Universidade do Minho
Escola de Engenharia
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Bernardo Almeida Leite Faria

Combining Paraconsistent And Dynamic Logic For Qiskit

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DIREITOS DE AUTOR E CONDIÇÕES DE UTILIZAÇÃO DO TRABALHO POR TERCEIROS

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STATEMENT OF INTEGRITY

I hereby declare having conducted this academic work with integrity. I confirm that I have not used plagiarism or any form of undue use of information or falsification of results along the process leading to its elaboration.

I further declare that I have fully acknowledged the Code of Ethical Conduct of the University of Minho.

ABSTRACT

This dissertation introduces a logic aimed at combining dynamic logic and paraconsistent logic for application to the quantum domain, to reason about quantum phase properties: *Paraconsistent Phased Logic Of Quantum Programs (PhLQP°)*.

In the design PhLQP°, firstly the dynamic was built first, *Phased Logic Of Quantum Programs (PhLQP)*. PhLQP is itself a dynamic logic capable of dealing with quantum phase properties, quantum measurements, unitary evolutions, and entanglements in compound systems , since it is a redesign of the already existing *Logic Of Quantum Programs (LQP)*, [14], over a representation of quantum states restricted to a space $\mathcal{B}$ equipped with only two computational basis, standard and Hadamard. As instances of applications of the logic PhLQP, there is a formal proof of the correctness of the Quantum Teleportation Protocol, of the 2-party and 4-party of the *Quantum Leader Election (QLE)* protocol, and of the *Quantum Fourier Transform (QFT)* operator for 1, 2 and 3 qubits .

On a second stage, PhLQP was extended with the connective $\circ$ known as the consistency operator, a typical connective of the paraconsistent logics *Logics of Formal Inconsistency (LFIs)*, [8, 21, 22]. The definition of consistent quantum state and a set of proper paraconsistent axioms for the quantum domain, *Fundamental Paraconsistent Quantum Axioms (FParQAxs)*, were provided.

An example of application of PhLQP° is the possibility of express and prove correctness of the universal quantum gate, the Deustch gate .

Keywords: Quantum Phase Properties, Dynamic Quantum Logic, Paraconsistent Dynamic Quantum Logic, PhLQP, PhLQP°, Quantum Teleportation Protocol, Quantum Leader Election Protocol, Quantum Fourier Transform , Deustch Gate.

RESUMO

Esta dissertação introduz uma lógica que tem como objectivo combinar lógica dinâmica e lógica paraconsistente com aplicação no domínio quântico, assim como expressar propriedades relacionadas com fases quânticas: PhLQP$^\circ$.

No projetar da PhLQP$^\circ$, primeiramente concebeu-se a sua componente dinâmica, PhLQP. PhLQP por si só é uma lógica capaz de lidar com propriedades de fases quânticas, evoluções unitárias, e entrelaçamento em sistemas compostos, uma vez que é um redesenhar da já existente LQP, [14], sobre uma representação de estados quânticos restrita a um espaço $\mathcal{B}$ munido de apenas duas bases computacionais, standard e Hadamard. Como instâncias de aplicação da lógica PhLQP, há uma prova formal para a correção do protocolo de Teletransporte Quântico, para o protocolo QLE para uma *party* quer de 2 quer de 4 agentes , e para o operador de QFT de 1, 2, e 3 qubits .

Numa segunda fase, PhLQP é extendida com a conectiva $\circ$, conhecida como operador de consistência, uma conectiva característica das LFIs, [8, 21, 22]. E a partir desta conectiva a definição de estado quântico consistente e um conjunto de axiomas paraconsistentes próprios para o domínio quântico, FParQAxs.

Um exemplo de aplicação da PhLQP$^\circ$ é a possibilidade de expressar e permitir correção para o comportamento da gate quântica universal, a Deutsch-gate.

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ACRONYMS

GLSSYMBOLS

$\mathcal{L}_{\text{PHLQP}^{\circ}}$ Language of Paraconsistent Phased Logic of Quantum Programs.

$\mathcal{L}_{\text{PHLQP}}$ Language of Phased Logic of Quantum Programs.

A

A-MACHINE automatic machine.

C

C-MACHINE choice machine.

CL classical propositional logic.

D

DFT Discrete Fourier Transform.

DTM Deterministic Turing Machine.

E

ECSQ "ex contracdictione sequitur quodlibet".

EPARTM Entangled Paraconsistent Turing Machine.

F

FOL First Order Logic.

FPARQAXS Fundamental Paraconsistent Quantum Axioms.

L

LFI Logic of Formal Inconsistency.

LFIS Logics of Formal Inconsistency.

LQA Logic of Quantum Actions.

LQP Logic Of Quantum Programs.

M

MBC most basic C-system.

N

NDTM Non-Deterministic Turing Machine.

P

PARTM Paraconsistent Turing Machine.

PDL Propositional Dynamic Logic.

PHLQP Phased Logic Of Quantum Programs.

PHLQP$^{\circ}$ Paraconsistent Phased Logic Of Quantum Programs.

PHQF Phased Quantum Frame.

PHQF$^{\circ}$ Paraconsistent Phased Quantum Frame.

PLQP Probabilistic Modal Dynamic Logic of Quantum Programs.

PRTM Probabilistic Turing Machine.

Q

QC Quantum Computation.

QCM Quantum Circuit Model.

QF Quantum Frame.

QFT Quantum Fourier Transform.

QI Quantum Information.

QL Quantum Logic.

QLE Quantum Leader Election.

QM Quantum Mechanics.

QPDSOL Quantum Probabilistic Dyadic Second-Order Logic.

QTM Quantum Turing Machine.

QTS Quantum Transition System.

T

TCR Tarskian consequence relation.

TM Turing Machine.

1

INTRODUCTION

1.1 CONTEXT

Since the origin of *Quantum Logic (QL)* until earlier 2000's, there was not a well-established link between traditional QL and *Quantum Computation (QC)*. This link was formed mainly by A.Baltag and S.Smets through a dynamic approach of traditional QL, [12].

In this way, it was enabled a relation between QL and QC models, such as the *Quantum Circuit Model (QCM)*. The QCM, as described by G.Nannicini in [41], is the model of computation implemented, for example, in IBM's Qiskit [38, 39], consisting of the following steps:

1. The quantum computing device has a quantum register that carry a *state* and is initialized in a predetermined way.

2. Subsequently, by applying certain *operations*, and combining them into an algorithm, it makes the state evolve.

3. When the computation ends, it is possible to acquire some information about the state of the quantum register through a special operation, designated by *measurement*.

The QCM must obey the conditions imposed by *Quantum Mechanics (QM)* as a theoretical framework.

QM on its own provides a formal framework (e.g. [3, 4, 5]) for modelling physical systems and describing these laws. There is a necessity of establish a link between the reality of the physical world and the mathematical formalism of QM . Such link is known as **the postulates of QM**. The postulates of QM, as approached in I.L. Chuang and M. A. Nielsen [42], are summarized in the sequel. The first postulate sets up the sphere of action of QM .

POSTULATE 1: For any *isolated* physical system there is a relatable complex vector space provided of inner product (thus, a Hilbert space) designated by *state space* of the system. A completed description of the system can be obtained from its *state vector*, which in typically presented as a unit vector .

The second postulate describes how a state $|\psi\rangle$ of a quantum mechanical system evolves over time.

POSTULATE 2: The evolution of a *closed* quantum system can be described by an *unitary transformation*. Therefore, a state $|\psi\rangle$ of the system at an instant of time t is related to the state $|\psi'\rangle$ at an instant of time t' by a unitary operator U which only depends on the instant of time t and t':

$$|\psi\rangle = U\,|\psi'\rangle \quad .$$

Finally, the third postulate describes the nature of the quantum measurement.

POSTULATE 3: Quantum measurements can be described by a set M of *measurement operators*. So when the system is being measured this set of operators acts on the state space, e.g. if the state of a quantum system is $|\psi\rangle$ promptly before the measurement then the outcome of the measurement occurs with a probability p given by:

$$p = \langle\psi|\, M^{\dagger} M\,|\psi\rangle \quad .$$

The inherent information to QCM is *Quantum Information (QI)*. QI is a more denser kind of information than the classical one, since it allows superposition state such $|\psi\rangle$ as a information state. $|\psi\rangle$ can be described as the following linear combination :

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \quad \text{such that} \quad |\alpha|^2 + |\beta|^2 = 1$$

In this way, if someone asks: " Is the qubit in the state $|0\rangle$ or in the state $|1\rangle$?", it would be very tempting to answer this question with: "The qubit is in both states". Therefore, it is possible to say that this information seems contradictory, although at a physical level such a contradiction vanishes when a quantum measurement is performed. On the other hand, before collapsing the superposition state with a quantum measurement it is not possible to predict with total certain its that such state will collapse.

This leads to the question whether a paraconsistent logics,e.g. LFIs, may help in reasoning about quantum systems. Some attempts to answer this question can be found in the work of J. C. Agudelo and W. Carnielli, in [6, 7], through the concept to *Paraconsistent Turing Machine (ParTM)*.

1.2 MOTIVATION

A quantum dynamic logic is a logic capable of characterize a quantum state by the transitions specified by the quantum program, providing a way to express the inherent quantum state to the quantum program over time. Although there is a certain variety of quantum dynamic logics [10, 11, 14, 18], non-so far embeds paraconsistent features in order to reason over QI, as well as a way of express quantum phases.

A paraconsistent logic is a logic capable of treat apparently contradictory information, e.g., quantum superposition states, as more "informative" information. In other words, it is a logic that can take profit from contradictions, to characterize information.

Combining paraconsistent features such as observed in LFIs ([6, 7, 8, 21]) with a quantum dynamic logic able of express quantum phases may perhaps be an option to express the behaviour of quantum systems/programs.

1.3 CONTRIBUTIONS

The contributions of this dissertation can be summarized as follows:

1. An exhaustive study about dynamic quantum logics, as well as of paraconsistent logics.
2. A dynamic quantum logic able to express phases of quantum states: PhLQP
3. A attempt of embedding paraconsistent features in a dynamic quantum logic: PhLQP$^{\circ}$.
4. A way to express the behaviour of the Deutsch-Gate through a dynamic paraconsistent quantum logic.

1.4 OUTLINE

The first section of Chapter 2 provides a contextualization for the chronological path leading from Quantum Logic to Dynamic Quantum Logic, as well as a review of literature with important background concepts related do the later. Section 2.2 consists of the development of the logic PhLQP, starting with the characterization of the quantum space and frames, its syntax, semantics and proof theory. The final section presents some worked examples showing PhLQP at work.

Chapter 3, in its Section 3.1, provides background for some preliminary concepts on propositional logics, along with background for paraconsistency in Section 3.2. Additionally, there is a contextualization and review of literature on paraconsistent logics in quantum computing in Section 3.3 Section 3.4 combines PhLQP with paraconsistent features. In the final Section 3.5 examples of inconsistency in quantum computing, related, for example, to the behaviour of the Deutsch gate, are discussed. Finally, Chapter 4 concludes and provides a number of suggestions for future work .

2

DYNAMIC LOGICS FOR QUANTUM PROGRAMS

2.1 FROM QUANTUM LOGIC TO DYNAMIC QUANTUM LOGIC

QL has its origins in Garret Birkhoff and John von Neumann article [19], where is stated that the logic inherent to the formalism of QM is a non-classical one, i.e. it is stated that QM forces to adopt a non-Boolean logic [25, 26, 37], since reasoning with a Boolean logic over quantum entities will give rise to quantum paradoxes, [3]. Consequently, a few decades after, in late 1950's, the article has caused the flourish of the question, whose attempt to understanding has bloomed in the 1970's among the scientific community, about if QM implies the necessity of giving up the fundamental principles of classical propositional logic, [34].

More recently, in earlier 2000's, a trend towards a dynamic approach of QL emerged mainly by A.Baltag and S.Smets in the paper [12], by holding the concept of quantum information systems in a dynamic way (i.e. in a such way that a *state* of a quantum system is characterized only by the quantum actions which can be successfully executed on this same *state*) through an axiomatization for a *Logic of Quantum Actions (LQA)* [15, 16, 17] called *Quantum Transition System (QTS)*. Therefore , in this way , this paper [12] has established a connection between the traditional QL and the proper requirements of QC by giving a perspective of quantum structures [35] that regards their fundamental *logical dynamics*, which includes the late 1990's developments in modal logic, e.g. [50].

QUANTUM TRANSITION SYSTEMS [12, 13]. QTSs , as Hilbert spaces structured as non-classical relational models of *Propositional Dynamic Logic (PDL)*, [36], are defined by a set of *states* Σ, and by a family of elementary transitions relations between states in Σ , $\longrightarrow \subseteq \Sigma \times \Sigma$. The states are intended to express *possible states of a physical system* , while the transition relations depict the transformations of state induced by *possible actions* that can be executed on the system. In a Hilbert space H, the "states" are *one-dimensional subspaces* of H denominated by *rays*, while the actions are expressed by certain *linear maps* on H. Moreover, there are two core kinds of elementary actions: " quantum tests" $\varphi?$ ($\{\xrightarrow{\varphi?}\}$) and "quantum gates " U ($\{\xrightarrow{U}\}$) . Tests express successful measurements of certain yes/no property φ, with a test φ? standing as a *projector* onto φ in a Hilbert space H. Quantum gates U correspond to *reversible evolutions* of the observed system, with them standing for *unitary transformations* in a Hilbert space H.

LOGIC OF QUANTUM ACTIONS [13, 17]. LQA is a *Boolean* logic. In LQA, the propositional connectives meet all the classical laws of propositional logic: characteristic formulas of dynamic logic refer to potential *properties of quantum states*, which in a Hilbert space H match to arbitrary unions of rays. The bivalence of LQA becomes from the fact of any such quantum property holding or not at a given state. The negation $\neg\varphi$ of a certain quantum property φ basically expresses the reality where the property φ does not hold . In this sense , there is a $\neg$- bivalent interpretation. Nonetheless, in LQA there are some expressible properties which aren't "testable" (i.e., there are properties that do not correspond to an "experimental" property) . In specific, the negation does not preserve the "testability" of a property, i.e. the negation of a testable property may not be testable. Also, in LQA the testable properties are the properties that can be expressed by negation free formulas. This is, any formula made without the use of Boolean negation implies a testable property.

A SPECIAL FEATURE OF LQA [13]. LQA has the special feature of embedding a reinterpretation of traditional QL through a *dynamic interpretation of the non-classical connectives of QL* . Therefore, through the definition of the *orthocomplement* $\sim\varphi$ of a property as the *impossibility of a successful test*, i.e. $\sim := [\varphi?]\bot$, the quantum join can be define by means of de Morgan law : $\varphi \sqcup \psi := \sim (\sim \varphi \wedge \sim \psi)$. Quantum join depicts all possible *superpositions* of states satisfying φ and states satisfying ψ . The "quantum implication ", also known as Sasaki hook, is basically obtained by the weakest precondition of a "test" : $\varphi \xrightarrow{S} \psi := [\varphi?]\psi$.

A "concrete" example of a QTS is a QF, and a "concrete" example of a LQA is the LQP. Both the notion of a QF and the LQP are presented in the paper [14] by A.Baltag and S.Smets.

THE LOGIC OF QUANTUM PROGRAMS [14] . LQP is a logic of a finitary syntax and a relational semantics which is able of dealing with elementary actions as unitary transformations , complex actions as quantum measurements, and quantum entanglements in compound systems. Therefore, LQP is a dynamic logic with the objective of reasoning about quantum information flow in quantum programs.

A PROBABILISTIC VARIATION OF LQP [10, 11, 18] A probabilistic variation of LQP is the *Probabilistic Modal Dynamic Logic of Quantum Programs (PLQP)*. PLQP by itself is a *Quantum Probabilistic Dyadic Second-Order Logic (QPDSOL)* . So, PLQP is a logic gifted with tensor operators to express inherent properties of compound systems (e.g. the capacity of expressing whether a state is separable or entangled), and with probabilistic predication formulas $P^{\geq r}(s)$ asserting that a quantum state in a state s will have a successful quantum test (i.e. a quantum test with the answer "yes") with a probability of at least r every time a quantum measurement is performed over property P. In PLQP, there is also two second-order quantifiers, one applicable over quantum testable properties, the other one over quantum "actions".

All the so far mentioned logics seem to fail at expressing quantum phase fundamental properties, leading to a main question: "How a logic capable of expressing quantum phase properties would be ?". In the next Section 2.2 it will be an attempt to answer this question by re-interpret the LQP over an unorthodox concept of quantum states. This is, quantum states will be seen more likewise "vectors" (instead of rays) and expressed by pairs. So, it is possible to have a very similar approach of interpreting quantum states as presented in the paper [40]: many states share the same ray.

2.2 Dynamic Quantum Logic

2.2.1 *The Quantum Space and Frames*

The quantum finite dimension space It is possible to characterize the quantum mechanics "workspace" as a finite dimension Hilbert space $\mathcal{H}$, similarly as described in the paper [2] by S.Abramsky and B.Coecke. However, the *state space* were the logic PhLQP presented in the sequel will work is denoted by $\mathcal{B}$.$\mathcal{B}$ is equipped with only two bases: the standard basis ($\{|0\rangle, |1\rangle\}$) and the Hadamard basis ($\{|+\rangle, |-\rangle\}$):

- $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$,
- $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$.

An element of $\mathcal{B}$ is a vector v with the form of $e^{i\theta}|\sigma\rangle$ such that $|\sigma\rangle \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$, and $0 \leq \theta < 2\pi$.

A *state* of the logic PhLQP will be described as the product of two sets: $\Sigma \times \Phi$. The set Σ will hold the information about the basis of the state/vector, and the set Φ will hold the information about the phases related to the state/vector.

Let ϕ be an element of Φ, $\phi \in \mathbb{Q}$ with $0 \leq \phi < 1$, it is possible to express a phase θ of a vector $v \in \mathcal{B}$ by:

$$\theta = 2\pi.\phi \ .$$

Such that a v is represented by:

$$v = e^{i\theta}|\sigma\rangle = e^{2\pi i.\phi}|\sigma\rangle, \quad \sigma \in \Sigma,\ \phi \in \Phi.$$

A vector $v \in \mathcal{B}$ represents a quantum state in the conventional sense, i.e., an element of and an Hilbert space, i.e., $v \in \mathcal{B} \subset \mathcal{H}$.

Example 2.2.1 (Representing quantum states I) *The following set of pairs* (σ, ϕ)*:*

$$\{(0,0),(1,\frac{1}{4}),(1,\frac{1}{2}),(-,\frac{1}{2}),(+,\frac{1}{4})\}$$

Represent the following quantum states :

$$\begin{aligned}\{e^{2\pi i.0}\,|0\rangle\,,\ e^{2\pi i.\frac{1}{4}}\,|1\rangle\ ,e^{2\pi i.\frac{1}{2}}\,|-\rangle\,,\ e^{2\pi i.\frac{1}{4}}\,|+\rangle\} =&\{|0\rangle\,,\ i\,|1\rangle\,,\ -|1\rangle\,,\ -|-\rangle\,,\ i\,|+\rangle\}\\ =&\{|0\rangle\,,\ i\,|1\rangle\,,\ -|1\rangle\,,\ -(\frac{|0\rangle-|1\rangle}{\sqrt{2}}),\ i(\frac{|0\rangle+|1\rangle}{\sqrt{2}})\}\ .\end{aligned}$$

For every *state space* $\mathcal{B}$ there is a unique morphism $f : \mathcal{B} \longrightarrow \Sigma \times \Phi$ denominated by universal arrow and denoted by the *product of morphisms* $\langle f_\Sigma, f_\Phi\rangle$ with $f_\Sigma : \mathcal{B} \longrightarrow \Sigma$ and $f_\Phi : \mathcal{B} \longrightarrow \Phi$. There are two *projection morphisms* $\pi_\Sigma : \Sigma \times \Phi \longrightarrow \Sigma$ and $\pi_\Phi : \Sigma \times \Phi \longrightarrow \Phi$ such that:

$$\pi_\Sigma \circ \langle f_\Sigma, f_\Phi\rangle = f_\Sigma \ \wedge\ \pi_\Phi \circ \langle f_\Sigma, f_\Phi\rangle = f_\Phi \quad .$$

Let be a vector $v \in \mathcal{B}$, it is possible to abbreviate $f(v)$ by $\overline{v}$.

Compound systems can be expressed by tensor products of the component systems. Consequently, in *compound systems*, *entanglement* can appear due to the fact of a vector in $\mathcal{B}_1 \otimes \mathcal{B}_2$ be given by the following type of expression :

$$\psi\ \otimes\ \gamma = \sum_{i=1}^{n} \alpha_i\ .\ (\psi_i\ \otimes\ \gamma_i)$$

Where α_i is the coefficient related to the probability of the state $\psi_i\ \otimes\ \gamma_i$ and $\psi\ \otimes\ \gamma \in \mathcal{B}_1 \otimes \mathcal{B}_2$, $\psi \in \mathcal{B}_1$, $\gamma \in \mathcal{B}_2$. Sequentially , there is :

$$\psi\ \otimes\ \gamma = e^{2\pi i((f_\Phi(\psi)+f_\Phi(\gamma))\ mod\ 1)}\,|f_\Sigma(\psi) f_\Sigma(\gamma)\rangle$$

With :

- $f_\Sigma(\psi) = f_\Sigma(\psi_1) f_\Sigma(\psi_2) \ldots f_\Sigma(\psi_n)$
- $f_\Sigma(\gamma) = f_\Sigma(\gamma_1) f_\Sigma(\gamma_2) \ldots f_\Sigma(\gamma_n)$

- $f_\Phi(\psi) = \sum_{i=1}^{n} f_\Phi(\psi_i)$
- $f_\Phi(\gamma) = \sum_{i=1}^{n} f_\Phi(\gamma_i)$

Example 2.2.2 (Representing quantum states II) *For the following quantum states* $\psi = e^{2\pi i(\frac{3}{2})}(\frac{|00\rangle+|01\rangle}{\sqrt{2}})$ *and* $\gamma = e^{2\pi i(\frac{1}{2})}\,|11\rangle$ *:*

- $f_\Sigma(\psi) = 1+$
- $f_\Sigma(\gamma) = 11$
- $f_\Phi(\psi) = \frac{3}{2}$
- $f_\Phi(\gamma) = \frac{1}{2}$

Which corresponds to the following set of pairs :

$$\{\ \underbrace{(1+, \frac{3}{2})}_{\psi}\ ,\ \underbrace{(11, \frac{1}{2})}_{\gamma}\} \quad .$$

There is :

$$\begin{aligned}\psi \otimes \gamma &= e^{2\pi i((\frac{3}{2}+\frac{1}{2}) \mod 1)}\,|1+11\rangle \\ &= e^{0}\,|1+11\rangle = |1+11\rangle \\ &= \frac{|1011\rangle + |1111\rangle}{\sqrt{2}} \quad .\end{aligned}$$

With the pair $(1+11,\ 0)$ *representing the quantum state given by* $\psi \otimes \gamma$ *.*

The adjoint map of a linear map $m : \mathcal{B} \longrightarrow \mathcal{B}$ is the linear map $m^\dagger : \mathcal{B} \longrightarrow \mathcal{B}$ such that for all $\psi, \gamma \in \mathcal{B}$:

$$\langle \gamma | m(\psi) \rangle_{\mathcal{B}} = \left\langle m^\dagger(\gamma) \middle| \psi \right\rangle_{\mathcal{B}} \quad .$$

A *unitary transformation* is a linear isomorphism $U : \mathcal{B} \longrightarrow \mathcal{B}$ such that $U^{-1} = U^\dagger : \mathcal{B} \longrightarrow \mathcal{B}$, and :

- $U^\dagger \circ U = U^{-1} \circ U = id_{\mathcal{B}}$.
- $U \circ U^\dagger = U \circ U^{-1} = id_{\mathcal{B}}$.

With $id_{\mathcal{B}}$ representing the identity map on $\mathcal{B}$:

$$id_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathcal{B}$$
$$id_{\mathcal{B}}(\psi) = \psi,\ \forall \psi \in \mathcal{B} \quad .$$

Unitary transformations preserve the *inner product*:

$$\langle U(\psi) | U(\psi') \rangle_{\mathcal{B}} = \left\langle UU^{\dagger}(\psi) \middle| \psi' \right\rangle_{\mathcal{B}} \quad .$$

Self-adjoint operators are linear transformations $M : \mathcal{B} \longrightarrow \mathcal{B}$ that hold $M = M^{\dagger}$.

The *elementary data transformations* are unitary transformations, therefore are *reversible*, e.g. quantum gates.

In a quantum system , a realizable *measurement* is represented by self-adjoint operators, such as projectors. Thereby, for any closed linear subspace $\mathcal{S} \subseteq \mathcal{B}$, a projector $P_{\mathcal{S}} : \mathcal{B} \longrightarrow \mathcal{B}$ onto $\mathcal{S}$ is obtained by :

$$P_{\mathcal{S}}(s + s') = s, \forall s \in \mathcal{S},\ s' \in \mathcal{S}^{\perp}$$

Also, projectors are linear, idempotent ($P_i \circ P_i = P_i$) , self-adjoint ($P_i = P_i^{\dagger}$) and mutually orthogonal ($P_i \circ P_j = 0,\ i \neq j$) . Therefore, a measurement can be interpreted as a set of projectors, although in a successful measurement just one of the projectors "survives" and the measurement's result is given by this projector.

QUANTUM FRAMES FOR SINGLE-SYSTEMS For a given space $\mathcal{B}$, it is possible to construct a PhQF, i.e. a QTS with the concept of phase notion,

$$\Sigma \times \Phi(\mathcal{B}) := (\Sigma \times \Phi, \{\xrightarrow{P?}\}_{P \in \mathcal{L}}, \{\xrightarrow{U}\}_{U \in \mathcal{U}})$$

by taking in to account the respective considerations:

1. For any non-zero vector $v \in \mathcal{B}$ it is possible to denote a state $s \in \Sigma \times \Phi$ by the vector which generated it, i.e. $s = \overline{v}$. A state s will be represented by a pair $(\sigma, \phi) \in \Sigma \times \Phi$. Consequently, two vectors that differ only in phase will be distinguishable by the second element of their corresponded pairs as states .

   Example 2.2.3 (Two vectors and two states) *Suppose that v and $w \in \mathcal{B}$ are two vectors with :*

$$v = |0\rangle \quad \textit{and} \quad w = e^{2\pi i.(\frac{1}{4})} |0\rangle$$

and two states s and $t \in \Sigma \times \Phi$ such that :

$$s = \overline{v} = (0,0) \quad and \quad t = \overline{w} = (0, \frac{1}{4})$$

Then the states are phase distinguishable : $\pi_\Phi(s) \neq \pi_\Phi(t)$.

Definition 2.2.1 (State equality) *For two states s and t are the same state if and only if*

$$\pi_\Sigma(s) = \pi_\Sigma(t) \quad and \quad \pi_\Phi(s) = \pi_\Phi(t) \quad .$$

2. Two states s and t in $\Sigma \times \Phi$ are orthogonal, $s \perp t$, if for any two vectors $v \in s$ and $w \in t$ are orthogonal. In other words:

$$\begin{aligned} \text{If} \quad \forall v \in s, \forall w \in t, \langle v|w\rangle = 0 \quad \text{with} \\ \langle f_\Sigma(v)|f_\Sigma(w)\rangle = 0 \quad \text{and} \quad f_\Phi(v) = f_\Phi(w) \end{aligned}$$

Or equivalently ,

$$\begin{aligned} \text{Iff} \quad \exists v \in s,\ w \in t \quad \text{with} \quad v \neq 0,\ w \neq 0 \\ \text{and} \quad \langle f_\Sigma(v)|f_\Sigma(w)\rangle = 0 \quad \text{and} \quad f_\Phi(v) = f_\Phi(w) \quad . \end{aligned}$$

Therefore, for a set of states $S \subseteq \Sigma \times \Phi$, it is possible to write :

$$\begin{aligned} S^\perp := \{t \in \Sigma \times \Phi \mid t \perp s\ , \forall s \in S\} \\ \text{or} \quad S^\perp := \{t \in \Sigma \times \Phi \mid \pi_\Sigma(t) \perp \pi_\Sigma(s) \quad \text{and} \quad \pi_\Phi(t) \perp \pi_\Phi(s)\ , \forall s \in S\} \quad . \end{aligned}$$

With :

- $\pi_\Sigma(s) \perp \pi_\Sigma(t)$ standing for the orthogonality relation between the first element of the pair of the state s and the first element of the pair of the state t, i.e. the relation of orthogonality between the basis of s and t;
- $\pi_\Phi(s) \perp \pi_\Phi(t)$ representing the orthogonality relation between the second element of the pair of the state s and the second element of the pair of the state t, i.e. the relation of orthogonality between the phases of s and t;
- $S^\perp$ identifies the orthogonal set of S, this is $S^\perp$ as the set of orthogonal states of the states of S .

Example 2.2.4 (The orthogonal set of S) *For a set of sets $S \subseteq \Sigma \times \Phi$ with :*

$$S = \{(0,0), (1, \frac{1}{4}), (+, \frac{1}{2})\}$$

$S^{\perp}$ will be defined as :

$$S^{\perp} = \{(1,0), (0, \frac{1}{4}), (-, \frac{1}{2})\}$$

Definition 2.2.2 (State Orthogonality) *For two states s and t are orthogonal if and only if*

$$\pi_{\Sigma}(s) = \sim \pi_{\Sigma}(t) \quad \text{and} \quad \pi_{\Phi}(s) = \pi_{\Phi}(t)$$

with:

- $\sim$ *denoting the orthocomplement ;*
- $\pi_{\Phi}(s) = \sim \pi_{\Phi}(t) \Leftrightarrow \pi_{\Phi}(s) = \pi_{\Phi}(t)$.

Remark 2.2.1 *The phase ϕ of a given state s is invariant to the orthogonality of the state s .*

Definition 2.2.3 (Biorthogonality) *The biorthogonal closure of S can be written by $\overline{S}$ and defined by :*

$$\overline{S} := (S^{\perp})^{\perp} = S^{\perp\perp}$$

3. A set of states $S \subseteq \Sigma \times \Phi$ is said to be a quantum testable property if and only if it is *biorthogonally closed*, i.e. if $S = \overline{S}$ (with $S \subseteq \overline{S}$ being always the case).

Theorem 2.2.1 (Quantum testability) *The quantum testability of a set of states $S \subseteq \Sigma \times \Phi$ depends only on the quantum non-testability in Σ .*

Proof 2.2.1 (Theorem 2.2.1) *For a given set $S \subseteq \Sigma \times \Phi$ it is possible to see S as a product of two sets, i.e. $S := S_{\Sigma} \times S_{\Phi}$ with $S_{\Sigma} \subseteq \Sigma$ and $S_{\Phi} \subseteq \Phi$. Therefore, the orthogonal set of S becomes :*

$$S^{\perp} = (S_{\Sigma} \times S_{\Phi})^{\perp} = S_{\Sigma}^{\perp} \times S_{\Phi}^{\perp}$$

Consequently, due to the Definition 2.2.2:

$$S_{\Phi} = S_{\Phi}^{\perp} \quad \textbf{\textit{being}} \text{ \textit{always the case}}$$
$$S_{\Sigma} = S_{\Sigma}^{\perp} \quad \textbf{\textit{not being}} \text{ \textit{always the case}}$$

Sequentially, from Definition 2.2.3:

$$S_\Phi = S_\Phi^{\perp\perp} = \overline{S_\Phi} \quad \textbf{\textit{being}} \textit{ always the case}$$
$$S_\Sigma = S_\Sigma^{\perp\perp} = \overline{S_\Sigma} \quad \textbf{\textit{not being}} \textit{ always the case}$$

Finally,

$$S = \overline{S} \implies S_\Sigma = \overline{S_\Sigma}$$

So, the Theorem 2.2.1 has been proven .

On the other hand, it is possible to denote by $\mathcal{L} \subseteq \mathcal{P}(\Sigma \times \Phi)$ the family of all quantum testable properties. And for the other sets , which are non-testable properties , it is possible to write $S \in \mathcal{P}(\Sigma \times \Phi) \backslash \mathcal{L}$.

4. The existence of a natural bijective correspondence between $\mathcal{L}$, the family of all quantum testable properties, and $\mathcal{W}$, the family of all closed liner subspaces W of $\mathcal{B}$: a bijection given by $S \mapsto W_S =: \bigcup S$. Consequently, under this one-to-one correspondence, the image of the biorthogonal closure $\overline{S}$ of a given set $S \subseteq \Sigma \times \Phi$ is the closed linear subspace $\overline{\bigcup S} \subseteq \mathcal{B}$ generated by the union $\bigcup S$ of all states in S.

   Remark 2.2.2 *This natural bijective correspondence :*

   $$S \mapsto W_S =: \bigcup S$$

   Can be seen in fact as a product of two natural bijective correspondence, i.e.:

   $$S_\Sigma \times S_\Phi \mapsto W_{S_\Sigma \times S_\Phi} =: \bigcup (S_\Sigma \times S_\Phi)$$

   With:

   - $S_\Sigma \mapsto W_{S_\Sigma} =: \bigcup S_\Sigma$
   - $S_\Phi \mapsto W_{S_\Phi} =: \bigcup S_\Phi$

5. For every testable property $S \in \mathcal{L}$, there is a partial map $S?$ on $\Sigma \times \Phi$, designated by quantum test.

Definition 2.2.4 (Quantum test) *Let $W = W_S = \bigcup S$ be the corresponding subspace of $\mathcal{B}$, the quantum test will be the map induced on states by the projector P_W onto the subspace W, this is :*

$$S?(\overline{x}) := \overline{P_W(x)} \in \Sigma \times \Phi, \text{ if } \overline{x} \notin S^{\perp} \text{ (i.e. if } P_W(x) \neq 0)$$
$$S?(\overline{x}) := \text{ undefined, otherwise .}$$

Equivalently, due to the Definition 2.2.2:

$$S?(\overline{x}) := \overline{P_W(x)} \in \Sigma \times \Phi, \text{ if } \pi_{\Sigma}(\overline{x}) \notin S_{\Sigma}^{\perp} \text{ (i.e. if } P_W(x) \neq 0)$$
$$S?(\overline{x}) := \text{ undefined, otherwise .}$$

It is possible to write $\xrightarrow{S?} \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ for the pair of binary relations which is the partial map $S?$, i.e. the map which is given by :

$$s \xrightarrow{S?} t := (\pi_{\Sigma}(s) \xrightarrow{S_{\Sigma}?} \pi_{\Sigma}(t), \pi_{\Phi}(s) \xrightarrow{S_{\Phi}?} \pi_{\Phi}(t))$$
$$\text{iff} \quad S?(s) = t := (S_{\Sigma}?(\pi_{\Sigma}(s)) = \pi_{\Sigma}(t),\ S_{\Phi}?(\pi_{\Phi}(s)) = \pi_{\Phi}(t))$$

Therefore, there is a family of pairs of binary relations indexed by the testable properties $S \in \mathcal{L}$.

6. For each unitary transformation U on $\mathcal{B}$, there is also a pair of binary relations:

$$\xrightarrow{U} \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi) := (\xrightarrow{U_{\Sigma}} \subseteq \Sigma \times \Sigma,\ \xrightarrow{U_{\Phi}} \subseteq \Phi \times \Phi)$$

which is given by :

$$s \xrightarrow{U} t \quad \text{iff} \quad (f_{\Sigma}(U(x)) \xrightarrow{U_{\Sigma}} f_{\Sigma}(y), f_{\Phi}(U(x)) \xrightarrow{U_{\Phi}} f_{\Phi}(y))$$

for non-zero vectors x and y such that : $f_{\Sigma}(x) \in \pi_{\Sigma}(s) \wedge f_{\Phi}(x) \in \pi_{\Phi}(s)$ and $f_{\Sigma}(y) \in \pi_{\Sigma}(t) \wedge f_{\Phi}(y) \in \pi_{\Phi}(t)$. Thus, it is possible to obtain a family of pairs of binary relations indexed by the unitary transformations $U \in \mathcal{U}$, with $\mathcal{U}$ standing for the set of unitary transformations on $\mathcal{B}$.

Thusly a PhQF is a PDL frame constructed over the top of a given $\mathcal{B}$ space, by consider a product of two one-dimensional subspaces as "states" (i.e. states as pairs), projectors as pairs of "tests" and unitary evolution as pairs of "actions".
Notice that by taking the concept of "states" as being pairs , it is possible to express phase-related properties, e.g., such as gates (unitary transformations) that only acts in the phase

domain.

By generalizing the earlier notions it is possible to say that every *linear operator* $F : \mathcal{B} \longrightarrow \mathcal{B}$ induces a partial map $F : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$ on states, and consequently on the subspaces, given by $F(\overline{x}) = \overline{F(x)}$ if $F(x) \neq 0$ (otherwise, undefined). Furthermore, due to linearity of the operators, it is possible to say that this map is well-defined for all the states.

Definition 2.2.5 (Adjoint and adjoint state) *For every map $F : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$ there is an adjoint $F^{\dagger} : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$, which is given by :*

$$F^{\dagger}(s) = s^{\dagger}$$

with :

- *s as the state given by $s = (\sigma, \phi)$;*
- *$s^{\dagger}$ as the adjoint state of s, with $s^{\dagger} = (\sigma, \phi^*)$ and ϕ^* as the conjugated phase of ϕ, i.e. :*

$$\phi^* := (1 - \phi) \ mod\ 1 \quad \forall \phi \in \Phi$$

Therefore, defined as the map on states induced by the adjoint of the linear operator F on $\mathcal{B}$, i.e. $F^{\dagger}$ on $\mathcal{B}$. On the other hand, for a unitary transformation U, the adjoint is the inverse: $U^{\dagger} = U^{-1}$.

