,-2,0]; ChP3 = CharacteristicPolynomial[q3]
Out[6]= Polynomial[1, −4, 0]

 In[7]:= Solve[PolyForm[ChP3,x] == 0, x]
Out[7]= {{x → 0}, {x → 4}}
```

We now present very briefly some results on the zeros of coquaternionic polynomials; more details can be seen in e.g. [13]. In what follows we restrict our attention to monic polynomials, i.e. to polynomials of the form (1) with $\mathsf{c}_n = 1$.[2]

Given a polynomial P, we denote by $Z(P)$ the *zero-set* of P, i.e. the set of all the zeros of P; we also define the *companion polynomial* of P, $\mathcal{C}_P$, as the polynomial given by

$$\mathcal{C}_P(x) = P(x)\overline{P}(x).$$

As was shown in [13], if $\mathsf{z} \in \mathbb{H}_{\mathrm{coq}}$ is a zero of P, then the characteristic polynomial of $[\![\mathsf{z}]\!]$ is a divisor of $\mathcal{C}_P$. When P is monic (or with a non-singular leading term), it can be shown easily that $\mathcal{C}_P$ is a polynomial of degree $2n$ with real coefficients and, as such, considered as a polynomial in $\mathbb{C}$, has $2n$ roots. If these roots are $\alpha_1, \overline{\alpha}_1, \ldots, \alpha_m$, $\overline{\alpha}_m \in \mathbb{C} \setminus \mathbb{R}$ and $r_1, r_2, \ldots, r_\ell \in \mathbb{R}$, where $\ell = 2(n-m)$, $(0 \leq m \leq n)$, then it is easy to conclude that the second-degree monic polynomials with real coefficients which divide $\mathcal{C}_P$ are the characteristic polynomials of the following quasi-similarity classes:

$$[\![\alpha_k]\!];\ k = 1, \ldots, m, \tag{2a}$$

$$[\![\mathsf{r}_{ij}]\!];\ i = 1, \ldots, \ell - 1,\ j = i + 1, \ldots, \ell, \tag{2b}$$

with

$$\mathsf{r}_{ij} = \frac{r_i + r_j}{2} + \frac{r_i - r_j}{2}\mathsf{j}. \tag{2c}$$

We thus have the following result concerning the zero-set of P:

$$Z(P) \subseteq \bigcup_k [\![\alpha_k]\!] \bigcup_{i,j} [\![\mathsf{r}_{ij}]\!],$$

where $[\![\alpha_k]\!]$ and $[\![\mathsf{r}_{ij}]\!]$ are the quasi-similarity classes defined by (2). We call the classes given by (2) the *admissible classes* (with respect to the zeros) of the polynomial P.

Two new functions to determine, respectively, the companion polynomial and the admissible classes of a coquaternionic polynomial – `CompanionPolynomial` and `AdmissibleClasses` – are now part of `CoqPolynomial`.

```
 In[8]:= P = Polynomial[1,Coquaternion[2,-1,2,-1],Coquaternion[1,1,4,1]]; CPP = CompanionPolynomial[P]
Out[8]= Polynomial[1, 4, 2, −12, −15]

