Global stability criteria for nonlinear differential systems with infinite delay and applications to BAM neural networks

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Abstract

For a general n-dimensional nonautonomous and nonlinear differential equation with infinite delay, we give sufficient conditions for its global asymptotic stability. The main stability criterion depends on the size of the delay on the linear part and the dominance of the linear terms over the nonlinear terms. We apply our main result to answer several open problems left by L. Berezansky et. al. [*Appl. Math. Comput.* 243 (2014) 899-910]. Using the obtained theoretical stability results, we get sufficient conditions for both the global asymptotic and global exponential stability of a bidirectional associative memory neural network model with delays which generalizes models recently studied. Finally, a numerical example is given to illustrate the novelty of our results.

Keywords: Nonlinear delay differential equation; Infinite delay; Asymptotic stability; Exponential stability; Bidirectional associative memory neural networks.

Mathematics Subject Classification: 34K20, 34K25, 92B20.

1 Introduction

Neural network models have many applications in various engineering and scientific areas such as signal and image processing, parallel computing, pattern recognition, content-addressable memory, optimization problems, and so on (see [10–12, 21, 40]). Consequently, the dynamic behavior of neural network models has been attracting the attention of many researchers, such as mathematicians, computer scientists, statisticians, and others.

The pioneer models describing artificial neural networks were presented and studied by Cohen and Grossberg [13], Hopfield [23], and Kosko [26] in the 80s of the last century. The following system of ordinary differential equations

$$\begin{cases} x_i'(t) = -x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij} f(y_j(t)) + \hat{I}_i \\ \\ y_i'(t) = -y_i(t) + \displaystyle\sum_{j=1}^{n} \hat{a}_{ji} f(x_i(t)) + I_i \end{cases} \qquad t \in \mathbb{R},\ i \in \{1, \dots, n\},$$

was presented by Kosko and it was the first of the so-called bidirectional associative memory (BAM) neural network models. Since then, the study of the dynamic behavior of BAM models has become an active research subject (see [1–4, 7, 17, 24, 27–31, 38, 39, 41–43] and the references therein)

Due to the transmission speed of signals between different neurons, Marcus and Westervelt [32] incorporated a discrete delay in a neural network model and observed that the delay can destabilize its dynamic behavior. In fact, as is was confirmed by Baldi and Atiya [5], the delay affects the neural network dynamic, and the stability of delay neural network models has been the goal of large research activity (see [1–4, 6, 7, 15, 17, 24, 27, 28, 30, 31, 33, 34, 37–39, 41–43] and the references therein).

In 2007, Gopalsamy [17] introduced delays in the negative feedback terms, known as either "forgetting" or leakage terms [21, 26]. The introduction of delays in the leakage terms has also a strong impact in the dynamic behaviors of neural network models and their stability analysis became an important and active research subject (see [1–4, 7, 24, 27, 28, 30, 31, 34, 37, 38, 43] and the references therein). Gopalsamy [17] studied the global stability for autonomous BAM models with delays in the leakage terms. Peng [34] added distributed delays in the leakage terms of BAM models and obtained sufficient conditions for the existence and global attractivity of a periodic solution. Liu [30] addressed time-varying delays in the leakage terms of BAM models and established conditions for the existence and global exponential stability of an equilibrium point. Balasubramaniam et. al. [3, 4] studied the stability of BAM models with fuzzy and impulsive effect. Lakshmanan et. al. [27] studied the stability of BAM models with probabilistic effect into the time-varying delays. Other important results have been obtained about BAM models with delays in the leakage terms such as: bifurcation results [24, 29, 44, 45]; asymptotic stability of uncertain models [37, 39]; stability of fractional-order models [1, 38]; stability of stochastic models [38]; exponential stability of models on time scales [43]; or stability of neutral type models [2, 3, 28].

Motivated by the above description, in this paper, we apply our theoretical results to establish an M-matrix condition for the global asymptotic stability of a nonautonomous BAM neural network model with possible unbounded discrete time-varying delays, infinite distributed delays, and finite delays in the leakage terms, which generalizes some BAM models recently studied (see models in [7, 30, 31, 39, 42]). Moreover, the same M-matrix condition assures the global exponential stability of the model in case of finite delays.

Although the applications in this work have an important role, the main motivation of this study was the list with nine open problems left by Berezansky et. al. in the last section of [7]. Concretely, in [7], the global exponential stability of the nonlinear nonautonomous differential system with finite delays

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n} h_{ij}(t, x_j(t-\tau_{ij}(t))), \qquad t \geq 0,\, i \in \{1\ldots,n\}, \tag{1.1}$$

was studied and, at the end of paper [7, page 909], the authors presented a list with nine points, where they described some particular cases and extensions of (1.1) which are important to study. In this paper, we focus our attention on points 2, 3, and 5, where Berezansky et. al. ask about the global stability of the following extensions of (1.1)

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t))), \qquad t \geq 0,\, i \in \{1\ldots,n\}, \tag{1.2}$$

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\int_{t-\varrho_j}^{t} K_{ij}(t,s) f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds, \qquad t \geq 0,\, i \in \{1\ldots,n\}, \tag{1.3}$$

and

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + \sum_{j=1}^{n}\int_{-\infty}^{t} K_{ij}(t,s) f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds, \qquad t \geq 0,\, i \in \{1\ldots,n\}, \tag{1.4}$$

respectively. In order to provide an answer to these open problems, we introduce the nonlinear system of nonautonomous differential equations with infinite delays

$$x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) + h_i\big(t, x(t-\tau_{i1}(t)), \ldots, x(t-\tau_{im}(t))\big) + f_i(t, x_t), \qquad i \in \{1,\ldots,n\}, \tag{1.5}$$

where the functions h_i deal with discrete time-varying delays, whereas the functions f_i deal with distributed delays. We split the nonlinear terms into two parcels by technical reasons. As we will see in Section 2, this way of writing the system allowed us to assume weaker hypotheses. We note that $n, m \in \mathbb{N}$, $x(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$, and see Section 2 for detailed notation.

Despite our main application being to BAM neural network models, system (1.5) is general enough to be applied to other types of neural network models or even to biological models such as Lotka-Volterra models [46].

In this work, we establish a global asymptotic stability criterion for system (1.5) without using a Lyapunov functional, a usual tool in the literature [2, 4, 17, 27, 28, 30, 31, 34, 37, 39, 42, 43]. Instead, our technique is based on some algebraic computations and convenient estimations obtained from imposed hypotheses over the size of delays in the linear part and the dominance of linear terms over the nonlinear terms.

We should say that there are several recent works where global asymptotic stability criteria for nonautonomous linear differential systems with delays are obtained [8, 9, 14]. System (1.5) can be seen as a perturbation of a linear delay differential system, but as (1.5) is nonautonomous, it is a difficult task to obtain its global stability from the stability of its linearization at an equilibrium point, if it exists.

The main novelties in this paper are:

i. The global stability criterion established for the general nonautonomous differential system, (1.5), including nonlinear terms with possible unbounded delays and linear terms with finite delay, Theorem 3.3. We remark that, recently, T. Faria [14] obtained global stability criteria for delay linear systems, while Berezansky and Braverman [6] obtained a global stability criterion for a general nonautonomous delay differential system, but with no delays in the linear terms;

ii. The resolution of three open problems left by Berezansky et.al. in [7], Theorem 4.2;

iii. The new global exponential stability criterion established for model (4.1), where a weaker M-matrix condition is assumed than the one used in [7], Theorem 4.4, Remark 4.2, and Remark 5.1;

vi. The new global stability criteria for the BAM neural network model (4.19) provided by Corollaries 4.5, 4.6, and 4.7.

This paper is divided into six sections. After the introduction, in Section 2 some definitions and notations are presented and the general system of delay differential equations (1.5) together with its phase space are introduced. In Section 3 the main global stability criteria of (1.5) are established. In Section 4, we apply the results in Section 3 to give an answer to points 2, 3, and 5 of the list of open problems presented in [7] and we apply the theoretical result to a BAM neural network model with possible unbounded time-varying delays, infinite distributed delays, and finite delays in the leakage terms, which generalizes several BAM models with delays present in recent literature. In Section 5, a numerical example is presented to illustrate the improvements and effectiveness of our results. Finally, in Section 6 some conclusions are given.

2 General model and notations

In this paper, we denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of positive integer numbers. For $n \in \mathbb{N}$, we consider the product $\mathbb{R}^n$ equipped with the norm $|x| = \max\{|x_i| : i \in \{1, \dots, n\}\}$, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

For $a, b \in \mathbb{R}$ with $b > a$, we consider the Banach space of continuous functions $\varphi : [a, b] \to \mathbb{R}^n$, denoted by $C([a, b]; \mathbb{R}^n)$, equipped with the norm

$$\|\varphi\| = \sup_{\theta \in [a,b]} |\varphi(\theta)|.$$

For $\tau > 0$, the space $C([-\tau, 0]; \mathbb{R}^n)$ is the suitable phase space for retarded functional differential equations with finite delay, τ (see [20]).

To define an adequate phase space for infinite delay differential equations, we consider the following space [18],

$$UC_g = \left\{ \varphi \in C((-\infty, 0]; \mathbb{R}^n) : \sup_{s \leq 0} \frac{|\varphi(s)|}{g(s)} < +\infty, \frac{\varphi(s)}{g(s)} \text{ is uniformly continuous on } (-\infty, 0] \right\},$$

where $n \in \mathbb{N}$ and $C((-\infty, 0]; \mathbb{R}^n)$ denotes the space of continuous functions $\varphi : (-\infty, 0] \to \mathbb{R}^n$ and $g : (-\infty, 0] \to [1, +\infty)$ is a continuous function verifying:

(g1) $g(0) = 1$ and g is nonincreasing;

(g2) $\lim_{u\to 0^-} \dfrac{g(s+u)}{g(s)} = 1$ uniformly on $(-\infty, 0]$;

(g3) $g(s) \to +\infty$ as $s \to -\infty$.

The space UC_g is a Banach space when it is equipped with the norm $||\varphi||_g = \sup_{s\leq 0} \dfrac{|\varphi(s)|}{g(s)}$, for $\varphi \in UC_g$.

In applications to neural network models, we restrict our attention to bounded initial conditions. Consequently, we need to consider the space $BC = BC((-\infty, 0]; \mathbb{R}^n)$ of bounded and continuous functions, $\varphi : (-\infty, 0] \to \mathbb{R}^n$, equipped with the norm $||\varphi|| = \sup_{s\leq 0} |\varphi(s)|$. It is clear that $BC \subseteq UC_g$ and, trivially, we have $\|\varphi\|_g \leq \|\varphi\|$ for all $\varphi \in BC$.

For $f : [0, +\infty) \times D \to \mathbb{R}^n$ being a continuous function, where $D \subseteq UC_g$ is an open set, we consider the general functional differential equation

$$x'(t) = f(t, x_t), \qquad t \geq 0, \tag{2.1}$$

where x_t denotes the function $x_t : (-\infty, 0] \to \mathbb{R}^n$ defined by $x_t(s) = x(t + s)$ for $s \in (-\infty, 0]$, with bounded initial conditions

$$x_{t_0} = \varphi, \qquad \text{with } t_0 \geq 0 \text{ and } \varphi \in BC. \tag{2.2}$$

In the sense of [18], the Banach space UC_g is an admissible phase space for (2.1) and the standard results of existence, uniqueness, and continuation of solutions are assured [19]. Consequently, the initial value problem (IVP) (2.1)-(2.2) has always at least one solution (see [19]). If, additionally, the function f is Lipschitz on the second variable, then we have a unique solution of IVP (2.1)-(2.2), denoted by $x(t, t_0, \varphi)$. In fact, from the uniqueness result in [19], we can conclude that the IVP (2.1)-(2.2) has a unique solution if there is a continuous function $L : [0, +\infty) \to [0, +\infty)$ such that

$$|f(t, \varphi) - f(t, \psi)| \leq L(t)\|\varphi - \psi\|_g, \qquad t \geq 0,\ \varphi, \psi \in BC. \tag{2.3}$$

Now, we recall here some stability definitions usual in the literature on neural networks with infinite delays [25].