Example 2.2.5 *For a given state $s = (1, \frac{1}{4})$, $F^{\dagger}$ will map the state s to a state $s^{\dagger} = (1, (1 - \frac{1}{4})) = (1, \frac{3}{4})$, i.e. $F^{\dagger}(s) = (1, \frac{3}{4})$.*

Remark 2.2.3 *The basis σ of a given state s is invariant to the adjoint of the state s .*

Definition 2.2.6 (Measurement relation) *For every state $s, t \in \Sigma \times \Phi$, $s \longrightarrow t$ if and only if $s \xrightarrow{S?} t$ for a property $S \in \mathcal{L}$. This is reaching a state t via performing a measurement on state s.*

Remark 2.2.4 *Notice that the measurement relation can be seen as a non-orthogonality relation: $s \longrightarrow t \quad iff \quad \pi_{\Sigma}(s) \not\perp \pi_{\Sigma}(t)$.*

Remark 2.2.5 *Since measurement can bee seen as a non-orthogonality relation, only the basis of a state will affect the measurement relation.*

Definition 2.2.7 (Quantum Action) *It is possible to define a quantum action as any relation $R \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ that can be written as an arbitrary union $R = \bigcup_i F_i$ of linear maps $F_i : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$.*

Moreover, a complete lattice with inclusion and a set-theoretic union $R \cup R'$ as *supremum* is established by the family of quantum actions. This family is closed under a *relational composition* $R;\ R'$ defined as :

$$R;\ R' := \{(s,t) \in (\Sigma \times \Phi) \times (\Sigma \times \Phi)\ : \exists w\ \in \Sigma \times \Phi,\ (s,w) \in R,\ (w,t) \in R'\}$$

and a iteration

$$R^* := \bigcup_{k \geq 0} R^n$$

Remark 2.2.6 *A quantum action represents an input-output relation of a quantum program.*

For every property $T \subseteq \Sigma \times \Phi$ and every quantum action $R \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$, there is

$$[R]T := \{s \in \Sigma \times \Phi : \forall t \in \Sigma \times \Phi(sRt \implies t \in T)\}$$

and on a similar way

$$\langle R \rangle T := (\Sigma \times \Phi) \backslash ([R]((\Sigma \times \Phi) \backslash T))$$

Also it is possible to write $R(T)$ as

$$R(T) := \{s \in \Sigma \times \Phi : \exists t \in T \quad \text{such that} \quad sRt\}$$

and use $R[T] := \overline{R(T)}$ to denote the biorthogonal closure of the image.
Finally, it is possible to write

$$\Box T := \{s \in \Sigma \times \Phi : \forall t(s \to t \implies t \in T)\}$$

and

$$\Diamond T := (\Sigma \times \Phi) \backslash (\Box((\Sigma \times \Phi) \backslash T)).$$

Definition 2.2.8 (Measurement Modalities) *$\Box T$ can be defined has the property T holding after any measurement such as a quantum test performed on the actual state. On the other hand, $\Diamond T$ can be defined as the property T being potentially satisfied,i.e. doing some quantum test in order to reach a state with property T.*

Definition 2.2.9 (Weakest precondition) *The weakest precondition can be expressed as $R[T]$ for a "program" R and a post-condition T. A particular case is $[S?]T$ which expresses the precondition that ensures the satisfaction of the property T in any state that results from a test of property S.*

Furthermore, $\langle S? \rangle T$ *expresses the fact of by performing a quantum test of property* S *on the actual state the possibility of reaching up in a state having property* T *.*

Definition 2.2.10 (Image and strongest post-condition) *The image of* T *via* R *is defined by* $R(T)$*, which corresponds to the strongest property among all testable quantum properties in* $\mathcal{P}((\Sigma \times \Phi) \times (\Sigma \times \Phi))$ *that after applying a program* R *will be ensured to hold, in case of a precondition* T *being hold at the input-state. Therefore, this is the "strongest post-condition".*

Lemma 2.2.1 *For all properties* $S \subseteq \Sigma \times \Phi$*, there is* $S^{\perp} = [S?]\varnothing = (\Sigma \times \Phi) \backslash \Diamond S$ *and* $\overline{S} = \Box \Diamond S$*.*

Proposition 2.2.1 ([14]) *For each property* $S \subseteq \Sigma \times \Phi$*, if* T *is testable, i.e.* $T \in \mathcal{L}$*, then* $\Box S, S^{\perp}, [S?]T \in \mathcal{L}$*, in other words, they are also testable. Also, it is possible to extend to each quantum relation* R *by take into consideration* $[R]T \in \mathcal{L}$*.*

Remark 2.2.7 *States are testable : for each state* $s \in \Sigma \times \Phi$ *, there is* $\{s\} \in \mathcal{L}$*.*

Proposition 2.2.2 *A given property* $S \subseteq \Sigma \times \Phi$ *is testable if and only if any of the following conditions, which are equivalent, holds :*

- $S = \overline{S} \implies S_{\Sigma} = \overline{S_{\Sigma}}$*;*
- $\exists T \in \Sigma \times \Phi$ *such that* $S = T^{\perp}$*;*
- $\exists T \in \Sigma \times \Phi$ *such that* $S = \Box T$*.*

Proof 2.2.2 (Proposition 2.2.2) *Proof for Proposition 2.2.2*

- $S = \overline{S} \implies S_{\Sigma} = \overline{S_{\Sigma}}$ *follows from the Theorem 2.2.1;*
- $\exists T \in \Sigma \times \Phi$ *such that* $S = T^{\perp}$ *follows from the fact that if* $T = S^{\perp}$*, then* $S = S^{\perp\perp} = S$*;*
- $\exists T \in \Sigma \times \Phi$ *such that* $S = \Box T$ *follows from the fact that if* $T = S$*, then* $S = \Box S = S$*.*

The family $\mathcal{L}$ of quantum testable properties is a complete lattice concerning to inclusion, its meet set intersection $S \cap T$ and its *quantum join*.

Definition 2.2.11 (Quantum Join) *A quantum join is the join of the biorthogonal closure of set-union* $S \sqcup T := \overline{S \cup T}$*. Moreover, for any arbitrary property* $S \subseteq \Sigma \times \Phi$*, there is* $\overline{S} = \bigsqcup\{\{s\} : s \in S\} = \bigcap\{T \in \mathcal{L} : S \subseteq T\}$ *,*

Remark 2.2.8 *The biorthogonal closure of* S*,* $\overline{S}$*, is the strongest testable property implied by the property* S*.*

Theorem 2.2.2 (Phased Quantum Frame) *For every PhQF* $\Sigma \times \Phi = \Sigma \times \Phi(\mathcal{B})$ *the following properties are held :*

1. *Partial functionality: If* $s \xrightarrow{S?=S_\Sigma?\times S_\Phi?} t$ *and* $s \xrightarrow{S?=S_\Sigma?\times S_\Phi?} v$ *then* $t = v$.

2. *Trivial tests:* $\xrightarrow{\emptyset?} = \emptyset$ *and* $\xrightarrow{\Delta_{\Sigma\times\Phi}?} = \Delta_{\Sigma\times\Phi}$, *with* $\Delta_{\Sigma\times\Phi} = \{(s,s) : s \in \Sigma \times \Phi\}$ *as the identity relation on* $(\Sigma \times \Phi) \times (\Sigma \times \Phi)$.

3. *Atomicity. States are testable, i.e.* $\{s\} \in \mathcal{L}$. *In other words , this is the same as saying that "distinguishable of states can be caused by tests ", i.e.*

$$\begin{aligned} &\textit{if} \quad s \neq t \quad \textit{then} \quad \exists P \in \mathcal{L} : \pi_\Sigma(s) \perp \pi_\Sigma(P),\ \pi_\Sigma(t) \not\perp \pi_\Sigma(P) \\ &\qquad\qquad \textit{or} \quad \pi_\Phi(s) \neq \pi_\Phi(P),\ \pi_\Phi(t) = \pi_\Phi(P) \end{aligned}$$

4. *Adequacy. A state cannot be changed by testing a true property :*

$$\textit{if} \quad s \in P \quad \textit{then} \quad s \xrightarrow{P?} s$$

5. *Repeatability. Every testable property holds after a successful test :*

$$\textit{if} \quad s \xrightarrow{P?} t \quad \textit{then} \quad t \in P$$

6. *Compatibility:*

$$\textit{If } S \textit{ and } T \textit{ are testable, i.e } S,T \in \mathcal{L} \quad \textit{and} \quad S?;T? = T?;S? \quad \textit{then} \quad S?;T? = (S \cap T)?.$$

7. *Self-Adjointness: if* $s \xrightarrow{P?} w \longrightarrow t$ *then* $\exists v \in \Sigma \times \Phi$ *such that* $t \xrightarrow{P?} v \longrightarrow s$.

8. *Proper Superposition. In a quantum system every two states can be properly superposed into a new state, a state of superposition:* $\forall s,t \in \Sigma \times \Phi\ \exists w \in \Sigma \times \Phi\ s \longrightarrow w \longrightarrow t$.

9. *Unitary Reversibility and Totality. Elementary unitary evolutions are totally bijective functions with their adjoint as their inverse:*

$$U;U^\dagger = U^\dagger;U = id$$

with id as the identity map .

10. *Orthogonality Preservation. Elementary unitary evolutions preserve (non)-orthogonality: For* $s,t,s',t' \in \Sigma \times \Phi$ *with* $s \xrightarrow{U} s'$ *and* $t \xrightarrow{U} t'$ *there is :* $s \longrightarrow t$ *iff* $s' \longrightarrow t'$.

Proof 2.2.3 (Theorem 2.2.2) *The proof will be done step-by-step, this is there will be a proof for each point of Theorem 2.2.2 . Therefore:*

1. *Partial functionality is a consequence of projectors being partially defined maps in $\mathcal{B}$. And therefore, is possible to conclude that states are equal by the condition of state equality, i.e. Definition 2.2.1.*

2. *Trivial tests come from the fact that by making a projection on the empty space outcomes the empty space by itself , and by making a projection of the total space keeps everything unchanged.*

3. *Atomicity derives from the fact that states are the product of two one-dimensional closed linear subspaces, i.e. atoms of the lattice of all closed linear subspaces.*

4. *Adequacy arises from the fact that for all $x \in W$ there is $P_W(x) = x$.*

5. *Repeatability comes from the fact that $P_W(x) \in W$ for all $x \in \mathcal{B}$.*

6. *Compatibility is a consequence of that of two projectors commuting, i.e. if $P_W \circ P_V = P_V \circ P_W$, then $P_W \circ P_V = P_{W \cap V}$.*

7. *Self-Adjointness can be seen as consequence of the fact that projectors are self-adjoint (i.e. $P = P^\dagger \implies P?^\dagger = P?$) and also sustained by Adjointness theorem, Theorem 2.2.3 stated below.*

8. *Proper Superpositions can be proved by taking into account the following cases:*
   a) *If $\pi_\Sigma(s) \not\perp \pi_\Sigma(t)$, i.e. let $s \longrightarrow t$ then $w = s \implies s \longrightarrow s \longrightarrow t$.*
   b) *If $\pi_\Sigma(s) \not\perp \pi_\Sigma(t)$, i.e. let $s \not\longrightarrow t$ then let $s = \overline{x},\ t = \overline{y}$ with $x, y \in \mathcal{B}$.*
   c) *Considering the superposition $x + y \in \mathcal{B}$ of x and y notice that $x + y \neq 0$ due to the fact that $x + y = 0 \implies x = -y \implies s = t$ would refute $\pi_\Sigma(s) \not\perp \pi_\Sigma(t)$.*
   d) *Sequentially notice that $x \not\perp (x + y)$ (Also, by considering the scenario of $x \perp (x + y)$, it will be $\langle x|x + y\rangle = 0$ and then $\langle x|x\rangle + \langle x|y\rangle = 0$; however $x \perp y$ only implies $\langle x|y\rangle = 0$. Therefore, from $\langle x|x\rangle = 0$ follows that $x = 0$, which yields a contradiction). In this way of thinking, there is also $y \not\perp (x + y)$.*

9. *Unitary Reversibility and Totality is a consequence of the definition of a unitary operator .*

10. *Orthogonality Preservation can be proven also as consequence of the definition of a unitary, but also by considering the Definition 2.2.2 of state orthogonality, e.g. by take into account the following Example 2.2.6*

    Example 2.2.6 *Suppose that for a given state $s = (1, 0)$ and a unitary evolution U such that:*

$$
\begin{gathered}
s \xrightarrow{U} s' = (0,0) \\
\therefore \quad \mathit{If} \quad s' \longrightarrow t' \quad \mathit{then} \quad t' = (0,0) \\
\mathit{And} \quad s \longrightarrow t = (1,0) \\
\because (1,0) = t \xrightarrow{U} t' = (0,0)
\end{gathered}
$$

Therefore, $s \longrightarrow t$ *iff* $s' \longrightarrow t'$.

End of proof .

Theorem 2.2.3 (Adjointness) *For a quantum map F and states* $s, w, t \in \Sigma \times \Phi$ *:*

If $s \xrightarrow{F} w \longrightarrow t$ *then there exists some state* $v \in \Sigma \times \Phi$ *such that* $t \xrightarrow{F^\dagger} v \longrightarrow s.$

Proof 2.2.4 (Theorem 2.2.3) *In order to proof Theorem 2.2.3, it will be considered the definition of adjoint map in Hilbert Space, i.e.* $\langle F^\dagger(x)|y\rangle = \langle x|F(y)\rangle$.

$$\therefore \left\langle F^\dagger(x)\middle|y\right\rangle = 0 \quad \textit{iff} \quad \langle x|F(y)\rangle = 0$$

Or equivalently,

$$f_\Sigma(F^\dagger(x)) \perp f_\Sigma(y) \quad \textit{iff} \quad f_\Sigma(x) \perp f_\Sigma(F(y))$$

$\forall x, y \in \mathcal{B}$.

Consequently, by considering the negation of both side and take into account the Definition 2.2.6, i.e. the fact that the measurement relation $s \longrightarrow t$ *can be seen as the non-orthogonality* $\pi_\Sigma(s) \not\perp \pi_\Sigma(t)$*, it is possible to obtain the following equivalence:*

$$\exists w(\overline{x} \xrightarrow{F} \overline{w} \longrightarrow \overline{y}) \quad \textit{iff} \quad \exists v(\overline{y} \xrightarrow{F^\dagger} \overline{v} \longrightarrow \overline{x})$$

Therefore, the Theorem 2.2.3 has been proven. As a consequence there are Corollary 2.2.3.1 and Corollary 2.2.3.2.

Corollary 2.2.3.1 *For every property* $P \subseteq \Sigma \times \Phi$ *and every linear map F there is :*

$$P \subseteq [F] \, \Box \, \langle F^\dagger \rangle \, \Diamond P$$

Corollary 2.2.3.2 *If F is a quantum map, then :*

$$F^\dagger(s^\dagger) = ([F]s^\perp)^\perp$$

Proof 2.2.5 (Corollary 2.2.3.2) *By considering the negation of the measurement relation, Definiton 2.2.6, as the orthogonality relation* $\perp$ *and Theorem 2.2.3 of adjointness there is :*

$$s \perp F^\dagger(t) \quad \textit{iff} \quad t^\dagger \perp F(s)$$

Or equivalently,

$$s \in (F^\dagger(t))^\perp \quad \textit{iff} \quad F(s) \in t^\dagger$$

Which follows that $(F^{\dagger}(t))^{\perp} = ([F]t^{\dagger})^{\perp}$ *. Also by the Definition 2.2.5 of adjoint and adjoited state,* $F^{\dagger}(t)$ *is a single state, i.e.* $t^{\dagger}$ *with* $t^{\dagger}$ *as a testable property. Therefore:*

$$F^{\dagger}(t) = (F^{\dagger}(t))^{\perp\perp} = ([F](t^{\dagger})^{\perp})^{\perp} = (t^{\perp\perp})^{\dagger} = t^{\dagger}$$

Thereby, with the results presented in Proof 2.2.5 it is possible generalize the the concept of adjoint to all quantum actions :

Definition 2.2.12 (Adjoint of a Quantum Action) *For every quantum action* $R \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ *it is possible to define a relation* $R^{\dagger} \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ *by :*

$$R^{\dagger}(s^{\dagger}) = ([R]s^{\perp})^{\perp}$$

Proposition 2.2.3 *For all quantum actions* $R, Z \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$*, states* $s, t \in (\Sigma \times \Phi)$ *and properties* $S \subseteq (\Sigma \times \Phi)$*, there are :*

1. $R^{\dagger}$ *is a quantum action.*
2. *if* $R = F$ *is a map then the adjoint of the quantum action* $R^{\dagger}$ *coincides with the Hermitian adjoint* $F^{\dagger}$ *, with* F *as a linear map.*
3. $s \perp R^{\dagger}(t)$ *iff* $t^{\dagger} \perp R(s)$ *.*
4. $(R; Z)^{\dagger} = Z^{\dagger}; R^{\dagger}$.
5. $(R \cup Z)^{\dagger} = R^{\dagger} \sqcup Z^{\dagger}$.
6. $R[S^{\dagger}] = ([R^{\dagger}]S^{\perp})^{\perp}$.

Proof 2.2.6 (Proposition 2.2.3) *Proof of Proposition 2.2.3.*

- *Point 1, 2 and 6 are a direct consequence from Definition 2.2.12.*
- *The proof of Point 3 is equal to Proof 2.2.5 by consider* R *instead of* F*.*
- *Point 4 follows from the Hermitian nature of the quantum actions.*
- *Point 5 follows from the Hermitian nature, as well as the union of* $R \cup Z$ *standing for e non-deterministic union of quantum action. Therefore, the following quantum join* $R^{\dagger} \sqcup Z^{\dagger}$*.*
- *Point 6 :* $R[S^{\dagger}] = ([R^{\dagger}]S^{\perp})^{\perp} = \overline{R(S^{\dagger})} = R[S^{\dagger}]$.

QUANTUM FRAMES FOR COMPOUND SYSTEMS Through the expansion of PhQF it is conceivable a quantum frame for compound systems, i.e. a quantum frame for quantum systems with more than one qubit . Therefore, by considering a two-dimensional B space and by establish a natural number $n \geq 2$ such that there is a set $N = \{1,\ 2,\ \ldots,\ n\}$, it is possible to define a *quantum frame for compound systems* as the quantum frame $\Sigma \times \Phi(\mathcal{B}_n)$ constructed on the following space :

$$\mathcal{B}_n = B^{\otimes n} = \underbrace{B \otimes B \otimes \cdots \otimes B}_{\text{n times}}$$

Consequently, it will be taken into account all the n copies of B as distinguishable by denoting $B^{(i)}$ as the i-th component of the tensor $B^{\otimes n}$, i.e.

$$\mathcal{B}_n = B^{\otimes n} = \bigotimes_{i=1}^{n} B^{(i)}$$

On the other hand, for a given set of indices $I \subseteq N$ there is :

$$\mathcal{B}_I = B^{\otimes I} = \bigotimes_{i \in I} B^{(i)}$$

Remark 2.2.9 *For the case where I = N there is $\mathcal{B}_I = \mathcal{B}_N = \mathcal{B}_n = \mathcal{B}$.*

Also, there is a canonical isomorphism between $\mathcal{B}$ and $\mathcal{B}^{(i)}$ denoted by $\varepsilon_i : B \longrightarrow B^{(i)}$. This kind of notation can be expanded to any set $I \subseteq N$ of indices of length $|I| = k$ in order to denote a canonical isomorphism between the spaces $B^{\otimes k}$ and $\mathcal{B}_I$, i.e. $\varepsilon_I : B^{\otimes k} \longrightarrow \mathcal{B}_I$. In this way of thinking, for each set $I \subseteq N$ there is a canonical isomorphism between the spaces $\mathcal{B}_I \otimes \mathcal{B}_{N \setminus I}$ and $\mathcal{B}$, i.e. $\mu_I : \mathcal{B}_I \otimes \mathcal{B}_{N \setminus I} \longrightarrow \mathcal{B}$. Sequentially, for every vector $|x\rangle \in B$, there is :

$$|x\rangle^{\otimes I} = \bigotimes_{i \in I} |x_i\rangle$$

Remark 2.2.10 *Let $|x\rangle^{\otimes I}$ be a vector with the following form $|x\rangle^{\otimes I} = e^{2\pi i \phi_x} |\sigma_x\rangle$, there is:*

- $\phi_x = (\sum_{i \in I} \phi_i) \mod 1$
- $\sigma_x \in \{0, 1, +, -\}^k$

Posteriorly, for a given set $I \subseteq N$ it is possible to say that a state $s \in \Sigma \times \Phi(\mathcal{B})$ has its I-qubits in a state $s_I = s' \in \Sigma \times \Phi(\mathcal{B}_{\mathcal{I}})$, if there are some vectors $\psi \in s,\ \psi' \in \mathcal{B}_I,\ \psi'' \in \mathcal{B}_{N \setminus I}$ such that :

$$\psi = \mu_I(\psi' \otimes \psi'')$$

Definition 2.2.13 (Local state) *If there is a given state s which is I-separated then there is a unique state s_I designated as the I-local state of s.*

Remark 2.2.11 *For the case where $I = \{i\}$, the local component $s_i \in \mathcal{B}_{\{i\}} = B^{(i)}$ is denominated the $\{i\}$-th coordinate of the state s.*

Furthermore, to refer to a state originated by a pair of qubits, by removing the Dirac's notation, there is, e.g., $(00, \phi) := \overline{e^{2\pi i.\phi}\,|00\rangle} = \overline{e^{2\pi i.\phi}(|0\rangle \otimes |0\rangle)}$. The Bell states will be written as follows:

- ${}^{\phi}\beta_{00} := \overline{\frac{e^{2\pi i.\phi}}{\sqrt{2}}(|00\rangle + |11\rangle)}$.
- ${}^{\phi}\beta_{01} := \overline{\frac{e^{2\pi i.\phi}}{\sqrt{2}}(|01\rangle + |10\rangle)}$.
- ${}^{\phi}\beta_{10} := \overline{\frac{e^{2\pi i.\phi}}{\sqrt{2}}(|00\rangle - |11\rangle)}$.
- ${}^{\phi}\beta_{11} := \overline{\frac{e^{2\pi i.\phi}}{\sqrt{2}}(|01\rangle - |10\rangle)}$.
- ${}^{\phi}\gamma := \overline{\frac{e^{2\pi i.\phi}}{2}(|00\rangle + |01\rangle\, |11\rangle + |10\rangle)}$

Proposition 2.2.4 ([14]) *Given two spaces $B^{(i)}$ and $B^{(j)}$ there is a bijective correlation ψ between the linear maps $F : B^{(i)} \longrightarrow B^{(j)}$ and the states $B^{(i)} \otimes B^{(j)}$. Therefore, by consider $\{\varepsilon^{(i)}_{\alpha}\}_{\alpha}$ and $\{\varepsilon^{(j)}_{\beta}\}_{\beta}$ as the basis of these spaces, the correlation ψ is given by the following mapping:*

$$F = \sum_{\alpha\beta} m_{\alpha\beta} \left\langle \varepsilon^{(i)}_{\alpha} \middle| - \right\rangle \, . \, \varepsilon^{(j)}_{\beta}$$

into the following state :

$$\psi(F) = \sum_{\alpha\beta} m_{\alpha\beta} \, .\varepsilon^{(i)}_{\alpha} \otimes \varepsilon^{(j)}_{\beta} \, .$$

Proposition 2.2.5 ([14]) *For $\mathcal{B} = B^{\otimes n}$ and $W = \{x \otimes |0\rangle^{\otimes(n-1)} \; : \; x \in B\}$, any linear map $F : \mathcal{B} \longrightarrow \mathcal{B}$ induces a linear map $F_{(1)} : B \longrightarrow B$ in such canonical way that is defined as the unique map on B that satisfy $F_{(1)}(x) = P_W \circ F(x \otimes |0\rangle^{\otimes(n-1)})$.*

Remark 2.2.12 *Any linear map $G : B \longrightarrow B$ can be denoted by $G = F_{(1)}$ for a linear map $F : \mathcal{B} \longrightarrow \mathcal{B}$.*

Consequently, it is possible to describe a compound state in $B^{(i)} \otimes B^{(j)}$ by a given linear map F on $\mathcal{B}$. Therefore, for the case where $F : \mathcal{B} \longrightarrow \mathcal{B}$ is such linear map and $F_{(1)} : B \longrightarrow B$ be a map as the map of Proposition 2.2.5, $F_{(1)}$ will induce a correlated map $F_{(1)}^{(ij)} : B^{(i)} \longrightarrow B^{(j)}$ by writing :

$$F_{(1)}^{(ij)} := \varepsilon_j \circ F_{(1)} \circ \varepsilon_i^{-1}$$

with ε_i standing for the canonical isomorphism between B and the given i-th component $B^{(i)}$ of $B^{\otimes n}$. Thusly , it is possible to specify the state $\overline{F}_{(ij)}$ by writing :

$$\overline{F}_{(ij)} = \overline{\psi(F_{(1)}^{(ij)})}$$

with ψ denoting the bijective correlation between $B^{(i)} \longrightarrow B^{(j)}$ and $B^{(i)} \otimes B^{(j)}$.

Proposition 2.2.6 ([14]) *For a linear map $F : \mathcal{B} \longrightarrow \mathcal{B}$, it is possible to say that the state $\overline{F}_{(ij)}$ is an entangled state attending to F, i.e. if $F_{(1)}(|x\rangle) = |y\rangle$ and also if $\overline{F}_{(ij)} \in B^{(i)} \otimes B^{(j)}$ is a state of system composed by two qubits , then by performing a measurement on qubit i with x_i as the outcome state will collapse the qubit j to a state y_j .*

Additionally, it is possible to expand the concept and notation of $\overline{F}_{(ij)}$ in order to define a property, i.e. a set of states $\overline{F}_{(ij)} \subseteq \Sigma \times \Phi = \Sigma \times \Phi(\mathcal{B})$ defined as the set of all states with $\{i, j\}$-qubits in the state $\overline{F}_{(ij)}$:

$$\begin{aligned} \overline{F}_{ij} &= \{s \in \Sigma \times \Phi : s_{\{i,j\}} = \overline{F}_{(ij)}\} \\ &= \{\overline{\mu_{\{i,j\}}(\psi \otimes \psi')} : \psi \in \overline{F}_{(ij)},\ \psi' \in \mathcal{B}_{N\setminus\{i,j\}}\} \subseteq \Sigma \times \Phi \end{aligned}$$

with $\mu_{\{i,j\}}$ representing the canonical isomorphism between $\mathcal{B}_{\{i,j\}} \otimes \mathcal{B}_{N\setminus\{i,j\}}$.

Remark 2.2.13 *Notice that $\overline{F}_{ij}$ it is merely the property of a state composed by n-qubits with its i-th and j-th separated from the other qubits and in a state that is entangled attending to F .*

Definition 2.2.14 (Local properties and separation) *For a set $I \subseteq N$, it is possible to define a property $S \subseteq \Sigma \times \Phi$ as local in I if it corresponds to a property of a subsystem composed by the I-qubits, i.e. if there is some property $S' \subseteq \Sigma \times \Phi(\mathcal{B}_I)$ so that :*

$$S' = \{s \in \Sigma \times \Phi : s_I \in S'\}$$

or equivalently:

$$S' = \{\overline{\mu_I(\psi \otimes \psi')} : \overline{\psi} \in S',\ \psi' \in \mathcal{B}_{N\setminus I}\}$$

Example 2.2.7 *$\overline{F}_{ij}$ is an $\{i, j\}$-local property.*

Thusly, the family of local properties shapes a complete lattice with inclusion and its join obtained by the union $S \cup T$. Also, the local states are the atoms of this lattice with the greatest element as the following property:

$$\top_I^{\Sigma\times\Phi} = \{s \in \Sigma \times \Phi \ : \ s \text{ is } I - \text{separated}\} = \bigcup\{S \subseteq \Sigma \times \Phi \ : \ S \text{ is } I - \text{local}\}$$

which defines separation.

Remark 2.2.14 *It is possible to say that a state s is I-separated iff* $s \in \top_I^{\Sigma\times\Phi}$.

Finally, notice that the local properties forms a family which is closed under union and intersection, however not under complementation.

Definition 2.2.15 (Local Maps) *For* $I \subseteq N$*, there is a linear map I-local map* $F : \mathcal{B} \longrightarrow \mathcal{B}$ *if F "acts" only on the I-qubits; this is, if there is a map* $G : \mathcal{B}_\mathcal{I} \longrightarrow \mathcal{B}_\mathcal{I}$ *such that:*

$$F \circ \mu_I(\psi \otimes \psi') = \mu_I(G(\psi) \otimes \psi')$$

A map $F : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$ *if it is the map induced on* $\Sigma \times \Phi$ *through I-local linear map on* $\mathcal{B}$.

Example 2.2.8 *All quantum gates that affect only the I-qubits are I-local maps, i.e. unitary transformations* $U_I : \mathcal{B} \longrightarrow \mathcal{B}$ *such that :*

$$U_I \circ \mu_I(\psi \otimes \psi') = \mu_I(U(\psi) \otimes \psi'), \quad \forall \psi, \psi'$$

for some $U : \mathcal{B}_I \longrightarrow \mathcal{B}_I$.

Also, the family of local maps is closed under composition.

Definition 2.2.16 (Local Actions) *It is possible to define a quantum action* $R \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ *as an arbitrary union of local maps, with the family of local actions forming a complete lattice equipped with inclusion, in which the join is obtained by the union* $R \cup R'$ *with the action:*

$$\top_I^{(\Sigma\times\Phi)\times(\Sigma\times\Phi)} := \bigcup\{F : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi \ : \ F \textit{ is an } I - \textit{local map}\}$$

as the greatest element.

Lemma 2.2.2 ("Teleportation Property [14]") *For a i-separated state s with is its i-th* s_i *qubit in the state* $x \in B$ *, there is two successive bipartite measurements* $\overline{G_{jk}}?$ *succeeded by* $\overline{F_{ij}}?$ *with the k-qubit being given by :*

$$(\overline{F_{ij}?} \circ \overline{G_{jk}?}(s))_k = G_{(1)} \circ F_{(1)}(x)$$

Lemma 2.2.3 (Entanglement Composition Lemma, [14]) *As presented as the main lemma in [28] and with the notation used in [14]. Given a set of four distinct indices $\{i, j, k, l\}$ and $F, G, H, U, V : B \longrightarrow B$ as single-qubit linear maps, there is:*

$$\overline{G_{jk}?} \circ V_k \circ U_j(\overline{F_{ij}} \cap \overline{H_{kl}}) \subseteq \overline{(H \circ U^{\dagger} \circ G \circ V \circ F)_{il}}$$

Sequentially, as mentioned in [14], Lemma 2.2.2 and Lemma 2.2.3 are used in [1, 28] (and also used in [2]), in order to have a main tool to explain a variety of quantum protocols such as teleportation and quantum gate teleportation. On the other hand, even by consider a PhQF, it is possible to say that the order which the operations U_j and V_k are applied is indeed **insignificant**. Since, this is a consequence of the following properties for local transformations, which are present in [14]:

Proposition 2.2.7 (Compatibility of local transformations affecting distinct sets of qubits, [14]) *For $I \cap J = \emptyset$, F_I as a I-local-map and G_J as a J-local map, there is:*

$$F_I \circ G_J = G_J \circ F_I$$

Example 2.2.9 (Proposition 2.2.7 and quantum circuits) *Suppose the following quantum circuits , Figure 1 and Figure 2:*

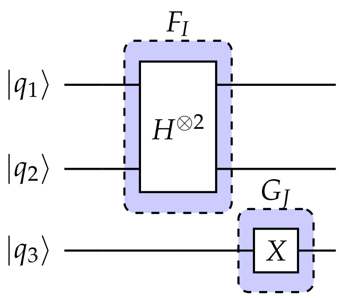

Figure 1: Example of a quantum circuit where Proposition 2.2.7 holds .

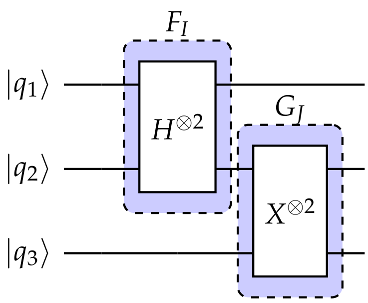

Figure 2: Example of a quantum circuit where Proposition 2.2.7 does not hold .

Both quantum circuits will have $N = \{1, 2, 3\}$ *as the set of all qubits* (q_1, q_2, q_3) *indices and the correspondent space defined by* $\mathcal{B} = B^{(1)} \otimes B^{(2)} \otimes B^{(3)}$ *.*

For the quantum circuit of the Figure 1 , $I = \{1, 2\}$ *and* $J = \{3\}$ *with* $I \cap J = \emptyset$ *. Also ,* $\mathcal{B}_I = B^{(1)} \otimes B^{(2)}$ *with* F_I *"acting" over* $\mathcal{B}_I$ *and* $\mathcal{B}_J = B^{(3)}$ *with* G_J *"acting" over* $\mathcal{B}_J$*. Therefore, for the instance of the example , it is possible to consider* F_I *as the action of the Hadamard gate* $H^{\otimes 2}$*,* G_J *as the action of the X-gate and notice that Proposition 2.2.7 holds.*

On the other hand, for the quantum circuit of the Figure 2, $I = \{1, 2\}$ *and* $J = \{2, 3\}$ *with* $I \cap J = \{2\} \neq \emptyset$*. As well,* $\mathcal{B}_I = B^{(1)} \otimes B^{(2)}$ *and* $\mathcal{B}_J = B^{(2)} \otimes B^{(3)}$*. Consequently, it is thinkable to see* F_I *as the action of the Hadamard gate* $H^{\otimes 2}$*,* G_J *as the action of the* $X^{\otimes 2}$*-gate and perceive that Proposition 2.2.7 does not hold since :*

$$|q_2\rangle \;\text{———}\boxed{H}\text{—}\boxed{X}\text{—} \;\neq\; |q_2\rangle \;\text{———}\boxed{X}\text{—}\boxed{H}\text{—} \;.$$

Proposition 2.2.8 (Agreement Property, [14]) *Let* $F_I, G_I : \Sigma \times \Phi \longrightarrow \Sigma \times \Phi$ *be two local I-local maps on states, with same domain (*$dom(F) = dom(G)$*). Then there is an agreement on their all non I-qubits output-states, i.e.:*

$$F(s)_{N \setminus I} = G(s)_{N \setminus I}, \quad \forall s \in \Sigma \times \Phi$$

when the identity of both sides exists.

Example 2.2.10 (An example of a quantum circuit where Proposition 2.2.8 is held) *Consider the following quantum circuit*

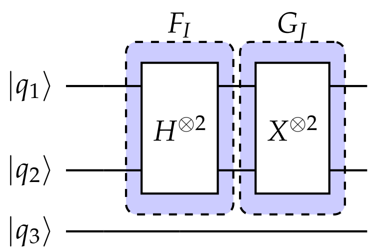

Figure 3: Example of a quantum circuit where Proposition 2.2.8 holds .

For the quantum circuit of the Figure 3, there is $N = \{1, 2, 3\}$ *as the set of all qubits* (q_1, q_2, q_3) *indices and the correspondent space defined by* $\mathcal{B} = B^{(1)} \otimes B^{(2)} \otimes B^{(3)}$ *. Also,* $I = \{1, 2\}$ *and* $N \setminus I = \{3\}$*.*
Consequently, $\mathcal{B}_I = B^{(1)} \otimes B^{(2)}$ *with* F_I *and* G_J *"acting" respectively over* $\mathcal{B}_I$ *and* $\mathcal{B}_{N \setminus I} = B^{(3)}$ *with* $F_{N \setminus I}$ *and* $G_{N \setminus I}$ *"acting" respectively over* $\mathcal{B}_{N \setminus I}$*.*
Therefore, for the instance of the example , it is also possible to consider $F_{N \setminus I}$ *and* $G_{N \setminus I}$ *(*$F_{N \setminus I}, G_{N \setminus I}$ *:*

$\Sigma \times \Phi \longrightarrow \Sigma \times \Phi$) as the action of two consecutive Identity-gate with Proposition 2.2.8 holding , since:

$$id_{\{3\}} = id_{\{3\}}$$

with $id_{\{3\}}$ being the identity on qubit q_3.

The necessity of a phased quantum frame The proper characteristics of a PhQF allows to express quantum phase properties , in opposition to the QF proposed by A.Baltag and S.Smets in [14]. In this sense , it is possible to say that a PhQF can express the inherent properties of quantum gates which "affects" only quantum phase related features. Therefore, it is possible to express unitary transformations that "act" over quantum phase properties such as Y and Z quantum gates (full expressiveness of the Z quantum gate).

Consequently, there are some examples in the sequel.

Example 2.2.11 (X-gate : PhQF vs QF) *By take into consideration the X-gate, Figure 4 :*

$$v = e^{2\pi i.(\phi_v)}|\sigma_v\rangle \;—\boxed{X}—\; w = e^{2\pi i.(\phi_w)}|\sigma_w\rangle$$

Figure 4: X-gate with input-vector $v \in \mathcal{B}$ and output-vector $w \in \mathcal{B}$.

And the input-vector v and output-vector w relation given by Table 1 :

Input Vector v	Output Vector w
$e^{2\pi i.(\phi_v)}\ \lvert 0\rangle$	$e^{2\pi i.(\phi_v)}\ \lvert 1\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert 1\rangle$	$e^{2\pi i.(\phi_v)}\ \lvert 0\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert +\rangle$	$e^{2\pi i.(\phi_v)}\ \lvert +\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert -\rangle$	$e^{2\pi i.(\phi_v)}\ \lvert -\rangle$

Table 1: Table of input-vector v and output-vector w relation for X-gate.

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- From the point of view of a PhQF :
  - $\overline{v} = (\sigma_v, \phi_v)$ and $\overline{w} = (\sigma_w, \phi_w)$ with $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 5:
- From the point of view of a QF:
  - $\overline{v} = \sigma_v$ and $\overline{w} = \sigma_w$ with $\overline{v}, \overline{w} \in \Sigma$.
  - Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 3 :

Input State $\overline{v}$	Output State $\overline{w}$
$(0,\ \phi_v)$	$(1, \phi_v)$
$(1,\ \phi_v)$	$(0, \phi_v)$
$(+,\ \phi_v)$	$(+, \phi_v)$
$(-,\ \phi_v)$	$(-, \phi_v)$

Table 2: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for X-gate by PhQF.

Input State $\overline{v}$	Output State $\overline{w}$
0	1
1	0
$+$	$+$
$-$	$-$

Table 3: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for X-gate by QF.

$$v = e^{2\pi i.(\phi_v)}|\sigma_v\rangle \;—\boxed{Y}—\; w = e^{2\pi i.(\phi_w)}|\sigma_w\rangle$$

Figure 5: Y-gate with input-vector $v \in \mathcal{B}$ and output-vector $w \in \mathcal{B}$.

Input Vector v	Output Vector w
$e^{2\pi i.(\phi_v)}\ \lvert 0\rangle$	$e^{2\pi i.((\phi_v+\frac{1}{4})\mod 1)}\ \lvert 1\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert 1\rangle$	$e^{2\pi i.((\phi_v+\frac{3}{4})\mod 1)}\ \lvert 0\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert +\rangle$	$e^{2\pi i.((\phi_v+\frac{3}{4})\mod 1)}\ \lvert -\rangle$
$e^{2\pi i.(\phi_v)}\ \lvert -\rangle$	$e^{2\pi i.((\phi_v+\frac{1}{4})\mod 1)}\ \lvert +\rangle$

Table 4: Table of input-vector v and output-vector w relation for Y-gate.

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Example 2.2.12 (Y-gate : PhQF vs QF) *By take into consideration the Y-gate, Figure 5 : And the input-vector v and output-vector w relation given by Table 4 :*

- *From the point of view of a PhQF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 5:*

Input State $\overline{v}$	Output State $\overline{w}$
$(0,\ \phi_v)$	$(1, (\phi_v + \frac{1}{4}) \mod 1)$
$(1,\ \phi_v)$	$(0, (\phi_v + \frac{1}{4}) \mod 1)$
$(+,\ \phi_v)$	$(-, (\phi_v + \frac{3}{4}) \mod 1)$
$(-,\ \phi_v)$	$(+, (\phi_v + \frac{1}{4}) \mod 1)$

Table 5: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for Y-gate by PhQF.

- *From the point of view of a QF:*

- $\overline{v} = \sigma_v$ *and* $\overline{w} = \sigma_w$ *with* $\overline{v}, \overline{w} \in \Sigma$.
- *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the Table 6 :*

Input State $\overline{v}$	Output State $\overline{w}$
0	1
1	0
$+$	$-$
$-$	$+$

Table 6: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for Y-gate by QF.