 In[9]:= Solve[PolyForm[CPP,x] == 0, x]
Out[9]= {{x → −2 − ⅈ}, {x → −2 + ⅈ}, {x → −√3}, {x → √3}}
```

[2] As far as the computation of zeros is concerned, considering this type of polynomials is equivalent to considering polynomials whose leading term is invertible.

In[10]:= `AdmissibleClasses[P]//TraditionalForm`

Out[10]= $\left\{-2-\mathbb{i}, \sqrt{3}\ \mathbb{j}\right\}$

The results given in the following theorem show how to find the zeros of a polynomial P belonging to one of its admissible classes.

Theorem 1 ([13, Theorem 3.14]). *Let $P(x)$ be a monic polynomial of the form (1) and let $[\![\mathsf{q}]\!] = [\![q_0 + \underline{\mathsf{q}}]\!]$ be an admissible class of $P(x)$. Also, let $\mathsf{a} + \mathsf{b}x$, with $\mathsf{b} = b_0 + b_1\mathsf{i} + b_2\mathsf{j} + b_3\mathsf{k}$, be the remainder of the right division of $P(x)$ by the characteristic polynomial of $[\![\mathsf{q}]\!]$.*

1. *If $\det \mathsf{b} \neq 0$, then $[\![\mathsf{q}]\!]$ contains only one zero of P, given by*

$$\mathsf{z} = -\mathsf{b}^{-1}\mathsf{a}.$$

2. *If $\mathsf{a} = \mathsf{b} = 0$, then $[\![\mathsf{q}]\!] \subseteq Z(P)$.*

3. *If $\mathsf{b} \neq 0, \det \mathsf{b} = 0$ and the equation $\mathsf{a} + \mathsf{b}x = 0$ has a real solution γ_0 satisfying*

$$(q_0 - \gamma_0)^2 = -\det(\underline{\mathsf{q}}),$$

then the zeros of P in $[\![\mathsf{q}]\!]$ form the following line in the hyperplane $x_0 = q_0$,

$$\mathcal{L} = \left\{q_0 + \alpha\mathsf{i} + (k_2\alpha + k_1(q_0-\gamma_0))\,\mathsf{j} + (-k_1\alpha + k_2(q_0-\gamma_0))\,\mathsf{k} : \alpha \in \mathbb{R}\right\},$$

with k_1 and k_2 given by

$$k_1 = -\frac{b_0b_2 + b_1b_3}{b_0^2 + b_1^2} \quad \text{and} \quad k_2 = \frac{b_1b_2 - b_0b_3}{b_0^2 + b_1^2}. \tag{3}$$

4. *If $\mathsf{b} \neq 0, \det \mathsf{b} = 0$ and the equation $\mathsf{a} + \mathsf{b}x = 0$ has a non-real solution $\gamma = \gamma_0 + \gamma_1\mathsf{i}$, then the class $[\![\mathsf{q}]\!]$ contains only one zero of P, given by*

$$\mathsf{z} = q_0 + (\beta + \gamma_1)\mathsf{i} + (k_2\beta + k_1(q_0-\gamma_0))\,\mathsf{j} + (-k_1\beta + k_2(q_0-\gamma_0))\,\mathsf{k},$$

where

$$\beta = \frac{\det(\underline{\mathsf{q}}) + (q_0-\gamma_0)^2 - \gamma_1^2}{2\gamma_1}$$

and k_1 and k_2 are given by (3).

5. *If none of the above conditions holds, then there are no zeros of P in $[\![\mathsf{q}]\!]$.*

In cases 1. and 4., we say that the zero z is an *isolated zero* of P; in case 2., we say that the class $[\![\mathsf{q}]\!]$ (or any of its elements) is a *hyperboloidal zero* of P and in case 3. we call the line $\mathcal{L}$ (or any of its elements) a *linear zero* of P.

With the understanding that all the zeros belonging to the same quasi-similarity class are counted as one zero, we have the following result.

Theorem 2 ([13, Theorem 3.10]). *A polynomial of degree n in $\mathbb{H}_{\mathrm{coq}}[x]$, has at most, $n(2n-1)$ zeros.*

A function `NumZeros` indicating the number of zeros of each type that a given polynomial possesses is now available in `CoqPolynomial`.

Example 1. The polynomials P and Q given by

$$\begin{aligned}
P(x) &= x^4 - (\mathsf{j}+\mathsf{k})x^3 + (-8+4\mathsf{i}-2\mathsf{j}-5\mathsf{k})x^2 + (9+4\mathsf{i}+7\mathsf{j}+10\mathsf{k})x \\
&\qquad - (5+5\mathsf{i}+7\mathsf{j}+\mathsf{k}), \\
Q(x) &= x^5 + (2\mathsf{i}-\mathsf{j}-4\mathsf{k})x^4 + (2\mathsf{i}-\mathsf{j}-4\mathsf{k})x^3 + (-21+35\mathsf{i}-10\mathsf{j}+37\mathsf{k})x^2 \\
&\qquad + (-63+7\mathsf{i}+16\mathsf{j}+46\mathsf{k})x + 6 - 88\mathsf{i} + 74\mathsf{j} - 48\mathsf{k}
\end{aligned}$$

are examples of polynomials achieving the maximum number of zeros allowed by the respective degrees (28 for a polynomial of degree 4, and 45 for a polynomial of degree 5).

```
In[11]:= P=Polynomial[1,- Coquaternion[0,0,1,1],Coquaternion[-8,4,-2,-5]],Coquaternion[9,4,7,10],
            - Coquaternion[5,5,7,1]];
          NumZeros[P]
```

Out[11]= $\{28, 0, 0\}$

The output indicates that P has 28 isolated zeros, no linear zeros and no hyperboloidal zeros.