Definition 2.1. *Assume that all solutions of* (2.1) *are defined on* $\mathbb{R}$.
The functional differential equation (2.1) *is called*

(i) stable *if for all* $t_0 > 0$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for* $\varphi, \widetilde{\varphi} \in BC$,

$$\|\varphi - \widetilde{\varphi}\| < \delta \Rightarrow |x(t, t_0, \varphi) - x(t, t_0, \widetilde{\varphi})| < \varepsilon, \qquad t \geq t_0;$$

(ii) global attractive *if, for any* $t_0 \geq 0$ *and* $\varphi, \widetilde{\varphi} \in BC$,

$$\lim_{t\to+\infty} \big(x(t, t_0, \varphi) - x(t, t_0, \widetilde{\varphi})\big) = 0;$$

(iii) globally asymptotically stable *if it is stable and global attractive;*

(iv) globally exponentially stable *if there are* $C, \lambda > 0$ *such that, for all* $t_0 \geq 0$ *and* $\varphi, \widetilde{\varphi} \in BC$,

$$|x(t, t_0, \varphi) - x(t, t_0, \widetilde{\varphi})| \leq C\mathrm{e}^{-\lambda(t-t_0)}\|\varphi - \widetilde{\varphi}\|, \qquad t \geq t_0.$$

Fix a continuous function $g : (-\infty, 0] \to [1, +\infty)$ satisfying (g1)-(g3). With UC_g as the phase space, we consider the following family of functional differential equations with finite delays in the linear terms and infinite delays in the nonlinear terms,

$$x_i'(t) = -a_i(t)x_i(t - \tau_i(t)) + h_i\big(t, x(t - \tau_{i1}(t)), \ldots, x(t - \tau_{im}(t))\big) + f_i(t, x_t), \qquad t \geq 0, \tag{2.4}$$

for all $i \in \{1,\dots,n\}$, where $n,m \in \mathbb{N}$, $a_i : [0,+\infty) \to (0,+\infty)$, $\tau_i : [0,+\infty) \to [0,+\infty)$, $\tau_{ip} : [0,+\infty) \to [0,+\infty)$, $h_i : [0,+\infty) \times \mathbb{R}^{nm} \to \mathbb{R}$, and $f_i : [0,+\infty) \times UC_g \to \mathbb{R}$ are continuous functions, for $i \in \{1,\dots,n\}$ and $p \in \{1,\dots,m\}$. We recall that functions h_i deal with possible unbounded discrete time-varying delays whereas the functions f_i deal with infinite distributed delays.

In this paper, the next hypotheses will be assumed for (2.4):

(A1) for each $i \in \{1,\dots,n\}$, there is $\overline{\tau}_i \geq 0$ such that

$$\tau_i(t) \leq \overline{\tau}_i, \qquad t \geq 0;$$

(A2) for each $i \in \{1,\dots,n\}$ and $p \in \{1,\dots,m\}$,

$$\lim_{t\to+\infty} (t - \tau_{ip}(t)) = +\infty;$$

(A3) for each $i \in \{1,\dots,n\}$, there is a continuous function $H_i : [0,+\infty) \to [0,+\infty)$ such that

$$|h_i(t,u) - h_i(t,v)| \leq H_i(t)|u - v|, \qquad t \geq 0,\ u,v \in \mathbb{R}^{nm};$$

(A4) for each $i \in \{1,\dots,n\}$, there is a continuous function $F_i : [0,+\infty) \to [0,+\infty)$ such that

$$|f_i(t,\varphi) - f_i(t,\phi)| \leq F_i(t)\|\varphi - \phi\|_g, \qquad t \geq 0,\ \varphi,\phi \in BC;$$

(A5) for each $i \in \{1,\dots,n\}$

$$\limsup_{t\to+\infty} \left(\frac{F_i(t) + H_i(t)}{a_i(t)} + \int_{t-\tau_i(t)}^{t} \big[a_i(w) + F_i(w) + H_i(w)\big]dw \right) < 1.$$

From hypotheses (A3) and (A4), it is easy to see that Lipschitz condition (2.3) holds, thus the IVP (2.4)-(2.2) has a unique solution $x(t,t_0,\varphi)$. Moreover, it is defined on $\mathbb{R}$, [22].

Now, we state some notations. We denote $\overline{\tau} = \max\{\overline{\tau}_1,\dots,\overline{\tau}_n\}$. For a vector $d = (d_1,\dots,d_n) \in \mathbb{R}^n$ with $d_i \neq 0$ for all $i \in \{1,\dots,n\}$, we denote by d^{-1} the vector $d^{-1} = \big(d_1^{-1},\dots,d_n^{-1}\big)$. In case $d_i > 0$ for all $i \in \{1,\dots,n\}$, we say that $d = (d_1,\dots,d_n)$ is a positive vector and we denote it by $d > 0$. For $d = (d_1,\dots,d_n) \in \mathbb{R}^n$ and $q = (q_1,\dots,q_n) \in \mathbb{R}^n$, we denote $d \cdot q = (d_1q_1,\dots,d_nq_n) \in \mathbb{R}^n$.

For a function $h : \mathbb{R} \to \mathbb{R}$, we denote by $h'_+(t)$ the right-hand derivative of h at $t \in \mathbb{R}$.

If $h : X \to \mathbb{R}$, with $X \subseteq \mathbb{R}$, is a bounded function, then we state the notations $\overline{h} = \sup_{s\in X} h(s)$ and $\underline{h} = \inf_{s\in X} h(s)$.

Some stability results in the next section involve the concept of non-singular M-matrix. Thus we recall the definition here.

Definition 2.2. *Let $\mathcal{M} = [m_{ij}]$ be a square real matrix with nonpositive off-diagonal entries, i.e. $m_{ij} \leq 0$ for all $i \neq j$.*

The matrix $\mathcal{M}$ is called non-singular M-matrix *if all the eigenvalues have positive real part.*

There exist several equivalent properties to identify a non-singular M-matrix and we indicate Chapter 5 of [16] to consult them and to study further properties. In this paper we are going to use the following property [16, Theorem 5.1.]: If $\mathcal{M} = [m_{ij}]_{i,j=1}^n$ is a non-singular M-matrix, then there is $d = (d_1,\dots,d_n) > 0$ such that $\mathcal{M}d > 0$.

Given $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ two square real matrices, we write $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i,j \in \{1,\dots,n\}$.

3 Global asymptotic stability

In this section, we study the global stability of the family of functional differential equations presented in (2.4). Mainly, we establish sufficient conditions for the global asymptotic stability of (2.4).

First we prove that system (2.4) is stable.

Theorem 3.1. *Assume (A3)-(A5).*

Then system (2.4) *is stable.*

Proof. From hypothesis (A5) there is $T > 0$ such that

$$-a_i(t) + F_i(t) + H_i(t) + a_i(t) \int_{t-\tau_i(t)}^{t} \big[a_i(w) + F_i(w) + H_i(w)\big] dw < 0, \qquad t \geq T,\ i \in \{1, \dots, n\}. \tag{3.1}$$

Let $t_0 \geq 0$ and $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{\mathrm{e}^{L_{t_0}(T_0 - t_0)}}$, where $L_{t_0} = \max\limits_{i \in \{1,\dots,n\}} \left\{ \sup\limits_{t \in [t_0, T_0]} \big\{ a_i(t) + H_i(t) + F_i(t) \big\} \right\}$ and $T_0 = \max\{T, t_0 + \overline{\tau}\}$.

Let $\varphi, \widetilde{\varphi} \in BC$ such that $\|\varphi - \widetilde{\varphi}\| < \delta$. Consider the solutions $x(t) = (x_1(t), \dots, x_n(t)) = x(t, t_0, \varphi)$ and $y(t) = (y_1(t), \dots, y_n(t)) = x(t, t_0, \widetilde{\varphi})$ of (2.4) and define $z(t) = (z_1(t), \dots, z_n(t))$ where $z_i(t) = |x_i(t) - y_i(t)|$ for all $i \in \{1, \dots, n\}$ and $t \geq t_0$.

The goal is to show that $|z(t)| < \varepsilon$, for all $t \geq t_0$. By uniqueness of a solution of IVP (2.4)-(2.2), the situation is trivial if $\varphi = \widetilde{\varphi}$. Thus we may assume that $\varphi \neq \widetilde{\varphi}$.

For all $i \in \{1, \dots, n\}$ and $t \geq t_0$, we have

$$\begin{aligned}
z_i'(t) &= \mathrm{sign}(x_i(t) - y_i(t))(x_i'(t) - y_i'(t)) \\
&= \mathrm{sign}(x_i(t) - y_i(t))\big[-a_i(t)x_i(t - \tau_i(t)) + a_i(t) y_i(t - \tau_i(t)) \\
&\quad + h_i(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) - h_i(t, y(t - \tau_{i1}(t)), \dots, y(t - \tau_{im}(t))) \\
&\quad + f_i(t, x_t) - f_i(t, y_t)\big] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.2)\\
&\leq -a_i(t)\mathrm{sign}(x_i(t) - y_i(t))(x_i(t) - y_i(t)) \\
&\quad + a_i(t)\mathrm{sign}(x_i(t) - y_i(t))\big[x_i(t) - y_i(t) - (x_i(t - \tau_i(t)) - y_i(t - \tau_i(t)))\big] \\
&\quad + |h_i(t, x(t - \tau_{i1}(t)), \dots, x(t - \tau_{im}(t))) - h_i(t, y(t - \tau_{i1}(t)), \dots, y(t - \tau_{im}(t)))| \\
&\quad + |f_i(t, x_t) - f_i(t, y_t)|,
\end{aligned}$$

and from hypotheses (A3) and (A4) we obtain, for $t \geq t_0 + \overline{\tau}$,

$$\begin{aligned}
z_i'(t) &\leq -a_i(t) z_i(t) + a_i(t) \left| \int_{t-\tau_i(t)}^{t} x_i'(w) - y_i'(w) dw \right| + H_i(t) \max_p \big\{ |x(t - \tau_{ip}(t)) - y(t - \tau_{ip}(t))| \big\} \\
&\quad + F_i(t) \|x_t - y_t\|_g \\
&= -a_i(t) z_i(t) + a_i(t) \bigg| \int_{t-\tau_i(t)}^{t} -a_i(w)\big[x_i(w - \tau_i(w)) - y_i(w - \tau_i(w))\big] \\
&\quad + \big[h_i(w, x(w - \tau_{i1}(w)), \dots, x(w - \tau_{im}(w))) - h_i(w, y(w - \tau_{i1}(w)), \dots, y(w - \tau_{im}(w)))\big] \\
&\quad + \big[f_i(w, x_w) - f_i(w, y_w)\big] dw \bigg| + H_i(t) \max_p \big\{ |z(t - \tau_{ip}(t))| \big\} + F_i(t) \|z_t\|_g \\
&\leq -a_i(t) z_i(t) \\
&\quad + a_i(t) \int_{t-\tau_i(t)}^{t} \left(a_i(w) z_i(w - \tau_i(w)) + H_i(w) \max_p \big\{ |z(w - \tau_{ip}(w))| \big\} + F_i(w) \|z_w\|_g \right) dw \\
&\quad + H_i(t) \max_p \big\{ |z(t - \tau_{ip}(t))| \big\} + F_i(t) \|z_t\|_g. \qquad\qquad\qquad\qquad\qquad\qquad (3.3)
\end{aligned}$$

Now define $\omega : [t_0, +\infty) \to \mathbb{R}$ by

$$\omega(t) = \sup \big\{ |z(r)| : r \in (-\infty, t] \big\}.$$

It is easy to see that ω is a continuous nondecreasing function, there exists $\omega'_+(t)$ for all $t \in [t_0, +\infty)$, and $\omega(t_0) = \|\varphi - \widetilde{\varphi}\| \neq 0$. For each $t \in (t_0, +\infty)$ define

$$J_t = \big\{j \in \{1, \dots, n\} : z_j(t) = \omega(t)\big\}.$$

By easy computations, we conclude that:

- If $J_t = \emptyset$, then $\omega'_+(t) = \omega'(t) = 0$;
- If $J_t \neq \emptyset$, then $\omega'_+(t) = \max\{z'_j(t) : j \in J_t\}$.