Example 2.2.13 (Z-gate : PhQF vs QF) *Also , by take into consideration the Z-gate, Figure 6 :*

$$v = e^{2\pi i.(\phi_v)}|\sigma_v\rangle \;—\boxed{Z}—\; w = e^{2\pi i.(\phi_w)}|\sigma_w\rangle$$

Figure 6: Z-gate with input-vector $v \in \mathcal{B}$ and output-vector $w \in \mathcal{B}$.

And the input-vector v and output-vector w relation given by Table 7 :

Input Vector v	Output Vector w
$e^{2\pi i.(\phi_v)}\,|0\rangle$	$e^{2\pi i.((\phi_v+\frac{1}{4}) \mod 1)}\,|0\rangle$
$e^{2\pi i.(\phi_v)}\,|1\rangle$	$e^{2\pi i.((\phi_v+\frac{3}{4}) \mod 1)}\,|1\rangle$
$e^{2\pi i.(\phi_v)}\,|+\rangle$	$e^{2\pi i.(\phi_v)}\,|-\rangle$
$e^{2\pi i.(\phi_v)}\,|-\rangle$	$e^{2\pi i.(\phi_v)}\,|+\rangle$

Table 7: Table of input-vector v and output-vector w relation for Z-gate.

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- *From the point of view of a PhQF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the Table 8:*

Input State $\overline{v}$	Output State $\overline{w}$
$(0,\ \phi_v)$	$(0, \phi_v)$
$(1,\ \phi_v)$	$(1, (\phi_v + \frac{1}{2}) \mod 1)$
$(+,\ \phi_v)$	$(-, \phi_v)$
$(-,\ \phi_v)$	$(+, \phi_v)$

Table 8: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for Z-gate by PhQF.

- *From the point of view of a QF:*
  - $\overline{v} = \sigma_v$ *and* $\overline{w} = \sigma_w$ *with* $\overline{v}, \overline{w} \in \Sigma$.
  - *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the Table 9 :*

Input State $\overline{v}$	Output State $\overline{w}$
0	0
1	1
$+$	$-$
$-$	$+$

Table 9: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for Z-gate by QF.

Attending to Table 8 and Table 9, it is possible to notice that QF can not express all properties inherent to the Z-gate , in a contrary way of what occurs with the expressiveness of PhQF.

Example 2.2.14 (H-gate : PhQF vs QF) *By consider the H-gate, Figure 7 :*

$$v = e^{2\pi i.(\phi_v)}|\sigma_v\rangle \;—\boxed{H}—\; w = e^{2\pi i.(\phi_w)}|\sigma_w\rangle$$

Figure 7: H-gate with input-vector $v \in \mathcal{B}$ and output-vector $w \in \mathcal{B}$.

And the input-vector v and output-vector w relation given by Table 10 :

Input Vector v	Output Vector w
$e^{2\pi i.(\phi_v)}\ \|0\rangle$	$e^{2\pi i.(\phi_v)}\ \|+\rangle$
$e^{2\pi i.(\phi_v)}\ \|1\rangle$	$e^{2\pi i.(\phi_v)}\ \|-\rangle$
$e^{2\pi i.(\phi_v)}\ \|+\rangle$	$e^{2\pi i.(\phi_v)}\ \|0\rangle$
$e^{2\pi i.(\phi_v)}\ \|-\rangle$	$e^{2\pi i.(\phi_v)}\ \|1\rangle$

Table 10: Table of input-vector v and output-vector w relation for H-gate.

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- *From the point of view of a PhQF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 11:*

Input State $\overline{v}$	Output State $\overline{w}$
$(0,\ \phi_v)$	$(+, \phi_v)$
$(1,\ \phi_v)$	$(-, \phi_v)$
$(+,\ \phi_v)$	$(0, \phi_v)$
$(-,\ \phi_v)$	$(1, \phi_v)$

Table 11: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for H-gate by PhQF.

- *From the point of view of a QF:*
  - $\overline{v} = \sigma_v$ *and* $\overline{w} = \sigma_w$ *with* $\overline{v}, \overline{w} \in \Sigma$.
  - *Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 12 :*

Input State $\overline{v}$	Output State $\overline{w}$
0	$+$
1	$-$
$+$	0
$-$	1

Table 12: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for H-gate by QF.

$$v = e^{2\pi i.(\phi_v)}|\sigma_v\rangle \;-\!\!\boxed{R_k}\!\!-\; w = e^{2\pi i.(\phi_w)}|\sigma_w\rangle$$

Figure 8: R_k-gate with input-vector $v \in \mathcal{B}$ and output-vector $w \in \mathcal{B}$.

Example 2.2.15 (R_k-gate : PhQF vs QF) *By consider the R_k-gate, Figure 8:*
And the input-vector v and output-vector w relation given by Table 13 :

Input Vector $\overline{v}$	Output Vector $\overline{w}$
$e^{2\pi i.(\phi_v)}\ \|0\rangle$	$e^{2\pi i.(\phi_v)}\ \|0\rangle$
$e^{2\pi i.(\phi_v)}\ \|1\rangle$	$e^{2\pi i.((\phi_v+\frac{1}{2^k}) \mod 1)}\ \|1\rangle$
$e^{2\pi i.(\phi_v)}\ \|+\rangle$	$\frac{1}{\sqrt{2}}(e^{2\pi i.(\phi_v)}\ \|0\rangle + e^{2\pi i.((\phi_v+\frac{1}{2^k}) \mod 1)}\ \|1\rangle)$
$e^{2\pi i.(\phi_v)}\ \|-\rangle$	$\frac{1}{\sqrt{2}}(e^{2\pi i.(\phi_v)}\ \|0\rangle + e^{2\pi i.((\phi_v+\frac{1}{2^k}+\frac{1}{2}) \mod 1)}\ \|1\rangle)$

Table 13: Table of input-vector v and output-vector w relation for R_k-gate.

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- *From the point of view of a PhQF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 14:*

Input State $\overline{v}$	Output State $\overline{w}$
$(0,\ \phi_v)$	$(0, \phi_v)$
$(1,\ \phi_v)$	$(1, (\phi_v + \frac{1}{2^k}) \mod 1)$
$(+,\ \phi_v)$	$(0, \phi_v)$ or $(1, (\phi_v + \frac{1}{2^k}) \mod 1)$
$(-,\ \phi_v)$	$(0, \phi_v)$ or $(1, (\phi_v + \frac{1}{2^k} + \frac{1}{2}) \mod 1)$

Table 14: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for R_k-gate by PhQF.

- *From the point of view of a QF:*
  - $\overline{v} = \sigma_v$ *and* $\overline{w} = \sigma_w$ *with* $\overline{v}, \overline{w} \in \Sigma$.
  - *Input-state $\overline{v}$ and output-state $\overline{w}$ relation given by the Table 15 :*

Example 2.2.16 (CR_k-gate : PhQF vs QF) *By consider the CR_k-gate, Figure 9, it is possible to build the Table 16 that describes the output vector w for a non-entangled input vector $v = v_1 \otimes v_2$.*

Input State $\overline{v}$	Output State $\overline{w}$
0	0
1	1
$+$	$+$
$-$	$-$

Table 15: Table of input-state $\overline{v}$ and output-state $\overline{w}$ relation for R_k-gate by QF.

Input Vector v_1	Input Vector v_2	Output Vector w
$e^{2\pi i.(\phi_{v_1})}\ket{0}$	v_2	$e^{2\pi i.(\phi_{v_1})}\ket{0} \otimes v_2 = e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}(\ket{0} \otimes \ket{\sigma_{v_2}})$
v_1	$e^{2\pi i.(\phi_{v_2})}\ket{0}$	$v_1 \otimes e^{2\pi i.(\phi_{v_2})}\ket{0} = e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}(\ket{\sigma_{v_1}} \otimes \ket{0})$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$e^{2\pi i.(\phi_{v_1})}\ket{1} \otimes e^{2\pi i.((\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{1} = e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}(\ket{1} \otimes \ket{1})$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{+}$	$e^{2\pi i.(\phi_{v_1})}\ket{1} \otimes \frac{1}{\sqrt{2}}(e^{2\pi i.(\phi_{v_2})}\ket{0} + e^{2\pi i.((\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{1}) = \frac{1}{\sqrt{2}}(e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}\ket{10} + e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{11})$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{-}$	$e^{2\pi i.(\phi_{v_1})}\ket{1} \otimes \frac{1}{\sqrt{2}}(e^{2\pi i.(\phi_{v_2})}\ket{0} + e^{2\pi i.((\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{1}) = \frac{1}{\sqrt{2}}(e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}\ket{10} + e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}+\frac{1}{2}) \bmod 1)}\ket{11})$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$\frac{1}{\sqrt{2}}(e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}\ket{01} + e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{11})$
$e^{2\pi i.(\phi_{v_1})}\ket{-}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$\frac{1}{\sqrt{2}}(e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}\ket{01} + e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}+\frac{1}{2}) \bmod 1)}\ket{11})$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{+}$	$\frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}(\ket{00} + \ket{01} + \ket{10}) + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{11}$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{-}$	$\frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}(\ket{00} + \ket{10}) + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2}) \bmod 1)}\ket{01} + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2}+\frac{1}{2^k}) \bmod 1)}\ket{11}$
$e^{2\pi i.(\phi_{v_1})}\ket{-}$	$e^{2\pi i.(\phi_{v_2})}\ket{+}$	$\frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}(\ket{00} + \ket{01}) + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2}) \bmod 1)}\ket{10} + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}+\frac{1}{2}) \bmod 1)}\ket{11}$
$e^{2\pi i.(\phi_{v_1})}\ket{-}$	$e^{2\pi i.(\phi_{v_2})}\ket{-}$	$\frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \bmod 1)}\ket{00} + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2}) \bmod 1)}(\ket{10} + \ket{01}) + \frac{1}{2}e^{2\pi i.((\phi_{v_1}+\phi_{v_2}+\frac{1}{2^k}) \bmod 1)}\ket{11}$

Table 16: Table of a non-entangled input-vectors v_1, v_2 and output-vector w relation for CR_k-gate.

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For entangled input-vectors v_1, v_2 compounding the input-vector v as standing for a Bell state, as well an output-vector w, there is the Table 17.

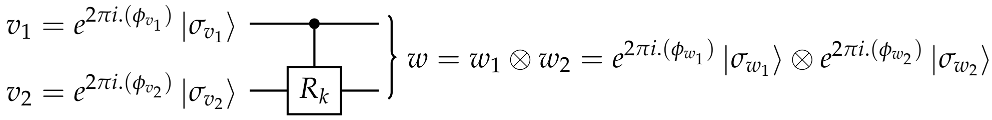

Figure 9: CR_k-gate with separable input-vectors $v_1 \in B^{(1)}$ and $v_2 \in B^{(2)}$, as well as an non-entangled output-vector $w \in \mathcal{B} = B^{(1)} \otimes B^{(2)}$.

Input Vector v	Output Vector w
${}^{\phi}\beta_{00}$	$\frac{1}{\sqrt{2}}(e^{2\pi i.\phi}\|00\rangle + e^{2\pi i.((\phi+\frac{1}{2^k}) \mod 1)}\|11\rangle)$
${}^{\phi}\beta_{01}$	${}^{\phi}\beta_{01}$
${}^{\phi}\beta_{10}$	$\frac{1}{\sqrt{2}}(e^{2\pi i.\phi}\|00\rangle + e^{2\pi i.((\phi+\frac{1}{2^k}+\frac{1}{2}) \mod 1)}\|11\rangle)$
${}^{\phi}\beta_{11}$	${}^{\phi}\beta_{11}$
${}^{\phi}\gamma$	$\frac{1}{2}(e^{2\pi i.\phi}\|00\rangle + e^{2\pi i.((\phi+\frac{1}{2^k}) \mod 1)}\|11\rangle) + \frac{1}{\sqrt{2}}{}^{\phi}\beta_{01}$

Table 17: Table of a non-entangled input-vector v and output-vector w relation for CR_k-gate.

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- *From the point of view of a PhQF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the:*
    1. *Table 18 for* $\overline{v} = \overline{v_1 \otimes v_2}$ *as a non-entangled state :*
    2. *Table 19 for* $\overline{v}$ *as an entangled state, a Bell state :*
- *From the point of view of a QF :*
  - $\overline{v} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the:*
    1. *Table 20 for* $\overline{v} = \overline{v_1 \otimes v_2}$ *as a non-entangled state :*
    2. *Table 21 for* $\overline{v}$ *as an entangled state, a Bell state :*

Example 2.2.17 (*CNOT*-gate : PhQF vs QF) *By consider the CR_k-gate, Figure 10, it is possible to build the Table 22 that describes the output vector w for a non-entangled input vector $v = v_1 \otimes v_2$.*

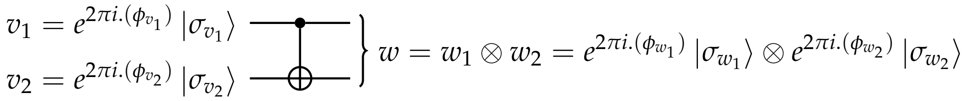

Figure 10: CR_k-gate with separable input-vectors $v_1 \in B^{(1)}$ and $v_2 \in B^{(2)}$, as well as an non-entangled output-vector $w \in \mathcal{B} = B^{(1)} \otimes B^{(2)}$.

Input State $\overline{v_1}$	Input State $\overline{v_2}$	Output State $\overline{w}$
$(0, \phi_{v_1})$	$(\sigma_{v_2}, \phi_{v_2})$	$(0\sigma_{v_2}, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(\sigma_{v_1}, \phi_{v_1})$	$(0, \phi_{v_2})$	$(\sigma_{v_1}0, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(1, \phi_{v_1})$	$(1, \phi_{v_2})$	$(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$
$(1, \phi_{v_1})$	$(+, \phi_{v_2})$	$(10, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$
$(1, \phi_{v_1})$	$(-, \phi_{v_2})$	$(10, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k} + \frac{1}{2}) \mod 1)$
$(1, \phi_{v_1})$	$(1, \phi_{v_2})$	$(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$
$(+, \phi_{v_1})$	$(1, \phi_{v_2})$	$(01, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$
$(-, \phi_{v_1})$	$(1, \phi_{v_2})$	$(01, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k} + \frac{1}{2}) \mod 1)$
$(+, \phi_{v_1})$	$(+, \phi_{v_2})$	$(00, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(01, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(10, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$
$(+, \phi_{v_1})$	$(-, \phi_{v_2})$	$(00, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(01, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2}) \mod 1)$ or $(10, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k} + \frac{1}{2}) \mod 1)$
$(-, \phi_{v_1})$	$(+, \phi_{v_2})$	$(00, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(01, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(10, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k} + \frac{1}{2}) \mod 1)$
$(-, \phi_{v_1})$	$(-, \phi_{v_2})$	$(00, (\phi_{v_1} + \phi_{v_2}) \mod 1)$ or $(01, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2}) \mod 1)$ or $(10, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2}) \mod 1)$ or $(11, (\phi_{v_1} + \phi_{v_2} + \frac{1}{2^k}) \mod 1)$

Table 18: Table of a non-entangled input-states $\overline{v_1}, \overline{v_2}$ and output-state $\overline{w}$ relation for CR_k-gate by PhQF.

.

Input State $\overline{v}$	Output State $\overline{w}$
${}^{\phi}\beta_{00}$	$(00, \phi)$ or $(11, (\phi + \frac{1}{2^k}) \mod 1)$
${}^{\phi}\beta_{01}$	${}^{\phi}\beta_{01}$
${}^{\phi}\beta_{00}$	$(00, \phi)$ or $(11, (\phi + \frac{1}{2^k}) \mod 1)$
${}^{\phi}\beta_{11}$	${}^{\phi}\beta_{11}$
${}^{\phi}\gamma$	$(00, \phi)$ or $(11, (\phi + \frac{1}{2^k}) \mod 1)$ or ${}^{\phi}\beta_{01}$

Table 19: Table of a entangled input-state$\overline{v}$ and output-vector $\overline{w}$ relation for CR_k-gate by PhQF.

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- *From the point of view of a PhQF*
  - $\overline{v} = \overline{v_1 \otimes v_2} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
  - *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the Table 23:*
- *From the point of view of a QF*

Input State $\overline{v_1}$	Input State $\overline{v_2}$	Output State $\overline{w}$
0	σ_{v_2}	$0\sigma_{v_2}$
σ_{v_1}	0	σ_{v_1}
1	1	11
1	$+$	$1+$
1	$-$	$1+$
$+$	1	$+1$
$-$	1	-1
$+$	$+$	$++$
$+$	$-$	$+-$
$-$	$+$	$-+$
$-$	$-$	$--$

Table 20: Table of a non-entangled input-states $\overline{v_1}, \overline{v_2}$ and output-state $\overline{w}$ relation for CR_k-gate by QF.

.

Input State $\overline{v}$	Output State $\overline{w}$
${}^{\phi}\beta_{00}$	${}^{\phi}\beta_{00}$
${}^{\phi}\beta_{01}$	${}^{\phi}\beta_{01}$
${}^{\phi}\beta_{00}$	${}^{\phi}\beta_{00}$
${}^{\phi}\beta_{11}$	${}^{\phi}\beta_{11}$
${}^{\phi}\gamma$	${}^{\phi}\gamma$

Table 21: Table of a entangled input-state $\overline{v}$ and output-vector $\overline{w}$ relation for CR_k-gate by QF.

.

Input Vector v_1	Input Vector v_2	Output Vector w
$e^{2\pi i.(\phi_{v_1})}\ket{0}$	v_2	$e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \mod 1)}\ket{0\sigma_{v_2}}$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{0}$	$e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \mod 1)}\ket{11}$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \mod 1)}\ket{10}$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{+}$	$e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \mod 1)}\ket{1+}$
$e^{2\pi i.(\phi_{v_1})}\ket{1}$	$e^{2\pi i.(\phi_{v_2})}\ket{-}$	$e^{2\pi i.((\phi_{v_1}+\phi_{v_2}) \mod 1)}\ket{1-}$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{0}$	$\ket{^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{00}}$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$\ket{^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{01}}$
$e^{2\pi i.(\phi_{v_1})}\ket{-}$	$e^{2\pi i.(\phi_{v_2})}\ket{0}$	$\ket{^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{10}}$
$e^{2\pi i.(\phi_{v_1})}\ket{-}$	$e^{2\pi i.(\phi_{v_2})}\ket{1}$	$\ket{^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{11}}$
$e^{2\pi i.(\phi_{v_1})}\ket{+}$	$e^{2\pi i.(\phi_{v_2})}\ket{+}$	$\ket{^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\gamma}$

Table 22: Table of a non-entangled input-vectors v_1, v_2 and output-vector w relation for $CNOT$-gate.

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- $\overline{v} = \overline{v_1 \otimes v_2} = (\sigma_v, \phi_v)$ *and* $\overline{w} = (\sigma_w, \phi_w)$ *with* $\overline{v}, \overline{w} \in \Sigma \times \Phi$.
- *Input-state* $\overline{v}$ *and output-state* $\overline{w}$ *relation given by the Table 24:*

Input State $\overline{v_1}$	Input State $\overline{v_2}$	Output State $\overline{w}$
$(0, \phi_{v_1})$	$(\sigma_{v_2}, \phi_{v_2})$	$(0\sigma_{v_2}, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(1, \phi_{v_1})$	$(0, \phi_{v_2})$	$(11, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(1, \phi_{v_1})$	$(1, \phi_{v_2})$	$(10, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(1, \phi_{v_1})$	$(+, \phi_{v_2})$	$(1+, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(1, \phi_{v_1})$	$(-, \phi_{v_2})$	$(1-, (\phi_{v_1} + \phi_{v_2}) \mod 1)$
$(+, \phi_{v_1})$	$(0, \phi_{v_2})$	${}^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{00}$
$(+, \phi_{v_1})$	$(1, \phi_{v_2})$	${}^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{01}$
$(-, \phi_{v_1})$	$(0, \phi_{v_2})$	${}^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{10}$
$(-, \phi_{v_1})$	$(1, \phi_{v_2})$	${}^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\beta_{11}$
$(+, \phi_{v_1})$	$(+, \phi_{v_2})$	${}^{((\phi_{v_1}+\phi_{v_2}) \mod 1)}\gamma$

Table 23: Table of a non-entangled input-states $\overline{v_1}, \overline{v_2}$ and output-state $\overline{w}$ relation for *CNOT*-gate by PhQF.

.

Input State $\overline{v_1}$	Input State $\overline{v_2}$	Output State $\overline{w}$
0	σ_{v_2}	$0\sigma_{v_2}$
1	0	11
1	1	10
1	$+$	$1+$
1	$-$	$1-$
$+$	0	${}^0\beta_{00}$
$+$	1	${}^0\beta_{01}$
$-$	0	${}^0\beta_{00}$
$-$	1	${}^0\beta_{01}$
$+$	$+$	${}^0\gamma$

Table 24: Table of a non-entangled input-states $\overline{v_1}, \overline{v_2}$ and output-state $\overline{w}$ relation for *CNOT*-gate by QF.

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Moreover, it is noticeable that PhQF's expressiveness is more "completed" than the QF's expressiveness.

Finally, attending to the notion of a PhQF, it is possible to construct a logic that captures the concept of quantum phase: PhLQP , which consists of an extension of LQP ([14]), and consequently an extension of PDL ([36]).

2.2.2 *Syntax of PhLQP*

In order to make the language of PhLQP there is a natural number n such that is possible to have a set of indices $N = \{1, 2, \ldots, n\}$, a set $\mathcal{Q}$ of *propositional variables*, a set $\mathcal{C}$ of *propositional constants*, and a set $\mathcal{U}$ of program constants. Moreover, this last set $\mathcal{U}$ denotes *basic programs*, which can be seen as *quantum gates* (i.e. *unitary transformations*)

Each program constant $U \in \mathcal{U}$ has an aggregate index I, which consists of a sequence of distinct indices in N (i.e. $I \subseteq N$), that establish the set of qubits which a quantum gate U_I is acting . Therefore, it is possible to write U_I for an I-local quantum gate. Concretely, for every $i, j \leq n$, there are some special programs constants: $CNOT_{ij}, X_i, Y_i, Z_i, R_{k_i}, \cdots \in \mathcal{U}$.

For the set $\mathcal{C}$ there are two special propositional constants: $(1, \phi), (+, \phi)$. These two constants denote the following states respectively :

$$e^{2\pi(\phi)i} \, |1\rangle^{\otimes n} = e^{2\pi(\phi)i} (\underbrace{|1\rangle \otimes |1\rangle \cdots \otimes |1\rangle}_{n \ times})$$
$$e^{2\pi(\phi)i} \, |+\rangle^{\otimes n} = e^{2\pi(\phi)i} (\underbrace{|+\rangle \otimes |+\rangle \cdots \otimes |+\rangle}_{n \ times})$$

The syntax of PhLQP is an extension of the syntax of LQP, with a set of propositional *formulas* and a set of *programs*, defined through mutual induction:

$$\varphi ::= \top_I \mid \ p \ \mid c \ \mid \neg\varphi \mid \varphi^\dagger \mid \varphi \wedge \psi \mid [\pi]\varphi$$
$$\pi ::= \top_I \mid \varphi? \mid U \mid \pi^\dagger \mid \pi \cup \pi \mid \pi; \pi$$

Where I denotes sequences of distinct indices in $N = \{1, 2, \ldots, n\}$. The sentence $\top_I$ stands for *I-separation*: it is true if and only if the qubits in I form a separated subsystem. Therefore, $\top_I$ is the notation for the greatest element $\top_I^{\Sigma\times\Phi}$ of the lattice of I-local properties.

Remark 2.2.15 *The sentence $\top := \top_N$ expresses the "always true" proposition (*verum*), i.e. the "top" of the lattice of all properties.*

Furthermore, $\neg\varphi$ and $\varphi \wedge \varphi$ represent the classical negation and conjunction, sequentially. Elements of the set of propositional *formulas* are pairs :

$$\varphi := (\varphi_\Sigma, \varphi_\Phi).$$

And have an *adjoint* defined by :

$$\varphi^\dagger := (\phi_\Sigma^*, \phi_\Phi^*) = (\phi_\Sigma, \phi_\Phi^*).$$

Also, $[\pi]\varphi$ expresses *the weakest precondition that ensures that property φ will hold after program π is executed*.

On the other hand, for programs: $\top_I$ represents the *trivial I-local action* $\top_I^{(\Sigma\times\Phi)\times(\Sigma\times\Phi)}$, which acts only on a state formed by the I-qubits subsystem and changing it to another randomly selected I-qubits subsystem , while living unchanged the $N \setminus I$-qubits subsystem. The significance of quantum test $\varphi?$, adjoint $\pi^\dagger$, union $\pi \cup \pi$ and composition $\pi; \pi$ is given by the corresponded operations on quantum actions.

expanding the elementary language of PhLQP It is possible to extend the language of PhLQP through the definitions of operations for a *classical disjunction* :

$$\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi) \quad ,$$

and for a *classical implication*:

$$\varphi \to \psi := \neg\varphi \vee \psi \quad .$$

Also, the introduction of constants *verum* $\top := \top_N$ and *falsum* $\bot := \neg\top$. The *classical dual* of $[\pi]\varphi$ as :

$$\langle\pi\rangle\varphi := \neg[\pi]\neg\varphi \quad .$$

Consequently, the well known *measurement modalities* $\Box$ and $\Diamond$ can be defined by writing :

$$\Diamond\varphi := \langle\varphi?\rangle\top \quad \text{and} \quad \Box\varphi := \neg\Diamond\neg\varphi \quad .$$

The *orthocomplement* is defined as

$$\sim \varphi := \Box\neg\varphi,$$

or equivalently as

$$\sim \varphi := [\varphi?]\bot.$$

Moreover, through the definition of *orthocomplement* it is possible to define a binary operation known as *quantum join* by:

$$\varphi \sqcup \psi :=\sim (\sim \varphi \wedge \sim \psi).$$

Notice that *quantum join* expresses *superposition of quantum states*: $\varphi \sqcup \psi$ is true for whichever state is a superposition of states satisfying φ or ψ. Therefore, any logical operator when applied over a element of this set is applied over both components of the (above) pair.

Example 2.2.18 (Applying unary operators) *For instance of the example, by take into account the following operators:* $\Box, \Diamond$ *and* $\sim$ *:*

- $\Box\varphi = (\Box\varphi_\Sigma, \Box\varphi_\Phi)$
- $\Diamond\varphi = (\Diamond\varphi_\Sigma, \Diamond\varphi_\Phi)$
- $\sim \varphi = (\sim \varphi_\Sigma, \sim \varphi_\Phi) = (\sim \varphi_\Sigma, \varphi_\Phi)$

Example 2.2.19 (Applying binary operators) *By considering the binary operation of quantum join there is :*

$$\varphi \ \sqcup \ \psi = (\varphi_\Sigma \ \sqcup \ \psi_\Sigma \ , \varphi_\Phi \ \sqcup \ \psi_\Phi)$$

For programs it is possible to introduce some concepts and notations: a program π is *deterministic* if π is built without the use of non-deterministic choice $\cup$ or of the non-deterministic program $\top_I$. In addition, it is possible to write:

$$SWAP_{ij} := CNOT_{ij}\ ;\ CNOT_{ji}\ ;\ CNOT_{ij}$$

for the program that swaps the i^{th} and j^{th} Σ-components of any $\{i, j\}$-separated input state. At last, for the *identity* map:

$$id := \top?$$

Remark 2.2.16 *Notice that $id = (id_\Sigma, id_\Phi) = (\top_\Sigma?, \top_\Phi?) = \top?$*

ORDER, EQUIVALENCE, ORTHOGONALITY,I-EQUIVALENCE, TESTABILITY, LOCALITY, SEPARATION. There is the possibility of incorporate the relations of *logical equivalence*, *being weaker than*,*I-equivalence* between formulas, the properties of *locality* and *testability*, and also the concept of *I-component* by writing the following formulas:

$$\varphi \leq \psi := \Box\Box(\varphi \to \psi) = (\Box\Box(\varphi_\Sigma \to \psi_\Sigma), \Box\Box(\varphi_\Phi \to \psi_\Phi))$$
$$\varphi = \psi := \Box\Box(\varphi \leftrightarrow \psi) = (\Box\Box(\varphi_\Sigma \leftrightarrow \psi_\Sigma), \Box\Box(\varphi_\Phi \leftrightarrow \psi_\Phi))$$
$$\varphi \perp \psi := \varphi \leq \sim \psi$$
$$\varphi_I := \top_I \wedge \langle \top_{N \setminus I} \rangle \varphi$$
$$\varphi =_I \psi := \varphi \leq \top_I \wedge \psi \leq \top_I \ \wedge \varphi =_I \psi_I$$
$$I(\varphi) := \varphi = \varphi_I$$

Consequently, notice that the double-box modality agrees with the universal modality : as matter of fact $\varphi \leq \psi$ stands for φ as *logically weaker* than ψ, while $\varphi = \psi$ for the equivalence of both formulas. $T(\varphi)$ as "φ is testable" and $I(\varphi)$ as "φ is I-local". Moreover, it is possible to read φ_I as the "I-component", i.e. a state which fulfils this sentence iff (it is I-separated and) its I-subsystem is (a subsystem of some state) satisfies φ.

Remark 2.2.17 *For $I = \{i\}$, it is possible to write $\varphi_i := \varphi_I$.*

Furthermore, $\varphi =_I \psi$ is readable as "φ is I-equivalent to ψ", and meaning that both φ and ψ are I-separated and have the same I-component. Lastly, φ is I-separated iff $\varphi \leq \top_I$.

Remark 2.2.18 *Every I-component φ_I is I-local.*

Definition 2.2.17 (Special Local States) *It is possible to define another propositional constants for special local states by writing:*

$$(0,\phi)_i := \sim (1,\phi)_i \quad \textit{and} \quad (-,\phi)_i := \sim (+,\phi)_i$$

Definition 2.2.18 (Image and Strongest Post-condition) *It is also definable the strongest testable post-condition $\pi[\varphi]$ ensured by (applying a program) π on (any state satisfying a given precondition) φ as:*

$$\pi[\varphi] := \sim [\pi^{\dagger}] \sim \varphi^{\dagger}$$

Therefore, for φ *testable* and π *deterministic*, the strongest postcondition $\pi[\varphi]$ agrees with the image $\pi(\varphi)$ of φ via π. The Definition 2.2.18 can be extended to *all programs which consist of finite unions of deterministic programs*, by writing for all *testable* formulas $\theta : \pi(\theta) = \pi[\theta]$ if π is deterministic, and $(\pi \cup \pi')(\theta) = \pi(\theta) \vee \pi'(\theta)$ in rest.

NOTATION For any sequence $I \subseteq N$ of indices and any vector $\vec{c} = (c(i))_{i\in I} \in \{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}^{|I|}$, there is :

$$\vec{c}_I := \Big(\bigwedge_{i\in I} c(i)_{\Sigma_i} \ , \ (\sum_{i\in I} c(i)_{\phi_i}) \mod 1 \Big)$$

Example 2.2.20 *Let $I = \{1,2\} \subseteq N = \{1,2,3\}$ and $c(i)_i \in \{\underbrace{(0,\frac{3}{4})}_{c(1)_1}, \underbrace{(+,\frac{1}{2})}_{c(2)_2}, \underbrace{(-,\frac{3}{4})}_{c(3)_3}\}$, then :*

$$\begin{aligned} \vec{c}_I = \vec{c}_{\{1,2\}} &= \Big(\bigwedge_{i\in\{1,2\}} c(i)_{\Sigma_i} \ , (\sum_{i\in\{1,2\}} c(i)_{\phi_i}) \mod 1 \Big) \\ &= (0_1 \wedge +_2 \ , \ (\frac{3}{4} + \frac{1}{2}) \mod 1) = (0+, \frac{1}{4}) \end{aligned}$$

THE UNARY MAPS INDUCED BY A PROGRAM. It is captured in the above syntax the construction $F_{(1)}$, where a linear map F on $B^{\otimes n}$ is used to describe a unary map $F_{(1)}$. So , it is possible to write:

$$(0,\phi)_i! := (0,\phi)_i? \ \cup \ ((1,\phi)_i?; X_i),$$

and

$$(0,\phi)_I! := (0,\phi)_{i_1}!; (0,\phi)_{i_2}!; \ldots ; (0,\phi)_{i_k}!,$$

where, $I = (i_1, i_2, \ldots, i_k)$. Notice that this maps any qubit in I to $(0,\phi)$. Also, it is writeable:

$$((0,\phi)_I? := (0,\phi)_{i_1}!; (0,\phi)_{i_2}!; \ldots ; (0,\phi)_{i_k})?.$$

At last, it is possible to define:

$$\pi_{(i)} := (0,\phi)_{N\setminus\{i\}}!; \pi; (0,\phi)_{N\setminus\{i\}}?$$

as the map which encodes a single qubit transformation. Additionally, there is a $B_i \to B_j$ -version of $\pi_{(1)}$:

$$\pi_{ij} := swap_{1i}; \pi_{(1)}; swap_{1j}$$

LOCAL PROGRAMS. There is the necessity of separate *local programs*, i.e. the programs that "act on only the qubits in a given set $I \subseteq N$ ". So, it is possible to write a formula $I(\pi)$ that stands for "program π is I-local":

$$I(\pi) := \bigwedge_{\vec{c},\vec{d},\vec{d}'} \left(\vec{d}_{N\setminus I} =_{N\setminus I} \pi(\vec{c}_I \wedge \vec{d}_{N\setminus I}) =_I \pi(\vec{c}_I \wedge \vec{d}'_{N\setminus I})\right)$$

where the conjunction holds for all $\vec{c} \in \{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}^{|I|}$ and all $\vec{d},\vec{d}' \in \{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}^{n-|I|}$.

ENTANGLEMENT IN RELATION TO π . To characterize states that are "entangled according to π" , there is the following formula:

$$\overline{\pi}_{ij} := \top_{ij} \wedge \bigwedge_{c\in\{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}} ([c_i?](\pi_{ij}(c_i))_j \wedge (\sim c_i \to \pi_{ij}(c_i) = \bot) \wedge (c_i^\dagger \to \pi_{ij}(c_i) = \bot)).$$

Consequently, the next formula verifies

$$c_i?(\overline{\pi}_{ij}) =_j \pi_{ij}(c_i)$$

for every $c_i \in \{(0,\phi)_i,(1,\phi)_i,(+,\phi)_i,(-,\phi)_i\}$.

2.2.3 *Semantics of PhLQP*

An PhLQP-model is a *multi-partite quantum frame* $\Sigma \times \Phi = \Sigma \times \Phi(\mathcal{B})$ based on an n-dimensional space $\mathcal{B}$ and equipped with a *valuation function* which maps every variable p into a set of states $\|p\| \subseteq \Sigma \times \Phi$. It will be use the valuation map in order to give an interpretation $\|\varphi\| \subseteq \Sigma \times \Phi$ for all formulas of PhLQP, i.e. sets of states in $\Sigma \times \Phi$. On the other hand, it is also possible to give an interpretation $\|\pi\| \subseteq (\Sigma \times \Phi) \times (\Sigma \times \Phi)$ for all programs. This two interpretations are defined through *mutual recursion*.

INTERPRETATION OF PROGRAMS.

$$\|\top_I\| := \top_I^{(\Sigma\times\Phi)\times(\Sigma\times\Phi)} \quad , \quad \|\varphi?\| := \|\varphi\|? \quad , \quad \|U\| := U$$
$$\|\pi^\dagger\| := \|\pi\|^\dagger \quad , \quad \|\pi_1 \cup \pi_2\| := \|\pi_1\| \ \cup \ \|\pi_2\| \quad , \quad \|\pi_1;\pi_2\| := \|\pi_1\|;\|\pi_2\|$$

The interpretation $\|\pi\|$ allows the possibility of extend the notation $\xrightarrow{\pi}$ to all programs , by writing:

$$s \xrightarrow{\pi} t \quad \text{iff} \quad (s,t) \in \|\pi\|$$

INTERPRETATION OF FORMULAS. Furthermore, by extending the valuation $\|p\|$ from propositional variables to all formulas, by establishing for the others:

$$\|(0,\phi)\| = e^{2\pi i\phi}\,|0\rangle^{\otimes n} \quad , \quad \|(1,\phi)\| = e^{2\pi i\phi}\,|1\rangle^{\otimes n} \quad , \quad \|(+,\phi)\| = e^{2\pi i\phi}\,|+\rangle^{\otimes n}$$
$$\|(-,\phi)\| = e^{2\pi i\phi}\,|-\rangle^{\otimes n} \quad , \quad \|\varphi\| = \|\varphi_\Sigma\| \times \|\varphi_\Phi\| \quad , \quad \|\varphi\wedge\psi\| = \|\varphi\| \cap \|\psi\|$$
$$\|\neg\varphi\| = \Sigma \setminus \|\varphi_\Sigma\| \times \Phi \quad , \quad \|\varphi^\dagger\| = \|\varphi_\Sigma\| \times \|\varphi^*_\Phi\| \quad , \quad \|[\pi]\varphi\| = [\|\pi\|]\|\varphi\| \quad , \quad \|\top_I\| = \top_I^{\Sigma\times\Phi}$$

Definition 2.2.19 (Inference Relation $\models$) *for a state w and a formula φ the inference relation $\models$ is defined through the valuation function by :*

$$w \models \varphi \quad \textit{iff} \quad w \in ||\varphi||$$

Proposition 2.2.9 ([14]) *The interpretation of a testable formula is a testable property. The interpretation of every I-local formula is an I-local property.*

Lemma 2.2.4 ([14])

$$\| \sim \varphi\| = \|\varphi\|^\perp \ , \ \|[\varphi?]\psi\| = [\|\varphi\|?]\|\psi\| \ , \ \|\Box\varphi\| = \Box\ \|\varphi\| \ , \overline{\|\varphi\|} = \|\sim\sim\varphi\|$$

Lemma 2.2.5 *Also,* $\overline{\|\varphi\|} = \|\varphi^{\dagger\dagger}\|$.

Proof 2.2.7 (Lemma 2.2.5) *Consider:*

- *for the term* $\overline{\|\varphi\|}$,

$$\begin{aligned}\overline{\|\varphi\|} &= \overline{\|\varphi_\Sigma\|} \times \overline{\|\varphi_\Phi\|} \\ &= \|\varphi_\Sigma\| \times \|\varphi_\Phi\| \quad .\end{aligned}$$

- *for the term* $\|\varphi^{\dagger\dagger}\|$,

$$\begin{aligned}\|\varphi^{\dagger\dagger}\| &= \|\varphi_\Sigma\| \times \|\varphi^{**}_\Phi\| \\ &= |\varphi_\Sigma\| \times \|\varphi_\Phi\| \quad .\end{aligned}$$

Therefore $\overline{\|\varphi\|} = \|\varphi^{\dagger\dagger}\|$.