```
In[12]:= Q=Polynomial[1,Coquaternion[0,2,-1,-4],Coquaternion[-7,-8,13,5],Coquaternion[-21,35,-10,37],
            Coquaternion[-63,7,16,46],Coquaternion[6,-88,74,-48]];
           NumZeros[Q]
```

Out[12]= $\{45, 0, 0\}$

Example 2. The polynomial $R(x) = x^2 - (1 + \mathsf{j} + \mathsf{k})$ illustrates an important difference between coquaternionic polynomials and complex (or quaternionic) polynomials: there exist polynomials with no zeros, i.e., the Fundamental Theorem of Algebra is not valid in $\mathbb{H}_{\rm coq}[x]$.

```
In[13]:= R = Polynomial[1,0,-Coquaternion[1,0,1,1]];NumZeros[R]
```

Out[13]= $\{0, 0, 0\}$

A function to compute and classify the zeros of a coquaternion polynomial – `Zeros` – was recently added to `CoqPolynomial`. This function implements an algorithm, proposed in [13], which makes use of formulas (2) for the identification of the admissible classes, followed by the application of the results of Theorem 1 to determine and classify the zeros in each class (if existing). We now present several examples of application of this function. Note that the output of the function `Zeros` is a list composed of three lists, containing, respectively, the isolated zeros, the linear zeros and (representatives of) the hyperboloidal zeros of the input polynomial.

Example 3. We consider again the polynomials given in Example 1 and present some of their zeros.

```
In[14]:= {zIP, zLP, zHP} = Zeros[P];
          (first4zP = Take[zIP,4])//TraditionalForm
```

Out[14]= $\{-2 - \frac{7\mathbb{i}}{8} - \frac{15\mathbb{j}}{8} - \frac{\mathbb{k}}{2}, -1 + \frac{6\mathbb{i}}{7} - \frac{2\mathbb{j}}{7} + \frac{9\mathbb{k}}{7}, -\frac{1}{2} + \frac{5\mathbb{i}}{2} - \frac{3\mathbb{j}}{2} + \frac{5\mathbb{k}}{2},$
$-\frac{1}{2} + \frac{1}{46}(163 - 72\sqrt{3})\mathbb{i} - \frac{3}{46}(-13 + 8\sqrt{3})\mathbb{j} + \frac{1}{46}(-137 + 102\sqrt{3})\mathbb{k}\}$

```
In[15]:= Eval[P,first4zP]
```

Out[15]= $\{0, 0, 0, 0\}$

```
In[16]:= {zIQ, zLQ, zHQ} = Zeros[Q];
          (first4zQ = Take[zIQ,4])//TraditionalForm
```

Out[16]= $\left\{-\frac{5}{2} - \frac{157\mathbb{i}}{114} + \frac{113\mathbb{j}}{114} + \frac{41\mathbb{k}}{38}, -\frac{3}{2} + \frac{31\mathbb{i}}{13} - \frac{73\mathbb{j}}{26} + \frac{3\mathbb{k}}{13}, -1 + \frac{7\mathbb{i}}{9} - \frac{11\mathbb{j}}{9} - \frac{\mathbb{k}}{3}, -1 - \frac{650\mathbb{i}}{159} + \frac{430\mathbb{j}}{159} + \frac{194\mathbb{k}}{53}\right\}$

```
In[17]:= Eval[Q,first4zQ]
```

Out[17]= $\{0, 0, 0, 0\}$

Example 4. The three quadratic polynomials

$$S(x) = x^2 - (2 + 6\mathsf{i} + 3\mathsf{j} + 5\mathsf{k})x + 9\mathsf{i} + 4\mathsf{j} + 8\mathsf{k},$$
$$T(x) = x^2 - (2 + 2\mathsf{i} + 2\mathsf{k})x + 1 + 2\mathsf{i} + 2\mathsf{k}$$

and

$$U(x) = x^2 - 2x + 1$$

have the same companion polynomial, $(x-1)^4$, hence the same unique admissible class, $[\![1]\!]$, but different types of zeros: S has an isolated zero, T a linear zero and U a hyperboloidal zero.

```
In[18]:= S=Polynomial[1,-Coquaternion[2,6,3,5],Coquaternion[0,9,4,8]];
         T=Polynomial[1,-Coquaternion[2,2,0,2],Coquaternion[1,2,0,2]];
         U=Polynomial[1,-2,1];
```

```
In[19]:= Simplify[PolyForm[CompanionPolynomial[S],x]]
```

Out[19]= $(-1+x)^4$

```
In[20]:= AdmissibleClasses[S]//QSimplify
```

Out[20]= $\{1\}$

```
In[21]:= QSimplify[AdmissibleClasses[T]]=={1}&&QSimplify[AdmissibleClasses[U]]=={1}
```

Out[21]= `True`

```
In[22]:= {zIS, zLS, zHS} = Zeros[S]
```

Out[22]= $\{\{\texttt{Coquaternion}[1,5,3,4]\},\{\},\{\}\}$