We claim that, for $T_0 = \max\{T, t_0 + \overline{\tau}\}$ and T defined in (3.1), we have:

1. $J_t = \emptyset$, for all $t > T_0$;
2. $\omega(t) \leq \mathrm{e}^{L_{t_0}(t-t_0)}\omega(t_0)$, for all $t \in [t_0, T_0]$.

By claim 1, the function ω is constant on $[T_0, +\infty)$ and, together with claim 2, we conclude that

$$|z(t)| \leq \omega(t) \leq \mathrm{e}^{L_{t_0}(T_0-t_0)}\omega(t_0) = \mathrm{e}^{L_{t_0}(T_0-t_0)}\|\varphi - \widetilde{\varphi}\| < \varepsilon, \qquad t \geq t_0, \tag{3.4}$$

which means that system (2.4) is stable.

To complete the proof, it remains to prove claims 1 and 2.

Claim 1.

By contradiction assume that there is $t > T_0$ such that $J_t \neq \emptyset$. For $i \in J_t$, we have $z_i(t) = \omega(t)$, thus

$$z_i(t) > 0, \qquad z'_i(t) \geq 0, \qquad \text{and} \qquad z_i(t) \geq |z(r)| \text{ for all } r \in (-\infty, t]. \tag{3.5}$$

Consequently, from (3.1), (3.3), and (3.5), we conclude that

$$\begin{aligned} z'_i(t) &\leq -a_i(t)z_i(t) + a_i(t)\int_{t-\tau_i(t)}^{t} \big[a_i(w)z_i(t) + H_i(w)z_i(t) + F_i(w)z_i(t)\big]dw + H_i(t)z_i(t) + F_i(t)z_i(t) \\ &= \left[-a_i(t) + a_i(t)\int_{t-\tau_i(t)}^{t} \big[a_i(w) + H_i(w) + F_i(w)\big]dw + H_i(t) + F_i(t)\right] z_i(t) < 0, \end{aligned}$$

which contradicts (3.5).

Claim 2.

Let $t \in (t_0, T_0]$ be such that $J_t \neq \emptyset$. Choosing $i \in J_t$ such that $\omega'_+(t) = z'_i(t)$, we have

$$z_i(t) > 0, \qquad z_i(t) \geq |z(r)| \text{ for all } r \in (-\infty, t]. \tag{3.6}$$

From (3.2), (3.6), and hypotheses (A3) and (A4), we have

$$\omega'_+(t) = z'_i(t) \leq \big(a_i(t) + H_i(t) + F_i(t)\big)z_i(t) \leq L_{t_0}z_i(t) = L_{t_0}\omega(t).$$

As $\omega'_+(t) = 0$ for all $t > t_0$ such that $J_t = \emptyset$, we obtain

$$\omega'_+(t) \leq L_{t_0}\omega(t), \qquad t \in (t_0, T_0].$$

Consequently

$$\omega(t) \leq \mathrm{e}^{L_{t_0}(t-t_0)}\omega(t_0), \qquad t \in [t_0, T_0].$$

□

From the proof of the previous result, mainly from the inequality (3.4), trivially we obtain the following result.

Corollary 3.2. *Assume (A3)-(A5).*

If a_i, H_i, and F_i are bounded functions, then there is $C \geq 1$ such that, for all $t_0 \geq 0$ and $\varphi, \widetilde{\varphi} \in BC$, the solutions $x(t, t_0, \varphi)$ and $x(t, t_0, \widetilde{\varphi})$ of (2.4) *satisfy*

$$|x(t, t_0, \varphi) - x(t, t_0, \widetilde{\varphi})| \leq C\|\varphi - \widetilde{\varphi}\|, \quad t \geq t_0.$$

Proof. Let $C = \mathrm{e}^{L\max\{T,\overline{\tau}\}}$, where $L = \max\limits_{i\in\{1,\dots,n\}}\left\{\sup\limits_{t\geq 0}\{a_i(t) + H_i(t) + F_i(t)\}\right\}$ and T comes from (A5) as in (3.1). For $t_0 \geq 0$ and $T_0 = \max\{T, t_0 + \overline{\tau}\}$, by (3.4), we obtain

$$|x(t,t_0,\varphi) - x(t,t_0,\widetilde{\varphi})| \leq \mathrm{e}^{L_{t_0}(T_0 - t_0)}\|\varphi - \widetilde{\varphi}\| \leq \mathrm{e}^{L\max\{T,\overline{\tau}\}}\|\varphi - \widetilde{\varphi}\| = C\|\varphi - \widetilde{\varphi}\|, \quad t \geq t_0,\ \varphi, \widetilde{\varphi} \in BC.$$

□

Now we are in a position to establish the global asymptotic stability of (2.4).

Theorem 3.3. *Assume (A1)-(A5).*

If, for each $i \in \{1, \dots, n\}$, there is $\underline{a}_i > 0$ such that

$$a_i(t) \geq \underline{a}_i, \quad t \geq 0, \tag{3.7}$$

then system (2.4) *is globally asymptotically stable.*

Proof. From Theorem 3.1, we only need to prove the global attractivity of (2.4).

Let $t_0 \geq 0$ and $\varphi, \widetilde{\varphi} \in BC$.

As in the proof of Theorem 3.1, considering the solutions $x(t) = (x_1(t), \dots, x_n(t)) = x(t, t_0, \varphi)$ and $y(t) = (y_1(t), \dots, y_n(t)) = x(t, t_0, \widetilde{\varphi})$ of (2.4), and defining $z(t) = (z_1(t), \dots, z_n(t))$, where $z_i(t) = |x_i(t) - y_i(t)|$ for all $i \in \{1, \dots, n\}$ and $t \geq t_0$, the inequality (3.3) holds for all $i \in \{1, \dots, n\}$ and $t \geq t_0$.

Now, define

$$v = \max\left\{\limsup_{t\to+\infty} z_i(t) : i \in \{1, \dots, n\}\right\}.$$

As $z(t) \geq 0$ for $t \in [t_0, +\infty)$, we have $v \in [0, +\infty]$. We need to prove that $v = 0$. This is done into two steps. In step 1 we show that $v \neq +\infty$ and in step 2 we show that $v = 0$.

Step 1. To prove that $v \neq +\infty$.

Assume that $v = +\infty$. Thus there are $i \in \{1, \dots, n\}$ and an increasing real sequence $(t_k)_{k\in\mathbb{N}}$ on $(t_0 + \overline{\tau}, +\infty)$ such that $\lim\limits_{k} t_k = +\infty$ and

$$z_i(t_k) = \|z_{t_k}\| > 0, \quad z_i'(t_k) \geq 0, \quad k \in \mathbb{N}. \tag{3.8}$$

From (3.3) and (3.8), for each $k \in \mathbb{N}$, we have

$$\begin{aligned} z_i'(t_k) \leq\ & -a_i(t_k)\|z_{t_k}\| \\ & + a_i(t_k)\int_{t_k - \tau_i(t_k)}^{t_k}\Big(a_i(w)\|z_{t_k}\| + H_i(w)\|z_{t_k}\| + F_i(w)\|z_w\|_g\Big)dw \\ & + H_i(t_k)\|z_{t_k}\| + F_i(t_k)\|z_{t_k}\|_g. \end{aligned}$$

Consequently, as $\|\phi\|_g \leq \|\phi\|$ for all $\phi \in BC$, we obtain

$$z_i'(t_k) \leq \left(-a_i(t_k) + a_i(t_k)\int_{t_k - \tau_i(t_k)}^{t_k}\big[a_i(w) + H_i(w) + F_i(w)\big]dw + H_i(t_k) + F_i(t_k)\right)\|z_{t_k}\|. \tag{3.9}$$

From hypothesis (A5), there is $T > t_0$ such that

$$\frac{F_i(t) + H_i(t)}{a_i(t)} + \int_{t-\tau_i(t)}^{t}\big[a_i(w) + F_i(w) + H_i(w)\big]dw < 1, \quad t \geq T,$$

which implies

$$-a_i(t) + F_i(t) + H_i(t) + a_i(t)\int_{t-\tau_i(t)}^{t} \big[a_i(w) + F_i(w) + H_i(w)\big]dw < 0, \qquad t \geq T. \tag{3.10}$$

For large k, we have $t_k > T$ and by (3.9) together with (3.10) we conclude that

$$z_i'(t_k) < 0,$$

and this contradicts (3.8). Thus $v \in [0, +\infty)$.

Step 2. To prove that $v = 0$.

From step 1, we have $v \in [0, +\infty)$ and consequently $z(t)$ is bounded. Choose $i \in \{1, \dots, n\}$ such that $v = \limsup_{t\to+\infty} z_i(t)$. By the fluctuation lemma [36], there is an increasing real sequence $(t_k)_{k\in\mathbb{N}}$ on $(t_0 + \overline{\tau}, +\infty)$ such that

$$\lim_k t_k = +\infty, \qquad \lim_k z_i(t_k) = v, \qquad \text{and} \qquad \lim_k z_i'(t_k) = 0. \tag{3.11}$$

Fix $\varepsilon > 0$ and choose $T > t_0$ such that $|z(t)| < v + \varepsilon$ for all $t \geq T$, and $\|\varphi - \tilde{\varphi}\|/g(t_0 - T) < v + \varepsilon$.

As $t - 2\overline{\tau}_i \to +\infty$, as $t \to +\infty$, then there is $k_1 \in \mathbb{N}$ such that $t_k - 2\overline{\tau}_i > T$ for all $k \geq k_1$.

By (A2), $t - \tau_{ip}(t) \to +\infty$ as $t \to +\infty$, thus $\inf_{\omega\in[t_k-\overline{\tau}_i, t_k]} \{\omega - \tau_{ip}(\omega)\} \to +\infty$ as $k \to +\infty$, then there is $k_2 \in \mathbb{N}$ such that $\omega - \tau_{ip}(\omega) > T$ for all $\omega \in [t_k - \overline{\tau}_i, t_k]$, $p \in \{1, \dots, m\}$, and $k \geq k_2$.

Define $k_0 = \max\{k_1, k_2\}$. Trivially, we have $t_{k_0} - 2\overline{\tau}_i > T$.

Defining $\mathcal{Z} = \max_{s\in[t_0, t_{k_0}]} |z(s)|$, from condition (g3) we conclude that there exists $s^* < 0$ such that $\dfrac{\mathcal{Z}}{g(s)} < v + \varepsilon$ for any $s \in (-\infty, s^*)$.