Proposition 2.2.10 ([14]) *For every formula φ , there are the following equivalences:*

1. $\|\varphi\|$ *is testable*
2. φ *is semantically equivalent to* $\sim\sim \varphi$
3. φ *is semantically equivalent to some formula* $\Box\psi$
4. φ *is semantically equivalent to some formula* $\sim \psi$

Proposition 2.2.11 *Moreover, for every formula φ:*

1. φ *is semantically equivalent to* $\varphi^{\dagger\dagger}$
2. φ *is semantically equivalent to some formula* $\psi^{\dagger}$

Proof 2.2.8 (2.2.11) *To proof Proposition 2.2.11.*

1. *For φ is semantically equivalent to φ, there is the need to show that $\|\varphi\| = \|\varphi^{\dagger\dagger}\|$, so:*

$$\begin{aligned}\|\varphi\| &= \|\varphi^{\dagger\dagger}\| \\ &= \|\varphi_{\Sigma}\| \times \|\varphi^{**}_{\Phi}\| \\ &= \|\varphi_{\Sigma}\| \times \|\varphi_{\Phi}\| \\ &= \|\varphi\| \quad .\end{aligned}$$

2. *By consider the above equality, it is easy to check that φ is semantically equivalent to the $\varphi^{\dagger\dagger}$, i.e. $\psi = \varphi^{\dagger}$.*

2.2.4 *Proof Theory for PhLQP*

Here, it will be presented the axioms and rules of PhLQP, which are based on LQP's proof theory, [14].

AXIOMS FOR SINGLE SYSTEMS Therefore, for single systems there are the following rules and axioms:

Substitution Rule. From $\vdash \Theta$ infer $\vdash \Theta[p/\varphi]$
and for the dynamic modalities $[\pi]$:
Kripke Axiom. $\vdash [\pi](p \to q) \to ([\pi]p \to [\pi]q)$
Necessitation Rule. From $\vdash p$ infer $\vdash [\pi]p$
Also, attending to $\Box p$:

Test Generalization Rule. If p does not occur either in φ or in ψ, then:
from $\vdash \varphi \to [q?]\psi$ infer $\vdash \varphi \to \Box\psi$
Testability Axiom. $\vdash \Box p \to [q?]p$

Moreover, Testability can be expressed by its dual form by writing $\langle q?\rangle p \to \Diamond p$ (or $\langle q?\rangle p \to \langle p?\rangle\top$), which gives the following interpretation: if a measurement can actualize the property inherent to p, then it is possible to test directly the property p. Additionally, the Test Generalization Rule codifies the reality of $\Box$ as a universal quantifier over all possible measurements.

Remark 2.2.19 *Notice that the above axioms and rules can actually be seen as pairs of axioms and rules . In other words, there is:*

$$\vdash := (\vdash_\Sigma , \vdash_\Phi)$$

Furthermore, by expanding the notation :

- ***Substitution Rule.***

$$\textit{From} \quad (\vdash_\Sigma \Theta_\Sigma , \vdash_\Phi \Theta_\Phi) \quad \textit{infer} \quad (\vdash_\Sigma \Theta_\Sigma[p_\Sigma/\varphi_\Sigma] , \vdash_\Phi \Theta_\Phi[p_\Phi/\varphi_\Phi])$$

- ***Kripke Axiom.***

$$(\vdash_\Sigma [\pi_\Sigma](p_\Sigma \to q_\Sigma) \to ([\pi_\Sigma]p_\Sigma \to [\pi_\Sigma]q_\Sigma) , \vdash_\Phi [\pi_\Phi](p_\Phi \to q_\Phi) \to ([\pi_\Phi]p_\Phi \to [\pi_\Phi]q_\Phi))$$

- ***Necessitation Rule.***

$$\textit{From} \quad (\vdash_\Sigma p_\Sigma , \vdash_\Phi p_\Phi) \quad \textit{infer} \quad (\vdash_\Sigma [\pi_\Sigma]p_\Sigma , \vdash_\Phi [\pi_\Phi]p_\Phi)$$

- ***Test Generalization Rule.*** *If (p_Σ, p_Φ) does not occur either in $(\varphi_\Phi, \varphi_\Sigma)$ or in (ψ_Φ, ψ_Σ), then:*

$$\textit{from} \quad (\vdash_\Sigma \varphi_\Sigma , \vdash_\Phi \varphi_\Phi) \to ([q_\Sigma?]\psi_\Sigma , [q_\Phi?]\psi_\Phi)$$

- ***Testability Axiom.***

$$(\vdash_\Sigma \Box p_\Sigma , \vdash_\Phi \Box p_\Phi) \to ([q_\Sigma?]p_\Sigma , [q_\Phi?]p_\Phi)$$

Other PhLQP axioms are:
Σ-Partial Functionality. $\vdash \neg[p?]q \to [p?]\neg q$
Φ-Partial Functionality. $\vdash [p^\dagger?]q \to [p?]q^\dagger$
Adequacy. $\vdash p \wedge q \to \langle p?\rangle q$
Repeatability. $\vdash T(p) \to [p?]p$
Proper Superpositions. $\vdash \langle\pi\rangle\Box\Box p \to [\pi']p$

Σ-Unitary Functionality. $\vdash \neg[U]q \leftrightarrow [U]\neg q$
Φ-Unitary Functionality. $\vdash [U^{\dagger}]q \leftrightarrow [U]q^{\dagger}$
Unitary Bijectivity 1. $\vdash p \leftrightarrow [U;U^{\dagger}]p$
Unitary Bijectivity 2. $\vdash p \leftrightarrow [U^{\dagger};U]p$
Adjointness. $\vdash p \to [\pi]\Box\langle\pi^{\dagger}\rangle\Diamond p$

Proposition 2.2.12 *Testability is a closure equipped with conjunctions, weakest preconditions , $\Box$-sentences, orthocomplements, adjoints, and strongest postconditions:*

- $\vdash T(p) \wedge T(q) \to T(p \wedge q)$
- $\vdash T(p) \to T([\pi]p)$
- $\vdash T(\Box p)$
- $\vdash T(\sim p)$
- $\vdash T(p^{\dagger})$
- $\vdash T([\pi]p)$

A formula φ is said to be *testable* if the theorem

$$\vdash T(\varphi)$$

is provable in PhLQP. Moreover, notice that Proposition 2.2.12 provides a plainly syntactical way of verify testability:

Corollary 2.2.3.3 *Every formula of the formula φ that has one of the following formulas $\Box\varphi$, $\sim \varphi$, $\varphi^{\dagger}$ or $\top$, or which can be acquired by using only conjunctions $\theta \wedge \varphi$ and weakest preconditions $[\pi]\varphi$, is testable.*

Proposition 2.2.13 (Quantum Logic, Weak Modularity or *Quantum Modus Ponens*, [14]) *In a similar way to LQP, all the axioms and rules of traditional Quantum Logic are assured by the testable formulas of PhLQP. With its axioms it is possible to prove the "Quantum Modus Ponens":*

$$\varphi \wedge [\varphi?]\psi \leq \psi.$$

Also, this rule is equivalent to the well-known condition of quantum logic, Weak Modularity:

$$\varphi \wedge (\sim \varphi \sqcup (\varphi \wedge \psi)) \leq \psi.$$

Theorem 2.2.4 (Soundness) *All previous presented axioms are sound.*

Proof 2.2.9 (Theorem 2.2.4) *The following rules and axioms:* ***Substitution Rule, Kripke Axiom,Necessitation Rule,Test Generalization Rule and Testability Axiom*** *are typical axioms of PDL, [36]. Therefore, it will be assumed that they are already sound. For the proof of soundness for the axioms in the sequel, notice that:*

- $\varphi \to \psi := \neg\varphi \vee \psi \quad .$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi) \quad .$
- *So,* $\|\varphi \to \psi\| = \|\neg\varphi \vee \psi\| = \|\neg(\varphi \wedge \neg\psi)\| \quad .$

Let $w \in \Sigma \times \Phi$ *be a state:*

1. ***Soundness of*** Σ***-Partial Functionality:***

$$\|\neg[p?]q \to [p?]\neg q\| = \|\neg([p?]q \wedge \neg[p?]\neg q)\|$$

Now, just consider $\|([p_\Sigma?]q_\Sigma \wedge \neg[p_\Sigma?]\neg q_\Sigma)\|$*:*

$$\begin{aligned}\|([p_\Sigma?]q_\Sigma \wedge \neg[p_\Sigma?]\neg q_\Sigma)\| &= \|[p_\Sigma?]q_\Sigma\| \ \cap \ \|\neg[p_\Sigma?]\neg q_\Sigma\| \\ &= \|[p_\Sigma?]q_\Sigma\| \ \cap \ \Sigma \setminus \|[p_\Sigma?]\neg q_\Sigma\| \\ &= [\|p_\Sigma?\|]\|q_\Sigma\| \cap \ \Sigma \setminus ([\|p_\Sigma?\|]\|\neg q_\Sigma\|) \\ &= [\|p_\Sigma?\|]\|q_\Sigma\| \cap \ \Sigma \setminus ([\|p_\Sigma?\|]\|q_\Sigma\|) \\ &= \emptyset\end{aligned}$$

So,

$$\begin{aligned}\Sigma \setminus \|([p_\Sigma?]q_\Sigma \wedge \neg[p_\Sigma?]\neg q_\Sigma)\| \times \Phi &= \Sigma \setminus \emptyset \times \Phi \\ &= \Sigma \times \Phi \quad .\end{aligned}$$

2. ***Soundness of*** Φ***-Partial Functionality:***

$$\|[p^\dagger?]q \to [p?]q^\dagger\| = \|\neg(\neg[p^\dagger?]q \wedge \neg[p?]q^\dagger)\|$$

Now, just consider $\|(\neg[p^\dagger?]q \wedge \neg[p?]q^\dagger)\|$*:*

- *For the term* $[p^\dagger?]q$*:*

$$\begin{aligned}\|[p^\dagger?]q\| &= [\|p^\dagger?\|]\|q\| \\ &= [\|p^\dagger\|?]\|q\| \\ &= \Sigma \times [\|p^*_\Phi\|?]\|q_\Phi\| \quad .\end{aligned}$$

- *For the term* $[p?]q^{\dagger}$*:*

$$\begin{aligned}\|[p?]q^{\dagger}\| &= [\|p?\|]\|q^{\dagger}\| \\ &= [\|p\|?]\|q^{\dagger}\| \\ &= \Sigma \times [\|p_{\Phi}\|?]\|q^{*}_{\Phi}\| \quad .\end{aligned}$$

-

$$\begin{aligned}\|(\neg[p^{\dagger}?]q \wedge \neg[p?]q^{\dagger})\| &= \|\neg[p^{\dagger}?]q\| \ \cap \ \|\neg[p?]q^{\dagger}\| \\ &= (\Sigma \setminus [\|p^{*}_{\Sigma}\|?]\|q_{\Sigma}\| \times [\|p^{*}_{\Phi}\|?]\|q_{\Phi}\|) \\ &\cap (\Sigma \setminus [\|p_{\Sigma}\|?]\|q^{*}_{\Sigma}\| \times [\|p_{\Phi}\|?]\|q^{*}_{\Phi}\|) \\ &= (\Sigma \setminus [\|p_{\Sigma}\|?]\|q_{\Sigma}\| \times [\|p^{*}_{\Phi}\|?]\|q_{\Phi}\|) \\ &\cap (\Sigma \setminus [\|p_{\Sigma}\|?]\|q_{\Sigma}\| \times [\|p_{\Phi}\|?]\|q^{*}_{\Phi}\|) \\ &= \Sigma \setminus [\|p_{\Sigma}\|?]\|q_{\Sigma}\| \times \emptyset \\ &= \emptyset \quad .\end{aligned}$$

So,

$$\begin{aligned}\|[p^{\dagger}?]q \rightarrow [p?]q^{\dagger}\| &= \|\neg(\neg[p^{\dagger}?]q \wedge \neg[p?]q^{\dagger})\| \\ &= \Sigma \setminus \emptyset \times \Phi \\ &= \Sigma \times \Phi \quad .\end{aligned}$$

3. ***Soundness of Adequacy:***

$$\begin{aligned}\|p \wedge q \rightarrow \langle p?\rangle q\| &= \|\neg(p \wedge q) \vee \langle p?\rangle q\| \\ &= \|\neg((p \wedge q) \wedge \neg\langle p?\rangle q)\|\end{aligned}$$

Now, just consider $(p \wedge q) \wedge \neg\langle p?\rangle q$):

$$
\begin{aligned}
\|(p \wedge q) \wedge \neg\langle p?\rangle q\| &= \|p \wedge q\| \cap \|\neg\langle p?\rangle q\| \\
&= \|p\| \cap \|q\| \cap \|[p?]\neg q\| \\
&= \|p\| \cap \|q\| \cap [\|p?\|]\|\neg q\| \\
&= \|p\| \cap \|q\| \cap [\|p\|?]\|\neg q\| \\
&= \big(\|p_\Sigma\| \cap \|q_\Sigma\| \cap [\|p_\Sigma\|?]\Sigma \setminus \|q_\Sigma\|\big) \\
&\times \big(\|p_\Phi\| \cap \|q_\Phi\| \cap [\|p_\Phi\|?]\Phi\big) \\
&= \emptyset \times \big(\|p_\Phi\| \cap \|q_\Phi\| \cap [\|p_\Phi\|?]\Phi\big) \\
&= \emptyset \quad .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|\neg((p \wedge q) \wedge \neg\langle p?\rangle q)\| &= \Sigma \setminus \emptyset \times \Phi \\
&= \Sigma \times \Phi \quad .
\end{aligned}
$$

4. ***Soundness Repeatability:***

$$
\begin{aligned}
\|T(p) \rightarrow [p?]p\| &= \|\neg T(p) \vee [p?]p\| \\
&= \|\neg(T(p) \wedge \neg[p?]p)\|
\end{aligned}
$$

For the term $\|T(p) \wedge \neg[p?]p\|$:

$$
\begin{aligned}
\|T(p) \wedge \neg[p?]p\| &= \|T(p)\| \cap \|\neg[p?]p\| \\
&= \|T(p)\| \cap \big(\Sigma \setminus ([\|p_\Sigma?\|]p_\Sigma\|)\big) \times ([\|p_\Phi?\|]\|p_\Phi\|)\big) \\
&= \emptyset \quad .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|T(p) \rightarrow [p?]p\| &= \Sigma \setminus \emptyset \times \Phi \\
&= \Sigma \times \Phi \quad .
\end{aligned}
$$

5. ***Soundness of Proper Superpositions:***

$$\begin{aligned}\|\langle\pi\rangle\Box\Box p \to [\pi']p\| &= \|\neg[\pi]\neg\Box\Box p \to [\pi']p\| \\ &= \|\neg(\neg[\pi]\neg\Box\Box p \wedge \neg[\pi']p)\|\end{aligned}$$

Consider just the term $(\neg[\pi]\neg\Box\Box p \wedge \neg[\pi']p)$*:*

- *For the term* $\neg[\pi]\neg\Box\Box p$ *:*

$$\begin{aligned}\|\neg[\pi]\neg\Box\Box p\| &= \Sigma \setminus ([\|\pi_\Sigma\|]\Sigma \setminus \Box\Box\|p_\Sigma\|) \times ([\|\pi_\Phi\|]\Box\Box\|p_\Phi\|) \\ &= [\|\pi_\Sigma\|]\Box\Box\|p_\Sigma\| \times [\|\pi_\Phi\|]\Box\Box\|p_\Phi\| \quad .\end{aligned}$$

- *For the term* $\neg[\pi']p$ *:*

$$\|\neg[\pi']p\| = \Sigma \setminus [\|\pi'_\Sigma\|\|p_\Sigma\|] \times \|\pi'_\Phi\|\|p_\Phi\| \quad .$$

- *For* $\neg[\pi]\neg\Box\Box p \wedge \neg[\pi']p$*:*

$$\begin{aligned}\|\neg[\pi]\neg\Box\Box p \wedge \neg[\pi']p\| &= \|\neg[\pi]\neg\Box\Box p\| \cap \|\neg[\pi']p\| \\ &= \big([\|\pi_\Sigma\|]\Box\Box\|p_\Sigma\| \cap \Sigma \setminus [\|\pi'_\Sigma\|]\|p_\Sigma\|\big) \\ &\times \big([\|\pi_\Phi\|]\Box\Box\|p_\Phi\| \cap [\|\pi'_\Phi\|]\Box\Box\|p_\Phi\|\big) \\ &= \emptyset \quad .\end{aligned}$$

So,

$$\begin{aligned}\|\langle\pi\rangle\Box\Box p \to [\pi']p\| &= \Sigma \setminus \emptyset \times \Phi \\ &= \Sigma \times \Phi \quad .\end{aligned}$$

6. ***Soundness of*** Σ***-Unitary Functionality:***

$$\begin{aligned}\|\neg[U]q \leftrightarrow [U]\neg q\| &= \|(\neg[U]q \to [U]\neg q) \wedge ([U]\neg q \to \neg[U]q)\| \\ &= \|(\neg[U]q \to [U]\neg q)\| \ \cap \ \|([U]\neg q \to \neg[U]q)\| \\ &= \|\neg(\neg[U]q \wedge \neg[U]\neg q)\| \ \cap \ \|\neg([U]\neg q \wedge [U]q)\|\end{aligned}$$

- *For the term* $(\neg[U]q \wedge \neg[U]\neg q)$*:*

$$\begin{aligned}\|(\neg[U]q \wedge \neg[U]\neg q)\| &= \|\neg[U]q\| \cap \|\neg[U]\neg q\| \\ &= (\Sigma \setminus \|[U_\Sigma]q_\Sigma\| \times \|[U_\Phi]q_\Phi\|) \ \cap \ (\Sigma \setminus \Sigma \setminus \|[U_\Sigma]q_\Sigma\| \times \|[U_\Phi]q_\Phi\|) \\ &= (\Sigma \setminus \|[U_\Sigma]q_\Sigma\| \times \|[U_\Phi]q_\Phi\|) \ \cap \ (|[U_\Sigma]q_\Sigma\| \times \|[U_\Phi]q_\Phi\|) \\ &= \varnothing \quad .\end{aligned}$$

- *For the term* $([U]\neg q \wedge [U]q)$*:*

$$\begin{aligned}\|([U]\neg q \wedge [U]q)\| &= \|[U]\neg q\| \cap \|[U]q\| \\ &= [\|U\|]\|\neg q\| \ \cap \ [\|U\|]\|q\| \\ &= ([U_\Sigma]\Sigma \setminus \|q_\Sigma\| \times [U_\Phi]\|q_\Phi\|) \ \cap \ ([U_\Sigma]\|q_\Sigma\| \times [U_\Phi]\|q_\Phi\|) \\ &= \varnothing \quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\|\neg[U]q \leftrightarrow [U]\neg q\| &= \|\neg(\neg[U]q \wedge \neg[U]\neg q)\| \ \cap \ \|\neg([U]\neg q \wedge [U]q)\| \\ &= (\Sigma \setminus \varnothing \times \Phi) \ \cap \ (\Sigma \setminus \varnothing \times \Phi) \\ &= \Sigma \times \Phi \quad .\end{aligned}$$

7. ***Soundness of*** Φ***-Unitary Functionality:***

$$\begin{aligned}\|[U^\dagger]q \leftrightarrow [U]q^\dagger\| &= \|[U^\dagger]q \rightarrow [U]q^\dagger\| \ \cap \ \|[U]q^\dagger \rightarrow [U^\dagger]q\| \\ &= \|\neg([U^\dagger]q \wedge \neg[U]q^\dagger)\| \ \cap \ \|\neg([U]q^\dagger \wedge \neg[U^\dagger]q)\|\end{aligned}$$

- *For the term* $([U^\dagger]q \wedge \neg[U]q^\dagger)$*:*

$$\begin{aligned}\|[U^\dagger]q \wedge \neg[U]q^\dagger\| &= \|[U^\dagger]q\| \ \cap \ \|\neg[U]q^\dagger\| \\ &= [\|[U^\dagger]\|]\|q\| \ \cap \ \|\neg[U]q^\dagger\| \\ &= [\|U\|^\dagger]\|q\| \ \cap \ (\Sigma \setminus \|[U_\Sigma]q_\Sigma\|\|) \times (\|[U_\Phi]q^*_\Phi\|) \\ &= ([U^\dagger_\Sigma]\|q_\Sigma\| \times [U^\dagger_\Phi]\|q_\Phi\|) \ \cap \ (\Sigma \setminus \|[U_\Sigma]q_\Sigma\|\|) \times (\|[U_\Phi]q^*_\Phi\|) \\ &= \varnothing \quad .\end{aligned}$$

- *For the term* $([U]q^\dagger \wedge \neg[U^\dagger]q)$*:*

$$\begin{aligned}
\|[U]q^\dagger \wedge \neg[U^\dagger]q\| &= \|[U]q^\dagger\| \ \cap \ \|\neg[U^\dagger]q\| \\
&= [\|U\|]\|q^\dagger\| \ \cap \ \Sigma \setminus \|[U^\dagger_\Sigma]q_\Sigma\| \times [\|U^\dagger_\Phi\|]\|q_\Phi\| \\
&= [U]\|q\|^\dagger \ \cap \ \Sigma \setminus [U^\dagger_\Sigma]\|q_\Sigma\| \times [U^\dagger_\Phi]\|q_\Phi\| \\
&= [U_\Sigma]\|q_\Sigma\| \times [U_\Phi]\|q^*_\Phi\| \ \cap \ \Sigma \setminus [U^\dagger_\Sigma]\|q_\Sigma\| \times [U^\dagger_\Phi]\|q_\Phi\| \\
&= \emptyset \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|[U^\dagger]q \wedge \neg[U]q^\dagger\| &= \Sigma \setminus \emptyset \times \Phi \ \cap \ \Sigma \setminus \emptyset \times \Phi \\
&= \Sigma \times \Phi \quad .
\end{aligned}$$

8. ***Unitary Bijectivity 1:***

$$\begin{aligned}
\|p \leftrightarrow [U;U^\dagger]p\| &= \|p \to [U;U^\dagger]p\| \ \cap \ \|[U;U^\dagger]p \to p\| \\
&= \|\neg(p \wedge \neg[U;U^\dagger]p)\| \ \cap \ \|\neg([U;U^\dagger]p \wedge \neg p)\|
\end{aligned}$$

- *For the term* $(p \wedge \neg[U;U^\dagger]p)$*:*

$$\begin{aligned}
\|p \wedge \neg[U;U^\dagger]p\| &= \|p\| \ \cap \ \|\neg[U;U^\dagger]p\| \\
&= \|p\| \ \cap \ \Sigma \setminus \|[U_\Sigma;U^\dagger_\Sigma]p_\Sigma\| \times \|[U_\Phi;U^\dagger_\Phi]p_\Phi\| \\
&= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus [\|U_\Sigma\|;\|U^\dagger_\Sigma\|]\|p_\Sigma\| \times [\|U_\Phi\|;\|U^\dagger_\Phi\|]\|p_\Phi\| \\
&= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus [\underbrace{U_\Sigma;U^\dagger_\Sigma}_{=id_\Sigma}]\|p_\Sigma\| \times [\underbrace{U_\Phi;U^\dagger_\Phi}_{=id_\Phi}]\|p_\Phi\| \\
&= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus \|p_\Sigma\| \times \|p_\Phi\| \\
&= \emptyset \quad .
\end{aligned}$$

- *For the term* $([U;U^\dagger]p \wedge \neg p)$*:*

$$\begin{aligned}
\|[U;U^\dagger]p \wedge \neg p\| &= \|[U,U^\dagger]p\| \ \cap \ \|\neg p\| \\
&= [\|U;U^\dagger\|]\|p\| \ \cap \ \|\neg p\| \\
&= [\underbrace{U;U^\dagger}_{=id}]\|p\| \ \cap \ \|\neg p\| \\
&= \|p\| \ \cap \|\neg p\| \\
&= \emptyset \quad .
\end{aligned}$$

So,

$$\begin{aligned}\|p \leftrightarrow [U;U^\dagger]p\| &= \Sigma \setminus \emptyset \times \Phi \ \cap \ \Sigma \setminus \emptyset \times \Phi \\ &= \Sigma \times \Phi.\end{aligned}$$

9. ***Unitary Bijectivity 2:***

$$\begin{aligned}\|p \leftrightarrow [U^\dagger;U]p\| &= \|p \rightarrow [U^\dagger;U]p\| \ \cap \ \|[U^\dagger;U]p \rightarrow p\| \\ &= \|\neg(p \wedge \neg[U^\dagger;U]p)\| \ \cap \ \|\neg([U^\dagger;U]p \wedge \neg p)\|\end{aligned}$$

- *For the term* $(p \wedge \neg[U^\dagger;U]p)$*:*

$$\begin{aligned}\|p \wedge \neg[U^\dagger;U]p\| &= \|p\| \ \cap \ \|\neg[U^\dagger;U]p\| \\ &= \|p\| \ \cap \ \Sigma \setminus \|[U_\Sigma^\dagger;U_\Sigma]p_\Sigma\| \times \|[U_\Phi^\dagger;U_\Phi]p_\Phi\| \\ &= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus [\|U_\Sigma^\dagger\|;\|U_\Sigma\|]\|p_\Sigma\| \times [\|U_\Phi^\dagger\|;\|U_\Phi\|]\|p_\Phi\| \\ &= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus [\underbrace{U_\Sigma^\dagger;U_\Sigma}_{=id_\Sigma}]\|p_\Sigma\| \times [\underbrace{U_\Phi^\dagger;U_\Phi}_{=id_\Phi}]\|p_\Phi\| \\ &= \|p_\Sigma\| \times \|p_\Phi\| \ \cap \ \Sigma \setminus \|p_\Sigma\| \times \|p_\Phi\| \\ &= \emptyset \quad .\end{aligned}$$

- *For the term* $([U^\dagger;U]p \wedge \neg p)$*:*

$$\begin{aligned}\|[U^\dagger;U]p \wedge \neg p\| &= \|[U^\dagger,U]p\| \ \cap \ \|\neg p\| \\ &= [\|U^\dagger;U\|]\|p\| \ \cap \ \|\neg p\| \\ &= [\underbrace{U^\dagger;U}_{=id}]\|p\| \ \cap \ \|\neg p\| \\ &= \|p\| \ \cap \|\neg p\| \\ &= \emptyset \quad .\end{aligned}$$

So,

$$\begin{aligned}\|p \leftrightarrow [U^\dagger;U]p\| &= \Sigma \setminus \emptyset \times \Phi \ \cap \ \Sigma \setminus \emptyset \times \Phi \\ &= \Sigma \times \Phi.\end{aligned}$$

10. ***Soundness of Adjointness:***

$$\begin{aligned}\|p \to [\pi]\Box\langle\pi^\dagger\rangle\Diamond p\| &= \|\neg(p \wedge [\pi]\Box\langle\pi^\dagger\rangle\Diamond p)\| \\ &= \|\neg(p \wedge \neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p)\|\end{aligned}$$

- *For the term $[\pi^\dagger]\Box\neg p$:*

$$\begin{aligned}\|[\pi^\dagger]\Box\neg p\| &= [\|\pi\|^\dagger]\|\Box\neg p\| \\ &= [\|\pi\|^\dagger]\Box\|\neg p\| \\ &= ([\|\pi_\Sigma\|^\dagger]\Box\Sigma \setminus \|p_\Sigma\|) \times ([\|\pi_\Phi\|^\dagger]\Box\|p_\Phi\|) \quad .\end{aligned}$$

- *For the term $\Box\neg[\pi^\dagger]\Box\neg p$:*

$$\begin{aligned}\|\Box\neg[\pi^\dagger]\Box\neg p\| &= \Box\|\neg[\pi^\dagger]\Box\neg p\| \\ &= \Box\Sigma \setminus ([\|\pi_\Sigma\|^\dagger]\Box\Sigma \setminus \|p_\Sigma\|) \times ([\|\pi_\Phi\|^\dagger]\Box\|p_\Phi\|) \quad .\end{aligned}$$

- *For the term $[\pi]\Box\neg[\pi^\dagger]\Box\neg p$:*

$$\begin{aligned}\|[\pi]\Box\neg[\pi^\dagger]\Box\neg p\| &= [\|\pi\|]\|\Box\neg[\pi^\dagger]\Box\neg p\| \\ &= ([\|\pi_\Sigma\|]\Box\Sigma \setminus ([\|\pi_\Sigma\|^\dagger]\Box\Sigma \setminus \|p_\Sigma\|)) \times ([\|\pi_\Phi\|^\dagger]\Box\|p_\Phi\|) \quad .\end{aligned}$$

- *For the term $\neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p$:*

$$\|\neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p\| = \Sigma \setminus ([\|\pi_\Sigma\|]\Box\Sigma \setminus ([\|\pi_\Sigma\|^\dagger]\Box\Sigma \setminus \|p_\Sigma\|)) \times ([\|\pi_\Phi\|^\dagger]\Box\|p_\Phi\|) \quad .$$

- *For the term $(p \wedge \neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p)$:*

$$\begin{aligned}\|(p \wedge \neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p)\| &= \|p\| \ \cap \ \|\neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p)\| \\ &= (\|p_\Sigma\| \times \|p_\Phi\|) \ \cap \ \ (\Sigma \setminus ([\|\pi_\Sigma\|]\Box\Sigma \setminus ([\|\pi_\Sigma\|^\dagger]\Box\Sigma \setminus \|p_\Sigma\|)) \\ &\times ([\|\pi_\Phi\|^\dagger]\Box\|p_\Phi\|)) = \varnothing \quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\|p \to [\pi]\Box\langle\pi^\dagger\rangle\Diamond p\| &= \|\neg(p \wedge \neg[\pi]\Box\neg[\pi^\dagger]\Box\neg p)\| \\ &= \Sigma \setminus \varnothing \times \Phi \\ &= \Sigma \times \Phi \quad .\end{aligned}$$

Proposition 2.2.14 ([14]) *The formula $\pi[\varphi]$ expresses the strongest testable postcondition assured by performing the program π on any state satisfying φ. This is, for every testable ψ :*

$$\pi[\varphi] \leq \psi \quad \textit{iff} \quad \varphi \leq [\pi]\psi$$

Proposition 2.2.15 (Adjointness Theorem) *For all testable formulas φ, ψ, there is:*

$$\varphi \perp \pi[\psi] \quad \textit{iff} \quad \pi^{\dagger}[\varphi] \perp \psi^{\dagger}$$

Proof 2.2.10 (Proposition 2.2.15) *Consider two testable formulas φ and ψ, if*

$$\varphi \perp \pi[\psi]$$

holds, then:

$$\varphi_{\Sigma} =\sim \pi_{\Sigma}[\psi_{\Sigma}] \quad \textit{with} \quad \varphi_{\Phi} = \pi_{\Phi}[\psi_{\Phi}] \quad .$$

On the other hand, if

$$\pi^{\dagger}[\varphi] \perp \psi^{\dagger}$$

holds, then:

$$\pi_{\Sigma}[\varphi_{\Sigma}] =\sim \psi_{\Sigma} \quad \textit{with} \quad \pi^{\dagger}_{\Phi}[\varphi_{\Phi}] = \psi^{\dagger}_{\Phi} \quad ,$$

which is equivalent to have

$$\varphi_{\Sigma} =\sim \pi_{\Sigma}[\psi_{\Sigma}] \quad \textit{with} \quad \varphi_{\Phi} = \pi_{\Phi}[\psi_{\Phi}] \quad .$$

So,

$$\varphi \perp \pi[\psi] \quad \textit{iff} \quad \pi^{\dagger}[\varphi] \perp \psi^{\dagger} \quad .$$

AXIOMS FOR COMPOUND SYSTEMS For the compound systems:

Axioms for the trivial I-local programs. The program $\top_I$ stands for the weakest I-local program, this is :

$$\vdash I(\pi) \rightarrow \langle \pi \rangle p \leq \langle \top_I \rangle p$$

and

$$\vdash I(\top_I)$$

Consequently, it is possible to obtain :

$$\sim \top_I = \perp$$

Example 2.2.21 *By considering the identity program id, which is $I-local$ for every program, there is for the application of the above axiom:*

$$\pi = (\pi_\Sigma, \pi_\Phi) = id = (id_\Sigma, id_\Phi)$$

with:

$$\top = \langle id \rangle \top \leq \langle \top_I \rangle \top \quad or \quad (\top_\Sigma, \top_\Phi) = \langle (id_\Sigma, id_\Phi) \rangle (\top_\Sigma, \top_\Phi) \leq \langle (\top_{I_\Sigma}, \top_{I_\Phi}) \rangle (\top_\Sigma, \top_\Phi) \quad ,$$

i.e.

$$\top = \langle \top_I \rangle \top \quad or \quad (\top_\Sigma, \top_\Phi) = \langle (\top_\Sigma, \top_\Phi) \rangle (\top_\Sigma, \top_\Phi) \quad ,$$

and so,

$$\sim \top_I = [\top_I ?] \bot = \neg \langle \top_I \rangle \top = \neg \top = \bot$$

or

$$\sim (\top_{I_\Sigma}, \top_{I_\Phi}) = [(\top_{I_\Sigma}, \top_{I_\Phi})?](\bot_\Sigma, \bot_\Phi) = \neg \langle (\top_{I_\Sigma}, \top_{I_\Phi}) \rangle (\top_\Sigma, \top_\Phi) = \neg (\top_\Sigma, \top_\Phi) = (\bot_\Sigma, \bot_\Phi) \quad .$$

Other consequence consists of *the formula T_I standing for the I-local property*, i.e. there is :

$$\vdash I(\top_I)$$

and

$$\vdash I(p) \to p \leq \top_I \quad .$$

In a syntactical way , it is possible to define an *"I-local state"* as any sentence φ such that

$$\vdash I(\varphi) \wedge \varphi \neq \bot \wedge (I(p) \wedge \bot \neq p \leq \varphi \to p = \varphi)$$

for some p not occurring on φ. This is, it is verifiable that these are propositions which are atoms of the lattice of (consistent) I-local properties.

Local States Axiom. *Testable local properties* are *"local states"*: if $I \neq N$ then

$$\vdash T(p) \wedge I(p) \wedge I(q) \wedge \bot \neq q \leq p \to q = p$$

or

$$\vdash T((p_\Sigma, p_\Phi)) \wedge I((p_\Sigma, p_\Phi)) \wedge I((q_\Sigma, q_\Phi)) \wedge (\bot_\Sigma, \bot_\Phi) \neq (q_\Sigma, q_\Phi) \leq (p_\Sigma, p_\Phi) \to (q_\Sigma, q_\Phi) = (p_\Sigma, p_\Phi)$$

Basic-State Testability Axiom. *Basic local states* such as $c_i, \overline{\pi_{ij}}$ are *testable*: If $i, j \in N, c \in \{(0,\phi), (1,\phi), (+,\phi), (-,\phi)\}$ and π is a deterministic program, then :

$$\vdash T(c_i) \wedge T(\overline{\pi_{ij}}) \quad \text{or} \quad \vdash T((c_\Sigma, c_\Phi)_i) \wedge T(\overline{(\pi_\Sigma, \pi_\Phi)_{ij}})$$

Therefore, attending to the last two above axioms, all constants with a construction as $\vec{c_I}$ ($\vec{c} \in \{(0,\phi), (1,\phi), (+,\phi), (-,\phi)\}^{|I|}$) are testable I-local states. On the other hand, if π is deterministic then $\overline{\pi_{ij}}$ is a testable $\{i, j\}$-local state.

Furthermore, there is an inference rule which states that the lattice of local properties is *atomistic*:

Local Atomicity Rule. *Local properties are unions of testable local properties* : if $I \neq N$ and p does not occur in φ, ψ or θ, then :

$$\text{from} \quad \vdash \psi \ \wedge \ T(p_I) \ \wedge p_I \leq \varphi \rightarrow p_I \leq \theta$$

$$\big(\text{from} \quad \vdash (\psi_\Sigma, \psi_\Phi) \ \wedge \ T((p_{I_\Sigma}, p_{I_\Phi})) \ \wedge \ (p_{I_\Sigma}, p_{I_\Phi}) \leq (\varphi_\Sigma, \varphi_\Phi) \rightarrow (p_{I_\Sigma}, p_{I_\Phi}) \leq (\theta_\Sigma, \theta_\Phi)\big)$$

$$\text{infer} \quad \vdash \psi \ \wedge \ I(\varphi) \rightarrow \varphi \leq \theta$$

$$\big(\text{infer} \quad \vdash (\psi_\Sigma, \psi_\Phi) \ \wedge \ I((\varphi_\Sigma, \varphi_\Phi)) \rightarrow (\varphi_\Sigma, \varphi_\Phi) \leq (\theta_\Sigma, \theta_\Phi)\big)$$

Therefore,

Corollary 2.2.4.1 *For $I \neq N$, every local state is testable. This is : if $I \neq N$ and p does not occur in φ , then :*

$$\textit{from} \quad \vdash I(\varphi) \wedge \varphi \neq \bot \wedge (I(p) \wedge \bot \neq p \leq \varphi \rightarrow p = \varphi)$$

$$\big(\textit{from} \quad \vdash I((\varphi_\Sigma, \varphi_\Phi)) \wedge (\varphi_\Sigma, \varphi_\Phi) \neq (\bot_\Sigma, \bot_\Phi) \wedge (I((p_\Sigma, p_\Phi)) \wedge$$
$$(\bot_\Sigma, \bot_\Phi) \neq (p_\Sigma, p_\Phi) \leq (\varphi_\Sigma, \varphi_\Phi) \rightarrow (p_\Sigma, p_\Phi) = (\varphi_\Sigma, \varphi_\Phi)\big)$$

it is inferable

$$\vdash T(\varphi) \quad (\vdash T((\varphi_\Sigma, \varphi_\Phi)))$$

Separation Axiom ([14]) If a state is either I-separated and J-separated, then it is $N \setminus I$-separated, $I \cup J$-separation and $I \cap J$-separated as well :

$$\vdash \top_I \ \wedge \ \top_J \rightarrow \top_{N \setminus I} \ \wedge \ \top_{I \cup J} \ \wedge \ \top_{I \cap J}$$

Sequentially, the below axioms express the fact that $(+,\phi)_i$ and $(-,\phi)_i$ are proper axioms of $(0,\phi)_i$ and $(1,\phi)_i$:

Proper Superposition Axioms:

$$\vdash (+,\phi)_i \rightarrow \Diamond(0,\phi)_i \ \wedge \ \Diamond(1,\phi)_i$$

and

$$\vdash (-,\phi)_i \to \Diamond(0,\phi)_i \;\wedge\; \Diamond(1,(\phi+\frac{1}{2}) \mod 1)_i \quad .$$

The succeeding axiom expresses the property of linear operators on $\mathcal{B}$ being entirely determined by their values on all the states $|x\rangle_1 \otimes \ldots |x\rangle_n$, with

$$|x\rangle_i \in \{e^{2\pi i\phi_{x_i}}\,|0\rangle_i\,,\; e^{2\pi i\phi_{x_i}}\,|1\rangle_i\,, e^{2\pi i\phi_{x_i}}\,|+\rangle_i\,,\; e^{2\pi i\phi_{x_i}}\,|-\rangle_i\} :$$

Determinacy Axiom of Deterministic Programs. For deterministic programs π, π':

$$\vdash \bigwedge_{\vec{c}\in\{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}^n} \big(\pi(\vec{c}_N) = \pi'(\vec{c}_N) \to \pi(p) = \pi'(p)\big)$$

The next axiom is a semantic equivalent of Proposition 2.2.6 :

Entanglement Axiom. If π is deterministic and $i \neq j$, then :

$$\vdash T(p_i) \to p_i?(\overline{\pi}_{ij}) =_j \pi_{ij}(p_i)$$

$$\Big(\;\vdash T\big((p_\Sigma, p_\Phi)_i\big) \to \big(p_{i_\Sigma}?(\overline{\pi_{ij_\Sigma}}), p_{i_\Phi}?(\overline{\pi_{ij_\Phi}})\big) =_j \big(\pi_{ij_\Sigma}(p_{i_\Sigma}), \pi_{ij_\Phi}(p_{i_\Phi})\big)\Big)$$

Consequently, attending to the earlier axioms it is possible to define a proof-theoretic concept of locality , as already done for testability . So, a formula φ is I-local if $\vdash I(\varphi)$ is a theorem; likewise, a program π is I-local if $\vdash I(\pi)$ is a theorem.