```
In[23]:= {zIT, zLT, zHT} = Zeros[T]
```

Out[23]= $\{\{\},\{\texttt{Coquaternion}[1,\mathbb{c}_1,0,\mathbb{c}_1]\},\{\}\}$

```
In[24]:= {zIU, zLU, zHU} = Zeros[U]
```

Out[24]= $\{\{\},\{\},\{1\}\}$

```
In[25]:= PolynomialZeroQ[S,zIS[[1]]]
```

Out[25]= `True`

```
In[26]:= PolynomialZeroQ[T,zLT[[1]]]
```

Out[26]= `True`

One can easily check that any element in the class $[\![1]\!]$ is, in fact, a zero of the polynomial U.

```
In[27]:= Assuming[Det[Coquaternion[1,C[1],C[2],C[3]]]==Det[ToCoquaternion[1],
          PolynomialZeroQ[U,Coquaternion[1,C[1],C[2],C[3]]]]]
```

Out[27]= `True`

Example 5. The five polynomials considered in this example are all polynomials of second degree and illustrate the very different behaviors that (even simple) polynomials may have, in what concerns their zero-sets.

1. *One linear zero*

```
In[28]:= Zeros[Polynomial[1,Coquaternion[0,1,1,0],0]]//TraditionalForm
```

Out[28]= $\{\{\},\{\mathbb{i}\mathbb{c}_1+\mathbb{j}\mathbb{c}_1\},\{\}\}$

2. *Two isolated zeros and one linear zero*

```
In[29]:= Zeros[Polynomial[1,Coquaternion[-2,1,1,0],Coquaternion[0,-2,-2,0]]]//TraditionalForm
```

Out[29]= $\{\{-\mathbb{i}-\mathbb{j},2\},\{1+\mathbb{i}\mathbb{c}_1+\mathbb{j}\mathbb{c}_1-\mathbb{k}\},\{\}\}$

3. *Two linear zeros*

```
In[30]:= Zeros[Polynomial[1,Coquaternion[0,0,2,0],Coquaternion[-1,0,-2,0]]]
```

Out[30]= $\{\{\},\{1+\mathbb{i}\mathbb{c}_1-2\mathbb{j}-\mathbb{k}\mathbb{c}_1,1+\mathbb{i}\mathbb{c}_1+\mathbb{k}\mathbb{c}_1\},\{\}\}$

4. *Two isolated zeros and two linear zeros*

```
In[31]:= Zeros[Polynomial[1, Coquaternion[-2,-1,1,1],Coquaternion[1,1,-1,-1]]]//TraditionalForm
```

Out[31]= $\left\{\{1, 1+\mathbb{i}-\mathbb{j}-\mathbb{k}\}, \left\{\frac{1}{2}+\mathbb{i}\mathbb{c}_1-\frac{\mathbb{j}}{2}-\mathbb{k}\mathbb{c}_1, \frac{3}{2}+\mathbb{i}\mathbb{c}_1-\mathbb{j}\mathbb{c}_1-\frac{\mathbb{k}}{2}\right\}, \{\}\right\}$

5. *Two isolated zeros and one hyperboloidal zero*

```
In[32]:= Zeros[Polynomial[1,-3,2]]//TraditionalForm
```

Out[32]= $\left\{\{1, 2\}, \{\}, \left\{\frac{3}{2}+\frac{\mathbb{i}}{2}\right\}\right\}$

The following theorem states a sufficient condition for the existence of linear zeros of a polynomial.

Theorem 3 ([13, Theorem 3.16])**.** *Let $P(x)$ be a polynomial of degree n whose companion polynomial has m real simple zeros $r_1, r_2, \ldots, r_m$, $m \le 2n$, and let $P_r(x) = P(x)(x-r)$ with $r \in \mathbb{R}$, $r \ne r_i; i = 1, \ldots, m$. Then, $P_r(x)$ has (at least) m linear zeros.*

Example 6. We illustrate the result of the previous theorem with a simple example. Consider the polynomial $P(x) = x^2 + (-1+3\mathsf{i}-2\mathsf{j}+\mathsf{k})x - 2\mathsf{i} - 2\mathsf{k}$. The companion polynomial of P has two simple real roots $r_1 = 0$ and $r_2 = 2$ (and two complex conjugate roots $\pm\sqrt{2}\mathsf{i}$). This polynomial has two isolated zeros:

```
In[33]:= Clear[P]; P=Polynomial[1,Coquaternion[-1,3,-2,2],Coquaternion[0,-2,0,-2]];