We show that, for $k \in \mathbb{N}$ such that $t_k > t_{k_0} + \overline{\tau}_i - s^*$ and $\omega \in [t_k - \overline{\tau}_i, t_k]$, we have

$$\|z_\omega\|_g \leq v + \varepsilon. \tag{3.12}$$

Let $\omega \in [t_k - \overline{\tau}_i, t_k]$ with $t_k > t_{k_0} + \overline{\tau}_i - s^*$.

$$\begin{aligned}
\|z_\omega\|_g &= \sup_{s\leq 0} \frac{|z(\omega + s)|}{g(s)} \\
&= \max\left\{ \sup_{s\in[t_0-\omega, 0]} \frac{|z(\omega + s)|}{g(s)}, \sup_{s\in(-\infty, t_0-\omega]} \frac{|z(\omega + s)|}{g(s)} \right\}
\end{aligned}$$

On the one hand, we have

$$\sup_{s\in(-\infty, t_0-\omega]} \frac{|z(\omega + s)|}{g(s)} = \sup_{s\in(-\infty, t_0]} \frac{|z(s)|}{g(s - \omega)} \leq \frac{\|\varphi - \widetilde{\varphi}\|}{g(t_0 - \omega)} \leq \frac{\|\varphi - \widetilde{\varphi}\|}{g(t_0 - T)} < v + \varepsilon.$$

On the other hand,

$$\begin{aligned}
\sup_{s\in[t_0-\omega, 0]} \frac{|z(\omega + s)|}{g(s)} &\leq \max\left\{ \sup_{s\in[t_0-\omega, t_{k_0}-\omega]} \frac{|z(\omega + s)|}{g(s)}, \sup_{s\in[t_{k_0}-\omega, 0]} \frac{|z(\omega + s)|}{g(s)} \right\} \\
&\leq \max\left\{ \frac{\mathcal{Z}}{g(t_{k_0} - t_k + \overline{\tau}_i)}, \sup_{s\in[t_{k_0}, \omega]} \frac{|z(s)|}{g(s - \omega)} \right\} \\
&\leq \max\left\{ v + \varepsilon, \sup_{s\in[t_{k_0}, \omega]} \frac{v + \varepsilon}{g(s - \omega)} \right\} = v + \varepsilon.
\end{aligned}$$

Consequently, condition (3.12) holds.

From (3.3) and (3.12), for all $k \in \mathbb{N}$ such that $t_k > t_{k_0} + \overline{\tau}_i - s^*$, we have

$$
\begin{aligned}
z_i'(t_k) &\leq -a_i(t_k)z_i(t_k)\\
&\quad + a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\left(a_i(w)z_i(w-\tau_i(w)) + H_i(w)\max_p\left\{|z(w-\tau_{ip}(w))|\right\} + F_i(w)\|z_w\|_g\right)dw\\
&\quad + H_i(t_k)\max_p\left\{|z(t_k-\tau_{ip}(t_k))|\right\} + F_i(t_k)\|z_{t_k}\|_g\\
&\leq -a_i(t_k)z_i(t_k)\\
&\quad + a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\big(a_i(w)(v+\varepsilon) + H_i(w)(v+\varepsilon) + F_i(w)(v+\varepsilon)\big)dw\\
&\quad + H_i(t_k)(v+\varepsilon) + F_i(t_k)(v+\varepsilon)\\
&= -a_i(t_k)z_i(t_k) + (v+\varepsilon)\left(a_i(t_k)\int_{t_k-\tau_i(t_k)}^{t_k}\big[a_i(w)+H_i(w)+F_i(w)\big]dw + H_i(t_k) + F_i(t_k)\right),
\end{aligned}
$$

thus

$$
z_i(t_k) + \frac{z_i'(t_k)}{a_i(t_k)} - (v+\varepsilon)\left(\int_{t_k-\tau_i(t_k)}^{t_k}\big[a_i(w)+H_i(w)+F_i(w)\big]dw + \frac{H_i(t_k)+F_i(t_k)}{a_i(t_k)}\right) \leq 0. \quad (3.13)
$$

From (3.7), we know that $\liminf_k a_i(t_k) > 0$. Finally, from (3.11), (3.13), and letting $k \to +\infty$ and $\varepsilon \to 0^+$, we obtain

$$
v - v\left(\limsup_{t\to+\infty}\left(\frac{F_i(t)+H_i(t)}{a_i(t)} + \int_{t-\tau_i(t)}^{t}\big[a_i(w)+F_i(w)+H_i(w)\big]dw\right)\right) \leq 0.
$$

From (A5), we must have $v = 0$. □

Remark 3.1. We should remark that, under conditions assuring the existence of a unique solution of IVP (2.4)-(2.2) defined on $\mathbb{R}$ and using similar arguments as those in the proofs of Theorems 3.1 and 3.3, we obtain that the zero solution of (2.4) is global asymptotic stability if we assume (A1), (A2), (A5), and

(A3*) for each $i \in \{1,\dots,n\}$, there is a continuous function $H_i : [0,+\infty) \to [0,+\infty)$ such that

$$
|h_i(t,u)| \leq H_i(t)|u|, \qquad t \geq 0,\ u \in \mathbb{R}^{nm};
$$

(A4*) for each $i \in \{1,\dots,n\}$, there is a continuous function $F_i : [0,+\infty) \to [0,+\infty)$ such that

$$
|f_i(t,\varphi)| \leq F_i(t)\|\varphi\|_g, \qquad t \geq 0,\ \varphi \in BC.
$$

4 Applications to BAM networks

In this section, first, we use the results in the previous section to obtain new stability criteria for a theoretical delay differential system which generalizes several systems introduced by Berezansky, Braverman, and Idels [7]. Then, we apply them to establish new criteria for the global stability of a BAM neural network model with unbounded time-varying delays, infinite distributed delays, and finite delays in the leakage terms.

To provide an answer to points 2, 3, and 5 of the list of open problems in [7, page 909], we are going to establish sufficient conditions for the global asymptotic stability of the following generalization of systems (1.2), (1.3), and (1.4)

$$
\begin{aligned}
x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) &+ \sum_{j=1}^{n}\sum_{k=1}^{K} h_{ijk}(t, x_j(t-\tau_{ijk}(t)))\\
&+ \sum_{j=1}^{n}\int_{-\infty}^{t} K_{ij}(t,s)f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds, \qquad t \geq 0,\ i \in \{1\dots,n\},
\end{aligned} \quad (4.1)
$$

where $n, K \in \mathbb{N}$ and $a_i : [0,+\infty) \to (0,+\infty)$, $h_{ijk}, K_{ij} : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $f_{ij} : \mathbb{R}^2 \to \mathbb{R}$, $\tau_i, \tau_{ijk} : [0,+\infty) \to [0,+\infty)$, and $\varrho_{ij} : \mathbb{R} \to [0,+\infty)$ are continuous functions, for all $i,j \in \{1,\dots,n\}$ and $k \in \{1,\dots,K\}$.

For system (4.1) we assume the following hypotheses:

(a1) there exist $\overline{\varrho} > 0$ and $\overline{\tau}_i > 0$, such that

$$\tau_i(t) \le \overline{\tau}_i \quad \text{ and } \quad \varrho_{ij}(t) \le \overline{\varrho}, \qquad t \ge 0,\, i,j \in \{1,\dots,n\};$$

(a2) for each $i,j \in \{1,\dots,n\}$ and $k \in \{1,\dots,K\}$

$$\lim_{t\to+\infty} (t - \tau_{ijk}(t)) = +\infty;$$

(a3) for each $i,j \in \{1,\dots,n\}$ and $k \in \{1,\dots,K\}$, there exists a continuous function $H_{ijk} : [0,+\infty) \to [0,+\infty)$ such that

$$|h_{ijk}(t,u) - h_{ijk}(t,v)| \le H_{ijk}(t)|u-v|, \qquad t \ge 0,\ u,v \in \mathbb{R};$$

(a4) for each $i,j \in \{1,\dots,n\}$, there exists $F_{ij} > 0$ such that

$$|f_{ij}(t,u) - f_{ij}(t,v)| \le F_{ij}|u-v|, \qquad t \ge 0,\ u,v \in \mathbb{R};$$

(a5) for each $i,j \in \{1,\dots,n\}$, there exist $\kappa_{ij} : [0,+\infty) \to [0,+\infty)$ and $g_{ij} : [0,+\infty) \to [0,+\infty)$ continuous functions such that

$$|K_{ij}(t,s)| \le \kappa_{ij}(t) g_{ij}(t-s), \qquad t \ge 0,\ s \le t,$$

and

$$\int_0^{+\infty} g_{ij}(t)dt = 1;$$

(a6) there is $d = (d_1,\dots,d_n) > 0$ such that, for each $i \in \{1,\dots,n\}$

$$\limsup_{t\to+\infty} \left[a_i(t)^{-1} \sum_{j=1}^n \frac{d_j}{d_i} \left(\kappa_{ij}(t) F_{ij} + \sum_{k=1}^K H_{ijk}(t) \right) \right.$$
$$\left. + \int_{t-\tau_i(t)}^t a_i(w) + \sum_{j=1}^n \frac{d_j}{d_i} \left(\kappa_{ij}(w) F_{ij} + \sum_{k=1}^K H_{ijk}(w) \right) dw \right] < 1.$$

To build the convenient phase space of (4.1), we need an auxiliary lemma to define a function g satisfying (g1)-(g3). The lemma is essentially the same one published in [15, Lemma 4.1.] and the proof follows the same steps. Thus we decided to omit it.

Lemma 4.1. *Let $m \in \mathbb{N}$ and consider $\eta_i : (-\infty,0] \to \mathbb{R}$, $i \in \{1,\dots,m\}$, bounded and nondecreasing functions such that*

$$\int_{-\infty}^0 d\eta_i(s) < \alpha, \qquad i \in \{1,\dots,m\},$$

for some $\alpha > 0$.

Then, for all $\tau \ge 0$, there exists a continuous function $g : (-\infty,0] \to [1,+\infty)$ satisfying (g1)-(g3) and

$$\int_{-\infty}^0 g(s-\tau) d\eta_i(s) < \alpha, \qquad i \in \{1,\dots,m\}.$$

Applying Theorem 3.3, we obtain the following global stability criterion for system (4.1).

Theorem 4.2. *Assume (a1)-(a6) and* (3.7).

Then system (4.1) *is globally asymptotically stable.*

Proof. From (a6), we conclude that there is $\alpha > 0$ such that

$$\limsup_{t\to+\infty}\left[a_i(t)^{-1}\sum_{j=1}^{n}\frac{d_j}{d_i}\left(\kappa_{ij}(t)(1+\alpha)F_{ij}+\sum_{k=1}^{K}H_{ijk}(t)\right)\right.$$
$$\left.+\int_{t-\tau_i(t)}^{t} a_i(w)+\sum_{j=1}^{n}\frac{d_j}{d_i}\left(\kappa_{ij}(w)(1+\alpha)F_{ij}+\sum_{k=1}^{K}H_{ijk}(w)\right)dw\right]<1. \quad (4.2)$$

for all $i \in \{1,\dots,n\}$.

With the change of variables $y_i(t) = d_i^{-1}x_i(t)$ system (4.1) takes the form

$$\begin{aligned} y_i'(t) &= -a_i(t)y_i(t-\tau_i(t)) + \sum_{j=1}^{n}\sum_{k=1}^{K}\frac{1}{d_i}h_{ijk}(t, d_jy_j(t-\tau_{ijk}(t))) \\ &\quad + \sum_{j=1}^{n}\int_{-\infty}^{t}\frac{1}{d_i}K_{ij}(t,s)f_{ij}(s, d_jy_j(s-\varrho_{ij}(s)))ds, \qquad t\geq 0,\ i\in\{1\dots,n\}, \end{aligned} \quad (4.3)$$

Trivially, system (4.1) is globally asymptotically stable if and only if system (4.3) is globally asymptotically stable.