Proposition 2.2.16 ([14]) *Some formula of the form φ_I is always I-local. Some formula of the form $\overline{\pi_{ij}}$ is $\{i,j\}$-local. For φ and ψ as I-local formulas and π as an I-local program, $\varphi \vee \psi$, $\varphi \wedge \neg\psi$ and $\varphi \wedge [\pi]\varphi$ are I-local. For φ I-local and φ J-local, $\varphi \wedge \psi$ is $I \cup J$-local.*

Proposition 2.2.17 ([14]) *For φ as a testable I-local formula, $\varphi?$ is an I-local program. $\top_I$ is I-local. For π and π' as I-local, then $\pi \cup \pi'$, $\pi;\pi'$ are I-local.*

Proposition 2.2.18 ([14]) *Local programs act locally. This is:*

$$\vdash I(\pi) \wedge p =_I q \to p =_{N\setminus I} \pi(p) =_I \pi(q)$$

$$\big(\vdash I((\pi_\Sigma, \pi_\Phi)) \wedge (p_\Sigma, p_\Phi) =_I (q_\Sigma, q_\Phi) \to (p_\Sigma, p_\Phi) =_{N\setminus I} (\pi_\Sigma, \pi_\Phi)((p_\Sigma, p_\Phi)) =_I (\pi_\Sigma, \pi_\Phi)((q_\Sigma, q_\Phi))\big)$$

Proposition 2.2.19 ([14]) *Systems composed of equal parts are equal:*

$$\vdash p =_I q \wedge p =_J q \to p =_{I\cup J} q$$

$$\big(\vdash (p_\Sigma, p_\Phi) =_I (q_\Sigma, q_\Phi) \wedge (p_\Sigma, p_\Phi) =_J (p_\Sigma, p_\Phi) \to (p_\Sigma, p_\Phi) =_{I\cup J} (q_\Sigma, q_\Phi)\big)$$

Proposition 2.2.20 ([14])

$$\vdash p_I \perp q \leftrightarrow p_I \perp q_I$$

$$\big(\vdash (p_{I_\Sigma}, p_{I_\Phi}) \perp (q_\Sigma, q_\Phi) \leftrightarrow (p_{I_\Sigma}, p_{I_\Phi}) \perp (q_{I_\Sigma}, q_{I_\Phi})\big)$$

Proposition 2.2.21 (Dual Local Atomicity Rule,[14]) *If $I \neq N$, φ and θ are I-separated, as well as p does not take place in φ, ψ or θ, then:*

$$\textit{from} \quad \vdash \psi \ \wedge \ T(p_I) \ \wedge \ p_I \perp \varphi \rightarrow p_I \perp \theta$$

$$\big(\textit{from} \quad \vdash (\psi_\Sigma, \psi_\Phi) \ \wedge \ T((p_{I_\Sigma}, p_{I_\Phi})) \ \wedge \ (p_{I_\Sigma}, p_{I_\Phi}) \perp (\varphi_\Sigma, \varphi_\Phi) \rightarrow (p_{I_\Sigma}, p_{I_\Phi}) \perp (\theta_\Sigma, \theta_\Phi)\big)$$

$$\textit{infer} \quad \vdash \psi \ \wedge \ T(\varphi_I) \ \wedge \ T(\theta_I) \rightarrow \varphi =_I \theta$$

$$\big(\textit{infer} \quad \vdash (\psi_\Sigma, \psi_\Phi) \ \wedge \ T((\varphi_{I_\Sigma}, \varphi_{I_\Phi})) \ \wedge \ T((\theta_{I_\Sigma}, \theta_{I_\Phi})) \rightarrow (\varphi_{I_\Sigma}, \varphi_{I_\Phi}) =_I (\theta_{I_\Sigma}, \theta_{I_\Phi})\big)$$

Proof 2.2.11 (Proposition 2.2.21) *By take into consideration the Proposition 2.2.20 it is possible to rewrite :*

$$\vdash \psi \ \wedge \ T(p_I) \ \wedge \ p_I \perp \varphi_I \rightarrow p_I \perp \theta_I$$

$$\big(\vdash (\psi_\Sigma, \psi_\Phi) \ \wedge \ T((p_{I_\Sigma}, p_{I_\Phi})) \ \wedge \ (p_{I_\Sigma}, p_{I_\Phi}) \perp (\varphi_{I_\Sigma}, \varphi_{I_\Phi}) \rightarrow (p_{I_\Sigma}, p_{I_\Phi}) \perp (\theta_{I_\Sigma}, \theta_{I_\Phi})\big)$$

Also through the I-locality of p_I:

$$\vdash \psi \ \wedge \ T(p_I) \ \wedge \ p_I \leq (\top_I \wedge \ \sim \varphi_I) \rightarrow p_I \leq (\top_I \wedge \ \sim \theta_I)$$

$$\big(\vdash (\psi_\Sigma, \psi_\Phi) \wedge T((p_{I_\Sigma}, p_{I_\Phi})) \wedge (p_{I_\Sigma}, p_{I_\Phi}) \leq (\top_I \wedge (\sim \varphi_{I_\Sigma}, \varphi_{I_\Phi})) \rightarrow (p_{I_\Sigma}, p_{I_\Phi}) \leq (\top_I \wedge (\sim \theta_{I_\Sigma}, \theta_{I_\Phi}))\big)$$

Consider now $\psi \ \wedge \ T(\varphi_I) \ \wedge \ T(\theta_I)$. Then

$$\top_I \wedge \ \sim \varphi_I = \top_I \ \wedge \ \neg(\top_I \ \wedge \ [\varphi_I?]\perp)$$

$$\big(\top_I \ \wedge \ (\sim \varphi_{I_\Sigma}, \varphi_{I_\Phi}) = \top_I \ \wedge \ \neg(\top_I \ \wedge \ [(\varphi_{I_\Sigma}, \varphi_{I_\Phi})?]\perp)\big) \quad .$$

and

$$\top_I \wedge \ \sim \theta_I = \top_I \ \wedge \ \neg(\top_I \ \wedge \ [\theta_I?]\perp)$$

$$\big(\top_I \ \wedge \ (\sim \theta_{I_\Sigma}, \theta_{I_\Phi}) = \top_I \ \wedge \ \neg(\top_I \ \wedge \ [(\theta_{I_\Sigma}, \theta_{I_\Phi})?]\perp)\big)$$

are I-local formulas, in consideration of φ_I and θ_I as testable I-local with $\varphi_I?$ and $\theta_I?$ as I-local programs . By applying the Local Atomicity Rule :

$$(\top_I \wedge \ \sim \varphi_I) \leq (\top_I \wedge \ \sim \theta_I)$$

$$\Big((\top_I \ \wedge \ (\sim \varphi_{I_\Sigma}, \varphi_{I_\Phi})) \leq (\top_I \ \wedge \ (\sim \theta_{I_\Sigma}, \theta_{I_\Phi}))\Big) \quad .$$

Through the orthocomplementation,

$$\sim(\top_I \wedge \sim\theta_I) \leq \sim(\top_I \wedge \sim\varphi_I)$$

$$\Big(\sim\big(\top_I \wedge (\sim\theta_{I_\Sigma}, \theta_{I_\Phi})\big) \leq \sim\big(\top_I \wedge (\sim\varphi_{I_\Sigma}, \varphi_{I_\Phi})\big)\Big) \quad .$$

Consequently,

$$\big(\sim\top_I \sqcup \sim\sim\theta_I\big) \leq \big(\sim\top_I \sqcup \sim\sim\varphi_I\big)$$

$$\Big(\big(\sim\top_I \sqcup (\sim\sim\theta_{I_\Sigma}, \theta_{I_\Phi})\big) \leq \big(\sim\top_I \sqcup (\sim\sim\varphi_{I_\Sigma}, \varphi_{I_\Phi})\big)\Big)$$

$$= \perp \sqcup \theta_I \leq \perp \sqcup \varphi_I$$

$$\big((\perp_\Sigma \sqcup \theta_{I_\Sigma}, \perp_\Phi \sqcup \theta_{I_\Phi}) \leq (\perp_\Sigma \sqcup \varphi_{I_\Sigma}, \perp_\Phi \sqcup \varphi_{I_\Phi})\big)$$

$$= \theta_I \leq \varphi_I$$

$$\big((\theta_{I_\Sigma}, \theta_{I_\Phi}) \leq (\varphi_{I_\Sigma}, \varphi_{I_\Phi})\big) \quad .$$

Therefore, attending to the Local States Axiom , the above formula implies $\theta_I = \varphi_I$ (due to the fact that both are testable I-local with $I \neq N$, they are local states). Also, because both θ_I and φ_I are I-separated , $\theta =_I \varphi$.

Theorem 2.2.5 (Compatibility of Programs Affecting Diferent Qubits [14]) *For I and J with $I \cap J = \emptyset$, as well as π and π' both deterministic, the following holds :*

$$\vdash I(\pi) \wedge J(\pi') \rightarrow \pi; \pi'(p) = \pi'; \pi(p)$$

$$\Big(\vdash I\big((\pi_\Sigma, \pi_\Phi)\big) \wedge J\big((\pi'_\Sigma, \pi'_\Phi)\big) \rightarrow \big(\pi_\Sigma; \pi'_\Sigma(p_\Sigma)\big), \pi_\Phi; \pi'_\Phi(p_\Phi)\big)\Big)$$

Proposition 2.2.22 (Dual Entanglement) *For π deterministic and $i \neq j$, there is*

$$\vdash T(q_j) \rightarrow q_j^\dagger?(\overline{\pi_{ij}}) =_i \pi_{ij}^\dagger(q_j)$$

$$\Big(\vdash T\big((q_\Sigma, q_\Phi)\big) \rightarrow (q_{j_\Sigma}?(\overline{\pi_{\Sigma_{ij}}}), q^*_{j_\Phi}?(\overline{\pi_{\Phi_{ij}}})) =_i (\pi^\dagger_{\Sigma_{ij}}(q_j), \pi^\dagger_{\Phi_{ij}}(q_j))\Big)$$

Proof 2.2.12 (Proposition 2.2.22) *Consider now that $T(q_j)$ and the necessity of verify that*

$$q_j^\dagger?(\overline{\pi_{ij}}) =_i \pi_{ij}^\dagger(q_j) \quad .$$

In this way, it is simple to notice that both sides are i-separated, i.e.

$$\big(q_j^\dagger?(\overline{\pi_{ij}})\big)_i \leq \top_i \;\wedge\; \big(\pi_{ij}^\dagger(q_j)\big)_i \leq \top_i \quad ,$$

and testable, since they are local states. Therefore, the conditions to apply the Dual Local Atomicity Rule (Proposition 2.2.21) are met, and so is possible to have :

$$\vdash T(p_i) \ \wedge \ p_i \perp q_j^\dagger?(\overline{\pi_{ij}}) \rightarrow p_i \perp \pi_{ij}^\dagger(q_j)$$

In order to show this , let p_i be such that $T(p_i)$ and $p_i \perp \pi_{ij}^\dagger(q_j)$. By the Adjointness Theorem (Proposition 2.2.15), there is $\pi_{ij} \perp q_j^\dagger$, and so $q_j^\dagger?(\pi_{ij}(p_i)) = \perp$. Consequently, attending to the Compatibility of Programs on Different Qubits (Theorem 2.2.5) :

$$p_i?(q_j^\dagger?(\overline{\pi_{ij}})) = (p_i?;q_j^\dagger?)(\overline{\pi_{ij}}) = (q_j^\dagger?;p_i?)(\overline{\pi_{ij}}) = q_j^\dagger?(p_i?(\overline{\pi_{ij}})) \quad .$$

And by the Entanglement Axiom :

$$q_j^\dagger?(p_i?(\overline{\pi_{ij}})) = q_j^\dagger?(\pi_{ij}(p_i)) = \perp \quad .$$

Then, it is possible to obtain:

$$p_i \perp q_j^\dagger?(\overline{\pi_{ij}}) \quad .$$

Finally, by using the Dual Local Atomicity Rule (Proposition 2.2.21):

$$\textit{from} \quad \vdash T(p_i) \ \wedge \ p_i \perp \pi_{ij}^\dagger(q_j) \rightarrow p_i \perp q_j^\dagger?(\overline{\pi_{ij}})$$

$$\textit{infer} \quad \vdash T\big(p_i \perp \pi_{ij}^\dagger(q_j)\big) \ \wedge \ T\big(p_i \perp q_j^\dagger?(\overline{\pi_{ij}})\big) \rightarrow q_j^\dagger?(\overline{\pi_{ij}}) =_i \pi_{ij}^\dagger(q_j) \quad .$$

Proposition 2.2.23 (Entanglement Preparation Lemma, [14])

$$\vdash \pi_{ij}(p_i) \perp q_j \rightarrow \overline{\pi_{ij}} \perp (p_i \wedge q_j)$$

$$\Big(\vdash \big(\pi_{\Sigma_{ij}}(p_{\Sigma_i}), \pi_{\Phi_{ij}}(p_{\Phi_i})\big) \perp \big(q_{j_\Sigma}, q_{j_\Phi}\big) \rightarrow \big(\overline{\pi_{\Sigma_{ij}}}, \overline{\pi_{\Phi_{ij}}}\big) \perp \big(p_{\Sigma_i} \wedge q_{\Sigma_j}, p_{\Phi_i} \wedge q_{\Phi_j}\big)\Big)$$

Theorem 2.2.6 (Teleportation Property) *For i, j, k as distinct indices, there is :*

$$\vdash (\overline{\varsigma_{jk}^\dagger}?;\overline{\pi_{ij}^\dagger}?)(p_i) =_k (\pi_{ij};\varsigma_{jk})(p_i)$$

$$\Big(\vdash \big((\overline{\varsigma_{\Sigma_{ij}}^\dagger}?;\overline{\pi_{\Sigma_{ij}}^\dagger}?)(p_{\Sigma_i}), (\overline{\varsigma_{\Phi_{ij}}^\dagger}?;\overline{\pi_{\Phi_{ij}}^\dagger}?)(p_{\Phi_i})\big) =_k \big((\pi_{\Sigma_{ij}}?;\varsigma_{\Sigma_{ij}}?)(p_{\Sigma_i}), (\varsigma_{\Phi_{ij}}?;\pi_{\Phi_{ij}}?)(p_{\Phi_i})\big)\Big)$$

Proof 2.2.13 (Theorem 2.2.6) *As done above , it is sufficient to verify that:*

$$\vdash T(q_k) \wedge q_k \perp (\pi_{ij};\varsigma_{jk})(p_i) \rightarrow q_k^\dagger \perp (\overline{\varsigma_{jk}^\dagger}?;\overline{\pi_{ij}^\dagger}?)(p_i)$$

Let q_k such that $T(q_k)$ and $q_k \perp (\pi_{ij};\varsigma_{jk})(p_i)$. Then, there is

$$q_k \perp \varsigma_{jk}(\pi_{ij}(p_i)) \quad ,$$

and by the Adjointness Theorem (Proposition 2.2.15) :

$$\varsigma^{\dagger}_{jk}(q_k) \perp (\pi_{ij}(p_i))^{\dagger} = \pi^{\dagger}_{ij}(p_i) \quad .$$

Consequently , by Dual Entanglement (Proposition 2.2.22) :

$$q^{\dagger}_k?(\overline{\varsigma^{\dagger}_{jk}}) =_j \varsigma^{\dagger\dagger}_{jk}(q_k) = \varsigma_{jk}(q_k) \quad ,$$

and so

$$q^{\dagger}_k?(\overline{\varsigma^{\dagger}_{jk}}) \perp \pi^{\dagger}_{ij}(p_i) \quad .$$

Also, by the Entanglement Preparation Lemma (Proposition 2.2.23):

$$\overline{\pi^{\dagger}_{ij}} \perp (p_i \wedge q^{\dagger}_k?(\overline{\varsigma^{\dagger}_{jk}})) \quad .$$

Therefore , by using the Theorem 2.2.5 on the Compatibility of Programs on Different Qubits , it is possible to get :

$$\begin{aligned} q^{\dagger}_k?((\overline{\varsigma^{\dagger}_{jk}}?;\overline{\pi^{\dagger}_{jk}}?)(p_i)) &= q^{\dagger}_k?(\overline{\pi^{\dagger}_{jk}}?(\overline{\varsigma^{\dagger}_{jk}}?(p_i))) = \overline{\pi^{\dagger}_{jk}}?(q^{\dagger}_k?(\overline{\varsigma^{\dagger}_{jk}}?(p_i))) \\ &=_{ijk} \overline{\pi^{\dagger}_{jk}}?(q^{\dagger}_k?(\overline{\varsigma^{\dagger}_{jk}}) \wedge p_i) = \perp \quad . \end{aligned}$$

Finally, as intended :

$$q^{\dagger}_k \perp (\overline{\varsigma^{\dagger}_{jk}}?;\overline{\pi^{\dagger}_{ij}}?)(p_i) \quad .$$

Corollary 2.2.6.1 ([14]) *For i,j,k as distinct indices :*

$$\vdash \overline{\pi_{ij}}?(p_i \ \wedge \ \overline{\varsigma_{jk}}?) =_k (\pi_{ij};\varsigma_{jk})(p_i)$$

$$\Big(\vdash (\overline{\pi_{\Sigma_{ij}}}?(p_{\Sigma_i} \ \wedge \overline{\varsigma_{\Sigma_{jk}}}?), \overline{\pi_{\Phi_{ij}}}?(p_{\Phi_i} \ \wedge \overline{\varsigma_{\Phi_{jk}}}?)) =_k ((\pi_{\Sigma_{ij}}?;\varsigma_{\Sigma_{ij}}?)(p_{\Sigma_i}), (\varsigma_{\Phi_{ij}}?;\pi_{\Phi_{ij}}?)(p_{\Phi_i})) \Big)$$

As proof-theoretic version of Entanglement Composition Lemma (Lemma 2.2.3):

Proposition 2.2.24 (Entanglement Composition Lemma, [14]) *If i,j,k,l are distinct indices, and $\pi, \pi', \pi'', \varsigma_1, \rho_1$ are programs with ς_1, ρ_1 as $\{1\}$-local programs, then :*

$$\vdash \overline{\pi_{ij}} \ \wedge \ \overline{\pi'_{kl}} \rightarrow [\varsigma_j;\rho_k;\overline{\pi''_{jk}}?]\overline{(\pi;\varsigma_1;\pi'';\rho^{\dagger}_1;\pi')_{il}}$$

$$\Big(\vdash (\overline{\pi_{\Sigma_{ij}}}? \ \wedge \overline{\pi'_{\Sigma_{kl}}}?, \overline{\pi_{\Phi_{ij}}}? \ \wedge \overline{\pi'_{\Phi_{kl}}}?) \rightarrow$$
$$([\varsigma_{\Sigma_j};\rho_{\Sigma_k};\overline{\pi''_{\Sigma_{jk}}}?]\overline{(\pi_\Sigma;\varsigma_{\Sigma_1};\pi'';\rho^{\dagger}_{\Sigma_1};\pi'_\Sigma)} \ , \ [\varsigma_{\Phi_j};\rho_{\Phi_k};\overline{\pi''_{\Phi_{jk}}}?]\overline{(\pi_\Phi;\varsigma_{\Phi_1};\pi'';\rho^{\dagger}_{\Phi_1};\pi'_\Phi)} \ ) \Big)$$

Remark 2.2.20 *$dom(\varphi)$ stands for the domain of a map π and is defined as $dom(\pi) := \langle\pi\rangle\top$.*

Theorem 2.2.7 (Agreement Property,[14]) *For two π, π' as I-local maps with the same domain and separated input-states , their output-states agree on all non I qubit : i.e. for $I \cap J = \emptyset$ and all deterministic programs π, π' :*

$$\vdash T(p) \wedge I(\pi) \wedge I(\pi') \wedge dom(\pi) = dom(\pi') \wedge \pi(p) \leq \top_I \wedge \pi'(p) \leq \top_I \rightarrow \pi(p) =_{N \backslash I} \pi'(p) \quad .$$

Characteristic Formulas To construct the next axioms of PhLQP , there are some characteristic formulas for binary states , by considering two qubits with indices i and j, Table 25 :

States	Characteristic Formulas
$\overline{e^{2\pi i.(\phi_{ij})}\ \lvert 00\rangle_{ij}} = \overline{e^{2\pi i.((\phi_i+\phi_j) \mod 1)}\ \lvert 0\rangle_i \otimes \lvert 0\rangle\ j}$	$\langle (0,\phi)_i?\rangle (0,\phi)_j \ \wedge\ [(1,\phi)_i?]\bot$
Bell States : ${}^{\phi_{ij}}\beta^{i,j}_{xy} = \overline{e^{2\pi i.(\phi_{ij})}\left(\lvert 0\rangle_i \otimes \lvert y\rangle_j + (-1)^x \lvert 1\rangle_i \otimes \lvert \tilde{y}\rangle_j \right)}$ with $\phi_{ij} = (\phi_i + \phi_j) \mod 1$, $\tilde{0} = 1$ and $\tilde{1} = 0$, $x, y \in \{0, 1\}$	$\langle (0,\phi)_i?\rangle (y_j, \phi_j) \wedge \langle (1,\phi)_i?\rangle (\tilde{y}_j, \phi_j)$ $\wedge \langle (+,\phi)_i?\rangle ((-)^x_j, \phi_j)$ where $(-)^x = -$ if $x = 1$ and $(-)^x = +$ if $x = 0$
${}^{\phi_{ij}}\gamma^{i,j} = {}^{\phi_{ij}}\beta^{i,j}_{00} + {}^{\phi_{ij}}\beta^{i,j}_{01} =$ $\overline{e^{2\pi i.(\phi_{ij})}\left(\lvert 00\rangle_{ij} + \lvert 01\rangle_{ij} + \lvert 10\rangle_{ij} + \lvert 11\rangle_{ij} \right)}$ with $\phi_{ij} = (\phi_i + \phi_j) \mod 1$	$\langle (0,\phi)_i?\rangle (+,\phi)_j \wedge \langle (1,\phi)_i?\rangle (+,\phi)_j$ $\wedge \langle (+,\phi)_i?\rangle (+,\phi)_j$

Table 25: States and respective characteristic formulas

Locality Axiom for Quantum Gates The special quantum gates of PhLQP are local, acting only on the specified qubits:

$$\begin{aligned} \vdash \{i\}(X_i) \wedge \{i\}(Y_i) \wedge \{i\}(Z_i) \wedge \{i\}(R_{k_i}) \wedge \{i,j\}(CNOT_{ij}) \\ \wedge \{i,j\}(CR_{k_{ij}}) \wedge \wedge \{i,j\}(SWAP_{ij}) \wedge \{i,j,k\}(TOFF_{ijk}) \end{aligned}$$

Also , it is required for X, Z, Y, H, R_k :

Characteristic Axioms for Quantum Gates : X, Z, Y, H, R_k **.**

$$\vdash (0,\phi)_i \to [X_i](1,\phi)_i;\ \vdash (1,\phi)_i \to [X_i](0,\phi)_i;$$
$$\vdash (+,\phi)_i \to [X_i](+,\phi)_i;\ \vdash (-,\phi)_i \to [X_i](-,\phi)_i;$$
$$\vdash (0,\phi)_i \to [Z_i](0,\phi)_i;\ \vdash (1,(\phi+\frac{1}{2}) \mod 1)_i \to [Z_i](1,\phi)_i;$$
$$\vdash (-,\phi)_i \to [Z_i](+,\phi)_i;\ \vdash (+,\phi)_i \to [Z_i](-,\phi)_i;$$
$$\vdash (0,(\phi+\frac{3}{4}) \mod 1)_i \to [Y_i](1,\phi)_i;\ \vdash (0,(\phi+\frac{1}{4}) \mod 1)_i \to [Y_i](1,\phi)_i;$$
$$\vdash (+,(\phi+\frac{1}{4}) \mod 1)_i \to [Y_i](-,\phi)_i;\ \vdash (-,(\phi+\frac{1}{4}) \mod 1)_i \to [Y_i](+,\phi)_i;$$
$$\vdash (+,\phi)_i \to [H_i](0,\phi)_i;\ \vdash (-,\phi)_i \to [H_i](1,\phi)_i;$$
$$\vdash (0,\phi)_i \to [H_i](+,\phi)_i;\ \vdash (1,\phi)_i \to [H_i](-,\phi)_i;$$
$$\vdash (0,\phi)_i \to [R_{k_i}](0,\phi)_i\ \vdash (1,(\phi+\frac{1}{2^k}) \mod 1)_i \to [R_{k_i}](1,\phi);$$
$$\vdash \Diamond(0,\phi)_i \ \wedge\ \Diamond(1,(\phi+\frac{1}{2^k}) \mod 1)_i \to [R_{k_i}](+,\phi)_i;$$
$$\vdash \Diamond(0,\phi)_i \ \wedge\ \Diamond(1,(\phi+\frac{1}{2^k}+\frac{1}{2}) \mod 1)_i \to [R_{k_i}](-,\phi)_i$$

Remark 2.2.21 (Notation) *For $x, y \in \{0,1\}$ and $i, j \in N$ as distinct indices, it is possible to write for the "Bell formulas ": ${}^{\phi_{ij}}\beta^{i,j}_{xy} := \overline{(Z_i^{(x,\phi_i)}; X_i^{(y,\phi_j)})}_{ij}$.*

Proposition 2.2.25 *The Bell states ${}^{\phi_{ij}}\beta^{ij}_{xy}$ are characterized by the logic Bell formulas ${}^{\phi_{ij}}\beta^{i,j}_{xy}$. This is , a state that satisfies one of these formulas corresponds to a Bell state .*

Proof 2.2.14 (Proposition 2.2.25) *It is only necessary to verify that the formulas ${}^{\phi_{ij}}\beta^{i,j}_{xy}$ imply the corresponding characteristic formulas in the Table 25. So, by using the Entanglement Axiom and the following theorems :*

$$\vdash (0,\phi)_i \leftrightarrow \langle Z_i^{(x,\phi_x)}; X_i^{(x,\phi_x)}\rangle (y,\phi_y)_i$$
$$\vdash (1,\phi)_i \leftrightarrow \langle Z_i^{(x,\phi_x)}; X_i^{(x,\phi_x)}\rangle (\tilde{y},\phi_y)_i$$
$$\vdash (+,\phi)_i \leftrightarrow \langle Z_i^{(x,\phi_x)}; X_i^{(x,\phi_x)}\rangle ((-)^x,\phi_{(-)^x})_i$$

Characteristic Axioms for $CNOT$-gate With the previous notation, it is possible to consider :

$$\vdash (0,\phi)_i \wedge c_j \to [CNOT_{ij}]c_j \quad ; \quad \vdash (1,\phi)_i \wedge (0,\phi)_j \to [CNOT_{ij}](1,\phi)_j$$
$$\vdash (1,\phi)_i \wedge (1,\phi)_j \to [CNOT_{ij}](0,\phi)_j \quad ; \quad \vdash (1,\phi)_i \wedge (+,\phi)_j \to [CNOT_{ij}](+,\phi)_j$$
$$\vdash (1,\phi)_i \wedge (-,\phi)_j \to [CNOT_{ij}](-,\phi)_j \quad ; \quad \vdash (+,\phi)_i \wedge (0,\phi)_j \to [CNOT_{ij}]^{\phi_{ij}}\beta^{i,j}_{00}$$
$$\vdash (-,\phi)_i \wedge (0,\phi)_j \to [CNOT_{ij}]^{\phi_{ij}}\beta^{i,j}_{10} \quad ; \quad \vdash (+,\phi)_i \wedge (1,\phi)_j \to [CNOT_{ij}]^{\phi_{ij}}\beta^{i,j}_{01}$$
$$\vdash (-,\phi)_i \wedge (1,\phi)_j \to [CNOT_{ij}]^{\phi_{ij}}\beta^{i,j}_{11} \quad ; \quad \vdash (+,\phi)_i \wedge (+,\phi)_j \to [CNOT_{ij}]^{\phi_{ij}}\gamma^{i,j}$$

Characteristic Axioms for CR_k-gate.

$$\vdash (0,\phi)_i \wedge c_j \to [CR_{k_{ij}}]c_j \quad ; \quad \vdash c_i \wedge (0,\phi)_j \to [CR_{k_{ij}}](0,\phi)_j$$
$$\vdash (1,\phi)_i \wedge (1,(\phi+\frac{1}{2^k}) \mod 1)_j \to [CR_{k_{ij}}](1,\phi)_j$$
$$\vdash (1,\phi)_i \wedge \big(\Diamond(0,\phi)_j \wedge \Diamond(1,(\phi+\frac{1}{2^k}) \mod 1)_j\big) \to [CR_{k_{ij}}](+,\phi)_j$$
$$\vdash (1,\phi)_i \wedge \big(\Diamond(0,\phi)_j \wedge \Diamond(1,(\phi+\frac{1}{2^k}+\frac{1}{2}) \mod 1)_j\big) \to [CR_{k_{ij}}](-,\phi)_j$$
$$\vdash (+,\phi)_i \wedge \big(\Diamond c_j \wedge \Diamond[R_{k_j}]c_j\big) \to [CR_{k_{ij}}]c_j \quad ; \quad \vdash (-,\phi)_i \wedge \big(\Diamond c_j \wedge \Diamond[R_{k_j}]c_j\big) \to [CR_{k_{ij}}]c_j$$
$$\vdash \Big(\Diamond\big((0,\phi)_i \wedge (0,\phi)_j \wedge (00,(\phi_i+\phi_j) \mod 1)_{ij}\big)\wedge$$
$$\Diamond\big((1,\phi)_i \wedge (1,(\phi+\frac{1}{2^k}) \mod 1)_j \wedge (11,(\phi_i+\phi_j) \mod 1)_{ij}\big)\Big) \to [CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{00}$$
$$\vdash \Big(\Diamond\big((0,\phi)_i \wedge (0,\phi)_j \wedge (00,(\phi_i+\phi_j) \mod 1)_{ij}\big)\wedge$$
$$\Diamond\big((1,\phi)_i \wedge (1,(\phi+\frac{1}{2^k}+\frac{1}{2}) \mod 1)_j \wedge (11,(\phi_i+\phi_j+\frac{1}{2}) \mod 1)_{ij}\big)\Big) \to [CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{10}$$
$$\vdash \Big(\Diamond\big((0,\phi)_i \wedge (1,\phi)_j \wedge (01,(\phi_i+\phi_j) \mod 1)_{ij}\big)\wedge$$
$$\Diamond\big((1,\phi)_i \wedge (0,\phi)_j \wedge (10,(\phi_i+\phi_j) \mod 1)_{ij}\big)\Big) \to [CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{01}$$
$$\vdash \Big(\Diamond\big((0,\phi)_i \wedge (1,\phi)_j \wedge (01,(\phi_i+\phi_j) \mod 1)_{ij}\big)\wedge$$
$$\Diamond\big((1,(\phi+\frac{1}{2}) \mod 1)_i \wedge (0,\phi)_j \wedge (10,(\phi_i+\phi_j) \mod 1)_{ij}\big)\Big) \to [CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{11}$$
$$\vdash \Diamond[CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{00} \wedge \Diamond[CR_{k_{ij}}]^{\phi_{ij}}\beta^{i,j}_{01} \to [CR_{k_{ij}}]^{\phi_{ij}}\gamma^{i,j}$$

Characteristic Axiom for $SWAP$-gate.

$$\vdash (y,\phi_x)_i \wedge (x,\phi_y)_j \to [SWAP_{ij}](x,\phi_x)_i$$

Proposition 2.2.26 ([14]) *For all $x, y \in \{0,1\} : \vdash \big(CNOT_{ij}; H_i((x,\phi_i)_i \wedge (y,\phi_j)_j)\big) = {}^{\phi_{ij}}\beta^{i,j}_{xy}$*

Corollary 2.2.7.1 ([14]) *For i, j, k as distinct indices , there is :*

$$\vdash \big(CNOT_{ij}; H_i((x,\phi_i)_i \wedge (y,\phi_j)_j))?\big)(p) =_k {}^{\phi_{ij}}\beta^{i,j}_{xy}(p)$$

2.3 WORKED EXAMPLES

2.3.1 *Quantum Teleportation Protocol*

Attending to [14, 42, 51], quantum teleportation is a technique built on the principle of teleporting the information and not the medium, i.e., the possibility of teleporting a state of a quantum system, without any quantum channel, but with a classical channel for classical communication. The workspace is $B \otimes B \otimes B$ with B standing for the bidimensional qubit space , and so $n = 3$. Subsequently, admit two agents Alice and Bob separated in space with each one having one qubit of an entangled EPR pair denoted by ${}^{\phi_{jk}}\beta^{j,k}_{00} \in B^{(j)} \otimes B^{(k)}$. Also, Alice detains an extra qubit $q_i \in B^{(i)}$ beyond her part of the EPR pair , in a unknown local state q_i . As depicted in Figure 11 by the QCM .

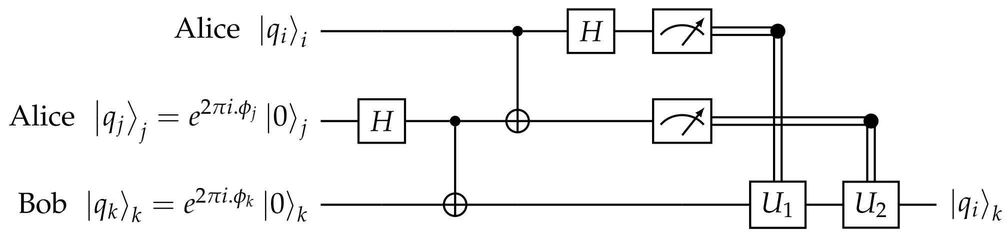

Figure 11: Quantum Teleportation implementation by QCM .

Remark 2.3.1 *q_1 is a testable i-local property, by reason of q_1 being a i-local state .*

Alice has the intention of "teleport" the state q_i to Bob, i.e. she will execute a program that will output a state satisfying $id_{i,k}(q_i)$. For so , Alice firstly entangles q_i with her component q_j of the EPR pair (i.e. she applies $CNOT_{i,j}$ gate on qubits q_i and q_j and then a Hadamard H_i on the q_i qubit component). On the other hand, Bob's qubit has its state "shaped" throughout the actions of Alice and when Alice does a measurement over her qubits , she will untangle the EPR pair that she shares with Bob . The initial state of Bob's qubit is previously known, and it is possible to determine which modifications occurred, upon the measurement results of Alice of her qubits q_i and q_j. Furthermore, the results of Alice's measurements establish the actions that Bob needs to execute to transfer his qubit q_k into the state $id_{i,k}(q_i)$, which is the state corresponding to the Alice's qubit q_i state previous protocol . Therefore, it is sufficient for Alice send Bob two classical bits encoding the result x_i of the first measurement and the result y_j of the second measurement . In other words, Bob will need to apply y

times the U_1-gate followed by x times the U_2-gate, in order to obligate his qubit q_k into the state $id_{i,k}(q_i)$.

By the PhLQP syntax, the quantum program can be expressed by :

$$\pi = \bigcup_{\substack{x,y \in \{0,1\} \\ \phi_y,\phi_x \in \Phi}} CNOT_{ij}; H_i; ((x,\phi_i)_i \wedge (y,\phi_j)_j)?; U_{1_k}^{(y,\phi_j)}; U_{2_k}^{(x,\phi_i)}$$

Now, consider that U_1 stands for a X gate and U_2 for a Z-gate , then :

$$\pi = \bigcup_{\substack{x,y \in \{0,1\} \\ \phi_y,\phi_x \in \Phi}} CNOT_{ij}; H_i; ((x,\phi_i)_i \wedge (y,\phi_j)_j)?; X_k^{(y,\phi_j)}; Z_k^{(x,\phi_i)}$$

with the following validity expressing the correctness of the protocol :

$$\vdash \pi(q_i \ \wedge \ {}^{\phi_{jk}}\beta_{00}^{j,k}) =_k id_{i,k}(q_i)$$

To demonstrate this, notice that by applying the Corollary 2.2.7.1 the validity becomes equivalent to :

$$\vdash \ ({}^{\phi_{i,j}}\beta_{xy}^{i,j}?; X_k^{(y,\phi_j)}; Z_k^{(x,\phi_i)})(q_i \ \wedge \ {}^{\phi_{jk}}\beta_{00}^{j,k}) =_k id_{i,k}(q_i) \quad .$$

By replacing the logical Bell formulas with the respective definitions ${}^{\phi_{ij}}\beta_{xy}^{i,j} := \overline{(Z_i^{(x,\phi_i)}; X_i^{(y,\phi_j)})}_{ij}$, there is the following equivalent validity :

$$\vdash \quad (\overline{(Z_i^{(x,\phi_i)}; X_i^{(y,\phi_j)})}_{ij}?; X_k^{(y,\phi_j)}; Z_k^{(x,\phi_i)})(q_i \ \wedge \ \overline{id}_{j,k}) =_k id_{i,k}(q_i) \quad ,$$

where $id_i = X_i^{(0,\phi_j)}; Z_i^{(0,\phi_i)}$ is the identity . Finally , the above validity follows from the application of Corollary 2.2.6.1 and the validity $Z_i^{(x,\phi_i)}; X_i^{(y,\phi_j)}; X_i^{(y,\phi_j)}; Z_i^{(x,\phi_i)} = id_i$, since $X_i^{(y,\phi_j)} = X_i^{(-y,\phi_j)}$ and $Z_i^{(x,\phi_i)} = Z_i^{(-x,\phi_i)}$.

2.3.2 *Quantum Leader Election Protocol*

The QLE protocol objectives to indiscriminately select a leader in a group of agents in such way that each agent has equal probability of be the selected leader. To such protocol there is numerous ways to solve this problem by recourse to quantum theory ,e.g., [31, 32, 47]. It will be chosen the proposed way by D'Hondt and Panangaden [31, 32] as the way to implement and verify such protocol, since it doesn't rely in a heavily basis on communication by omitting explicit communication , and avoiding the necessity of explicit a model of communication. So for a group of n agents , the inherent task to the QLE protocol entails the attribution to each agent the two states $\{|0\rangle, |1\rangle\}$ with $|1\rangle$ standing for the state "leader" and $|0\rangle$ standing for the state "follower" in such way that only and only one agent is assigned state "leader" and all others "follower". A protocol for such task is correct if and only if it at all times ends in states where each agent has the same probability of getting assigned to be the "leader". In [31] , D'Hondt and Panangaden provide a correct protocol for the task with quantum resources , as also showed that the required and enough condition for such protocol to be possible is to acquire the "" W-state" , i.e., by following [11] :

$$\frac{1}{\sqrt{n}} \sum_{j \in N} \bigotimes_{i \in N} |\delta(i,j)\rangle_i \quad \text{where} \quad \delta(i,j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{, otherwise .} \end{cases}$$

So the QLE protocol can be implanted by preparing n qubits $i \in N = \{0, \dots, n-1\}$ in the W-state , and give each agent one qubit . The agents measure their qubits in an individual way. The only one whose qubit collapses to $|1\rangle$ is appoint the state "leader" , and all the others , whose qubits collapse to $|0\rangle$, are declare "follower". However, a *Language of Phased Logic of Quantum Programs ($\mathcal{L}_{PhLQP}$)* can only express the total correctness of the QLE protocol when the probabilities involved are equal to $\frac{1}{n}$, since PhLQP is not equipped with probabilistic predication formulas. Let W_n denote the W-state for n qubits , it is possible express the full correctness of W_2-state , a 2-party QLE protocol as depicted in Figure 12 , and the the full correctness of W_4-state´, a 4-party QLE illustrated in Figure 13 .