In[34]:= Solve[PolyForm[CompanionPolynomial[P],x] == 0]
```

Out[34]= $\left\{\{x \to 0\}, \{x \to 2\}, \left\{x \to -\mathbb{i}\sqrt{2}\right\}, \left\{x \to \mathbb{i}\sqrt{2}\right\}\right\}$

```
In[35]:= NumZeros[P]
```

Out[35]= $\{2, 0, 0\}$

Multiplying P by the factor $(x-1)$, we obtain a polynomial with two linear zeros (and three isolated zeros):

```
In[36]:= Clear[P1]; P1=P**Polynomial[1,-1];

In[37]:= NumZeros[P1]
```

Out[37]= $\{3, 2, 0\}$

```
In[38]:= Zeros[P]//TraditionalForm
```

Out[38]= $\{-6\mathbb{i}+3\mathbb{j}-5\mathbb{k}, 1-\mathbb{i}+\mathbb{j}-\mathbb{k}\}, \{\}, \{\}\}$

```
In[39]:= Zeros[P1]//TraditionalForm
```

Out[39]= $\left\{\{-6\mathbb{i}+3\mathbb{j}-5\mathbb{k}, 1, 1-\mathbb{i}+\mathbb{j}-\mathbb{k}\}, \left\{\frac{1}{2}+\mathbb{i}\mathbb{c}_1+\frac{\mathbb{j}}{2}+\mathbb{k}\mathbb{c}_1, \frac{3}{2}+\mathbb{i}\mathbb{c}_1-\mathbb{j}\mathbb{c}_1-\frac{\mathbb{k}}{2}\right\}, \{\}\right\}$

A list of functions related to `CoqPolynomial` is presented in Table 2.

Observation: For all the polynomials considered in the previous examples, it was possible to compute the roots of the corresponding companion polynomial exactly, and hence, to obtain the zeros with infinite precision. In general, however, even for a polynomial P of moderate degree n, the zeros of $\mathcal{C}_P$ (which is a polynomial of degree $2n$), may only be obtained by using a numerical method. Since the classification and determination of the zeros of the polynomial depend critically on knowing of whether or not certain quantities are zero and also on the existence or not of a real or complex solution of a linear system, having approximate values for the representatives for the admissible classes may lead, in some cases, to a miss-classification and inappropriate determination of zeros and some care has to be taken. Ways to circumvent these problems require further investigation.

Table 2: Some functions of `CoqPolynomial`

Function	Description
`AdmissibleClasses[P]`	gives the admissible classes of the polynomial `P`
`CharacteristicPolynomial[q]`	gives the characteristic polynomial of (the class) of the coquaternion `q`
`CompanionPolynomial[P]`	gives the companion polynomial of the polynomial `P`
`Conjugate[P]`	gives the conjugate of the polynomial `P`
`Eval[P,q]`	computes `P(q)`
`NumZeros[P]`	gives the number of each type of zeros of the polynomial `P`
`PolyForm[P,x]`	gives the standard representation of the real polynomial `P` in the variable `x`
`Polynomial[`c_n`,`c_{n-1}`,...,`c_1`,`c_0`]`	represents the polynomial $c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$
`PolynomialZeroQ[P,q]`	gives `True` if `q` is a zero of `P` and `False` otherwise
`PSimplify[P]`	gives the polynomial `P` with simplified coefficients
`Zeros[P]`	gives the zeros of the polynomial `P`

4 The nth roots of a coquaternion

The nth roots of a coquaternion q can be obtained by solving the equation $x^n - \mathsf{q} = 0$, through the use of the algorithm described in the previous section. However, as already pointed out, this may involve a numerical method for finding the zeros of a $2n$th degree real polynomial. In order to avoid this issue and have explicit exact solutions to the problem, one can use the polar form of a non-real coquaternion q together with a De Moivre's formula. When $\mathsf{q} \in \mathbb{R}$ the technique described in Section 3 allows to obtain exact solutions.

This approach was used in [11], where the authors gave a complete characterization of the nth roots ($n \geq 2$) of a coquaternion, extending the results of [20].

We recall that a coquaternion q is called *space-like, light-like* or *time-like* if $\det \mathsf{q} < 0$, $\det \mathsf{q} = 0$ or $\det \mathsf{q} > 0$, respectively; the sets of such coquaternions is denoted by $\mathbb{S}$, $\mathbb{L}$ and $\mathbb{T}$, respectively. When two coquaternions belong to the same set $\mathbb{S}$, $\mathbb{L}$ or $\mathbb{T}$, we say that they have the same nature. We also adopt the notations

$$\mathbb{T}_{\mathbb{S}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{S}\}, \quad \mathbb{T}_{\mathbb{L}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{L}\}, \quad \mathbb{T}_{\mathbb{T}} = \{\mathsf{q} \in \mathbb{T} : \underline{\mathsf{q}} \in \mathbb{T}\},$$