For each $i,j \in \{1,\dots,n\}$, consider the function $\eta_{ij} : (-\infty, 0] \to \mathbb{R}$, defined by

$$\eta_{ij}(s) = \int_{-\infty}^{s} g_{ij}(-\nu)d\nu. \quad (4.4)$$

Since $\overline{\varrho} > 0$ and, by (a5), $\int_{-\infty}^{0} d\eta_{ij}(s) = \int_{-\infty}^{0} g_{ij}(-\nu)d\nu < 1+\alpha$ for all $i,j \in \{1,\dots,n\}$, Lemma 4.1 assures that there exists a function $g : (-\infty, 0] \to [1,+\infty)$ satisfying (g1)-(g3) and

$$\int_{-\infty}^{0} g(s-\overline{\varrho})d\eta_{ij}(s) < 1+\alpha. \quad (4.5)$$

Consider UC_g as the phase space of system (4.3).

System (4.3) is a particular case of (2.4). In fact, considering $m = nK$ and identifying each $p \in \{1,\dots,nK\}$ with (j,k), i.e. $p \equiv (j,k)$ for $j \in \{1,\dots,n\}$ and $k \in \{1,\dots,K\}$, system (4.3) is obtained if we take in (2.4)

$$\begin{aligned} h_i\left(t, u^{(1)},\dots,u^{(m)}\right) &= h_i\left(t, u^{(1,1)},\dots,u^{(1,K)},\dots,u^{(n,1)},\dots,u^{(n,K)}\right) \\ &= \sum_{j=1}^{n}\sum_{k=1}^{K}\frac{1}{d_i}h_{ijk}\left(t, d_ju_j^{(j,k)}\right), \end{aligned} \quad (4.6)$$

for $u^{(p)} \equiv u^{(j,k)} = \left(u_1^{(j,k)},\dots,u_n^{(j,k)}\right) \in \mathbb{R}^n$, $\tau_{ip}(t) \equiv \tau_{i(j,k)}(t) = \tau_{ijk}(t)$, and

$$f_i(t,\phi) = \sum_{j=1}^{n}\int_{-\infty}^{0}\frac{1}{d_i}K_{ij}(t,t+s)f_{ij}\big(t+s, d_j\phi_j(s-\varrho_{ij}(t+s))\big)ds, \quad (4.7)$$

for all $t \geq 0$, $\phi = (\phi_1,\dots,\phi_n) \in BC$, $p \in \{1,\dots,nK\} \equiv \{(1,1),\dots,(n,K)\}$, $k \in \{1,\dots,K\}$, and $i,j \in \{1,\dots,n\}$.

Now, in order to apply Theorem 3.3, it remains to verify hypotheses (A3), (A4), and (A5).

From (a3) trivially we obtain that each function h_i, $i \in \{1,\dots,n\}$, defined by (4.6), satisfies (A3) with

$$H_i(t) = \sum_{j=1}^{n}\sum_{k=1}^{K} \frac{d_j}{d_i} H_{ijk}(t), \qquad t \geq 0, \ .$$

For $i \in \{1,\dots,n\}$, $t \geq 0$, and $\varphi = (\varphi_1,\dots,\varphi_n), \phi = (\phi_1,\dots,\phi_n) \in BC$, from (g1), (a1), (a4) and (a5), function f_i, defined by (4.7), satisfies

$$\begin{aligned}
|f_i(t,\varphi) - f_i(t,\phi)| \ \leq \ & \sum_{j=1}^{n} \int_{-\infty}^{0} \frac{1}{d_i} |K_{ij}(t,t+s)| \, \big| f_{ij}\big(t+s, d_j\varphi_j(s-\varrho_{ij}(t+s))\big) \\
& - f_{ij}\big(t+s, d_j\phi_j(s-\varrho_{ij}(t+s))\big)\big| \, ds \\
\leq \ & \sum_{j=1}^{n} \int_{-\infty}^{0} \frac{1}{d_i} \kappa_{ij}(t) g_{ij}(-s) F_{ij} \, |d_j\varphi_j(s-\varrho_{ij}(t+s)) - d_j\phi_j(s-\varrho_{ij}(t+s))| \, ds \\
= \ & \sum_{j=1}^{n} \frac{d_j}{d_i} \kappa_{ij}(t) F_{ij} \int_{-\infty}^{0} g_{ij}(-s) \frac{|\varphi_j(s-\varrho_{ij}(t+s)) - \phi_j(s-\varrho_{ij}(t+s))|}{g(s-\varrho_{ij}(t+s))} \times \\
& \times g(s-\varrho_{ij}(t+s)) ds \\
\leq \ & \sum_{j=1}^{n} \frac{d_j}{d_i} \kappa_{ij}(t) F_{ij} \int_{-\infty}^{0} g_{ij}(-s) \|\varphi - \phi\|_g g(s-\overline{\varrho}) ds \\
= \ & \sum_{j=1}^{n} \frac{d_j}{d_i} \kappa_{ij}(t) F_{ij} \|\varphi - \phi\|_g \int_{-\infty}^{0} g_{ij}(-s) g(s-\overline{\varrho}) ds.
\end{aligned}$$

By definition of η_{ij} in (4.4), we obtain $\int_{-\infty}^{0} g_{ij}(-s) g(s-\overline{\varrho}) ds = \int_{-\infty}^{0} g(s-\overline{\varrho}) d\eta_{ij}(s)$ and by (4.5) we conclude that

$$|f_i(t,\varphi) - f_i(t,\phi)| \leq \left(\sum_{j=1}^{n} \frac{d_j}{d_i} \kappa_{ij}(t) F_{ij} (1+\alpha) \right) \|\varphi - \phi\|_g,$$

and (A4) holds. From (4.2) hypothesis (A5) also holds and Theorem 3.3 assures the global asymptotic stability of (4.3) and we conclude that system (4.1) is globally asymptotically stable. □

Remark 4.1. As system (4.1) includes systems (1.2), (1.3), and (1.4) as particular situations, then Theorem 4.2 provides an answer to points 2, 3, and 5 of the list of open problems left by Berezansky et.al. [7].

Now, we assume that a_i, H_{ijk}, and κ_{ij} are bounded functions for all $i,j \in \{1,\dots,n\}$ and $k \in \{1,\dots,K\}$. Consequently, there are $\overline{a}_i, \overline{H}_{ijk}, \overline{\kappa}_{ij} > 0$ such that

$$a_i(t) \leq \overline{a}_i, \quad H_{ijk}(t) \leq \overline{H}_{ijk}, \quad \text{and} \quad \kappa_{ij}(t) \leq \overline{\kappa}_{ij}, \quad t \geq 0, \ i,j \in \{1,\dots,n\}, \ k \in \{1,\dots,K\}. \tag{4.8}$$

We define the matrix $\mathcal{A}$ as follows

$$\mathcal{A} = [a_{ij}]_{i,j=1}^{n}, \qquad a_{ii} = 1 - \frac{\underline{a}_i(\overline{a}_i + \widetilde{L}_{ii})\overline{\tau}_i + \widetilde{L}_{ii}}{\underline{a}_i}, \qquad a_{ij} = -\frac{\underline{a}_i \widetilde{L}_{ij}\overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a}_i}, \ i \neq j, \tag{4.9}$$

where $\widetilde{L}_{ij} = \left(\overline{\kappa}_{ij} F_{ij} + \sum_{k=1}^{K} \overline{H}_{ijk} \right)$, for all $i,j \in \{1,\dots,n\}$.

Corollary 4.3. *Assume (a1)-(a5),* (3.7), *and* (4.8).

If the matrix $\mathcal{A}$ in (4.9) *is a non-singular M-matrix, then system* (4.1) *is globally asymptotically stable.*

Proof. In order to apply Theorem 4.2 to obtain the global asymptotic stability of system (4.1), it is enough to verify (a6).

As $\mathcal{A}$ is a non-singular M-matrix, then [16, Theorem 5.1.] there exists $(d_1, \dots, d_n) > 0$ such that

$$\begin{aligned} d_i a_{ii} > \sum_{\substack{j=1\\ j\neq i}}^n d_j(-a_{ij}) \quad &\Leftrightarrow \quad 1 - \frac{\underline{a}_i(\overline{a}_i + \widetilde{L}_{ii})\overline{\tau}_i + \widetilde{L}_{ii}}{\underline{a}_i} > \sum_{\substack{j=1\\ j\neq i}}^n \frac{d_j}{d_i} \frac{\underline{a}_i \widetilde{L}_{ij}\overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a}_i} \\ &\Leftrightarrow \quad \overline{a}_i\overline{\tau}_i + \sum_{j=1}^n \frac{d_j}{d_i} \frac{\underline{a}_i \widetilde{L}_{ij}\overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a}_i} < 1 \\ &\Leftrightarrow \quad \underline{a}_i^{-1} \sum_{j=1}^n \frac{d_j}{d_i}\widetilde{L}_{ij} + \overline{a}_i\overline{\tau}_i + \sum_{j=1}^n \frac{d_j}{d_i}\widetilde{L}_{ij}\overline{\tau}_i < 1, \qquad i \in \{1, \dots, n\}. \end{aligned} \tag{4.10}$$

From the definition of $\widetilde{L}_{ij}$ and the inequalities (3.7), (4.8), and (4.10), trivially we conclude that (a6) holds and the proof is concluded. □

The next result shows that, in case of finite delays, the hypotheses in Corollary 4.3 are enough to obtain the global exponential stability of the system

$$\begin{aligned} x_i'(t) = -a_i(t)x_i(t-\tau_i(t)) &+ \sum_{j=1}^n \sum_{k=1}^K h_{ijk}(t, x_j(t-\tau_{ijk}(t))) \\ &+ \sum_{j=1}^n \int_{t-\varrho_j}^t K_{ij}(t,s) f_{ij}(s, x_j(s-\varrho_{ij}(s)))ds, \qquad t \geq 0,\ i \in \{1\dots,n\}, \end{aligned} \tag{4.11}$$

with $\varrho_j > 0$, for $j \in \{1, \dots, n\}$.

Theorem 4.4. *Assume (a1), (a3)-(a5),* (3.7), (4.8)*, and there exists* $\tau^* > 0$ *such that*

$$\tau_{ijk}(t) \leq \tau^*, \qquad t \geq 0,\ i,j \in \{1, \dots, n\},\ k \in \{1, \dots, K\}. \tag{4.12}$$

If the matrix $\mathcal{A}$ *in* (4.9) *is a non-singular M-matrix, then system* (4.11) *is globally exponentially stable.*

Proof. As $\mathcal{A}$ is a non-singular M-matrix, then (see [16, Theorem 5.1.]) there is $d = (d_1, \dots, d_n) > 0$ such that (4.10) holds. Consequently, there is $\lambda > 0$, small enough, such that

$$1 > \lambda\left(\frac{1}{\underline{a}_i} + \overline{\tau}_i\right) + \overline{a}_i\overline{\tau}_i e^{\lambda\overline{\tau}_i} + e^{\lambda\gamma} \sum_{j=1}^n \frac{d_j}{d_i} \frac{\underline{a}_i \widetilde{L}_{ij}\overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a}_i}, \qquad i \in \{1, \dots, n\}, \tag{4.13}$$

where $\gamma = \max\{\tau^*, \varrho^*\}$, with $\varrho^* = \overline{\varrho} + \max\{\varrho_1, \dots, \varrho_n\}$.