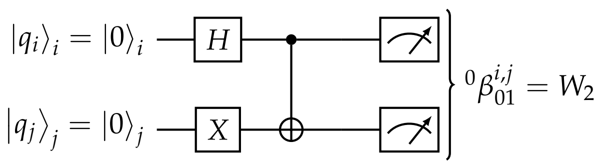

Figure 12: 2-party QLE implementation by QCM .

Attending to the PhLQP syntax , the 2-party QLE can be expressed as the program :

$$\pi_{QLE_2} = \bigcup CNOT_{ij}; H_i; X_j; (q_{i_i} \wedge q_{j_j})?$$

with the following validity asserting the correctness of the 2-party QLE protocol :

$$\vdash \pi_{QLE_2}((0,0)_i \wedge (0,0)_j) = {}^0\beta^{i,j}_{01}? \quad ,$$

or in an equivalent way :

$$\vdash (CNOT_{ij}; H_i; X_j)((0,0)_i \wedge (0,0)_j)? = {}^0\beta^{i,j}_{01}? \quad .$$

To show this, notice that from the characteristic axioms of the X-gate the above validity is proven to be equivalent to :

$$\vdash (CNOT_{ij}; H_i)((0,0)_i \wedge (1,0)_j)? = {}^0\beta_{01}^{i,j}? \quad .$$

And by applying Proposition 2.2.26 , there is the last validity :

$$\vdash {}^0\beta_{01}^{i,j}? = {}^0\beta_{01}^{i,j}? \quad .$$

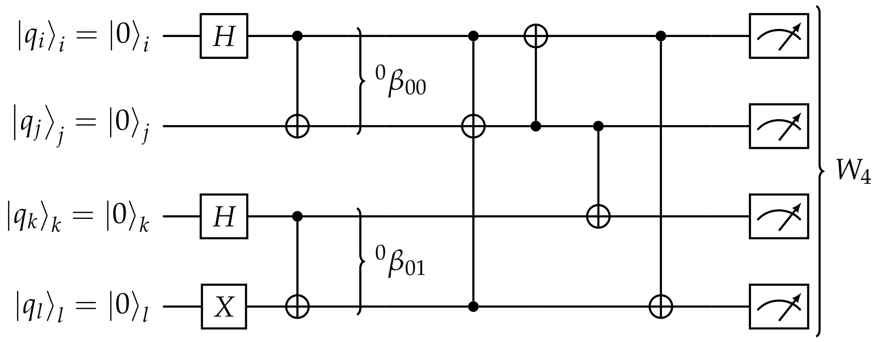

Figure 13: 4-party QLE implementation by QCM .

Following PhLQP syntax, it is also possible to write the 4-party-QLE as the following program :

$$\pi_{QLE_4} = \bigcup \pi''_{QLE_4}; \pi'_{QLE_4}; (q_{i_i} \wedge q_{j_j} \wedge q_{k_k} \wedge q_{l_l})? \quad ,$$

with

$$\pi'_{QLE_4} = \bigcup CNOT_{kl}; CNOT_{ij}; X_l; H_k; H_i(p)$$

and

$$\pi''_{QLE_4} = \bigcup CNOT_{il}; CNOT_{jk}; CNOT_{ji}; TOFF_{jil}(p) \quad .$$

Also with the following validity expressing the correctness of the 4-party-QLE protocol :

$$\vdash \underbrace{\left(\pi'_{QLE_4}((0,0)_i \wedge (0,0)_j \wedge (0,0)_k \wedge (0,0)_l) = {}^0\beta_{00}^{i,j} \wedge {}^0\beta_{01}^{k,l}\right)}_{1^{st} condition} \wedge \underbrace{\left(\pi''_{QLE_4}; \pi''_{QLE_4}(p) = id(p)\right)}_{2^{nd} condition} \quad .$$

Equivalently , for the 1^{st} condition of the validity there is :

$$(CNOT_{kl}; CNOT_{ij}; X_l; H_k; H_i)((0,0)_i \wedge (0,0)_j \wedge (0,0)_k \wedge (0,0)_l) \quad .$$

By consider the X-gate characteristic axiom, the above condition of the validity becomes :

$$(CNOT_{kl}; CNOT_{ij}; H_k; H_i)((0,0)_i \wedge (0,0)_j \wedge (0,0)_k \wedge (1,0)_l) \quad .$$

Consequently, by applying Proposition 2.2.26 with $\{i,j\}$-locality and $\{k,l\}$-locality :

$$^{0}\beta_{00}^{i,j} \wedge {}^{0}\beta_{01}^{k,l} \quad .$$

Also , for the 2^{nd} condition of the validity , there is the following equivalence :

$$(CNOT_{il}; CNOT_{jk}; CNOT_{ji}; TOFF_{jil}); (CNOT_{il}; CNOT_{jk}; CNOT_{ji}; TOFF_{jil})(id) \quad .$$

So, attending to the inherent property of quantum gates as unitary transformations, there is :

$$\begin{aligned}&(CNOT_{il}^{-1}; CNOT_{jk}^{-1}; CNOT_{ji}^{-1}; TOFF_{jil}^{-1}); (CNOT_{il}; CNOT_{jk}; CNOT_{ji}; TOFF_{jil})(p)\\ &= CNOT_{il}^{-1}; CNOT_{il}; CNOT_{jk}^{-1}; CNOT_{jk}; CNOT_{ji}^{-1}; CNOT_{ji}; TOFF_{jil}^{-1}; TOFF_{jil}(id)\end{aligned}$$

With the 2^{nd} condition of the validity becoming :

$$id_{il}; id_{jk}; id_{ji}; id_{jil}(p) = id(p) \quad .$$

2.3.3 *Quantum Fourier Transform*

By following [51], the QFT, the quantum analogue of the classical *Discrete Fourier Transform (DFT)*, can be approached as a quantum gate, and also with a unitary matrix representation. Therefore, it is possible to define the **QFT** operator as :

Definition 2.3.1 (QFT operator) *For a n-qubits register where $|q\rangle_n = |q_{n-1} \dots q_1 q_0\rangle$, there is*

$$\boldsymbol{QFT}\,|q\rangle_n = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i.(\frac{qk}{N})}\,|k\rangle_n \quad where \quad N = 2^n \tag{1}$$

with $\boldsymbol{QFT}^\dagger\ \boldsymbol{QFT} = \boldsymbol{QFT}\ \boldsymbol{QFT}^\dagger = id$.

A SINGLE-QUBIT QFT OPERATOR By using equation (1) , it is possible to write :

$$\mathbf{QFT}\,|0\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{2\pi i.(\frac{0}{2}(k))}\,|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\mathbf{QFT}\,|1\rangle = \frac{1}{\sqrt{2}} \sum_{k=0}^{1} e^{2\pi i.(\frac{1}{2}(k))}\,|k\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Remark 2.3.2 *It is noticeable that a single-qubit **QFT** operator is indistinguishable from a Hadamard gate .*

Proposition 2.3.1 (Single-qubit QFT operator characteristic axiom) *For $_1QFT$ denoting a single-qubit of index i **QFT** operator , there is :*

$$\vdash {}_1QFT_i(c) = H_i(c) \quad .$$

.

A TWO-QUBIT QFT OPERATOR . By the QCM it possible to implement the two-qubit **QFT** operator as depicted in Figure 14 . And by the equation (1) there is :

$$\mathbf{QFT}\,|0\rangle_2 = \frac{1}{\sqrt{4}}\sum_{k=0}^{3} e^{2\pi i.(\frac{0}{4}(k))}\,|k\rangle = \frac{1}{2}(|0\rangle_2 + |1\rangle_2 + |2\rangle_2 + |3\rangle_2) = \frac{1}{2}(|0\rangle + |1\rangle)\otimes(|0\rangle + |1\rangle)$$

$$\mathbf{QFT}\,|1\rangle_2 = \frac{1}{\sqrt{4}}\sum_{k=0}^{3} e^{2\pi i.(\frac{1}{4}(k))}\,|k\rangle = \frac{1}{2}(|0\rangle_2 + i\,|1\rangle_2 - |2\rangle_2 - i\,|3\rangle_2) = \frac{1}{2}(|0\rangle - |1\rangle)\otimes(|0\rangle + i\,|1\rangle)$$

$$\mathbf{QFT}\,|2\rangle_2 = \frac{1}{\sqrt{4}}\sum_{k=0}^{3} e^{2\pi i.(\frac{2}{4}(k))}\,|k\rangle = \frac{1}{2}(|0\rangle_2 - |1\rangle_2 + |2\rangle_2 - |3\rangle_2) = \frac{1}{2}(|0\rangle + |1\rangle)\otimes(|0\rangle - |1\rangle)$$

$$\mathbf{QFT}\,|3\rangle_2 = \frac{1}{\sqrt{4}}\sum_{k=0}^{3} e^{2\pi i.(\frac{3}{4}(k))}\,|k\rangle = \frac{1}{2}(|0\rangle_2 - |1\rangle_2 + |2\rangle_2 - |3\rangle_2) = \frac{1}{2}(|0\rangle - |1\rangle)\otimes(|0\rangle - i\,|1\rangle)$$

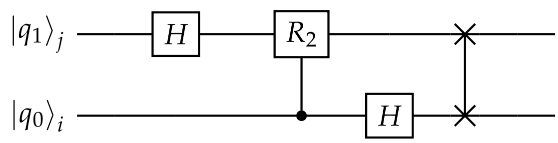

Figure 14: A two-qubit QFT gate implementation by QCM .

.

By consider the syntax PhLQP, the quantum program (of Figure 14) can be described as :

$$_2QFT_{ij} = \bigcup_{q_0,q_1\in\{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}} SWAP_{ij}; H_i; CR_{2_{ij}}; H_j; (q_{0_i}\wedge q_{1_i})$$

With the following validity asserting the correctness of the two-qubit QFT operator :

$$\vdash \underbrace{\left({}_2QFT^{\dagger}_{ij};{}_2QFT_{ij}(p) = id(p)\right)}_{1^{st}condition} \wedge \underbrace{\left(SWAP_{ij}(p);{}_2QFT_{ij} =_i {}_1QFT_i(p)\right)}_{2^{nd}condition}$$

For the 1^{st} condition of the validity , there is the following equivalence :

$$\begin{aligned}
&\big(SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)^{\dagger}; \big(SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p)\\
&= \big(H_j; CR_{2_{ij}}; H_i; SWAP_{ij}\big); \big(SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p)\\
&= \big(H_j^{-1}; CR_{2_{ij}}^{-1}; H_i^{-1}; SWAP_{ij}^{-1}; SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p)\\
&= \big(H_j^{-1}; CR_{2_{ij}}^{-1}; H_i^{-1}; id_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p)\\
&= \big(H_j^{-1}; CR_{2_{ij}}^{-1}; id_i; id_{ij}; CR_{2_{ij}}; H_j\big)(p)\\
&= \big(id_j; id_{ij}; id_i; id_{ij}\big)(p) = id(p)
\end{aligned}$$

Also, for the 2^{nd} condition of the validity, there is the subsequent equivalence :

$$SWAP_{ij}; \big(SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p) \quad .$$

which follows :

$$\big(SWAP_{ij}^{-1}; SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p) = \big(id_{ij}; H_i; CR_{2_{ij}}; H_j\big)(p) = \big(H_i; CR_{2_{ij}}; H_j\big)(p)$$

with

$$\big(H_i; CR_{2_{ij}}; H_j\big)(p) =_i H_i(p) \quad .$$

And from Proposition 2.3.1 there is :

$$\big(H_i; CR_{2_{ij}}; H_j\big)(p) =_i H_i(p) =_i {}_1QFT_i(p) \quad .$$

Proposition 2.3.2 (Two-qubit QFT operator characteristic axiom) *For ${}_2QFT$ denoting a $\{i, j\}$-local **QFT** operator , there is :*

$$\vdash {}_2QFT_{ij}(c) =_{ij} \big(SWAP_{ij}; H_i; CR_{2_{ij}}; H_j\big)(c) \quad .$$

A THREE - QUBIT QFT OPERATOR . By the QCM it possible to implement the three-qubit **QFT** operator shown in Figure 15 . And by the equation (1) there is :

$$\begin{aligned}
\mathbf{QFT}\,|0\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{0}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 + |1\rangle_3 + |2\rangle_3 + |3\rangle_3 + |4\rangle_3 + |5\rangle_3 + |6\rangle_3 + |7\rangle_3)\\
&= \frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathbf{QFT}\,|1\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{1}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 + \frac{\sqrt{2}}{2}(1+i)\,|1\rangle_3 + i\,|2\rangle_3 + \frac{\sqrt{2}}{2}(i-1)\,|3\rangle_3 \\
&\quad - |4\rangle_3 - \frac{\sqrt{2}}{2}(1+i)\,|5\rangle_3 - i\,|6\rangle_3 - \frac{\sqrt{2}}{2}(i-1)\,|7\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|0\rangle + \frac{\sqrt{2}}{2}(i+1)\,|1\rangle) \otimes (|0\rangle + i\,|1\rangle) \otimes (|0\rangle - |1\rangle) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathbf{QFT}\,|2\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{2}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 + i\,|1\rangle_3 - |2\rangle_3 - i\,|3\rangle_3 - |4\rangle_3 + i\,|5\rangle_3 - |6\rangle_3 - i\,|7\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|0\rangle + i\,|1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathbf{QFT}\,|3\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{3}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 + \frac{\sqrt{2}}{2}(i-1)\,|1\rangle_3 - i\,|2\rangle_3 + \frac{\sqrt{2}}{2}(i+1)\,|3\rangle_3 \\
&\quad - |4\rangle_3 - \frac{\sqrt{2}}{2}(i-1)\,|5\rangle_3 + i\,|6\rangle_3 - \frac{\sqrt{2}}{2}(i+1)\,|7\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|0\rangle + \frac{\sqrt{2}}{2}(i-1)\,|1\rangle) \otimes (|0\rangle - i\,|1\rangle) \otimes (|0\rangle - |1\rangle) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathbf{QFT}\,|4\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{4}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 - |1\rangle_3 + |2\rangle_3 - |3\rangle_3 + |4\rangle_3 - |5\rangle_3 + |6\rangle_3 - |7\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|0\rangle - |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \quad .
\end{aligned}$$

$$\begin{aligned}
\mathbf{QFT}\,|5\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{5}{8}(k))}\,|k\rangle &= \frac{1}{\sqrt{8}}(|0\rangle_3 - \frac{\sqrt{2}}{2}(i+1)\,|1\rangle_3 + i\,|2\rangle_3 + \frac{\sqrt{2}}{2}(1-i)\,|3\rangle_3 \\
&\quad - |4\rangle_3 + \frac{\sqrt{2}}{2}(i+1)\,|5\rangle_3 - i\,|6\rangle_3 - \frac{\sqrt{2}}{2}(1-i)\,|7\rangle_3) \\
&= \frac{1}{2\sqrt{2}}(|0\rangle - \frac{\sqrt{2}}{2}(i+1)\,|1\rangle) \otimes (|0\rangle + i\,|1\rangle) \otimes (|0\rangle - |1\rangle) \quad .
\end{aligned}$$

$$\mathbf{QFT}\,|6\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{6}{8}(k))}\,|k\rangle = \frac{1}{\sqrt{8}}(|0\rangle_3 - i\,|1\rangle_3 - |2\rangle_3 + i\,|3\rangle_3 + |4\rangle_3 - i\,|5\rangle_3 - |6\rangle_3 + i\,|7\rangle_3)$$
$$= \frac{1}{2\sqrt{2}}(|0\rangle - i\,|1\rangle)\otimes(|0\rangle - |1\rangle)\otimes(|0\rangle + |1\rangle) \quad .$$

$$\mathbf{QFT}\,|7\rangle_3 = \frac{1}{\sqrt{8}}\sum_{k=0}^{7} e^{2\pi i.(\frac{7}{8}(k))}\,|k\rangle = \frac{1}{\sqrt{8}}(|0\rangle_3 + \frac{\sqrt{2}}{2}(i-1)\,|1\rangle_3 - i\,|2\rangle_3 - \frac{\sqrt{2}}{2}(1+i)\,|3\rangle_3$$
$$- |4\rangle_3 - \frac{\sqrt{2}}{2}(i-1)\,|5\rangle_3 + i\,|6\rangle_3 + \frac{\sqrt{2}}{2}(1+i)\,|7\rangle_3)$$
$$= \frac{1}{2\sqrt{2}}(|0\rangle + \frac{\sqrt{2}}{2}(1-i)\,|1\rangle)\otimes(|0\rangle - i\,|1\rangle)\otimes(|0\rangle - |1\rangle) \quad .$$

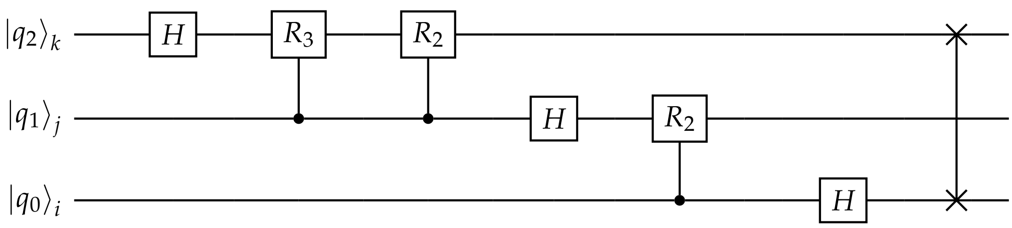

Figure 15: A three-qubit QFT gate implementation by QCM .

By consider the syntax PhLQP, the quantum program (of Figure 15) can be expressed as :

$$_3QFT_{ijk} = \bigcup_{q_0,q_1,q_2\in\{(0,\phi),(1,\phi),(+,\phi),(-,\phi)\}} SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k;(q_{0_i}\wedge q_{1_j}\wedge q_{2_k})$$

With the following validity expressing the correctness of the two-qubit QFT operator :

$$\vdash \underbrace{\big({}_3QFT^{\dagger}_{ijk};{}_3QFT_{ijk}(p) = id(p)\big)}_{1^{st}condition} \wedge \underbrace{\big(CR_{3_{jk}};CR_{2_{jk}};SWAP_{ik};{}_3QFT_{ijk}(p) =_{ij} SWAP_{ij};{}_2QFT_{ij}(p)\big)}_{2^{nd}condition}$$
$$\wedge \underbrace{\big(SWAP_{ik};{}_3QFT_{ijk}(p) =_k {}_1QFT_i(p)\big)}_{3^{rd}condition} \quad .$$

For the 1^{st} condition of the validity , there is the following equivalence :

$$\begin{aligned}
&\big(SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)^{\dagger};\big(SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k;CR_{3_{jk}};CR_{2_{jk}};H_j;CR_{2_{ij}};H_i;SWAP_{ik}\big);\big(SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};H_j^{-1};CR_{2_{ij}}^{-1};H_i^{-1};SWAP_{ik}^{-1};SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};H_j^{-1};CR_{2_{ij}}^{-1};H_i^{-1};id_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};H_j^{-1};CR_{2_{ij}}^{-1};id_i;id_{ik};CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};H_j^{-1};id_{ij};id_i;id_{ik}H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};id_j;id_{ij};id_i;id_{ik};CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_k^{-1};id_{jk};id_{jk};id_j;id_{ij};id_i;id_{ik};H_k\big)(p)\\
&=\big(id_k;id_{jk};id_{jk};id_j;id_{ij};id_i;id_{ik}\big)(p)=id(p)\quad .
\end{aligned}$$

Sequentially, for the 2^{nd} condition of the validity, there is the below equivalence :

$$CR_{3_{jk}};CR_{2_{jk}};SWAP_{ik};\big(SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\quad ,$$

which follows :

$$\begin{aligned}
&\big(CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};SWAP_{ik}^{-1};SWAP_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};id_{ik};H_i;CR_{2_{ij}};H_j;CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_i;CR_{2_{ij}};H_j;CR_{3_{jk}}^{-1};CR_{2_{jk}}^{-1};CR_{2_{jk}};CR_{3_{jk}};H_k\big)(p)\\
&=\big(H_i;CR_{2_{ij}};H_j;id_{jk};id_{jk};H_k\big)(p)=\big(H_i;CR_{2_{ij}};H_j;H_k\big)(p)\quad .
\end{aligned}$$

By consider $\{i,j\}$-locality, it follows:

$$\big(H_i;CR_{2_{ij}};H_j;H_k\big)(p)=_{ij}\big(H_i;CR_{2_{ij}};H_j\big)(p)\quad .$$

Consequently,

$$\big(H_i;CR_{2_{ij}};H_j\big)(p)=_{ij} id_{ij};\big(H_i;CR_{2_{ij}};H_j\big)(p)=_{ij} SWAP_{ij}^{-1};SWAP_{ij};\big(H_i;CR_{2_{ij}};H_j\big)(p)\quad .$$

And by the Proposition 2.3.2 the above expression is equivalent to :

$$SWAP_{ij};\big(SWAP_{ij};H_i;CR_{2_{ij}};H_j\big)(p)=_{ij} SWAP_{ij};{}_2QFT_{ij}(p)\quad .$$

At last, for the 3^{rd} condition , the next equivalence holds :

$$\big(SWAP_{ik}; SWAP_{ik}; H_i; CR2_{ij}; H_j; CR2_{jk}; CR3_{jk}; H_k\big)(p) \quad .$$

So,

$$\big(id_{ik}; H_i; CR2_{ij}; H_j; CR2_{jk}; CR3_{jk}; H_k\big)(p) = \big(H_i; CR2_{ij}; H_j; CR2_{jk}; CR3_{jk}; H_k\big)(p) \quad .$$

And by the i-locality , there is :

$$\big(H_i; CR2_{ij}; H_j; CR2_{jk}; CR3_{jk}; H_k\big)(p) =_i H_i(p) \quad .$$

Finally, by Proposition 2.3.1 there is :

$$H_i(p) = {}_1QFT_i(p) \quad .$$

Proposition 2.3.3 (Three-qubit QFT operator characteristic axiom) *For ${}_3QFT$ denoting a $\{i, j, k\}$-local **QFT** operator , there is :*

$$\vdash SWAP_{ik}; H_i; CR2_{ij}; H_j; CR2_{jk}; CR3_{jk}; H_k(c) =_{ijk} {}_3QFT_{ijk}(c) \quad .$$

3

COMBINING PARACONSISTENT AND DYNAMIC LOGIC FOR QUANTUM PROGRAMS

3.1 SOME PRELIMINARY DEFINITIONS ON PROPOSITIONAL LOGICS

To present (propositional) paraconsistent logics, first, there is the need of introduce some definitions about *propositional logic*. In this way, some notation will be introduced, accordingly to [8].

$\mathcal{L}$ denotes some *propositional language*,i.e. a structure consisting of a set of primitives, called the *propositional variables* of $\mathcal{L}$, and a finite set $\mathcal{C}(\mathcal{L})$ of *logical connectives*. Such connectives have its arity expressed by a certain natural number m. Given $\mathcal{L}$, it is definable:

- Each formula of $\mathcal{L}$ is inductively developed from other formulas as follows:
  - All propositional variables and all propositional constants are a formula. Such formulas are named *atomic*.
  - For $\psi_1, \ldots, \psi_m$, as formulas, and $\heartsuit$ as an m-ary logical connective of $\mathcal{L}$, the $\heartsuit(\psi_1, \ldots, \psi_m)$ is a formula . For $m > 0$ the formula is said to be *complex*.

  For illustration, in case of unary operators it will be written $\heartsuit\psi$ as an alternative of $\heartsuit(\psi)$, and when $\heartsuit$ is a binary connective it will be written $\psi\heartsuit\varphi$ for $\heartsuit(\psi, \varphi)$. A set of variables $Var = \{p_1, p_2, \ldots\}$. A set of formulas of $\mathcal{L}$ denoted by $\mathcal{W}(\mathcal{L})$, as well as φ, ψ, τ to vary over the elements of $\mathcal{W}(\mathcal{L})$.

- A propositional language $\mathcal{L}$ will be identified with $\mathcal{C}(\mathcal{L})$. So, if $\mathcal{L}_1$ is a sub-language of $\mathcal{L}_2$ then $\mathcal{C}(\mathcal{L}_1) \subseteq \mathcal{C}(\mathcal{L}_2)$.

- $2^{\mathcal{W}(\mathcal{L})}$ stands for the power set of $\mathcal{W}(\mathcal{L})$, which elements are called *theories*. It will be use the symbols $\mathcal{T}, \mathcal{S}$ to vary over theories , and Γ, Δ to vary over *finite* theories. $Var(\varphi)$ denotes the set of variables that appear in φ. $\mathcal{T} \cup \{\psi\}$ will be abbreviate by $\mathcal{T}, \psi$, and $\mathcal{T} \cup \mathcal{T}'$ by $\mathcal{T}, \mathcal{T}'$. The notation $\emptyset$ for the empty set will often be omitted, i.e., $\vdash \psi$ instead of $\emptyset \vdash \psi$.

- An $\mathcal{L}$-*substitution* θ is a finite set of pairs $\{(\psi_1, p_1), \ldots, (\psi_m, p_m)\}$, $\psi_1, \ldots, \psi_m$, are formulas of $\mathcal{L}$, and $p1, \ldots, p_m$ are m distinct variables. Given a substitution $\theta = \{(\psi_1, p_1), \ldots, (\psi_m, p_m)\}$ as well as formula τ, it will be denoted by $\theta(\tau)$ or by $\tau[\psi/p_1, \ldots, \psi_m/p_m]$ the formula which is acquired by replacing each occurrence of p_i in it by ψ_i, $i \in \{1, \ldots, m\}$. Given a theory $\mathcal{T}$ and a substitution θ, it will be written by $\theta(\mathcal{T})$, for the set $\{\theta(\tau) | \tau \in \mathcal{T}\}$.

In earlies 1940's, Tarski introduced the subsequent essential concept:

Definition 3.1.1 (tcr [8, 48]) *A* Tarskian consequence relation (tcr) *for a language $\mathcal{L}$ consists of a binary relation $\vdash$ between theories in $2^{\mathcal{W}(\mathcal{L})}$ and formulas in $\mathcal{W}(\mathcal{L})$, satisfying the subsequent three conditions:*

$$
\begin{array}{lrl}
[R] & \textit{Reflexity:} & \{\psi\} \vdash_{\mathbf{L}} \psi. \\
[M] & \textit{Monotocity:} & \textit{if } \mathcal{T} \vdash_{\mathbf{L}} \psi \textit{ and } \mathcal{T} \subseteq \mathcal{T}', \textit{ then } \quad \mathcal{T}' \vdash_{\mathbf{L}} \psi. \\
[C] & \textit{Cut:} & \textit{if } \mathcal{T} \vdash_{\mathbf{L}} \psi \textit{ and } \mathcal{T}', \psi \vdash_{\mathbf{L}} \varphi, \textit{ then } \quad \mathcal{T}, \mathcal{T}' \vdash_{\mathbf{L}} \psi.
\end{array}
$$

Definition 3.1.2 (Extra properties of a tcr [8]) *Let $\vdash$ be a tcr relation for $\mathcal{L}$.*

- *$\vdash$ is structural if for every $\mathcal{L}$- substitution θ and every $\mathcal{T}$ and ψ: if $\mathcal{T} \vdash \psi$ then $\theta(\mathcal{T}) \vdash \theta(\psi)$.*
- *$\vdash$ is consistent if there exist some non-empty theory $\mathcal{T}$ as well as some formula ψ such that $\mathcal{T} \nvdash \psi$.*
- *$\vdash$ is finitary if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$, there is a finite theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.*

Definition 3.1.3 (Propositional logic [8]) *Defining a propositional logic :*

- *A propositional logic consists of a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, where $\mathcal{L}$ stands for a propositional language, and $\vdash_{\mathbf{L}}$ is a structural and consistent tcr for $\mathcal{L}$.*
- *A logic $\mathbf{L}$ is finitary if so is $\vdash_{\mathbf{L}}$.*

Definition 3.1.4 (Fragment of a logic [8]) *Let $\mathbf{L_1} = \langle \mathcal{L}_1, \vdash_{\mathbf{L_1}} \rangle$, and consider $\mathcal{L}_2 \subseteq \mathcal{L}_1$. The $\mathcal{L}_2$-fragment of $\mathbf{L}_1$ is the logic $\mathbf{L_2} = \langle \mathcal{L}_2, \vdash_{\mathbf{L_2}} \rangle$, where $\vdash_{\mathbf{L_2}}$ is the restriction of $\vdash_{\mathbf{L_1}}$ to $\mathcal{L}_2$.*

CLASSICAL LOGIC The ***classical propositional logic (CL)*** is the most standard and valuable logic. From the semantic perspective, CL is established on the employment of two *truth values* : t (standing for truth) and f (standing for falsity), and also a *truth-functional* interpretations of the connectives, Definition 3.1.5.

Definition 3.1.5 (Bivalent interpretation [8]) *Let $\mathcal{L}$ be a propositional language.*

- *A two-valued truth table for an n-ary connective $\heartsuit$ of $\mathcal{L}$ is an n-ary function $\tilde{\heartsuit} : \{t, f\}^n \to \{t, f\}$. $\tilde{\heartsuit}$ stands for an interpretation of $\heartsuit$.*

  Remark 3.1.1 *Notice that if $n = 0$ then $\tilde{\heartsuit} \in \{t, f\}$.*

- *A bivalent interpretation of $\mathcal{L}$ is a function* **L** *that assigns a two valued truth table to each primitive connective of $\mathcal{L}$.*

- $\mathbf{F}(\heartsuit)$ *is a bivalent interpretation for $\heartsuit$, abbreviated by $\tilde{\heartsuit}$ if it is function $\nu : \mathcal{W}(\mathcal{L}) \to \{t, f\}$ such that for every n-ary primitive connective $\heartsuit$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$,*

  $$\nu\big(\heartsuit(\psi_1, \ldots, \psi_n)\big) = \tilde{\heartsuit}\big(\nu(\psi_1), \ldots, \nu(\psi_n)\big).$$

  The set of **F***-valuations for $\mathcal{L}$ is denoted by $\mathbf{\Lambda_{2_F}}$.*

- *A classical (bi-valued) semantics for $\mathcal{L}$ is a pair $\langle \mathbf{\Lambda_{2_F}}, \models_{\mathbf{F}} \rangle$ in which* **F** *is a bivalent interpretation of* **L***, and the satisfaction relation $\models_{\mathbf{F}}$ is defined by $\nu \models_{\mathbf{F}} \psi$ if $\nu(\psi) = t$. Also, it is possible to write $\vdash_{2_\mathbf{F}}$ for the consequence relation that is induced by the denotational semantics $\langle \mathbf{\Lambda_{2_F}}, \models_{\mathbf{F}} \rangle$.*

Definition 3.1.6 (Containment in CL [8]) *For a propositional logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$:*

- *Let* **F** *be a bivalent interpretation of $\mathcal{L}$.* **L** *is* **F***-contained in classical logic if $\varphi_1, \ldots, \varphi_n \vdash_{2_\mathbf{F}} \psi$ for every $\varphi_1, \ldots, \varphi_n \in \mathcal{W}(\mathcal{L})$ such that $\varphi_1, \ldots, \varphi_n \in \vdash_{\mathbf{L}} \psi$.*

- **L** *is contained in classical logic if it is* **F***-contained in it for some bivalent interpretation* **F** *for $\mathcal{L}$.*

As presented in [8], it is consider the following version of **CL**:

Definition 3.1.7 ($\mathcal{L}_{CL}$, CL) *Let $\mathcal{L}_{CL}$ be a language.*

- *$\mathcal{L}_{CL}$ is the language of $\{\to, \wedge, \vee, \neg\}$, where $\neg$ stands for an unary connective, while the other connectives are binary.*

- $\mathbf{F_{CL}}$ *is the bivalent interpretation of $\mathcal{L}_{CL}$ defined by $\mathbf{F_{CL}}(\heartsuit) = \tilde{\heartsuit}$, where for every $x, y \in \{t, f\}$:*

  $$x \mathrel{\tilde{\to}} y = \begin{cases} f & \text{if} \quad x = t \quad \text{and} \quad y = f, \\ t & \text{, otherwise .} \end{cases}$$

  $$x \mathrel{\tilde{\wedge}} y = \begin{cases} t & \text{if} \quad x = t \quad \text{and} \quad y = t, \\ f & \text{, otherwise .} \end{cases}$$

$$x \tilde{\vee} y = \begin{cases} t & \textit{if} \quad x = t \quad \textit{or} \quad y = t, \\ f & \textit{, otherwise} . \end{cases}$$

$$\tilde{\neg}\, x = \begin{cases} t & \textit{if} \quad x = f, \\ f & \textit{if} \quad x = t. \end{cases}$$

- ***CL** is the logic* $\langle \mathcal{L}_{\mathbf{CL}}, \vdash_{\mathbf{CL}} \rangle$*, where* $\vdash_{\mathbf{CL}}$ *is the relation obtained through* $\mathbf{F}_{\mathbf{CL}}$.

Also, there is an especially important fragment of **CL**, the positive fragment of **CL**: $\mathbf{CL}^+$. This fragment by itself is a propositional logic, i.e., Definition 3.1.8.

Definition 3.1.8 ($\mathcal{L}^+_{\mathbf{CL}}$ and $\mathbf{CL}^+$ [8]) $\mathbf{CL}^+$ *is the* $\mathcal{L}^+_{\mathbf{CL}}$*-fragment of* **CL***, where* $\mathcal{L}^+_{\mathbf{CL}} = \{\rightarrow, \wedge, \vee\}$.

Now, it will be enumerated the fundamental properties that $\neg$ has related to $\vdash_{\mathbf{CL}}$.

Definition 3.1.9 (Consistent and inconsistent theories [8]) *A theory* $\mathcal{T}$ *in* $\mathcal{L}_{CL}$ *is inconsistent if there exists a formula* ψ *such that* $\mathcal{T} \vdash_{CL} \psi$ *and* $\mathcal{T} \vdash_{CL} \neg\psi$ *hold. Otherwise,* $\mathcal{T}$ *is consistent.*

Also, there is the following Proposition 3.1.1.

Proposition 3.1.1 ([8]) *Let* $\mathcal{T}$ *be a theory and* ψ, φ *be formulas in* $\mathcal{L}_{CL}$.

1. *If* $\mathcal{T}, \psi \vdash_{CL} \varphi$ *and* $\mathcal{T}, \neg\psi \vdash_{CL} \varphi$ *then* $\mathcal{T} \vdash_{CL} \varphi$.
2. *If* $\mathcal{T}$ *is inconsistent then* $\mathcal{T} \vdash_{CL} \varphi$*. This is:* $\psi, \neg\psi \vdash_{CL} \varphi$.
3. $\psi \vdash_{CL} \neg\neg\psi$.
4. $\neg\neg\psi \vdash_{CL} \psi$.
5. *If* $\mathcal{T}, \psi \vdash_{CL} \varphi$ *then* $\mathcal{T}, \neg\varphi \vdash_{CL} \neg\psi$.
6. *If* $\mathcal{T}, \psi \vdash_{CL} \varphi$*, and* $\mathcal{T}, \psi \vdash_{CL} \neg\varphi$*, then* $\mathcal{T} \vdash_{CL} \neg\psi$.

HILBERT-TYPE SYSTEMS Hilbert-type proof systems shape the simplest and very commonly used class of proof systems. Inherent to proof systems are two fundamental properties: *soundness* and *completeness*.

Definition 3.1.10 (Soundness and completeness [8]) *A proof system P is sound/complete for* $\vdash$ *(P is sound/complete for its logic):*

- *If P is sound, everything in P that is provable is true.*
- *If P is complete, everything in P that is true has a proof.*

Definition 3.1.11 (Hilbert-type proof systems [8]) *Let* $\mathcal{L}$ *stand for a propositional language.*

- *For a finite set $\mathcal{S}$ of rules in $\mathcal{L}$. A Hilbert-type proof system $H_{\mathcal{S}}$ for $\mathcal{L}$, is the set of all triples $\langle \mathcal{T}, \psi, d \rangle$ such that:*
  - *d is a finite sequence $\varphi_1, \ldots, \varphi_m$ of formulas in $\mathcal{L}$.*
  - *$\varphi_m = \psi$.*
  - *Each φ_i (where $i \in \{1, \ldots, m\}$) is either an element of $\mathcal{T}$, or stands for a the conclusion of an application of some rule in $\mathcal{S}$ whose premises are included in $\{\psi_1, \ldots, \psi_{i-1}\}$.*

Definition 3.1.12 (*HCL* [8]) *HCL is the Hilbert-type proof system for CL. HCL holds the following axioms :*

$$
\begin{array}{ll}
[\wedge_{\rightarrow}] & \psi \wedge \varphi \rightarrow \psi,\ \psi \wedge \varphi \rightarrow \varphi \\
[\rightarrow_{\wedge}] & \psi \rightarrow (\varphi \rightarrow \psi \wedge \varphi) \\
[\rightarrow_{\vee}] & \psi \rightarrow (\psi \vee \varphi),\ \varphi \rightarrow \psi \vee \varphi \\
[\vee_{\rightarrow}] & (\psi \rightarrow \tau) \rightarrow ((\varphi \rightarrow \tau) \rightarrow (\psi \vee \varphi \rightarrow \tau)) \\
[t] & \neg\psi \vee \psi \\
[\neg \rightarrow] & \neg\psi \rightarrow (\psi \rightarrow \varphi)
\end{array}
$$

Remark 3.1.2 *HCL is sound and complete for* **CL** *(c.f. [8, Theorem 1.90]).*

At last, the Hilbert proof system for $\mathbf{CL}^+$, HCL^+.

Definition 3.1.13 (HCL^+ [8]) *HCL^+ is the Hilbert-type proof system for the purely positive fragment of CL. HCL^+ holds the following axioms :*

$$
\begin{array}{ll}
[\wedge_{\rightarrow}] & \psi \wedge \varphi \rightarrow \psi,\ \psi \wedge \varphi \rightarrow \varphi \\
[\rightarrow_{\wedge}] & \psi \rightarrow (\varphi \rightarrow \psi \wedge \varphi) \\
[\rightarrow_{\vee}] & \psi \rightarrow (\psi \vee \varphi),\ \varphi \rightarrow \psi \vee \varphi \\
[\vee_{\rightarrow}] & (\psi \rightarrow \tau) \rightarrow ((\varphi \rightarrow \tau) \rightarrow (\psi \vee \varphi \rightarrow \tau)) \\
[\rightarrow 3] & ((\psi \rightarrow \varphi) \rightarrow \psi) \rightarrow \psi
\end{array}
$$

Remark 3.1.3 *HCL^+ is sound and complete for* $\mathbf{CL}^+$ *(c.f. [8, Theorem 1.90]).*

3.2 PARACONSISTENT LOGICS

Traditional logics, as CL, admit the principle of explosion, *"ex contracdictione sequitur quodlibet" (ECSQ)*, i.e. "If one states something is both true and not true, one can logically draw any conclusion ", [8, 45] :

$$\psi, \neg\psi \vdash \varphi. \tag{2}$$

In opposition to traditional logics, paraconsistent logics reject (2). However, this rejection in paraconsistent logics, which have a disjunction $\vee$ in its languages, implies the need of reject also at least one of the two subsequent principles:

- *The introduction of disjunction*: from ψ infer $\psi \vee \varphi$.
- *The disjunctive syllogism*: from $\neg\psi$ and $\psi \vee \varphi$ infer φ .