$$\mathbb{L}_{\mathbb{S}} = \{\mathsf{q} \in \mathbb{L} : \underline{\mathsf{q}} \in \mathbb{S}\}, \quad \mathbb{L}_{\mathbb{L}} = \{\mathsf{q} \in \mathbb{L} : \underline{\mathsf{q}} \in \mathbb{L}\}.$$

We observe that since $\det \mathsf{q} \geq \det \underline{\mathsf{q}}$, if q is space-like, $\underline{\mathsf{q}}$ is of the same nature and a light-like coquaternion can not have a time-like vector part. The function `Nature` to classify coquaternions is included in `Coquaternions` (see [12] for more details).

Any coquaternion $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \in \mathbb{S} \cup \mathbb{T}$, such that $\underline{\mathsf{q}} \notin \mathbb{L}$, has a polar representation in one of the forms

$$\mathsf{q} = \begin{cases} \|\mathsf{q}\|\big(\sinh\phi_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\cosh\phi_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{S}, \\ \|\mathsf{q}\|\big(\operatorname{sgn} \mathrm{q}_0 \cosh\psi_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\sinh\psi_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{T}_{\mathbb{S}}, \\ \|\mathsf{q}\|\big(\cos\theta_{\mathsf{q}} + \boldsymbol{\omega}_{\underline{\mathsf{q}}}\sin\theta_{\mathsf{q}}\big), & \text{if } \mathsf{q} \in \mathbb{T}_{\mathbb{T}}, \end{cases} \tag{4a}$$

where sgn is the usual sign function,

$$\sinh\phi_{\mathsf{q}} = \frac{q_0}{\|\mathsf{q}\|}, \quad \sinh\psi_{\mathsf{q}} = \frac{\|\underline{\mathsf{q}}\|}{\|\mathsf{q}\|}, \quad \cos\theta_{\mathsf{q}} = \frac{q_0}{\|\mathsf{q}\|}, \quad \sin\theta_{\mathsf{q}} = \frac{\|\underline{\mathsf{q}}\|}{\|\mathsf{q}\|}, \ \theta_{\mathsf{q}} \in (0,\pi) \tag{4b}$$

and

$$\boldsymbol{\omega}_{\underline{\mathsf{q}}} = \frac{\underline{\mathsf{q}}}{\|\underline{\mathsf{q}}\|} \tag{4c}$$

is a unit coquaternion satisfying $\boldsymbol{\omega}^2_{\underline{\mathsf{q}}} = 1$, if $\underline{\mathsf{q}} \in \mathbb{S}$ and $\boldsymbol{\omega}^2_{\underline{\mathsf{q}}} = -1$, if $\underline{\mathsf{q}} \in \mathbb{T}$.

We recall that the polar form (4) is implemented in `Coquaternions` through the use of the function `PolarForm[q]` which gives a list of one of the forms

$$\begin{aligned}&\{\mathsf{nat} \to \mathsf{Spacelike}, r \to \|\mathsf{q}\|, \varphi \to \phi_{\mathsf{q}}, \omega \to \boldsymbol{\omega}_{\underline{\mathsf{q}}}\},\\ &\{\mathsf{nat} \to \mathsf{TimeSpacelike}, r \to \|\mathsf{q}\|, \varphi \to \psi_{\mathsf{q}}, \omega \to \boldsymbol{\omega}_{\underline{\mathsf{q}}}, \mathsf{sgn} \to \operatorname{sgn} q_0\},\\ &\{\mathsf{nat} \to \mathsf{TimeTimelike}, r \to \|\mathsf{q}\|, \varphi \to \theta_{\mathsf{q}}, \omega \to \boldsymbol{\omega}_{\underline{\mathsf{q}}}\},\end{aligned}$$

according to (4a)-(4c). The function `FromPolarForm` reads a list of one of the above forms and gives the corresponding coquaternion. For more details on the use of these functions we refer to the work [12].

In what follows, we use ζ_k and η_k to denote the complex nth $(n \geq 2)$ roots of 1 and -1, respectively, i.e.,

$$\zeta_k = \cos\tfrac{2k\pi}{n} + \mathsf{i}\sin\tfrac{2k\pi}{n};\ k = 0, 1, 2 \ldots, n-1$$

and

$$\eta_k = \cos\tfrac{(2k+1)\pi}{n} + \mathsf{i}\sin\tfrac{(2k+1)\pi}{n};\ k = 0, 1, 2 \ldots, n-1.$$

The nth roots of a coquaternion q depend on the nature of q, on the parity of n and (eventually) on the sign of the real part of q. Since these roots are the zeros of the equation $x^n = \mathsf{q}$ and linear zeros never occur in the case of polynomials of this form, the roots of a coquaternion q are either isolated or hyperboloidal.

The new function `RootsN[q]` implements the algorithm described in [11, Theorems 3.1, 3.2, 4.3] for obtaining the nth roots of the coquaternion q. The output is a list with two lists: the first one contains the isolated roots, while the second one contains representatives of the hyperboloidal roots (compare with the function `Zeros` described in Section 3). To offer a glimpse of the diversity of roots that a coquaternion may have, we now present several examples.

Example 7. The coquaternion $\mathsf{q} = 1 + \mathsf{i} + \mathsf{j} + \mathsf{k}$ has a positive real part and is in $\mathbb{L}_{\mathbb{S}}$. Therefore its nth roots are in $\mathbb{L}_{\mathbb{S}}$ and the number of roots depends on the parity of n.