With the change of variables $y_i(t) = d_i^{-1}e^{\lambda t}x_i(t)$ system (4.11) takes the form

$$\begin{aligned} y_i'(t) = -a_i(t)e^{\lambda\tau_i(t)}y_i(t-\tau_i(t)) + \lambda y_i(t) &+ \sum_{j=1}^n\sum_{k=1}^K \frac{e^{\lambda t}}{d_i} h_{ijk}\left(t, d_j e^{-\lambda(t-\tau_{ijk}(t))} y_j(t-\tau_{ijk}(t))\right) \\ + \sum_{j=1}^n \frac{e^{\lambda t}}{d_i} \int_{-\varrho_j}^0 & K_{ij}(t,t+s) f_{ij}\left(t+s, d_j e^{-\lambda(s+t-\varrho_{ij}(t+s))} y_j(t+s-\varrho_{ij}(t+s))\right) ds, \end{aligned} \tag{4.14}$$

for $t \geq 0$, and $i \in \{1\dots,n\}$.

Consider UC_g the phase space of (4.14), where $g : (-\infty, 0] \to [1, +\infty)$ is defined by

$$g(s) = \begin{cases} 1, & s \in [-\varrho^*, 0] \\ -s+1-\varrho^*, & s \in (-\infty, -\varrho^*) \end{cases}. \tag{4.15}$$

System (4.14) is a particular case of (2.4). In fact, system (4.14) is obtained if we take in (2.4) $m = nK$, identifying again each $p \in \{1, \dots, nK\}$ with (j, k), i.e. $p \equiv (j, k)$ for $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, K\}$,

$$h_i \left(t, u^{(1,1)}, \dots, u^{(1,K)}, \dots, u^{(n,1)}, \dots, u^{(n,K)}\right) = \sum_{j=1}^{n} \sum_{k=1}^{K} \frac{\mathrm{e}^{\lambda t}}{d_i} h_{ijk} \left(t, d_j \mathrm{e}^{-\lambda(t-\tau_{ijk}(t))} u_j^{(j,k)}\right), \quad (4.16)$$

for $u^{(p)} \equiv u^{(j,k)} = \left(u_1^{(j,k)}, \dots, u_n^{(j,k)}\right) \in \mathbb{R}^n$, $\tau_{ip}(t) \equiv \tau_{i(j,k)}(t) = \tau_{ijk}(t)$, and

$$\begin{aligned} f_i(t, \phi) &= \lambda \phi_i(0) \\ &\quad + \sum_{j=1}^{n} \frac{\mathrm{e}^{\lambda t}}{d_i} \int_{-\varrho_j}^{0} K_{ij}(t, t+s) f_{ij} \left(t+s, d_j \mathrm{e}^{-\lambda(t+s-\varrho_{ij}(t+s))} \phi_j(s - \varrho_{ij}(t+s))\right) ds, \end{aligned}$$

for all $t \geq 0$, $\phi = (\phi_1, \dots, \phi_n) \in BC$, $p \in \{1, \dots, nK\} \equiv \{(1,1), \dots, (n,K)\}$, $k \in \{1, \dots, K\}$, and $i, j \in \{1, \dots, n\}$.

In order to apply Corollary 3.2, we need to show that hypotheses (A3), (A4), and (A5) hold.

From (a3), (4.8), and (4.12), it is easy to show that each function h_i, defined by (4.16), satisfies (A3) with

$$H_i(t) = \sum_{j=1}^{n} \frac{d_j \mathrm{e}^{\lambda \tau^*}}{d_i} \left(\sum_{k=1}^{K} \overline{H}_{ijk}\right), \qquad t \geq 0,\ i \in \{1, \dots, n\}. \quad (4.17)$$

For each $i \in \{1, \dots, n\}$, $t \geq 0$, and $\varphi = (\varphi_1, \dots, \varphi_m), \phi = (\phi_1, \dots, \phi_n) \in BC$, from (a1), (a4), (b5), (4.8), and (4.15) we have

$$\begin{aligned} |f_i(t, \varphi) - f_i(t, \phi)| &\leq \lambda |\varphi_i(0) - \phi_i(0)| + \sum_{j=1}^{n} \int_{-\varrho_j}^{0} \left(\frac{\mathrm{e}^{\lambda t}}{d_i} |K_{ij}(t, t+s)| F_{ij} \mathrm{e}^{-\lambda(t+s-\varrho_{ij}(t+s))} \times \right. \\ &\qquad \left. \times d_j \left|\varphi_j(s - \varrho_{ij}(t+s)) - \phi_j(s - \varrho_{ij}(t+s))\right| \right) ds \\ &\leq \lambda |\varphi_i(0) - \phi_i(0)| + \sum_{j=1}^{n} \int_{-\varrho_j}^{0} \left(\frac{d_j}{d_i} \kappa_{ij}(t) g_{ij}(-s) F_{ij} \mathrm{e}^{-\lambda(s-\overline{\varrho})} \times \right. \\ &\qquad \left. \times \left|\varphi_j(s - \varrho_{ij}(t+s)) - \phi_j(s - \varrho_{ij}(t+s))\right| \right) ds \\ &\leq \left(\lambda + \sum_{j=1}^{n} \frac{d_j}{d_i} \kappa_{ij}(t) F_{ij} \mathrm{e}^{\lambda \varrho^*} \int_{-\varrho_j}^{0} g_{ij}(-s) ds \right) \|\varphi - \phi\|_g \\ &\leq \left(\lambda + \sum_{j=1}^{n} \frac{d_j}{d_i} \overline{\kappa}_{ij} F_{ij} \mathrm{e}^{\lambda \varrho^*} \right) \|\varphi - \phi\|_g, \end{aligned}$$

and (A4) holds with $F_i(t) = \lambda + \sum_{j=1}^{n} \frac{d_j \mathrm{e}^{\lambda \varrho^*}}{d_i} \overline{\kappa}_{ij} F_{ij}$.

From (a1), (3.7), (4.8), (4.13), and (4.17), for each $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} &\limsup_{t \to +\infty} \left(\frac{F_i(t) + H_i(t)}{a_i(t) \mathrm{e}^{\lambda \tau_i(t)}} + \int_{t-\tau_i(t)}^{t} \left[a_i(w) \mathrm{e}^{\lambda \tau_i(w)} + F_i(w) + H_i(w) \right] dw \right) \\ &\leq \limsup_{t \to +\infty} \left(\frac{\lambda + \mathrm{e}^{\lambda \gamma} \sum_{j=1}^{n} \frac{d_j}{d_i} \widetilde{L}_{ij}}{\underline{a_i}} + \int_{t-\overline{\tau}_i}^{t} \left(\overline{a}_i \mathrm{e}^{\lambda \overline{\tau}_i} + \lambda + \mathrm{e}^{\lambda \gamma} \sum_{j=1}^{n} \frac{d_j}{d_i} \widetilde{L}_{ij} \right) dw \right) \\ &= \lambda \left(\frac{1}{\underline{a_i}} + \overline{\tau}_i \right) + \overline{a}_i \overline{\tau}_i \mathrm{e}^{\lambda \overline{\tau}_i} + \mathrm{e}^{\lambda \gamma} \sum_{j=1}^{n} \frac{d_j}{d_i} \frac{\underline{a_i} \widetilde{L}_{ij} \overline{\tau}_i + \widetilde{L}_{ij}}{\underline{a_i}} < 1, \end{aligned}$$

and hypothesis (A5) holds.

Now, from Corollary 3.2, there is $C \geq 1$ such that, for all $t_0 \geq 0$ and $\psi, \widetilde{\psi} \in BC$, the solutions $y(t, t_0, \psi)$ and $y(t, t_0, \widetilde{\psi})$ of (4.14) satisfy

$$|y(t, t_0, \psi) - y(t, t_0, \widetilde{\psi})| \leq C\|\psi - \widetilde{\psi}\|, \quad t \geq t_0.$$

Let $t_0 \geq 0$ and $\varphi, \widetilde{\varphi} \in BC$ and consider solutions $x(t) = x(t, t_0, \varphi)$ and $\widetilde{x}(t) = x(t, t_0, \widetilde{\varphi})$ of (4.11). Consequently, $y(t) = \mathrm{e}^{\lambda t} d^{-1} \cdot x(t)$ and $\widetilde{y}(t) = \mathrm{e}^{\lambda t} d^{-1} \cdot \widetilde{x}(t)$ are solutions of (4.14) and we obtain

$$\begin{aligned}
|y(t) - \widetilde{y}(t)| \leq C\|y_{t_0} - \widetilde{y}_{t_0}\| \quad &\Leftrightarrow \quad |\mathrm{e}^{\lambda t} d^{-1} \cdot x(t) - \mathrm{e}^{\lambda t} d^{-1} \cdot \widetilde{x}(t)| \leq C\|\mathrm{e}^{\lambda(t_0+\cdot)} d^{-1} \cdot \varphi - \mathrm{e}^{\lambda(t_0+\cdot)} d^{-1} \cdot \widetilde{\varphi}\| \\
&\Rightarrow \quad \mathrm{e}^{\lambda t} \min_i \{d_i^{-1}\} |x(t) - \widetilde{x}(t)| \leq C\mathrm{e}^{\lambda t_0} \max_i \{d_i^{-1}\} \|\varphi - \widetilde{\varphi}\| \\
&\Leftrightarrow \quad |x(t, t_0, \varphi) - \widetilde{x}(t, t_0, \widetilde{\varphi})| \leq C \frac{\max_i\{d_i\}}{\min_i\{d_i\}} \mathrm{e}^{-\lambda(t-t_0)} \|\varphi - \widetilde{\varphi}\|,
\end{aligned}$$

for all $t \geq t_0$. Thus system (4.11) is globally exponentially stable. □

Remark 4.2. In [7], the global exponential stability of (1.1) was obtained assuming finite delays, i.e. condition (4.12) holds, bounded linear coefficients, i.e. $0 < \underline{a}_i \leq a_i(t) \leq \overline{a}_i$ for all $t \geq 0$ and $i \in \{1, \ldots, n\}$, functions h_{ij} satisfying

$$|h_{ij}(t, u)| \leq \widetilde{H}_{ij}|u|, \quad t \geq 0, \ u \in \mathbb{R}, \ i, j \in \{1, \ldots, n\},$$

for some $\widetilde{H}_{ij} > 0$, and the matrix $\mathcal{A}^*$, defined by

$$\mathcal{A}^* = \left[a^*_{ij}\right]_{i,j=1}^n, \quad a^*_{ii} = 1 - \frac{\overline{a}_i(\overline{a}_i + \widetilde{H}_{ii})\overline{\tau}_i + \widetilde{H}_{ii}}{\underline{a}_i}, \quad a^*_{ij} = -\frac{\overline{a}_i\widetilde{H}_{ij}\overline{\tau}_i + \widetilde{H}_{ij}}{\underline{a}_i}, \ i \neq j, \tag{4.18}$$

being a non-singular M-matrix. In case $K = 1$ and $F_{ij} = 0$ because system (1.1) has no distributed delays, we have $\mathcal{A}^* \leq \mathcal{A}$ and, from [16, Theorem 5.7], we conclude that if $\mathcal{A}^*$ is a non-singular M-matrix, then $\mathcal{A}$ is also a non-singular M-matrix. Since the reverse is false, as it is illustrated by the numerical example in the last section, then Theorem 4.4 improves and extends the main result in [7] (see also Remark 3.1). We should remark that we deal with continuous functions, whereas [7] deals with Lebesgue mensurable functions. Consequently, our improvements and extensions refer only to models with continuous coefficients.