The introduction of disjunction stands for a rule of inference of propositional logic that allows to introduce disjunctions to logical proofs. This is: if ψ is true, then $\psi \vee \varphi$ should be also true.

The disjunctive syllogism is a rule of inference of propositional logic that allows to eliminate disjunctions in logical proofs. In other words, if is told that ψ is true, then $\neg\psi \vee \varphi$ is true by φ.

3.2.1 *Negation and Paraconsistency*

In (2), it is obvious that the idea of paraconsistency hinge on a associated notion of negation , i.e. a notion of the connective $\neg$. Also, from now on, consider for the definitions in the sequel: a logic $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, where $\vdash_{\mathbf{L}}$ stands for a tcr and $\mathcal{L}$ for a propositional language.

Definition 3.2.1 (Explosive negation [8]) *Let $\mathcal{L}$ be a language with a unary connective $\neg$. $\neg$ is* ***explosive*** *if $\neg\varphi, \varphi \vdash_{\mathbf{L}} \psi$ for every φ and ψ.*

Now, it will be define the notion of paraconsistency in precise terms.

Definition 3.2.2 ((strong) pre-$\neg$-paraconsistency [8]) *Let $\mathcal{L}$ be a language with a unary connective $\neg$.*

- $\mathbf{L}$ *is pre-$\neg$-paraconsistent if $\neg$ is not explosive in it .*
- $\mathbf{L}$ *is said to be strongly pre-$\neg$-paraconsistent if there are variables p, q such that $p, \neg p \nvdash_{\mathbf{L}} \neg q$.*

Definition 3.2.3 ((strong) $\neg$-paraconsistency [8]) *Let $\mathcal{L}$ be a language with a unary connective $\neg$.* $\mathbf{L}$ *is (strongly) $\neg$-paraconsistent if it is (strongly) pre-$\neg$-paraconsistent, with $\neg$ is as a negation of* $\mathbf{L}$.

Definition 3.2.4 (Non-explosive logic [8]) **L** *is non explosive if* $\mathcal{T} \nvdash_{\mathbf{L}} q$ *for every theory* $\mathcal{T}$ *such that* $Var(\mathcal{T}) \neq Var$ *and every variable* q *such that* $q \notin Var(\mathcal{T})$. **L** *is strongly non exploding if* $\mathcal{T} \nvdash_{\mathbf{L}} \psi$ *for every* ψ *such that* $Var(\mathcal{T}) \cap Var(\mathcal{T}) = \emptyset$ *and* $\nvdash_{\mathbf{L}} \psi$.

Remark 3.2.1 *Classical logic is* $\neg$*-explosive.*

Definition 3.2.5 (classical $\neg$-paraconsistent logic [8]) *Let* $\mathcal{L}_{CL} \subseteq \mathcal{L}$. **L** *stands for a classical* $\neg$*-paraconsistent logic if it is an axiomatic extension of* $\mathbf{CL}^+$ *which is pre-*$\neg$*-paraconsistent and* $\neg$*-contained in CL.*

3.2.2 *Logics of Formal Inconsistency*

Newton da Costa is the founder of one of the oldest and presently most well-known lines of study of paraconsistency: the "Brazilian school". This line's approach has begun with the hierarchy of paraconsistent calculi $\{\mathbf{C}_n \mid 0 < n < \omega\}$ developed by him in the 1960's [29, 33], holding the theoretical concepts that led to the development of a considerable family of paraconsistent logics called LFIs (cf. [21, 22, 23] and [8, Chapter 8]) .

The central idea behind this line of study is that propositions can be classified into two kinds : the *normal* ones (or *consistent* ones) and *abnormal* ones (or *inconsistent* ones). Where contradictions are allowed for the inconsistent propositions. LFIs are characterized as logics that are provided of resources to express this meta-theoretical concept of consistency of formulas within the object language. As a result, a *Logic of Formal Inconsistency (LFI)* is capable of distinguish between propositions for which the explosion (the consistent/normal ones), and those for which does not hold (the inconsistent/abnormal ones).

A C-SYSTEM AS A UNIQUE TYPE OF LFI A C-system is a unique type of LFI which utilizes a unary connective $\circ$ in order to express the (in)consistency of its formulas. Therefore, in da Costa's most basic system the consistency[1] of a sentence ψ is expressed as $\neg(\psi \wedge \neg\psi)$.

Definition 3.2.6 (LFI [8]) *Let* $\mathcal{L}$ *be a language with negation* $\neg$. **L** *is a LFI with reverence to* $\neg$ *if there occurs a non-empty set* $\bigcirc(p)$ *of formulas in* $\mathcal{L}$ *with the following properties :*

- **L** *is* $\neg$*-paraconsistent .*
- *There exists some formulas* ψ_0, φ_0 *such that :*
  - $\bigcirc(\psi_0), \psi_0 \nvdash_{\mathbf{L}} \varphi_0$,
  - $\bigcirc(\psi_0), \neg\psi_0 \nvdash_{\mathbf{L}} \varphi_0$.
- $\bigcirc(\psi), \psi, \neg\psi \vdash_{\mathbf{L}} \varphi$ *for every* ψ *and* φ.

1 or *normality* in [8, Chapter 8]

where $\bigcirc(\psi) = \{\varphi[\psi/p] \mid \varphi \in \bigcirc(p)\}$.

A C-system can be characterized with respect to some base logic , which such a C-system extends, such as the positive fragment of a **CL**, $\mathbf{CL}^+$. Where $\mathbf{CL} = \langle \mathcal{L}_{CL}, \vdash_{\mathbf{CL}} \rangle$ stands for an arbitrary classical logic with $\mathcal{L}_{CL}$ as its language and $\vdash_{\mathbf{CL}}$ as its tcr.

Definition 3.2.7 (Consistency operator [8]) *Let $\mathcal{L}_{CL} \subseteq \mathcal{L}$, $\mathbf{CL}^+ \subseteq \mathbf{L}$, and $\circ$ be a unary connective of $\mathcal{L}$. It is possible to say , in* **L**, *that $\circ$ is a consistency operator for $\neg$ if :*

- $[n_0]$ $\quad \vdash_{\mathbf{L}} (\circ\psi \wedge \neg\psi \wedge \psi) \rightarrow \varphi$ *for every* $\psi, \varphi \in \mathcal{W}(\mathcal{L})$.
- $[n_1]$ $\quad \nvdash_{\mathbf{L}} (\circ p \wedge \neg p) \rightarrow q$ *whenever p and q are distinct propositional variables .*
- $[n_2]$ $\quad \nvdash_{\mathbf{L}} (\circ p \wedge p) \rightarrow q$ *whenever p and q are distinct propositional variables .*

Definition 3.2.8 (C-system [8]) *Let* **L** *be a like in Definition 3.2.7, as well as $\circ$ be a unary connective of $\mathcal{L}$.*

- **L** *is a $\circ$-C-system if it is classical $\neg$-paraconsistent logic in which $\circ$ is a consistency operator for $\neg$. If $\circ$ is a primitive of connective $\mathcal{L}$, then* **L** *is a strict $\circ$-C-system for $\neg$.*
- **L** *is a (strict) C-system if it is a (strict) $\circ$-C-system for some $\circ$.*

Notice that due to the condition $[n_0]$, the conditions $[n_1], [n_2]$ can be reformulated as $[n'_1], [n'_2]$ along with a condition of pre-paraconsistency $[n'_3]$:

$[n'_1]$ $\qquad \nvdash_{\mathbf{L}} (\circ p \wedge \neg p) \rightarrow p.$

$[n'_2]$ $\qquad \nvdash_{\mathbf{L}} (\circ p \wedge \neg p) \rightarrow \neg p.$

$[n'_3]$ $\qquad \nvdash_{\mathbf{L}} (\circ p \wedge \neg p) \rightarrow \circ p.$

By combining conditions $[n'_1], [n'_2], [n'_3]$ and $[n_0]$ it is possible to say that every formula follows from the set $\{p, \neg p, \circ p\}$, although no element from this set follows from the other two. Therefore, the set which contains these four conditions is entirely symmetric with concern to $\neg$ and $\circ$. In specific, the $\mathcal{L}_{\mathbf{CL}}$- fragment of **L** is contained in **CL**. Moreover , it is possible to expand this condition to $\{\wedge, \vee, \rightarrow, \neg, \circ\}$ in such manner that the symmetry between $\neg$ and $\circ$ is broken , and the intuition that classical logic is established over the idea that all formulas are consistent is reinforced: Definition 3.2.9.

Definition 3.2.9 (Regular C-system [8]) *Let* **L** *and $\circ$ be defined by Definition 3.2.7.*

- *$\circ$ is a regular consistency operator (for $\neg$) if it satisfies condition $[n_0]$ from Definition 3.2.7, along with the subsequent condition :*
  - $[c_0]$ **L** *is* **F***-contained in* **CL** *(Definition 3.1.6) for some bivalent $\neg$-interpretation* **F** *such that $\mathbf{F}(\circ) = \lambda x.t$ (i.e., $(\mathbf{F}(\circ))(t) = (\mathbf{F}(\circ))(f) = t$).*

- **L** *is a regular* $\circ$*-C-system if it is a pre-*$\neg$*-paraconsistent axiomatic extension of* $\mathbf{CL}^+$ *in which* $\circ$ *is a regular consistency operator, and* $\neg$ *is complete.*

- **L** *is a regular C-system if it is regular* $\circ$*-C-system for some* $\circ$*.*

Where the t *and* f *stands for the two truth values:* t *for truth and* f *for falsity.*

The following theorem (Theorem 3.2.1) gives the most essential property of regular consistency operators. It shows the straightforward perception that classical logic presumes that all formulas are consistent. Additionally, it shows that in the presence of a regular consistency operator, any classical proof can be replicated in any regular C-system by adding the assumptions concerning consistency of formulas.

Theorem 3.2.1 (Consistent formulas [8]) *Let* **L** *and* $\circ$ *be like in Definition 3.2.7. Consider that* $\circ$ *is a regular consistency operator for* $\neg$*. Let* $\mathcal{T} \cup \{\psi\}$ *be a set of formulas in* $\mathcal{L}_{\mathbf{CL}}$*. Then* $\mathcal{L} \vdash_{\mathbf{CL}} \psi$ *iff there exists a finite set* Δ *of subformulas of* $\mathcal{T} \cup \{\psi\}$ *such that* $\circ\Delta \cup \mathcal{T} \vdash_{\mathbf{L}} \psi$ *(where* $\circ\Delta = \{\circ\psi \mid \psi \in \Delta\}$*).*

Corollary 3.2.1.1 ([8]) *Let* **L** *and* $\circ$ *be as in Definition 3.2.7.*

1. *If* $\circ$ *is a regular consistency operator then it is a consistency operator.*

2. *If* **L** *is a regular (*$\circ$*-)C-system then it is a (*$\circ$*-)C-system.*

A STRONGER DEFINITION OF CONSISTENCY OPERATOR As reasoned in [9] and discussed in [8, Chapter 8] , there is an alternative more stronger condition (which is organic) to impose on $\circ$ and $\neg$.

Definition 3.2.10 (Strong consistency operator [8, 9]) *Let* **L** *and* $\circ$ *be like in Definition 3.2.7* $\circ$ *is a strong consistency operator (for* $\neg$*) if every instance of the subsequent axioms is theorem of* **L***:*

$$[b] \qquad (\circ\varphi \wedge \neg\varphi \wedge \varphi) \to \psi,$$

$$[k] \qquad \circ\varphi \vee (\neg\varphi \wedge \varphi).$$

By consider the intuitive meaning of $\circ\varphi$ as "φ is consistent" , the axiom $[b]$ suggests that no formula is both consistent and contradictory . On the other hand, axiom $[k]$ acts as complement of axiom $[b]$ by stating that every formula is either consistent or contradictory. So, the combination of the axioms $[b]$ and $[k]$ captures the principle of the desired meaning of $\circ$, and so demanding their combination is natural.

Definition 3.2.11 (Strong C-System [8]) *Let* **L** *and* $\circ$ *be like in Definition 3.2.7 .*

- **L** *is a strong* $\circ$*-C-system if it is a classical* $\neg$*-paraconsistent logic in which* $\circ$ *stands for a strong consistency operator for* $\neg$*, and* $\neg$ *is complete .*

- **L** *is a strong C-system if it is strong* $\circ$*-C-system for some* $\circ$*.*

Definition 3.2.12 (Bottom element [8]) *A formula F is said to be bottom element for a logic* **L** *in* $\mathcal{L}$ *if* $\vdash_{\mathbf{L}} F \to \varphi$ *for every* $\varphi \in \mathcal{W}(\mathcal{L})$*.*

THE MOST BASIC C-SYSTEMS The most basic systems of the are **B** and **Bk** .

Let be a language $\mathcal{L}_{\mathbf{C}} = \{\wedge, \vee, \to, \neg, \circ\}$.

Definition 3.2.13 (*HB, HBk,* B, Bk [8]) *The Hilbert-type system HB stands for the extension in* $\mathcal{L}_C$ *of HCLuN (Definition 3.2.14) with axiom* $[b]$*. The system HBk by itself is the extension of HB by adding the axiom* $[k]$*.* **B**, **Bk** *are the logics in* $\mathcal{L}_{\mathbf{C}}$ *induced by HB and HBk, respectively (i.e.,* $\mathbf{B} = \mathbf{L}_{HB}$*,*$\mathbf{Bk} = \mathbf{L}_{HBk}$*).*

Definition 3.2.14 (*HCLuN* [8]) *HCLuN is the system acquired from* HCL^{+}*(Definition 3.1.13) by adding the axiom:*

$$[t] \qquad \neg\varphi \vee \varphi \quad .$$

B is also known as "the most basic C-system", and so it is denominated by *most basic C-system (mbC)*, [21]. Nevertheless, in [8] is argue that **Bk** would be a more suitable candidate for mbC. The rationale behind this is explained in three main points, attending to Hilbert type proof systems. The first one, is that there is no axiom $[k]$, axiom which is important for the desired meaning of $\circ\varphi$ that is no less important that what is expressed by $[b]$. Secondly, **Bk** has a particular completeness property that **B** lacks, i.e.: there is a divergence between the minimality of **Bk** (Proposition) and the $\neg$-paraconsistent axiomatics extensions of **B**, which are $\neg$-contained in classical logic , although in which $\circ$ is not a consistency operator . And the third one: $[k]$ is a theorem of the majority of important C-systems , since it is derivable in **B** from three crucial axioms related to $\circ$, which will be mentioned in the sequel : $[i]$, $[l]$, and $[d]$.

PROPAGATION OF CONSISTENCY So far, it was mentioned extensions **Bk** with some of the most standard axioms regarding negation, $\neg$ connective. Additionality, there is special type of axioms: those that deals with consistency propagation, Definition 3.2.15. Such axioms are split among two sets: the *a*-axioms, and the *o*-axioms. The *a*-axioms are safer for holding strong paraconsistency, although they require that *every* immediate proper subformula of a complex formula φ to be consistent in order to make φ consistent. On the other hand, in the *o*-axioms it is enough that *some* immediate proper subformula of φ to be consistent in order to make φ consistent.

Definition 3.2.15 (Propagation axioms for the connectives of $\mathcal{L}_C$)

$[a_{\neg}]$	$\vdash \circ\varphi \to \circ\neg\varphi$
$[a_{\wedge}]$	$\vdash (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \wedge \psi)$
$[a_{\vee}]$	$\vdash (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \vee \psi)$
$[a_{\to}]$	$\vdash (\circ\varphi \wedge \circ\psi) \to \circ(\varphi \to \psi)$
$[a_{\circ}]$	$\vdash \circ\varphi \to \circ\circ\varphi$
$[o^1_{\wedge}]$	$\vdash \circ\varphi \to \circ(\varphi \wedge \psi)$
$[o^2_{\wedge}]$	$\vdash \circ\psi \to \circ(\varphi \wedge \psi)$
$[o^1_{\vee}]$	$\vdash \circ\varphi \to \circ(\varphi \vee \psi)$
$[o^2_{\vee}]$	$\vdash \circ\psi \to \circ(\varphi \vee \psi)$
$[o^1_{\to}]$	$\vdash \circ\varphi \to \circ(\varphi \to \psi)$
$[o^2_{\to}]$	$\vdash \circ\psi \to \circ(\varphi \to \psi)$

Also, one of the most essential usage of the *a*-axioms is that they can strengthen the key Theorem 3.2.1, in a significant manner.

Theorem 3.2.2 *([8]) For* **L** *as a regular* $\circ$*-C-system in which all the a-axioms (with the possible exception of* $a_\circ$*) are provable and* $\mathcal{T} \cup \{\psi\}$ *as a set of formulas in* $\mathcal{L}_{\mathbf{CL}}$*, there is* $\mathcal{T} \vdash_{\mathbf{CL}} \psi$ *iff there exists a finite set* Δ *of propositional variables which occur in* $\mathcal{T} \cup \{\psi\}$ *such that* $\circ\Delta \cup \mathcal{T} \vdash_{\mathbf{L}} \psi$*.*

Remark 3.2.2 *([8]) By consider the instance* $\circ\circ\varphi \vee (\circ\varphi \wedge \neg\circ\varphi)$ *of* $[k]$*, it is possible to show that* $\varphi \to \circ\circ\varphi$ *is equivalent to* $\circ\circ\varphi$ *in* **Bk**.

A SPECIAL DEFINITION OF CONSISTENCY OPERATOR In all the above mentioned C-systems there is a emphasis on an operator for *formal consistency*, instead of formal inconsistency . The motive for this is that inconsistency of φ implies that bot φ and $\neg\varphi$ are true, i.e. such is expressed by the sentence $\varphi \wedge \neg\varphi$. In fact, as mentioned in [8, Section 8.4], the content of two axioms concerning negation of *HBk* (i.e., $[b]$, $[k]$) states that each $\circ\varphi$ and $\varphi \wedge \neg\varphi$ is equivalent to the *classical negation* of the other: one of them is always true , but never both of them are true. The core idea behind da Costa's main system $\mathbf{C}_1$ regarding (in)consistency of formulas, the negation $\neg$ of the logic acts as classical negation, and consequently $\neg(\varphi \wedge \neg\varphi)$ can act as a definition of the consistency operator in $\mathcal{L}_{\mathbf{CL}}$. By consider da Costa's idea of consistency operator is possible to verify that the sentences

$\circ\varphi \to \neg(\varphi \wedge \neg\varphi)$ and $(\varphi \wedge \neg\varphi) \to \neg \circ \varphi$ are theorems of **B**([8]). So, what is crucial to make $\circ\varphi$ equivalent to the negation $\varphi \wedge \neg\varphi$, and vice versa, are the subsequent two axioms :

$$[i] \qquad \neg \circ \varphi \to (\varphi \wedge \neg\varphi),$$

$$[l] \qquad \neg(\varphi \wedge \neg\varphi) \to \circ\varphi.$$

Remark 3.2.3 ([8]) *Also, there is the alternative way of one consider $\neg\varphi \wedge \varphi$ as expressing consistency of φ with the use of $\neg(\neg\varphi \wedge \varphi)$ as the consistency operator.Thus, in this circumstance $[l]$ must be replaced by the following axiom:*

$$[d] \qquad \neg(\neg\varphi \wedge \varphi) \to \circ\varphi.$$

3.3 PARACONSISTENT LOGICS IN QUANTUM COMPUTING: PARACONSISTENT TURING MACHINES

In late 1930's article [49], A.M. Turing formalizes the presently known definition of *Turing Machine (TM)* and establishes a difference between two kinds of machines: *automatic machine (a-machine)* and *choice machine (c-machine)*. An a-machine is a machine where all its actions are entirely determined by its configuration: *Deterministic Turing Machine (DTM)*. On the other hand, a c-machine is a machine where its actions are merely in part determined by the machine configurations, i.e., in an ambiguous configuration this kind of machine cannot continue until certain outward operator selects an instruction to be executed: *Non-Deterministic Turing Machine (NDTM)*. As mentioned by J.C. Agudelo and W.Carnielli in [7] and asserted by P. Odifreddi in [43, p. 48], in a DTM is required the inexistency of ambiguous instructions, since ambiguous instructions can led to the undecidability problem of a presented (non-paraconsistent) *First Order Logic (FOL)*, cf. [20, 27, 49]. In [43], P.Odifreddi also defines the following concepts: NDTM and *Probabilistic Turing Machine (PrTM)*. He defines a NDTM as a machine that, towards an ambiguous configuration, indiscriminately chooses an instruction to be executed, and a PrTM as a machine that, towards an ambiguous configuration, chooses an instruction to be executed by following a probability distribution.

By focus in the NDTM, when a NDTM reaches an ambiguous configuration then its inherent theory (related to its FOL) can be contradictory, [6, Theorem 3]. So, for this reason, J.C. Agudelo and W.Carnielli proposes a model of computation called ParTM, [6, 7]. A ParTM is a NDTM such that:

- Contradictory instructions are permitted;

- Towards an ambiguous configuration the machine executes at once and parallelly all possible instructions, originating a multiplicity of states, of positions, and of symbols in certain cells of the tape;
- Instructions are executed in certain cells of the tape, and in the execution the instruction directs the present symbols in the cells in such way that they are not altered for the same cells in the subsequent instant of time;
- At the end of the computation, when there are no instructions to execute, every single cell of the tape can have multiple symbols[2], any choice of these symbols stands for an result of the computation .

Where the underlining logic is a LFI: $LFI1$, [24]. In $LFI1$, contradiction and inconsistency are characterized by means of the equivalence

$$\bullet\varphi \leftrightarrow (\varphi \wedge \neg\varphi),$$

where $\bullet$ stands for a inconsistency operator that can be related to $\circ\varphi$ by :

$$\bullet\varphi = \neg \circ \varphi \quad .$$

QUANTUM COMPUTATION AND PARACONSISTENT COMPUTATION As stated in [6, 7], there are two well known models of quantum computation: *Quantum Turing Machine (QTM)* and QCM. These two models are generalizations of the concept of TM and boolean circuits, respectively , by applying the laws of QM. By focusing in the concept of QTM, since QCM was already been characterized in, the generalization concerning QTM is done by replacing elements[3] of the classical TM for observables in a quantum system. The generalization concerning QTM is done by replacing elements of the classical TM for observables in a quantum system. This approach follows the postulates of QM, i.e., to a QTM is related a space state, the state of the QTM is provided by a vector state of the space state, and the evolution of QTM is described by a unitary operator. Certain restrictions are imposed to condition unitary in order to assure that the machine operates finitely, i.e. (cf. [6, 7, 30, 44]) :

- Just a single element of the system must be in motion for the period of time of each step;
- The motion should simply depend on the quantum state of a finite subsystem;
- The laws that specify the motion are required to be mathematical finite.

2 Instructions of a TM can be defined by quadruples of this symbols, cf. [6, 7].
3 such as current state,position and symbols on the tape

Remark 3.3.1 *Therefore, defining QTM lies fundamentally on the defining of a unitary operator with the above conditions.*

On the other hand, by comparing NDTMs and ParTMs. The difference between NDTMs and ParTMs is that in the scenario of multiple paths of computation the NDTMs will only "take profit" of one specific path of all the multiple paths, whereas ParTMs can "take profit" of all multiple paths of computation in only one computation, a certain computational parallelism, a *paraconsistent parallelism*. Also, in the same scenario, a QTM will behaviour by being simultaneously in an exponential number of configurations derived by the multiple paths and the number of computational steps, *quantum parallelism*. The simultaneously configurations of a QTM stand for superposition state of the machine (for a quantum superposition state). Furthermore, simultaneously configurations may possibly be *entangled* (for a quantum entangled state).

IT IS POSSIBLE TO SIMULATE QUANTUM COMPUTING VIA PARACONSISTENT TURING MACHINES? As asserted in [6, 7], it is possible to look at ParTM as a *uniform*[4] superposition of classical TM configurations. So, by this point of view ParTMs are like QTMs, i.e. it is possible to consider ParTMs as QTMs without amplitudes, which permits only to represent uniform superposition states. Still, in ParTMs, actions executed by distinct instructions mix indiscriminately, therefore every single combination of the singular elements in a ParTM is consider, which makes it unfeasible to represent (quantum) entangled states by merely contemplate the multiplicity of elements as a superposed state. One more difference between ParTMs and QTMs is that superposed states in the first ones do not provide a notion of relative phase (i.e., signs of basis states in uniform superpositions), an essential characteristic of quantum superposition required for quantum interference. Such a characteristic is the main mechanism for getting gain of quantum parallelism. However, ParTMs make profit from inconsistency conditions through paraconsistent parallelism, which appears to be a more powerful property than quantum interference (cf. [6]). Even so, ParTMs are able to simulate essential aspects of quantum computing; in specific, they are capable of simulate uniform non-entangled superposed quantum states and solve the Deustch and Deustch-Jozsa problems , while keeping the efficiency of quantum algorithms, although with some restrictions, [6, 7].

A MODEL OF PARACONSISTENT TURING MACHINE FOR QUANTUM ENTANGLEMENT Also, in [6], J.C. Agudelo and W.Carnielli defined another model of ParTMs , founded on a paraconsistent logic provided of a "non-separable" conjunction , which allows the simulation of uniform entangled states and stands for a improved approach for the model of QTMs:

4 A state of superposition is uniform if all the states in the superposition have non-null equal amplitude magnitude .

the *Entangled Paraconsistent Turing Machine (EParTM)* model. A EParTM is a NDTM such that:

- In an ambiguous configuration with n possible instructions of execution, the machine configuration will split up into n copies, executing a distinct instruction per copy;
- The set of the different configurations for a given instant of time t is called a *superposed configuration*;
- Only on the first two symbols corresponding to instructions are allowed inconsistency conditions;
- When there are no instructions to be executed, the machine will stop; and at this point the machine can be in a superposed configuration, with each configuration in superposition configuration representing a result of the computation.

Although, EParTMs are not able to express the notion of relative phase, the notion of relative phase can be incorporated in EParTMs, as demonstrated in [6]: such can be done by adding a sign indicating the relative phase of the configuration, and a new type of instructions to modify the relative phase.

3.4 A paraconsistent approach of PhLQP

This section is developed over two main points. The first one aims to define the notion of (in)consistent quantum state among the quantum space, as well as the concept of *Paraconsistent Phased Quantum Frame (PhQF°)*. The consistency of quantum state will be defined attending if it is a quantum state where superposition holds (i.e. Hadamard basis holds), and if it is a quantum state where multiple and distinct phases hold on a coefficient for the basis (i.e. if it is a normalized sum of exponentials that forms the respective coefficient). The second one intends to add paraconsistent features to PhLQP, resulting in a logic, the PhLQP°.

3.4.1 *The Quantum Space and Paraconsistent Quantum Frames*

Consider Subsection 2.2.1 and what follows in the sequel.

The set of states $\Sigma \times \Phi_\wedge$ Let $\Phi_\wedge$ standing for the Φ-subspace equipped with classical conjunction (i.e. the "$\wedge$" operator) in such way that is possible to built the definition of *phase vector*.

Definition 3.4.1 (Phase vector) *For a state $s \in \Sigma \times \Phi_\wedge$, if $\pi_{\Phi_\wedge}(s)$ has the following formula :*

$$\bigwedge_{i=1}^{n} \phi_i = \phi_1 \dots \phi_n = \vec{\phi}_n$$

with $n \in \mathbb{N}_{>0}$ and all ϕ_i not equal for $n > 1$, then $\pi_{\Phi_\wedge}(s)$ is a phase vector of length n , $\vec{\phi}_n$.

Definition 3.4.2 (The correspondent of a state with $\pi_{\Phi_\wedge}$-component as a phase vector) *Let $\vec{\phi}_n$ be a phase vector of length n, a state $s \in \Sigma \times \Phi_\wedge$ such that $s = (\sigma, \vec{\phi}_n)$, and a $\mathcal{B}$ space, then :*

$$s = (\sigma, \vec{\phi}_n) \mapsto \begin{cases} \frac{\sum_{i=1}^{n} e^{2\pi i(\phi_i)}}{\left|\sum_{i=1}^{n} e^{2\pi i(\phi_i)}\right|} \left|\sigma\right\rangle \in \mathcal{B} & \textit{if } \left|\sum_{i=1}^{n} e^{2\pi i(\phi_i)}\right| \neq 0 \\ \textit{undefined} \in \mathcal{B} & \textit{, otherwise} \end{cases}$$

Proposition 3.4.1 (A not-normalized quantum state) *If a state $s \in \Sigma \times \Phi_\wedge \mapsto$ undefined , then s stands for a non normalized quantum state .*

Example 3.4.1 (Some states of $\Sigma \times \Phi_\wedge$) *For the following set of states $\{(+, \frac{1}{2}\frac{1}{2}\frac{1}{4}), (0, \frac{1}{2}\frac{1}{4}), (0, \frac{1}{2}0)\}$ there is the following set of quantum states :*

$$\left\{ \frac{e^{2\pi i(\frac{1}{2})} + e^{2\pi i(\frac{1}{2})} + e^{2\pi i(\frac{1}{4})}}{\sqrt{5}} \left|1\right\rangle, \frac{e^{2\pi i(\frac{1}{2})} + e^{2\pi i(\frac{1}{4})}}{\sqrt{2}} \left|0\right\rangle, \textit{undefined} \right\}$$
$$= \left\{ \frac{i-2}{\sqrt{5}} \left|1\right\rangle, \frac{i-1}{\sqrt{2}} \left|0\right\rangle, \textit{undefined} \right\} \quad .$$

Remark 3.4.1 (Two special trigonometric cases of $\vec{\phi}_2$) *If $\vec{\phi}_2$ has the following form $\vec{\phi}_2 = \phi\phi^*$, then :*

$$\vec{\phi}_2 \mapsto \frac{\cos(2\pi.\phi)}{|\cos(2\pi.\phi)|} \quad \textit{with} \quad |\cos(2\pi.\phi)| \neq 0 \quad .$$

Also, if $\vec{\phi}_2$ has the following form $\vec{\phi}_2 = \phi((\phi^* + \frac{1}{2}) \mod 1)$, then :

$$\vec{\phi}_2 \mapsto i\frac{\sin(2\pi.\phi)}{|\sin(2\pi.\phi)|} \quad \textit{with} \quad |\sin(2\pi.\phi)| \neq 0 \quad .$$

Remark 3.4.2 (A special trigonometric case of $\vec{\phi}_4$) *If $\vec{\phi}_4$ has the following form $\vec{\phi}_4 = \phi\phi^*\phi((\phi^* + \frac{1}{2}) \mod 1)$, then :*

$$\vec{\phi}_4 \mapsto \cos(2\pi.\phi) + i\sin(2\pi.\phi) = e^{2\pi i.\phi} \quad .$$

THE ARITHMETIC OF THE PHASE VECTOR . First , some inherent notation to the phase vector . Let $\vec{\phi}_n$ be a phase vector of length n , $N = \{1, \dots, n\}$ a set of indices , then ${}^{(i)}\vec{\phi}_n$ refers to the i-th component of $\vec{\phi}_n$.

Example 3.4.2 (i-th component of $\vec{\phi}_n$) *Consider a state $s \in \Sigma \times \Phi_\wedge$ such that $s = (\sigma, \vec{\phi}_3) = (+, \frac{1}{2}\frac{1}{2}\frac{1}{4})$. So, $\pi_{\Phi_\wedge}(s) = \vec{\phi}_n = \vec{\phi}_3$ with , for example, ${}^{(1)}\pi_{\Phi_\wedge}(s) = {}^{(1)}\vec{\phi}_n = {}^{(1)}\vec{\phi}_3 = \frac{1}{2}$.*

Definition 3.4.3 (The sum operation) *For two phase vectors $\vec{\phi}_n, \vec{\phi'}_m$, the sum operation can be defined as :*

$$\vec{\phi}_n + \vec{\phi'}_m = \bigwedge_{j=1}^{m} \big(\bigwedge_{i=1}^{n} (\phi_i + \phi'_j) \big)$$

with

$$\vec{\phi}_n + \vec{\phi'}_m \mapsto \frac{\sum_{j=1}^{m} \big(\sum_{i=1}^{n} e^{((\phi_i + \phi'_j) \mod 1)} \big)}{\big| \sum_{j=1}^{m} \big(\sum_{i=1}^{n} e^{((\phi_i + \phi'_j) \mod 1)} \big) \big|} \quad .$$

Example 3.4.3 (Two cases of sum operation) *For a state $s = (\sigma, \vec{\phi}_3) = (+, \frac{1}{2}\frac{1}{2}\frac{1}{4})$, $\phi = \frac{1}{2}$, and $\vec{\phi}_2 = \frac{1}{3}\frac{1}{2}$. There is $(+, (\frac{1}{2}\frac{1}{2}\frac{1}{4}) + \frac{1}{2}) = (+, 00\frac{3}{4})$, and $(+, (\frac{1}{2}\frac{1}{2}\frac{1}{4}) + (\frac{1}{3}\frac{1}{2})) = (+, \frac{1}{3}\frac{1}{3}\frac{1}{4})$. Also :*

$$(+, 00\frac{3}{4}) \mapsto \frac{2-i}{\sqrt{5}} \left|+\right\rangle \quad ,$$

and

$$(+, (\frac{1}{3}\frac{1}{3}\frac{1}{4})) \mapsto \frac{(\frac{\sqrt{3}}{2} - 1) + (\frac{1}{2} + \sqrt{3})i}{5} \left|+\right\rangle \quad .$$

Definition 3.4.4 (The arithmetic of two states in superposition by a quantum join) *Let $s = (\sigma, \vec{\phi}_n)$ and $s' = (\sigma', \vec{\phi'}_m)$ be two states of $\Sigma \times \Phi_\wedge$, such that they are in a superposition expressed by their quantum join $s \sqcup s'$. There is :*

$$s \sqcup s' \mapsto \underbrace{\left(\frac{\sum_{i=1}^{n} e^{2\pi i (\phi_i)}}{\big| \sum_{i=1}^{n} e^{2\pi i (\phi_i)} + \sum_{i=1}^{m} e^{2\pi i (\phi'_i)} \big|} \right)}_{\alpha} \left|\sigma\right\rangle + \underbrace{\left(\frac{\sum_{i=1}^{m} e^{2\pi i (\phi'_i)}}{\big| \sum_{i=1}^{n} e^{2\pi i (\phi_i)} + \sum_{i=1}^{m} e^{2\pi i (\phi'_i)} \big|} \right)}_{\beta} \left|\sigma'\right\rangle \quad .$$

with

$$|\alpha|^2 + |\beta|^2 = 1 \quad .$$

Example 3.4.4 (A superposition of 2 states by a quantum join) *Consider two states $s, s' \in \Sigma \times \Phi_\wedge$ such that $s = (0, \frac{1}{8}\frac{7}{8})$ and $s' = (1, \frac{1}{8}((\frac{7}{8} + \frac{1}{2}) \mod 1))$. Then :*

$$s \sqcup s' \mapsto \cos(\pi/4) \left|0\right\rangle + i \sin(\pi/4) \left|1\right\rangle \quad .$$

Definition 3.4.5 (The arithmetic of n states in superposition by a quantum join) *It is possible to expand the Definition 3.4.4 for n states by writing :*

$$\bigsqcup_{i=1}^{n} s_i \mapsto \frac{1}{\big| \sum_{i=1}^{n} \big(\sum_{j=0}^{m} e^{2\pi i (^{(j)}\pi_{\Phi_\wedge}(s_i))} \big) \big|} \sum_{i=1}^{n} \big(\sum_{j=0}^{m} e^{2\pi i (^{(j)}\pi_{\Phi_\wedge}(s_i))} \big) \left|\pi_\Sigma(s_i)\right\rangle$$

with

$$\frac{\sum_{i=1}^{n} \big| \big(\sum_{j=0}^{m} e^{2\pi i (^{(j)}\pi_{\Phi_\wedge}(s_i))} \big) \big|^2}{\big| \sum_{i=1}^{n} \big(\sum_{j=0}^{m} e^{2\pi i (^{(j)}\pi_{\Phi_\wedge}(s_i))} \big) \big|^2} = 1 \quad .$$

Example 3.4.5 (A superposition of 3 states by a quantum join) *Now, consider three states $s, s', s'' \in \Sigma \times \Phi_\wedge$ such that $s = (100, 0), s' = (010, 0), s'' = (001, \frac{1}{4})$. Then :*

$$s \sqcup s' \sqcup s'' \mapsto \quad \frac{1}{|2+i|}|100\rangle + \frac{1}{|2+i|}|010\rangle + \frac{i}{|2+i|}|001\rangle$$
$$= \frac{1}{\sqrt{3}}|100\rangle + \frac{1}{\sqrt{3}}|010\rangle + \frac{i}{\sqrt{3}}|001\rangle \quad .$$

PARACONSISTENT QUANTUM FRAMES A PhQF° for single systems is a PhQF built over a set of states $\Sigma \times \Phi_\wedge$ (instead of a set of states $\Sigma \times \Phi$), i.e.:

$$\Sigma \times \Phi_\wedge(\mathcal{B}) := (\Sigma \times \Phi_\wedge, \{\xrightarrow{P?}\}_{P \in \mathcal{L}}, \{\xrightarrow{U}\}_{U \in \mathcal{U}})$$

for a given space $\mathcal{B}$. Where the Theorem 2.2.2 also holds, attending to the arithmetic of the phase vector. On the other hand, it is also possible to construct a PhQF° for compound systems, i.e.: $\Sigma \times \Phi_\wedge(\mathcal{B}_n)$.

3.4.2 *PhLQP as a Logic of Formal Inconsistency*

Now, remember the logic PhLQP (from Section 2.2), its syntax (Subsection 2.2.2), semantics (Subsection 2.2.3), and proof theory (Subsection 2.2.4). The notion of consistent quantum state will follow from axioms $[b]$ and $[k]$. Also, from $\sim := \Box\neg$, it is possible to write $[b]$ as $[b']$ and $[k]$ as $[k']$:

$[b']$ $$(\circ\varphi \wedge \sim \varphi \wedge \varphi) \rightarrow \psi,$$
$[k']$ $$\circ\varphi \vee (\sim \varphi \wedge \varphi).$$

Which follows from the definition of quantum join:

$[b']$ $$(\circ\varphi \wedge \sim (\varphi \sqcup \sim \varphi)) \rightarrow \psi,$$
$[k']$ $$\circ\varphi \vee \sim (\varphi \sqcup \sim \varphi).$$

Therefore, by consider the intuitive meaning of $\circ\varphi$ as the the " quantum state φ is consistent" , the axiom $[b']$ suggests that no quantum state is both a quantum state of superposition and consistent . On the other hand, axiom $[k']$ acts as complement of axiom $[b']$ by stating that every quantum state is either consistent or a quantum state of superposition.