```
 In[40]:= Nature2[q_Coquaternion]:=Nature/@{q,Vec[q]};
           SetAttributes[Nature2,Listable];

 In[41]:= q=Coquaternion[1,1,1,1];Nature2[q]
Out[41]= {Lightlike, Spacelike}

 In[42]:= ({iroots,hroots}=RootsN[q,2])//TraditionalForm
```

Out[42]= $\{\{-\frac{1}{\sqrt{2}}-\frac{\mathbb{i}}{\sqrt{2}}-\frac{\mathbb{j}}{\sqrt{2}}-\frac{\mathbb{k}}{\sqrt{2}}, \frac{1}{\sqrt{2}}+\frac{\mathbb{i}}{\sqrt{2}}+\frac{\mathbb{j}}{\sqrt{2}}+\frac{\mathbb{k}}{\sqrt{2}}\}, \{\}\}$

```
 In[43]:= Union[iroots^2]=={q}
Out[43]= True

 In[44]:= Nature2[iroots]
Out[44]= {{Lightlike, Spacelike}, {Lightlike, Spacelike}}

 In[45]:= ({iroots,hroots}=RootsN[q,3])//TraditionalForm
```

Out[45]= $\left\{\left\{\frac{1}{2^{2/3}}+\frac{\mathbb{i}}{2^{2/3}}+\frac{\mathbb{j}}{2^{2/3}}+\frac{\mathbb{k}}{2^{2/3}}\right\}, \{\}\right\}$

```
 In[46]:= Nature2[iroots]
Out[46]= {{Lightlike, Spacelike}}
```

Example 8. The coquaternion $\mathsf{q} = 3 + 2\mathsf{j}$ has a positive real part and is in $\mathbb{T}_{\mathbb{S}}$. The number and nature of the nth roots of q depend on the parity of n.

```
 In[47]:= q=Coquaternion[3,0,2,0];PolarForm[q]
```

Out[47]= $\{\mathtt{nat} \to \mathtt{Timelike\text{-}Spacelike}, r \to \sqrt{5}, \psi \to \mathtt{ArcSinh}\left[\frac{2}{\sqrt{5}}\right], \omega \to \mathtt{Coquaternion}[0, 0, 1, 0], \mathtt{sgn} \to 1\}$

```
 In[48]:= ({iroots,hroots}=RootsN[q,2])//TraditionalForm
```

Out[48]= $\{\{\frac{1}{2}(1+\sqrt{5})+\frac{1}{2}(-1+\sqrt{5})\mathbb{j}, \frac{1}{2}(-1-\sqrt{5})+\frac{1}{2}(1-\sqrt{5})\mathbb{j},$

$\frac{1}{2}(-1+\sqrt{5})+\frac{1}{2}(1+\sqrt{5})\mathbb{j}, \frac{1}{2}(1-\sqrt{5})+\frac{1}{2}(-1-\sqrt{5})\mathbb{j}\Big\}, \{\}\Big\}$

```
In[49]:= Tally[Nature2[iroots]
```

Out[49]= $\{\{\{$`Timelike, Spacelike`$\}, 2\}, \{\{$`Spacelike, Spacelike`$\}, 2\}\}$

```
In[50]:= Select[iroots,Nature2[#]=={Timelike,Spacelike}&]
```

Out[50]= $\left\{\frac{1}{2}(1+\sqrt{5})+\frac{1}{2}(-1+\sqrt{5})\mathbb{j}, \frac{1}{2}(-1-\sqrt{5})+\frac{1}{2}(1-\sqrt{5})\mathbb{j}\right\}$

Example 9. The nth roots of a non-real coquaternion q are all isolated. When $\mathsf{q} \in \mathbb{R}$ the situation is quite different. Here one can also find hyperboloidal roots. We compute now some of the roots of 1 and -1.