Now, we apply the previous results to establish global stability criteria for the following BAM neural network with unbounded time-varying delays, infinite distributed delays, and finite delays in the leakage terms:

$$\begin{cases}
x'_i(t) = -\hat{b}_i(t)x_i(t - \hat{r}_i(t)) + \displaystyle\sum_{j=1}^{k} c_{ij}(t)l_j(y_j(t)) + \sum_{j=1}^{k} d_{ij}(t)l_j(y_j(t - r_{ij}(t))) \\
\qquad\qquad + \displaystyle\sum_{j=1}^{k} e_{ij}(t) \int_{-\infty}^{t} \mathcal{K}_{ij}(t-s)g_j(y_j(s - \varrho_{ij}(s)))ds + \hat{I}_i(t), \quad i \in \{1, \ldots, \hat{k}\}, \\
y'_j(t) = -b_j(t)y_j(t - r_j(t)) + \displaystyle\sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{l}_i(x_i(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{l}_i(x_i(t - \hat{r}_{ji}(t))) \\
\qquad\qquad + \displaystyle\sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t) \int_{-\infty}^{t} \hat{\mathcal{K}}_{ji}(t-s)\hat{g}_i(x_i(s - \hat{\varrho}_{ji}(s)))ds + I_j(t), \quad j \in \{1, \ldots, k\}
\end{cases} \tag{4.19}$$

for $t \geq 0$, where $k, \hat{k} \in \mathbb{N}$, $b_i, \hat{b}_j : [0, +\infty) \to (0, +\infty)$, $\hat{r}_i, r_j, r_{ij}, \hat{r}_{ji} : [0, +\infty) \to [0, +\infty)$, $\varrho_{ij}, \hat{\varrho}_{ji} : \mathbb{R} \to [0, +\infty)$, $l_j, \hat{l}_i, g_j, \hat{g}_i : \mathbb{R} \to \mathbb{R}$, and $c_{ij}, \hat{c}_{ji}, d_{ij}, \hat{d}_{ji}, e_{ij}, \hat{e}_{ji}, I_j, \hat{I}_i : [0, +\infty) \to \mathbb{R}$ are continuous functions, and $\mathcal{K}_{ij}, \hat{\mathcal{K}}_{ji} : [0, +\infty) \to [0, +\infty)$ are piecewise continuous functions, for all $i \in \{1, \ldots, \hat{k}\}$ and $j \in \{1, \ldots, k\}$.

Model (4.19) is a generalization of some BAM neural network models studied in recent literature [7, 30, 31, 39, 42].

For model (4.19), consider the following hypotheses. For each $i \in \{1, \dots, \hat{k}\}$ and $j \in \{1, \dots, k\}$:

(BAM1) there are $\underline{b}_j, \underline{\hat{b}}_i, \overline{r}_j, \overline{\hat{r}}_i, \overline{\varrho} > 0$ such that

$$\underline{b}_j \leq b_j(t),\ \underline{\hat{b}}_i \leq \hat{b}_i(t),\ r_j(t) \leq \overline{r}_j,\ \hat{r}_i(t) \leq \overline{\hat{r}}_i,\ \varrho_{ij}(s) \leq \overline{\varrho},\ \text{and } \hat{\varrho}_{ji}(s) \leq \overline{\varrho}, \quad t \geq 0,\ s \in \mathbb{R};$$

(BAM2) the time-varying delay functions satisfy

$$\lim_{t \to +\infty} (t - r_{ij}(t)) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} (t - \hat{r}_{ji}(t)) = +\infty;$$

(BAM3) there are $L_j, \hat{L}_i, G_j, \hat{G}_i > 0$ such that

$$\left\{ \begin{array}{l} |l_j(u) - l_j(v)| \leq L_j |u - v| \\ \\ |\hat{l}_i(u) - \hat{l}_i(v)| \leq \hat{L}_i |u - v| \end{array} \right., \quad \text{and} \quad \left\{ \begin{array}{l} |g_j(u) - g_j(v)| \leq G_j |u - v| \\ \\ |\hat{g}_i(u) - \hat{g}_i(v)| \leq \hat{G}_i |u - v| \end{array} \right., \quad t \geq 0,\ u, v \in \mathbb{R};$$

(BAM4) The kernel functions verify

$$\int_0^{+\infty} \mathcal{K}_{ij}(s) ds = \int_0^{+\infty} \hat{\mathcal{K}}_{ji}(s) ds = 1.$$

Model (4.19) is a particular situation of (4.1). In fact, model (4.19) is obtained if we consider in (4.1) $n = \hat{k} + k$, $K = 2$,

$$\tau_i(t) = \left\{ \begin{array}{ll} \hat{r}_i(t), & i \in \{1, \dots, \hat{k}\} \\ r_{i-\hat{k}}(t), & i \in \{\hat{k}+1, \dots, \hat{k}+k\} \end{array} \right., \quad t \geq 0,$$

$$a_i(t) = \left\{ \begin{array}{ll} \hat{b}_i(t), & i \in \{1, \dots, \hat{k}\} \\ b_{i-\hat{k}}(t), & i \in \{\hat{k}+1, \dots, \hat{k}+k\} \end{array} \right., \quad t \geq 0,$$

$$h_{ij1}(t, u) = \left\{ \begin{array}{ll} c_{i(j-\hat{k})}(t) l_{j-\hat{k}}(u) + \frac{\hat{I}_i(t)}{k}, & i \in \{1, \dots, \hat{k}\},\ j \in \{\hat{k}+1, \dots, \hat{k}+k\} \\ \hat{c}_{(i-\hat{k})j}(t) \hat{l}_j(u) + \frac{I_{i-\hat{k}}(t)}{\hat{k}}, & i \in \{\hat{k}+1, \dots, \hat{k}+k\},\ j \in \{1, \dots, \hat{k}\} \\ 0, & \text{otherwise} \end{array} \right., \quad t \geq 0,\ u \in \mathbb{R},$$

$$h_{ij2}(t, u) = \left\{ \begin{array}{ll} d_{i(j-\hat{k})}(t) l_{j-\hat{k}}(u), & i \in \{1, \dots, \hat{k}\},\ j \in \{\hat{k}+1, \dots, \hat{k}+k\} \\ \hat{d}_{(i-\hat{k})j}(t) \hat{l}_j(u), & i \in \{\hat{k}+1, \dots, \hat{k}+k\},\ j \in \{1, \dots, \hat{k}\} \\ 0, & \text{otherwise} \end{array} \right., \quad t \geq 0,\ u \in \mathbb{R},$$

$$K_{ij}(t, s) = \left\{ \begin{array}{ll} e_{i(j-\hat{k})}(t) \mathcal{K}_{i(j-\hat{k})}(t-s), & i \in \{1, \dots, \hat{k}\},\ j \in \{\hat{k}+1, \dots, \hat{k}+k\} \\ \hat{e}_{(i-\hat{k})j}(t) \hat{\mathcal{K}}_{(i-\hat{k})j}(t-s), & i \in \{\hat{k}+1, \dots, \hat{k}+k\},\ j \in \{1, \dots, \hat{k}\} \\ 0, & \text{otherwise} \end{array} \right., \quad t \geq 0,\ t \geq s,$$

$$f_{ij}(s, u) = \left\{ \begin{array}{ll} g_{j-\hat{k}}(u), & i \in \{1, \dots, \hat{k}\},\ j \in \{\hat{k}+1, \dots, \hat{k}+k\} \\ \hat{g}_j(u), & i \in \{\hat{k}+1, \dots, \hat{k}+k\},\ j \in \{1, \dots, \hat{k}\} \\ 0, & \text{otherwise} \end{array} \right., \quad s, u \in \mathbb{R}.$$

Consequently, from Theorem 4.2, we trivially obtain the following stability criterion.

Corollary 4.5. *Assume (BAM1)-(BAM4).*

Model (4.19) *is globally asymptotically stable, if there are* $d=(d_1,\dots,d_k)>0$ *and* $\hat{d}=(\hat{d}_1,\dots,\hat{d}_{\hat{k}})>0$ *such that*

$$\begin{cases} \limsup\limits_{t\to+\infty}\left(\dfrac{\mathcal{H}_i(t,d)+\mathcal{G}_i(t,d)}{\hat{d}_i\hat{b}_i(t)}+\displaystyle\int_{t-\hat{r}_i(t)}^{t}\hat{b}_i(w)+\dfrac{\mathcal{H}_i(w,d)}{\hat{d}_i}+\dfrac{\mathcal{G}_i(w,d)}{\hat{d}_i}dw\right)<1, \\ \hfill i\in\{1,\dots,\hat{k}\} \\ \limsup\limits_{t\to+\infty}\left(\dfrac{\hat{\mathcal{H}}_i(t,\hat{d})+\hat{\mathcal{G}}_i(t,\hat{d})}{d_{i-\hat{k}}b_i(t)}+\displaystyle\int_{t-r_{i-\hat{k}}(t)}^{t}b_{i-\hat{k}}(w)+\dfrac{\hat{\mathcal{H}}_i(w,\hat{d})}{d_{i-\hat{k}}}+\dfrac{\hat{\mathcal{G}}_i(w,\hat{d})}{d_{i-\hat{k}}}dw\right)<1, \\ \hfill i\in\{\hat{k}+1,\dots,\hat{k}+k\} \end{cases}$$

where

$$\mathcal{H}_i(t,d)=\sum_{j=1}^{k}\left(|c_{ij}(t)|+|d_{ij}(t)|\right)L_jd_j,\qquad \mathcal{G}_i(t,d)=\sum_{j=1}^{k}|e_{ij}(t)|G_jd_j,\quad t\geq 0,\ i\in\{1,\dots,\hat{k}\};$$

$$\hat{\mathcal{H}}_i(t,\hat{d})=\sum_{j=1}^{\hat{k}}\left(|\hat{c}_{(i-\hat{k})j}(t)|+|\hat{d}_{(i-\hat{k})j}(t)|\right)\hat{L}_j\hat{d}_j,\qquad t\geq 0,\ i\in\{\hat{k}+1,\dots,\hat{k}+k\};$$

$$\hat{\mathcal{G}}_i(t,\hat{d})=\sum_{j=1}^{\hat{k}}|\hat{e}_{(i-\hat{k})j}(t)|\hat{G}_j\hat{d}_j,\qquad t\geq 0,\ i\in\{\hat{k}+1,\dots,\hat{k}+k\}.$$

Now, we assume that all coefficient functions are bounded, i.e. for all $i\in\{1,\dots,\hat{k}\}$ and $j\in\{1,\dots,k\}$ there are $\bar{\hat{b}}_i,\bar{b}_j,\bar{c}_{ij},\bar{\hat{c}}_{ji},\bar{d}_{ij},\bar{\hat{d}}_{ji},\bar{e}_{ij},\bar{\hat{e}}_{ji}>0$ such that

$$\begin{array}{llll} \hat{b}_i(t)\leq\bar{\hat{b}}_i, & b_j(t)\leq\bar{b}_j, & |c_{ij}(t)|\leq\bar{c}_{ij}, & |\hat{c}_{ji}(t)|\leq\bar{\hat{c}}_{ji}, \\ |d_{ij}(t)|\leq\bar{d}_{ij}, & |\hat{d}_{ji}(t)|\leq\bar{\hat{d}}_{ji}, & |e_{ij}(t)|\leq\bar{e}_{ij}, & \text{and } |\hat{e}_{ji}(t)|\leq\bar{\hat{e}}_{ji}, \end{array}\tag{4.20}$$

for all $t\geq 0$.