Also, it is natural to generalize this intuition to any kind of quantum superposition expressed by an arbitrary quantum join $\varphi \sqcup \psi$ which implies the notion of a strong consistency operator $\circ$ for the quantum domain.

A PARACONSISTENT EXPANSION OF PHLQP LANGUANGE This extension of PhLQP will be designated by PhLQP$^{\circ}$ with an inherent language: *Language of Paraconsistent Phased Logic of Quantum Programs ($\mathcal{L}_{PhLQP^{\circ}}$).* The action of the operator $\circ$ can be described by the following bivalent interpretation:

$$\circ : \Sigma \times \Phi_{\wedge} \longrightarrow \{\top, \bot\} \quad .$$

Definition 3.4.6 (Consistent state) *A state $s \in \Sigma \times \Phi_{\wedge}$ is consistent if both $\pi_{\Sigma}(s)$ and $\pi_{\Phi_{\wedge}}(s)$ are consistent, i.e. :*

$$\circ s := \circ\pi_{\Sigma}(s) \wedge \circ\pi_{\Phi_{\wedge}}(s) \quad .$$

Where :

$$\circ\pi_{\Sigma}(s) := \begin{cases} \bot & \textit{if} \quad \exists \pi_{\Sigma}(s_i).\ \pi_{\Sigma}(s_i) \in \{+, -\} \\ \top & \textit{, otherwise} \end{cases}$$

with s_i as an i-th component of the state s . And

$$\circ\pi_{\Phi_{\wedge}}(s) := \begin{cases} \bot & \textit{if } \pi_{\Phi_{\wedge}}(s) \textit{ is a phase vector } \vec{\phi}_n \textit{ with } \ n > 1 \\ \top & \textit{, otherwise} \end{cases}$$

Also, if s has the form $s = \bigsqcup_{i=1}^{n} s_i$ (from Definition 3.4.5), then $\circ s = \bot$.

Example 3.4.6 (Applying the consistency operator to a set of states) *Consider the following set of states $\{(0, \frac{1}{2}0), (1, 0), (+, 0), (-, \frac{1}{2}0\frac{1}{4})\}$, then by applying the operator $\circ$ to each state of the set there is $\{\bot, \top, \bot, \bot\}$, respectively.*

Notice that in the above Example 3.4.6 the state $(0, \frac{1}{2}0) \mapsto$ undefined $\in B$ is inconsistent, since is a superposition normalized to zero under the proposed arithmetic.

Proposition 3.4.2 (Inconsistency of a non-normalized quantum state) *If a state $s \in \Sigma \times \Phi_{\wedge} \mapsto$ undefined , then s is inconsistent (i.e. $\circ s = \bot$).*

The constants *verum* $\top$ and *falsum* $\bot$ are (always) consistent. Hence, for the constants *verum* $\top$ and *falsum* $\bot$, there is $\circ\top = \circ\bot = \top$.

PARACONSISTENT SEMANTICS OF PHLQP$^{\circ}$. An PhLQP$^{\circ}$-model is an extension with paraconsistent features of an PhLQP-model (Subsection 2.2.3) , by consider a set of states

$\Sigma \times \Phi_\wedge$ and the follow-up interpretation of paraconsistent formulas. By extending the valuation $\|p\|$ to the following paraconsistent formulas:

$$\| \circ \varphi\| = \| \circ \varphi_\Sigma\| \cap \| \circ \varphi_{\Phi_\wedge}\|;$$
$$\| \circ \varphi\| = \circ\|\varphi\|;$$
$$\| \circ \varphi \wedge \circ\psi\| = \circ\|\varphi\| \cap \circ\|\psi\|;$$
$$\| \circ [\pi]\varphi\| = \circ[\|\pi\|]\|\varphi\|$$

PROOF THEORY FOR PhLQP$^\circ$ The proof theory for PhLQP$^\circ$ is an extension of the proof theory for PhLQP, i.e., the proof theory for PhLQP with the axioms that will be presented in the sequel, by consider Definition 3.2.15. The PhLQP$^\circ$ is endowed with a set of proper paraconsistent axioms, the FParQAxs:

Definition 3.4.7 (The Fundamental Paraconsistent Quantum Axioms of PhLQP$^\circ$) *The FParQAxs are:*

Consistency non-dependency from orthogonality (FParQAx1): $\vdash \circ\psi = \circ \sim \psi$.

Consistency non-dependency from adjointness (FParQAx2): $\vdash \circ\psi = \circ\psi^\dagger$.

Consistency's density (FParQAx3): $\vdash \circ\psi \leq \circ \circ \psi$.

The (in)consistency of superposition (FParQAx4): $\vdash \circ(\varphi \sqcup \psi) \leq (\circ\varphi \sqcup \circ\psi)$.

Non-locality of consistency (FParQAx5): $\vdash (\circ\varphi \wedge \circ\psi) = \circ(\varphi \wedge \psi)$.

By take into account the above axioms, Definition 3.4.7 :

- The axiom $\vdash \circ\psi = \circ \sim \psi$ is sustained by the fact that the orthocomplement of a (quantum) state holds the same (in)consistency of that same (quantum) state, i.e., a consequence from $(0, \phi) =\sim (1, \phi)$ and $(-, \phi) =\sim (+, \phi)$.
- The axiom $\vdash \circ\psi = \circ\psi^\dagger$ expresses the reality that a phase vector of length $n > 1$ can never be induced by adjointness.
- The axiom $\vdash \circ\psi \leq \circ \circ \psi$ stands for the always consistent nature of the bivalent interpretation of the consistency operator. In other words, for a formula $\psi, \circ\psi \in \{\bot, \top\}$. Now, remember that $\circ\bot = \circ\top = \top$. So, $\circ \circ \psi = \top$ always holds for any formula ψ with :
$$\vdash \circ\psi \leq \top \quad .$$

- The axiom $\vdash \circ(\varphi \sqcup \psi) \leq (\circ\varphi \sqcup \circ\psi)$ asserts that any superposition of states is always inconsistent, even if is a superposition where all superposed states are consistent by themselves. By Definition 3.4.6, $\circ(\varphi \sqcup \psi) = \bot$ is always verified. So:

$$\vdash \bot \leq (\circ\varphi \sqcup \circ\psi) \quad ,$$

which follows :

$$\vdash \bot \leq \sim (\sim \circ\varphi \wedge \sim \circ\psi) \quad .$$

For instance, even if $\circ\psi = \circ\varphi = \top$, the axiom holds :

$$\vdash \bot \leq \top.$$

- The axiom says $\vdash (\circ\varphi \wedge \circ\psi) = \circ(\varphi \wedge \psi)$ that for a (quantum) compound system is enough to one of its components to be inconsistent to the entirely (quantum) compound system be inconsistent by itself, e.g., Example 3.4.7.

Example 3.4.7 (A depicting example for FParQAx5 axiom) *Let $s \in \Sigma \times \Phi_\wedge$ be a state where is no entanglement in a workspace corresponding to a space $\mathcal{B}_2 = B^{(1)} \otimes B^{(2)}$, i.e., s is a $\{1,2\}$-qubits state with no entanglement. So:*

$$\circ s_1 \wedge \circ s_2 = \circ s \quad .$$

Notice that $s = s_1 \wedge s_2$ and :

$$\circ s_1 \wedge \circ s_2 = \circ(s_1 \wedge s_2) \quad .$$

So far (as in the Example 3.4.7) s is a quantum state with no entanglement. And what happens if s is a state where entanglement occurs ?

THE INCONSISTENCY OF QUANTUM ENTANGLED STATES A quantum entangled state can be seen as a particular kind of superposition state, and somehow expressed by using or through a quantum join, e.g. :

$$R_{2_j}({}^0\beta_{01}^{i,j}) = (10,0)_{ij} \sqcup (01, \frac{1}{4})_{ij} \mapsto \frac{1}{\sqrt{2}}(|10\rangle_{ij} + i\,|01\rangle_{ij}) \quad .$$

Remark 3.4.3 *Therefore, by Definition 3.4.6 all quantum entanglement states are inconsistent.*

The **FParQAx5** axiom for an entanglement holds as follows :

$$\vdash \circ(s_1 \wedge s_2) = \bot \quad .$$

Where the above connective $\wedge$ must be interpreted as a non-separable one, i.e., as in [6] for EParTMs .

Theorem 3.4.1 (Soundness of FParQAxs) *The FParQAxs are sound.*

Proof 3.4.1 (Theorem 3.4.1) *Let $w \in \Sigma \times \Phi_\wedge$ be a state.*

1. ***Soundness of FParQAx1:***

$$\begin{aligned}\|\circ\psi = \circ \sim \psi\| &= \|\Box\Box\circ\psi \leftrightarrow \Box\Box\circ \sim \psi\| \\ &= \|\Box\Box\circ\psi \to \Box\Box\circ \sim \psi\| \ \cap \ \|\Box\Box\circ \sim \psi \to \Box\Box\circ\psi\| \\ &= \|\neg(\Box\Box\circ\psi \wedge \neg\Box\Box\circ \sim \psi)\| \ \cap \ \|\neg(\Box\Box\circ \sim \psi \wedge \neg\Box\Box\circ\psi)\|\end{aligned}$$

- *For the term $(\Box\Box\circ\psi \wedge \neg\Box\Box\circ \sim \psi)$:*

$$\begin{aligned}\|\Box\Box\circ\psi \wedge \neg\Box\Box\circ \sim \psi\| &= \|\Box\Box\circ\psi\| \ \cap \ \|\neg\Box\Box\circ \sim \psi\| \\ &= \Box\Box\circ\|\psi\| \ \cap \ \Sigma \setminus \|\Box\Box\circ \sim \psi_\Sigma\| \times \|\Box\Box\circ \sim \psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\circ\| \sim \psi_\Sigma\| \times \Box\Box\circ\|\psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma, \bot_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}, \bot_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

- *For the term $(\Box\Box\circ \sim \psi \wedge \neg\Box\Box\circ\psi)$:*

$$\begin{aligned}\|\Box\Box\circ \sim \psi \wedge \neg\Box\Box\circ\psi\| &= \|\Box\Box\circ \sim \psi\| \ \cap \ \|\neg\Box\Box\circ\psi\| \\ &= \Box\Box\circ\| \sim \psi\| \ \cap \ \Sigma \setminus \|\Box\Box\circ\psi_\Sigma\| \times \|\Box\Box\circ\psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma, \bot_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}, \bot_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\|\circ\psi = \circ \sim \psi\| &= \Sigma \setminus \emptyset \times \Phi_\wedge \ \cap \ \Sigma \setminus \emptyset \times \Phi_\wedge \\ &= \Sigma \times \Phi_\wedge \quad .\end{aligned}$$

2. ***Soundness of FParQAx2:***

$$\begin{aligned}\|\circ\psi = \circ\psi^\dagger\| &= \|\Box\Box\circ\psi \leftrightarrow \Box\Box\circ\psi^\dagger\| \\ &= \|\Box\Box\circ\psi \to \Box\Box\circ\psi^\dagger\| \ \cap \ \|\Box\Box\circ\psi^\dagger \to \Box\Box\circ\psi\| \\ &= \|\neg(\Box\Box\circ\psi \wedge \neg\Box\Box\circ\psi^\dagger)\| \ \cap \ \|\neg(\Box\Box\circ\psi^\dagger \wedge \neg\Box\Box\circ\psi)\|\end{aligned}$$

- *For the term* $(\Box\Box \circ \psi \wedge \neg \Box\Box \circ \psi^\dagger)$*:*

$$\begin{aligned}\|\Box\Box \circ \psi \wedge \neg\Box\Box \circ \psi^\dagger\| &= \|\Box\Box \circ \psi\| \;\cap\; \|\neg\Box\Box \circ \psi^\dagger\| \\ &= \Box\Box \circ \|\psi\| \;\cap\; \Sigma \setminus \|\Box\Box \circ \psi^\dagger_\Sigma\| \times \|\Box\Box \circ \psi^\dagger_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box \circ \|\psi^\dagger_\Sigma\| \times \Box\Box \circ \|\psi^\dagger_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma, \bot_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}, \bot_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

- *For the term* $(\Box\Box \circ \psi^\dagger \wedge \neg \Box\Box \circ \psi)$ *:*

$$\begin{aligned}\|\Box\Box \circ \psi^\dagger \wedge \neg\Box\Box \circ \psi\| &= \|\Box\Box \circ \psi^\dagger\| \;\cap\; \|\neg\Box\Box \circ \psi\| \\ &= \Box\Box \circ \|\psi^\dagger\| \;\cap\; \Sigma \setminus \|\Box\Box \circ \psi_\Sigma\| \times \|\Box\Box \circ \psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box \circ \|\psi_\Sigma\| \times \Box\Box \circ \|\psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma, \bot_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}, \bot_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\| \circ \psi = \circ \psi^\dagger\| &= \Sigma \setminus \emptyset \times \Phi_\wedge \;\cap\; \Sigma \setminus \emptyset \times \Phi_\wedge \\ &= \Sigma \times \Phi_\wedge \quad .\end{aligned}$$

3. ***Soundness of FParQAx3:***

$$\begin{aligned}\| \circ \psi \leq \circ \circ \psi\| &= \|\Box\Box \circ \psi \rightarrow \Box\Box \circ \circ \psi\| \\ &= \|\neg(\Box\Box \circ \psi \wedge \neg\Box\Box \circ \circ \psi)\|\end{aligned}$$

- *For the term* $(\Box\Box \circ \psi \wedge \neg \Box\Box \circ \circ \psi)$

$$\begin{aligned}\|\Box\Box \circ \psi \wedge \neg\Box\Box \circ \circ\psi\| &= \|\Box\Box \circ \psi\| \;\cap\; \|\neg\Box\Box \circ \circ\psi\| \\ &= \Box\Box \circ \|\psi\| \;\cap\; \Sigma \setminus \|\Box\Box \circ \circ\psi_\Sigma\| \times \|\Box\Box \circ \circ\psi_{\Phi_\wedge}\| \\ &= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

So,

$$\begin{aligned}\| \circ \psi \leq \circ \circ \psi\| &= \Sigma \setminus \emptyset \times \Phi_\wedge \\ &= \Sigma \times \Phi_\wedge \quad .\end{aligned}$$

4. ***Soundness of FParQAx4:***

$$\begin{aligned}\| \circ (\varphi \sqcup \psi) \leq (\circ\varphi \sqcup \circ\psi)\| &= \|\Box\Box \circ (\varphi \sqcup \psi) \to \Box\Box(\circ\varphi \sqcup \circ\psi)\| \\ &= \|\neg(\Box\Box \circ (\varphi \sqcup \psi) \wedge \neg\Box\Box(\circ\varphi \sqcup \circ\psi))\|\end{aligned}$$

- *For the term* $\Box\Box \circ (\varphi \sqcup \psi)$ *:*

$$\begin{aligned}\|\Box\Box \circ (\varphi \sqcup \psi)\| &= \Box\Box\| \circ (\varphi \sqcup \psi)\| \\ &= \Box\Box \circ \|(\varphi \sqcup \psi)\| \\ &= \Box\Box\{\bot\} \quad .\end{aligned}$$

- *For the term* $\neg\Box\Box(\circ\varphi \sqcup \circ\psi)$*:*

$$\begin{aligned}\|\neg\Box\Box(\circ\varphi \sqcup \circ\psi)\| &= \|\neg\Box\Box \sim (\sim \circ\varphi \wedge \sim \circ\psi)\| \\ &= \Sigma \setminus \|\Box\Box \sim (\sim \circ\varphi_\Sigma \wedge \sim \circ\psi_\Sigma)\| \times \|\Box\Box \sim (\sim \circ\varphi_\Phi \wedge \sim \circ\psi_\Phi)\| \\ &= \Sigma \setminus \Box\Box\{\bot_\Sigma, \top_\Sigma\} \times \Box\Box\{\bot_{\Phi_\wedge}, \top_{\Phi_\wedge}\} \quad .\end{aligned}$$

- *For the term* $(\Box\Box \circ (\varphi \sqcup \psi) \wedge \neg\Box\Box(\circ\varphi \sqcup \circ\psi))$*:*

$$\begin{aligned}\|(\Box\Box \circ (\varphi \sqcup \psi) \wedge \neg\Box\Box(\circ\varphi \sqcup \circ\psi))\| &= \|\Box\Box \circ (\varphi \sqcup \psi)\| \ \cap \ \|\neg\Box\Box(\circ\varphi \sqcup \circ\psi)\| \\ &= \Box\Box\{\bot\} \ \cap \ \Sigma \setminus \Box\Box\{\bot_\Sigma, \top_\Sigma\} \times \Box\Box\{\bot_{\Phi_\wedge}, \top_{\Phi_\wedge}\} \\ &= \emptyset \quad .\end{aligned}$$

Therefore,

$$\begin{aligned}\| \circ (\varphi \sqcup \psi) \leq (\circ\varphi \sqcup \circ\psi)\| &= \Sigma \setminus \emptyset \times \Phi_\wedge \\ &= \Sigma \times \Phi_\wedge \quad .\end{aligned}$$

5. ***Soundness of FParQAx5***

$$\begin{aligned}\|(\circ\varphi \wedge \circ\psi) = \circ(\varphi \wedge \psi)\| &= \|\Box\Box(\circ\varphi \wedge \circ\psi) \leftrightarrow \Box\Box \circ (\varphi \wedge \psi)\| \\ &= \|\Box\Box(\circ\varphi \wedge \circ\psi) \to \Box\Box \circ (\varphi \wedge \psi)\| \\ &\cap \ \|\Box\Box \circ (\varphi \wedge \psi) \to \Box\Box(\circ\varphi \wedge \circ\psi)\| \\ &= \|\neg(\Box\Box(\circ\varphi \wedge \circ\psi) \wedge \neg\Box\Box \circ (\varphi \wedge \psi))\| \\ &\cap \ \|\neg(\Box\Box \circ (\varphi \wedge \psi) \wedge \neg\Box\Box(\circ\varphi \wedge \circ\psi))\|\end{aligned}$$

- *For the term* $(\Box\Box(\circ\varphi \wedge \circ\psi) \wedge \neg\Box\Box \circ (\varphi \wedge \psi))$*:*

$$\begin{aligned}
\|\Box\Box(\circ\varphi \wedge \circ\psi) \wedge \neg\Box\Box \circ (\varphi \wedge \psi)\| &= \Box\Box\|(\circ\varphi \wedge \circ\psi)\| \\
&\cap\ \Sigma \setminus \Box\Box \circ \|(\varphi_\Sigma \wedge \psi_\Sigma)\| \times \Box\Box\|(\varphi_{\Phi_\wedge} \wedge \psi_{\Phi_\wedge})\| \\
&= \Box\Box(\circ\|\varphi\| \ \cap\ \circ\|\psi\|) \\
&\cap\ \Sigma \setminus \Box\Box\{\bot_\Sigma, \top_\Sigma\} \times \Box\Box\{\bot_{\Phi_\wedge}, \top_{\Phi_\wedge}\} \\
&= \Box\Box(\{\top, \bot\} \ \cap\ \{\top, \bot\}) \\
&\cap\ \Sigma \setminus \Box\Box\{\bot_\Sigma, \top_\Sigma\} \times \Box\Box\{\bot_{\Phi_\wedge}, \top_{\Phi_\wedge}\} \\
&= \emptyset \quad .
\end{aligned}$$

- *For the term* $(\Box\Box \circ (\varphi \wedge \psi) \wedge \neg\Box\Box(\circ\varphi \wedge \circ\psi))$*:*

$$\begin{aligned}
\|\Box\Box \circ (\varphi \wedge \psi) \wedge \neg\Box\Box(\circ\varphi \wedge \circ\psi)\| &= \Box\Box \circ \|(\varphi \wedge \psi)\| \\
&\cap\ \Sigma \setminus \Box\Box\|(\circ\varphi_\Sigma \wedge \circ\psi_\Sigma)\| \times \Box\Box\|(\circ\varphi_{\Phi_\wedge} \wedge \circ\psi_{\Phi_\wedge})\| \\
&= \Box\Box\{\top, \bot\} \ \cap\ \Sigma \setminus \Box\Box(\circ\|\varphi_\Sigma\| \ \cap\ \circ\|\psi_\Sigma\|) \\
&\times \Box\Box(\circ\|\varphi_{\Phi_\wedge}\| \ \cap\ \circ\|\psi_{\Phi_\wedge}\|) \\
&= \Box\Box\{\top, \bot\} \cap \Sigma \setminus \Box\Box\{\top_\Sigma, \bot_\Sigma\} \times \Box\Box\{\top_{\Phi_\wedge}, \bot_{\Phi_\wedge}\} \\
&= \emptyset \quad .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|(\circ\varphi \wedge \circ\psi) = \circ(\varphi \wedge \psi)\| &= \Sigma \setminus \emptyset \times \Phi_\wedge \ \cap\ \Sigma \setminus \emptyset \times \Phi_\wedge \\
&= \Sigma \times \Phi_\wedge \quad .
\end{aligned}$$

3.5 EXAMPLES OF INCONSISTENCY IN QUANTUM COMPUTING

3.5.1 *A Paraconsistent View on the Hadamard Gate*

The simplest example is the Hadamard gate, since it induces a state with a computational standard basis into a superposition state, or vice-versa. In other words, it induces inconsistency or breaks it . Therefore, consider a Hadamard gate with an input state s and an output state s' , Figure 16 .

$$|s\rangle \ —\boxed{H}— \ |s'\rangle$$

Figure 16: A Hadamard-gate with an input state s and an output state s'.

.

Attending to the behaviour of the Hadamard gate , it is noticeable that the paraconsistent validity is

$$\vdash \circ\pi_\Sigma(s) =\sim \circ\pi_\Sigma(s')$$

or equivalently

$$\vdash \circ\pi_\Sigma(s) =\sim \ \circ\pi_\Sigma(H(s))$$

expresses the correctness of the Hadamard gate . And by the characteristic dynamic axioms of the Hadamard gate in Subsection 2.2.4 , the above validity always holds .

Characteristic axiom for the paraconsistent behaviour of the H_i-gate:

$$\vdash \circ s =_i\sim \circ H_i(s).$$

3.5.2 *The Deustch Gate*

By following [46]. The Deutsch Gate , D_θ , is a three-qubit gate and is defined by :

$$\alpha\beta\,|q_0,q_1,q_2\rangle \mapsto \begin{cases} \alpha\beta i\cos(\theta)\,|q_0,q_1,q_2\rangle + \alpha\beta\sin(\theta)\,|q_0,q_1,1-q_2\rangle & \text{for } q_0 = q_1 = 1 \\ \alpha\,|q_0,q_1,q_2\rangle & \text{otherwise.} \end{cases}$$

with $q_0, q_1, q_2 \in \{0,1\}$,

$$\alpha = \frac{\sum_{i=1}^{n} e^{2\pi i(\phi_i)}}{|\sum_{i=1}^{n} e^{2\pi i(\phi_i)}|} \quad ,$$

$$\beta = \frac{\sum_{j=1}^{m} e^{2\pi i(\phi'_j)}}{|\sum_{j=1}^{m} e^{2\pi i(\phi'_j)}|} \quad ,$$

and $n, m \in \mathbb{N}_{n>0}$.

Attending to the behaviour of the D_θ it is perceptible that no matter what states are the qubits with $|q_0\rangle$, $|q_1\rangle$ as quantum basis , their states maintain the same, this is their basis as their phases are invariable. These qubits act as control-qubits. In fact, the TOFF-gate is a specific case of the D_θ-gate, i.e., $D_{\frac{\pi}{2}}$. The coefficient α is the coefficient associated to the phase vector $\vec{\phi}_n$ of the quantum basis $|q0, q1\rangle$, and β is the coefficient associated to the phase vector $\vec{\phi'}_m$ of the quantum basis $|q2\rangle$. Therefore , it is possible to write :

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \alpha\,|q_0,q_1\rangle \otimes \beta(i\cos(\theta)\,|q_2\rangle + \sin(\theta)\,|1-q_2\rangle)$$

for $q_0 = q_1 = 1$ with $q_2 \in \{0,1\}$. For $q_0 = 0 \vee q_1 = 0$, the D_θ act as a three-qubit identity gate . Now consider $q_0, q_1, q_2 \in \{0,1,+,-\}$. When $q_0, q_1 \in \{+,-\}$ with $q_2 \in \{0,1\}$, there is:

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \frac{1}{\sqrt{2}}\big(\alpha\,|q_0,q_1\rangle \otimes \beta(i\cos(\theta)\,|q_2\rangle + \sin(\theta)\,|1-q_2\rangle)\big) + \frac{\alpha\beta}{\sqrt{2}}\,|q_0,q_1,q_2\rangle \ .$$

With the conditions of $q_0 = q_1 = 1$ and $q_2 = +$:

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \alpha\,|q_0,q_1\rangle \otimes \frac{\beta}{\sqrt{2}}\big((i\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle) + (i\cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle)\big)\ .$$

Similar with $q_0 = q_1 = 1$ and $q_2 = -$, there is:

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \alpha\,|q_0,q_1\rangle \otimes \frac{\beta}{\sqrt{2}}\big((i\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle) - (i\cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle)\big)\ .$$

When $q_0, q_1 \in \{+,-\}$ and $q_2 = +$:

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \frac{\alpha}{\sqrt{2}}\,|q_0,q_1\rangle \otimes \frac{\beta}{\sqrt{2}}\big((i\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle) + (i\cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle)\big) + \frac{\alpha\beta}{\sqrt{2}}\,|q_0,q_1,q_2\rangle\ .$$

And at last , for $q_0, q_1 \in \{+,-\}$ and $q_2 = -$:

$$D_\theta(\alpha\beta\,|q_0,q_1,q_2\rangle) \mapsto \frac{\alpha}{\sqrt{2}}\,|q_0,q_1\rangle \otimes \frac{\beta}{\sqrt{2}}\big((i\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle) - (i\cos(\theta)\,|1\rangle + \sin(\theta)\,|0\rangle)\big) + \frac{\alpha\beta}{\sqrt{2}}\,|q_0,q_1,q_2\rangle\ .$$

AXIOMS FOR THE DYNAMIC BEHAVIOUR OF THE DEUTSCH GATE Let D_θ be denoted as $D_{2\pi.\phi_D}$ with $\theta = 2\pi.\phi_D$.

Locality for the Deutsch Gate . The Deutsch Gate acts only in the specified qubits :

$$\vdash \{i,j,k\}(D_{2\pi.\phi_{ijk}})\ .$$

Characteristic axioms for the dynamic behaviour of the $D_{2\pi.\phi_{D_{ijk}}}$-gate

$$\vdash (q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (q_2, \vec{\phi'}_m)_k \to [D_{2\pi.\phi_{D_{ijk}}}](q_2, \vec{\phi'}_m)_k \quad \text{for} \quad q_0 = 0 \vee q_1 = 0 \text{ and } q_2 \in \{0, 1, +, -\} ;$$

$$\vdash (11, \phi')_{ij} \wedge \big(\lozenge(q_2, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \wedge \lozenge(1 - q_2, \phi((\phi^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k\big) \to [D_{2\pi.\phi_{ijk}}](q_2, \vec{\phi'}_m)_k \quad \text{with} \quad q_2 \in \{0, 1\};$$

$$\vdash \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (q_2, \vec{\phi'}_m)_k)\big) \wedge \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (\lozenge(q_2, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(1 - q_2, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k))\big) \to [D_{2\pi.\phi_{ijk}}](q_2, \vec{\phi'}_m)_k \quad \text{with} \quad q_0, q_1 \in \{+, -\} \quad \text{and} \quad q_2 \in \{0, 1\} ;$$

$$\vdash (11, \phi')_{ij} \ \wedge \ (\lozenge(\lozenge(0, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(1, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k)) \wedge \lozenge(\lozenge(1, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(0, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k))) \to [D_{2\pi.\phi_{ijk}}](+, \vec{\phi'}_m)_k ;$$

$$\vdash (11, \phi')_{ij} \wedge (\lozenge(\lozenge(0, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(1, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k)) \wedge \lozenge(\lozenge(1, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4} + \frac{1}{2})_k \ \wedge \ \lozenge(0, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4} + \frac{1}{2})_k)))\big) \to [D_{2\pi.\phi_{ijk}}](-, \vec{\phi'}_m)_k;$$

$$\vdash \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (+, \vec{\phi'}_m)_k)\big) \wedge \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (\lozenge(\lozenge(0, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(1, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k)) \wedge \lozenge(\lozenge(1, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(0, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k)))\big) \to [D_{2\pi.\phi_{ijk}}](+, \vec{\phi'}_m)_k \quad \text{with} \quad q_0, q_1 \in \{+, -\};$$

$$\vdash \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (+, \vec{\phi'}_m)_k)\big) \wedge \lozenge\big((q_{0_i}q_{1_j}, \vec{\phi}_n)_{ij} \wedge (\lozenge(\lozenge(0, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4})_k \ \wedge \ \lozenge(1, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4})_k)) \wedge \lozenge(\lozenge(1, \vec{\phi'}_m + \phi_D\phi_D^* + \frac{1}{4} + \frac{1}{2})_k \ \wedge \ \lozenge(0, \vec{\phi'}_m + \phi_D(\phi_D^* + \frac{1}{2}) \mod 1) + \frac{3}{4} + \frac{1}{2})_k)))\big) \to [D_{2\pi.\phi_{ijk}}](-, \vec{\phi'}_m)_k \quad \text{with} \quad q_0, q_1 \in \{+, -\}.$$

THE PARACONSISTENCY OF THE DEUTSCH GATE Let $|q\rangle = \alpha\beta\,|q_0q_1q_2\rangle$ standing for the input state of the D_θ-gate and $q' = D_\theta(q)$ for its output state . As said before, when $q_1 = 0 \vee q_2 = 0$ the D_θ acts as identity-gate. So, the consistency of the output state is the same as the input state since the input state and the output state are the same state. On the other hand, if $q_0, q_1 \in \{1, +, -\}$, the output state D_θ will be always inconsistent by the Definition 3.4.6 . This is, D_θ-gate always induce a phase vector of length two on the $\pi_{\Phi_\wedge}(q)$-component of the input state q.

Characteristic axioms for the paraconsistent behaviour of the $D_{\theta_{ijk}}$-gate

$$\vdash \big(\pi_\Sigma(q) =_i 0 \vee \pi_\Sigma(q) =_j 0\big) \to \big(\circ D_{\theta_{ijk}}(q) = \circ q\big) \wedge \big(D_{\theta_{ijk}}(q) = id(q)\big) \quad ;$$

$$\vdash \big(\pi_\Sigma(q) \neq_i 0 \wedge \pi_\Sigma(q) \neq_j 0\big) \to \big(\circ D_{\theta_{ijk}}(q) = \bot\big) \quad .$$

A CONCRETE SCENARIO OF THE DEUTSCH GATE BEHAVIOUR Now , consider the Figure 17 as the possible QCM illustration for the D_θ-gate.

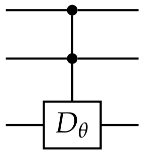

Figure 17: A possible quantum circuit illustration for the D_θ-gate.

And the quantum program depicted in Figure 18 . For the state $\left|q'_k\right\rangle$ there is :

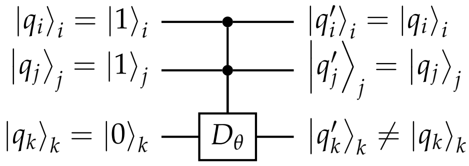

Figure 18: A simple quantum program with the D_θ-gate.

$$\left|q'_k\right\rangle = i\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle = i\cos(2\pi.\phi_D)\,|0\rangle + \sin(2\pi.\phi_D)\,|1\rangle\,.$$

By the PhLQP$^\circ$ syntax, the above quantum program can be expressed by:

$$\pi_D = D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) \quad .$$

With the following validity asserting the correctness of the program:

$$\vdash \underbrace{\left(\pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) =_{ij} id(q_{i_i} \wedge q_{j_j})\right)}_{1^{st}\ condition} \wedge \underbrace{\left(\pi_D; \pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = id(q_{i_i} \wedge q_{j_j} \wedge q_{k_k})\right)}_{2^{nd}\ condition}$$
$$\wedge \underbrace{\left(\circ\, \pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = \bot\right)}_{3^{rd}\ condition} \quad ,$$

for $q_i = q_j = (1,0)$ and $q_k = (0,0)$.

For the 1^{st} condition, there is :

$$\pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) =_{ij} D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j}) \quad .$$

Notice that $\{i, j\}$-qubits act as control qubits, and so their states q_i and q_j are invariant to the program, i.e.:

$$D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j}) =_{ij} q_{i_i} \wedge q_{j_j} = id(q_{i_i} \wedge q_{j_j}) \quad .$$

In concern to the 2^{nd} condition :

$$\pi_D; \pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = D_{2\pi.\phi_{ijk}}; D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) \quad .$$

And due to the inherent property of quantum gates:

$$D_{2\pi.\phi_{ijk}}; D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = D^{-1}_{2\pi.\phi_{ijk}}; D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = id(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) \quad .$$

At last, for the 3^{rd} condition :

$$\circ\pi_D(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = \circ D_{2\pi.\phi_{ijk}}(q_{i_i} \wedge q_{j_j} \wedge q_{k_k}) = \perp \quad .$$

Since $q_i = q_j = (1, 0)$.

4

CONCLUSIONS AND PROSPECT FOR FUTURE WORK

4.1 CONCLUSIONS

The aim of this dissertation was to design a variant of a dynamic logic with paraconsistent features to reason about quantum programs. With the dynamic nature of the logic the intention is to provide a basis for verifying quantum programs correctness. On the other hand, with paraconsistent features it is possible to treat non consistent information as potentially informative, e.g., quantum superposition states.

The work in this dissertation was started by the study of the concept of a dynamic logic applied to the quantum domain. It was possible to conclude that to characterize a quantum state by the transitions specified by a quantum program is a powerful tool to analyse the correctness of such program. However, in the studied dynamic logics [11, 14, 16, 17, 18], there was a gap in what concerns the expression of quantum phase related proprieties. In particular, it was seen that the interpretation of a quantum state as a "ray" in a Hilbert space seems too limited, and that a dynamic logic based on such an interpretation of quantum states is very unlikely to be able to reason about such proprieties. Therefore, with this point of view the first step to design the desired logic is to consider a new interpretation of quantum states and from there shape such a logic. So, a new interpretation of quantum state was proposed. Such unorthodox interpretation relies on looking at a quantum state in a more "vectorized" way and represents such state by a pair (σ, ϕ) in $\Sigma \times \Phi$. Where Σ is the set $\{0, 1, +, -\}$ that stands for the standard computational basis $\{|0\rangle, |1\rangle\}$ and for the Hadamard basis $\{|+\rangle, |-\rangle\}$, and Σ is the subspace of the quantum phases.

Later, upon this new interpretation to a quantum state there was the necessity of limit the subspace of the quantum phases, Φ, by defining the elements ϕ of Φ as follows : $\phi < 1, \phi \in \mathbb{Q}$. Where the restriction $\phi < 1$ is due to the fact of $e^{2\pi i} = e^0$ and $e^{2\pi n i} = e^{2\pi i}$, $n \in \mathbb{N}$. A modular arithmetic for operating over the elements of Φ, concerning the periodicity property of Euler's formulas , and so the periodicity of quantum phases, was used.

In this new line of thought of how to model a quantum state, it is proposed re-design of the (LQP) ([14]) designated by PhLQP, since LQP already provides a good foundation for verifying properties of quantum programs that not deal with quantum phase proprieties.

Actually, the difference between PhLQP and LQP as dynamic logics relies on the fact that PhLQP is able to provide a foundation for quantum programs correctness dealing with quantum phase related proprieties, which is not possible in LQP. The design of PhLQP was only feasible due to a special kind of QF proposed in this dissertation, the PhQF, which is a $\mathcal{B}$ space structured as non-classical relational models of PDL. Nevertheless, PhLQP is not endowed with probabilistic predication formulas, and therefore cannot address properties on expressing probabilistic coefficients or the probability of a certain quantum state occurring. Moreover, there is another inherent limitation to PhLQP: there is an impossibility of represent states with partial quantum phases. Thus, there is a need of improvement for this logic in expressing quantum states.

In the second part of this dissertation, a study of paraconsistent logics was made, [6, 7, 8, 21, 22]. This study showed that was profitable to bring paraconsistent features to PhLQP, since this brings an additional way of characterizing quantum states where superposition occurs, i.e. characterizing such states through their inconsistency. Consequently, with this in mind, it was expanded the notion of expressing quantum states through the notion of *phase vector* in a redefined subspace Φ, $\Phi_\wedge$. As a result, there is the set of states $\Sigma \times \Phi_\wedge$. Such set of states is gifted of proper arithmetic, the arithmetic of phase vector, which allows to express the vast majority of normalized quantum states, even if the quantum state is a superposition with trigonometric coefficients . Additionally, in the set $\Sigma \times \Phi_\wedge$ inconsistency at a quantum phase domain occurs, i.e. a quantum state can have two or more phases associated to the same basis.

A paraconsistent version of PhLQP was designed, the PhLQP$^\circ$. This logic is built over a PhQF$^\circ$, which is a PhQF with a set of states $\Sigma \times \Phi_\wedge$, instead of a set of states $\Sigma \times \Phi$. The language of PhLQP$^\circ$, $\mathcal{L}_{\text{PhLQP}^\circ}$, is an extension of the language of PhLQP$^\circ$, $\mathcal{L}_{\text{PhLQP}}$, adding the connective $\circ$ that stands for the operator of consistency to $\mathcal{L}_{\text{PhLQP}}$. Also, the proof theory for PhLQP$^\circ$ is an extension of the proof theory for PhLQP by adding to the latter a set of paraconsistent axioms concerning the quantum domain, the FParQAxs. Clearly PhLQP$^\circ$ is more expressive than PhLQP, since, for example , PhLQP$^\circ$ can express and proof correctness of the Deutsch gate , in contrast to PhLQP .

At last , the existence of PhLQP$^\circ$ is possible not only by combining dynamic logic with paraconsistent logic, but also by building this combination over the above mentioned interpretation of a quantum state .

4.2 PROSPECT FOR FUTURE WORK

In near future there is the intention to explore the following research lines.

1. The design of a probabilistic version of PhLQP$^\circ$.

By endowing PhLQP$^\circ$ with probabilistic formulas it will be possible to address the verification of correctness for a quantum program that relies on expressing probabilistic coefficients or a probability of a certain quantum state occurring, e.g. the correctness of the QLE protocol for all scenarios .

2. A comparative study between the traditional representation of a quantum state as a ″ray″ in a Hilbert Space and the representation of a quantum state proposed here .

   An exhaustive study of the two ways of representing a quantum state will allow to have a better idea of pros and cons, and if there is a possibility to redesign PhLQP$^\circ$ under the traditional way of representing a quantum state as a "ray".

3. Relation between quantum noise and inconsistency of quantum states.

   It will be interesting to study if there is a relation between inconsistent quantum states and quantum noise under the possibility of developing metrics to measure quantum noise, and so verify correctness of quantum programs under this circumstance.

4. The role of the Deustch gate on simulating the time-dependent Schrödinger equation .

   Since the Deustch gate can output superposition states that are a linear combination of two basis with trigonometric coefficients, it may well express general solutions for the Schrödinger equation. Therefore, providing a way of expressing and verify properties of quantum simulation programs.

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