```
In[51]:= roots=Table[RootsN[ToCoquaternion[1],k],{k,2,10}];

In[52]:= Map[Length,roots,{2}]
```

Out[52]= $\{\{2,1\},\{1,1\},\{2,2\},\{1,2\},\{2,3\},\{1,3\},\{2,4\},\{1,4\},\{2,5\}\}$

```
In[53]:= TableForm[
          Transpose@Prepend[Transpose@Take[roots,5],Range[2,6]]//
          TraditionalForm,TableDepth→2, TableHeadings→
          {None,{"n","Isolated roots","Hyperboloidal roots"}}]
```

Out[53]/TableForm=

n	Isolated roots	Hyperboloidal roots
2	$\{-1,1\}$	$\{\mathbb{j}\}$
3	$\{1\}$	$\left\{-\frac{1}{2}+\frac{\sqrt{3}\mathbb{i}}{2}\right\}$
4	$\{-1,1\}$	$\{\mathbb{j},\mathbb{i}\}$
5	$\{1\}$	$\left\{-\frac{1}{4}+\frac{\sqrt{5}}{4}+\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\mathbb{i}, -\frac{1}{4}-\frac{\sqrt{5}}{4}+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\mathbb{i}\right\}$
6	$\{-1,1\}$	$\left\{-\frac{1}{2}+\frac{\sqrt{3}\mathbb{i}}{2}, \mathbb{j}, \frac{1}{2}+\frac{\sqrt{3}\mathbb{i}}{2}\right\}$

```
In[54]:= roots=Table[RootsN[ToCoquaternion[-1],k],{k,2,10}];

In[55]:= Map[Length,roots,{2}]
```

Out[55]= $\{\{0,1\},\{1,1\},\{0,2\},\{1,2\},\{0,3\},\{1,3\},\{0,4\},\{1,4\},\{0,5\}\}$

```
In[56]:= TableForm[
          Transpose@Prepend[Transpose@Take[roots,5],Range[2,6]]//
          TraditionalForm,TableDepth→2,TableHeadings→
          {None,{"n","Isolated roots","Hyperboloidal roots"}}]
```

Out[56]/TableForm=

n	Isolated roots	Hyperboloidal roots
2	$\{\}$	$\{\mathbb{i}\}$
3	$\{-1\}$	$\left\{\frac{1}{2}+\frac{\sqrt{3}\mathbb{i}}{2}\right\}$
4	$\{\}$	$\left\{\frac{1}{\sqrt{2}}+\frac{\mathbb{i}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}+\frac{\mathbb{i}}{\sqrt{2}}\right\}$
5	$\{-1\}$	$\left\{\frac{1}{4}+\frac{\sqrt{5}}{4}+\sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}\mathbb{i}, \frac{1}{4}-\frac{\sqrt{5}}{4}+\sqrt{\frac{5}{8}+\frac{\sqrt{5}}{8}}\mathbb{i}\right\}$
6	$\{\}$	$\left\{\frac{\sqrt{3}}{2}+\frac{\mathbb{i}}{2}, \mathbb{i}, -\frac{\sqrt{3}}{2}+\frac{\mathbb{i}}{2}\right\}$

Table 3 contains a description of the functions related to the computation of the roots of a coquaternion.

5 Conclusions

This work presents a collection of Mathematica tools – `CoqPolynomial` – complementing the package – `Coquaternions` – introduced in [12]. `CoqPolynomial` contains several functions to manipulate coquaternionic polynomials, including the function `Zeros` for polynomial rootfinding. These packages are fundamental tools supporting the recent interests of the authors in the area and should be considered as work in progress; their current versions are available at the web-page http://w3.math.uminho.pt/Coquaternions.

Table 3: Functions associated with roots of a coquaternion

Function	Description
`FromPolarForm[l]`	gives the coquaternion whose "polar coordinates" are `l`
`LightlikeQ[q]`	gives `True` if `q` is light-like and `False` otherwise
`Nature[q]`	classifies `q` as space-like, light-like or time-like
`PolarForm[q]`	gives the polar form of `q` (in case of existence)
`RootsN[q,n]`	gives the `n`th roots of `q`
`SpacelikeQ[q]`	gives `True` if `q` is space-like and `False` otherwise
`TimelikeQ[q]`	gives `True` if `q` is time-like and `False` otherwise

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