We define the matrix $\mathcal{B}$ as follows

$$\mathcal{B}=\begin{bmatrix}\hat{\mathcal{D}} & \mathcal{P}\\ \hat{\mathcal{P}} & \mathcal{D}\end{bmatrix}_{(\hat{k}+k)\times(\hat{k}+k)},\tag{4.21}$$

where $\hat{\mathcal{D}}=diag\left(1-\bar{\hat{b}}_1\bar{\hat{r}}_1,\dots,1-\bar{\hat{b}}_{\hat{k}}\bar{\hat{r}}_{\hat{k}}\right)$, $\mathcal{D}=diag\left(1-\bar{b}_1\bar{r}_1,\dots,1-\bar{b}_k\bar{r}_k\right)$, $\mathcal{P}=[-p_{ij}]_{\hat{k}\times k}$ with $p_{ij}=\left(\bar{\hat{r}}_i+\underline{\hat{b}}_i^{-1}\right)[(\bar{c}_{ij}+\bar{d}_{ij})L_j+\bar{e}_{ij}G_j]$, and $\hat{\mathcal{P}}=[-\hat{p}_{ji}]_{k\times\hat{k}}$ with $\hat{p}_{ji}=\left(\bar{r}_j+\underline{b}_j^{-1}\right)[(\bar{\hat{c}}_{ji}+\bar{\hat{d}}_{ji})\hat{L}_i+\bar{\hat{e}}_{ji}\hat{G}_i]$.

As an immediate consequence of Corollary 4.3, we obtain

Corollary 4.6. *Assume (BAM1)-(BAM4) and* (4.20)*.*

If the matrix $\mathcal{B}$ *in* (4.21) *is a non-singular M-matrix, then model* (4.19) *is globally asymptotically stable.*

Finally, for the situation of BAM model (4.19) with finite delays, Theorem 4.4 allowed us to obtain the next result.

Corollary 4.7. *Assume (BAM1), (BAM3), (BAM4),* (4.20)*, and, for each* $i\in\{1,\dots,\hat{k}\}$ *and* $j\in\{1,\dots,k\}$, $\hat{\varrho}_i>0$, $\varrho_j>0$, *and there is* $r^*>0$ *such that*

$$r_{ij}(t)\leq r^*,\qquad \hat{r}_{ji}(t)\leq r^*.$$

If the matrix $\mathcal{B}$ in (4.21) *is a non-singular M-matrix, then model*

$$\begin{cases} x_i'(t) = -\hat{b}_i(t)x_i(t-\hat{r}_i(t)) + \sum_{j=1}^{k} c_{ij}(t)l_j(y_j(t)) + \sum_{j=1}^{k} d_{ij}(t)l_j(y_j(t-r_{ij}(t))) \\ \qquad + \sum_{j=1}^{k} e_{ij}(t)\int_{t-\hat{\varrho}_i}^{t} K_{ij}(t-s)g_j(y_j(s-\varrho_{ij}(s)))ds + \hat{I}_i(t), \quad i \in \{1,\dots,\hat{k}\}, \\ y_j'(t) = -b_j(t)y_j(t-r_j(t)) + \sum_{i=1}^{\hat{k}} \hat{c}_{ji}(t)\hat{l}_i(x_i(t)) + \sum_{i=1}^{\hat{k}} \hat{d}_{ji}(t)\hat{l}_i(x_i(t-\hat{r}_{ji}(t))) \\ \qquad + \sum_{i=1}^{\hat{k}} \hat{e}_{ji}(t)\int_{t-\varrho_j}^{t} \hat{K}_{ji}(t-s)\hat{g}_i(x_i(s-\hat{\varrho}_{ji}(s)))ds + I_j(t), \quad j \in \{1,\dots,k\} \end{cases} \tag{4.22}$$

is globally exponentially stable.

Remark 4.3. We remark that Corollary 4.7 is an extension of the main global exponential stability criterion in [30] to nonautonomous BAM neural network models with finite time-varying delays and finite distributed delays.

Remark 4.4. Model (4.22) has the BAM neural network model [7, model (3.1)] as a particular situation. Thus, for continuous coefficient and activation functions, Corollary 4.7 improves and extends the stability result given by [7, Theorem 3.4].

Remark 4.5. Using a suitable Lyapunov-Krasovskii functional, in [31] the authors established a different global exponential stability criterion for model (4.22) with constant coefficients.

5 Numerical example

Now, we present a numerical example to illustrate the novelty of some new stability criteria given in this work.

Example 5.1. The system

$$\begin{cases} x'(t) = -(6+\sin(t))x\left(t - \frac{|\sin(t)|}{9}\right) + c\sin(t)y(t) + dy(t-10-pt) + e\int_{-\infty}^{t} \mathrm{e}^{-t+s}y(s)ds \\ y'(t) = -(4+\cos(t))y\left(t - \frac{|\cos(t)|}{9}\right) + \hat{c}\arctan(x(t)) + \hat{d}\arctan\left(x(t-10-q\log(t+1))\right) \\ \qquad\qquad + \hat{e}\int_{-\infty}^{t} \mathrm{e}^{-t+s}x(s)ds \end{cases}, \tag{5.1}$$

where $p \in [0,1)$, $q \geq 0$, and $c, \hat{c}, d, \hat{d}, e, \hat{e} \in \mathbb{R}$, is a particular situation of (4.19). Here $k = \hat{k} = 1$, $\bar{\hat{b}}_1 = 7$, $\underline{\hat{b}}_1 = 5$, $\bar{b}_1 = 5$, $\underline{b}_1 = 3$, $\bar{\hat{r}}_1 = \bar{r}_1 = \frac{1}{9}$, and $L_1 = \hat{L}_1 = G_1 = \hat{G}_1 = 1$. In case $|c|+|d|+|e| = \frac{15}{14}$ and $|\hat{c}|+|\hat{d}|+|\hat{e}| = \frac{1}{2}$, the matrix $\mathcal{B}$ in (4.21) reads as

$$\begin{bmatrix} \frac{2}{9} & -\frac{3}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix},$$

which is a non-singular M-matrix (the eigenvalues are $\frac{3+\sqrt{7}}{9}$ and $\frac{3-\sqrt{7}}{9}$). Consequently, all hypotheses of Corollary 4.6 are satisfied, hence system (5.1) is global asymptotically stable (see the numerical simulation of three solutions $(x(t), y(t))$ of (5.1) with $p = \frac{1}{2}$, $q = 1$, $c = \frac{1}{14}$, $d = 1$, $e = \hat{e} = 0$, and $\hat{c} = \hat{d} = \frac{1}{4}$ in Figure 1. The blue graphs correspond to the first coordinate, $x(t)$, of three solutions, while the brown graphs correspond to the second coordinate, $y(t)$, of three solutions.). We should say that the Mathlab software, [35], was used to plot the numerical simulations of solutions.

In case of $p = q = 0$ and $e = \hat{e} = 0$, system (5.1) has finite delays. In this setting, if $|c| + |d| = \frac{15}{14}$ and $|\hat{c}| + |\hat{d}| = \frac{1}{2}$, then Corollary 4.7 (or Theorem 4.4) assures the global exponential stability of (5.1).

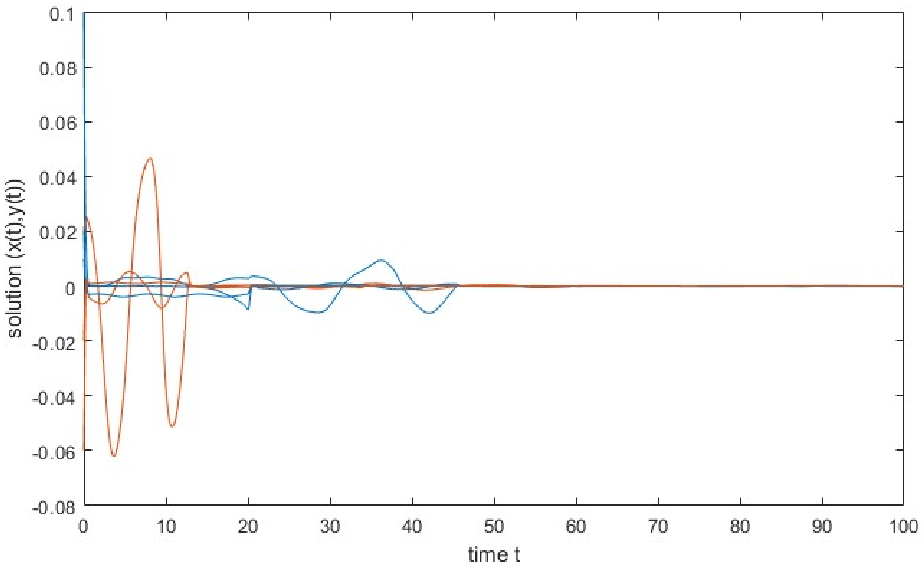

Figure 1: Numerical simulations of three solutions $(x(t), y(t))$ of system (5.1), where $p = \frac{1}{2}$, $q = 1$, $c = \frac{1}{14}$, $d = 1$, $e = \hat{e} = 0$, and $\hat{c} = \hat{d} = \frac{1}{4}$, with initial condition $\varphi(s) = (0.01+\sin(s), 0.01+0.01\cos(s))$, $\varphi(s) = (0.1\cos(s), -0.06e^{s})$, $\varphi(s) = (-0.02, 0.02)$ for $s \leq 0$, respectively, at $t_0 = 0$.

Remark 5.1. System (5.1) with $p = q = 0$, $c = \hat{c} = e = \hat{e} = 0$, $d = \frac{15}{14}$, and $\hat{d} = \frac{1}{2}$ reads as

$$\begin{cases} x'(t) = -(6 + \sin(t))x\left(t - \frac{|\sin(t)|}{9}\right) + \frac{15}{14}y(t-10) \\ y'(t) = -(4 + \cos(t))y\left(t - \frac{|\cos(t)|}{9}\right) + \frac{1}{2}\arctan\left(x(t-10)\right) \end{cases}, \tag{5.2}$$

which is a particular case of (1.1). In this case, matrix $\mathcal{A}$, defined by (4.9), and matrix $\mathcal{A}^*$, defined by (4.18), are

$$\mathcal{A} = \begin{bmatrix} \frac{2}{9} & -\frac{3}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix} \quad \text{and} \quad \mathcal{A}^* = \begin{bmatrix} -\frac{4}{45} & -\frac{8}{21} \\ -\frac{7}{27} & \frac{2}{27} \end{bmatrix},$$

respectively. Since $\mathcal{A}^*$ is not a non-singular M-matrix, then it is not possible to apply [7, Theorem 2.5.] to obtain the global exponential stability of the zero solution of (5.2). This particular example illustrate that Theorem 4.4 improves the main result in [7].

6 Conclusions

In this paper, we present a criterion for global asymptotic stability of a general family of functional differential equations with infinite delays (Theorem 3.3). With the theoretical result, we give answers to points 2, 3, and 5 of the list of open problems presented in [7] (Theorem 4.2). Moreover, considering the particular model studied in [7], model (1.1), our exponential stability criterion is better (Theorem 4.4 and Remark 4.2).

Regarding applications to neural network models, we obtained a global asymptotic stability criterion for a BAM neural network model with infinite delays, which generalizes some models in recent literature (Corollary 4.5). In the case of a model with finite delays, it is possible to obtain global exponential stability (Corollary 4.7).

The proof method based on non-singular M-matrices is easier to apply than the usual Lyapunov method and the hypotheses are normally easy to verify, as it is illustrated by the numerical example.

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