TOPOLOGICAL ASPECTS OF INCOMPRESSIBLE FLOWS

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Abstract. In this article we approach some of the basic questions in topological dynamics, concerning periodic points, transitivity, the shadowing and the gluing orbit properties, in the context of C^0 incompressible flows generated by Lipschitz vector fields. We prove that a C^0-generic incompressible and fixed-point free flow satisfies the periodic shadowing property, it is transitive and has a dense set of periodic points in the non-wandering set. In particular, a C^0-generic fixed-point free incompressible flow satisfies the reparametrized gluing orbit property. We also prove that C^0-generic incompressible flows satisfy the general density theorem and the weak shadowing property, moreover these are transitive.

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1. Introduction

1.1. **The Lipschitz vector fields.** Dynamists are always keen to integrate a vector field into a flow that can be understood from the dynamical point of view. It is well-known that, when working with vector fields, and in order to have Picard-Lindelöf uniqueness of integrability into a flow, Lipschitz continuity is enough. Nonetheless, if we weaken from Lipschitz to Hölder continuity then we lose the uniqueness of integrability, as the simple example $\dot{x} = 2\sqrt{|x|}$ with $x(0) = 0$ displaying two solutions $x(t) = t^2$ and $x(t) = 0$, shows. Ultimately, Lipschitz continuity is in the threshold with respect to the output of a dynamical system or not. Due to the success of the *Hyperbolic Dynamics theory* - note this theory is established in the differential aspects of the vector field namely in the use of properties of the first derivative to characterize local and global dynamics - in the 1960's the understanding of local and global aspects of flows lead to consider a smooth setup instead of a Lipschitz one. Nowadays there is a vast literature on the behavior of C^r ($r \geq 1$) vector fields, not only from the C^r stability focus but also from the C^r generic perspective. So it is natural to ask what can be said about the dynamics of flows derived from vector fields in the broader Lipschitz regularity class and endowed with a coarser topology. Our direction clearly points to a much larger world not only because C^r vector fields are Lipschitz vector fields but also grosser topologies can 'reach' more vector fields. Of course that there is no hope to obtain an interesting C^0-stability theory. Indeed while *topological stability* has a C^0 flavour, still good properties are achieved under topological stability if we consider C^1-dynamics with some robust properties.

As our goal is to understand properties like shadowing, transitivity, denseness of periodic points and the gluing orbit property we can go a long way by considering a C^0 approach. The space of Lipschitz continuous vector fields with a uniformly bounded Lipschitz constant is complete when endowed with the C^0-topology, hence a Baire space. In consequence, the space of Lipschitz continuous vector fields is Baire although not complete.

1.2. **The incompressible setting.** The study of flows which preserve a volume form is a central object not only in applied mathematics such as classical, fluid, continuum and quantum mechanics but also in pure aspects of mathematics like contact and symplectic geometry. We call such a flow an *incompressible flow* or a *volume-preserving* flow. The vector fields associated to incompressible flows are called *divergence-free* or *solenoidal* vector fields. Incompressible flows are free of friction and keep an invariant volume form unchanged as they evolve in time and for this reason are quite interesting from the ergodic theory perspective. Perturbations of incompressible flows within general ones is a delicate issue because perturbations must keep the incompressible characteristic of the flow which turn to be an additional difficulty.

1.3. **The dynamical properties.**

1.3.1. *Chaotic flow.* Throughout this paper, a flow is said to be *chaotic* if it has a dense set of periodic orbits, it is transitive and satisfies the shadowing property. Under this terminology, a chaotic flow satisfies the reparametrized gluing orbit property, which is a specification-like property in the continuous-time setting (see [6, 11]). There are several concepts of chaotic dynamics, some of which are related, including the presence of positive Lyapunov exponents, positive metric entropy, positive topological entropy, expansiveness, Li-Yorke pairs, etc. If the dynamics is not minimal and equicontinuous, then the gluing orbit property ensures that the dynamics has positive topological entropy and strong recurrence properties (see [9, 38]).

1.3.2. *Abundance of closed solutions.* The search for periodic trajectories for flows associated with systems of differential equations has been a central issue in dynamics since the work of Poincaré on celestial mechanics more than a century ago. Smale conjectured that there could not be, in compact spaces, dynamical systems (diffeomorphisms of dimension 2, or flows of dimension 3) with infinite periodic points. He himself helped to refute this conjecture, building the 'Smale horseshoe' (dimension 2 diffeomorphism) based on Levinson's suggestions about previous work by Cartwright and Littlewood. We recall that Shilnikov exhibited a similar phenomenon for three-dimensional flows. The abundance of periodic orbits on the manifold gives us information about the dynamical complexity of the system. A major result is the general density theorem which is a direct consequence of the combination of the closing lemma and the stability and persistence of non-degenerated (e.g. hyperbolic) closed orbits. The general density theorem, first established in the C^1-class by Pugh ([33]) in the late 1960's and much later by Pugh and Robinson for volume-preserving and symplectic diffeomorphisms as also for Hamiltonians (see [34]), gives us a residual subset on which periodic orbits are dense in the non-wandering set.

1.3.3. *Shadowing.* In brief terms the shadowing property, which consists of a reconstruction of a true orbit for the dynamics provided a set of points that form approximately an orbit, appears in many offshoots on dynamics. This concept is often related with how complex the system is (see [22]). Actually, the computational estimates, fitted with a certain small error allowed to the orbits, are meaningless if they cannot be realized by real orbits of the given dynamical system. The search for the genericity of shadowing is an old question in dynamics. Contrary to the case of smooth dynamics, where C^1-generically the shadowing property is characterized in terms of uniform hyperbolicity, a C^0-generic Lipschitz vector field satisfies the shadowing property (see [6] and the references therein).

1.3.4. *Transitivity.* An incompressible flow is transitive if it displays a dense orbit in the whole manifold. From the topological point of view, a transitive flow cannot be decomposed into more than a single component. It is the topological counterpart of an ergodic flow and a property explored since the classical work by Oxtoby and Ulam [31]. Nowadays we know that C^1-generic incompressible flows are transitive and even topologically mixing (see [4, 7]) but our ambition is to go further under the C^0 hypothesis, and investigate wether C^0-perturbations are sufficient to create robust obstructions to transitivity. The main concept on which we proceed with our goal is called gluing orbit property, which implies on a strong transitivity.

1.3.5. *Gluing orbit property.* The gluing orbit property is a weakening of the notion of specification and briefly means that any finite pieces of orbits can be shadowed by a true orbit where the time lag between the pieces of orbits is bounded above by a constant that depends only on the shadowing distance. Such condition is embracing, as it is satisfied by transitive hyperbolic dynamics, minimal rotations on compact abelian groups and certain classes of partially hyperbolic diffeomorphisms [11, 10, 38]. Precise definitions will be given in the sequel. The gluing orbit property constitutes a useful tool to describe the thermodynamic formalism, large deviations, recurrence and multifractal analysis (see e.g. [9, 11, 12, 36, 39] and references therein). In the context of flows, one needs to reparametrize true orbits of the dynamics that shadow the prescribed pieces of orbits.

1.4. **Objectives of our study.** Our main goal here is to study the abundance of transitivity-like properties of C^0 incompressible flows generated by Lipschitz vector fields and to establish a weaker counterpart of Oxtoby-Ulam theorem: *C^0-generic flows are strongly transitive*, generalizing [6] for the conservative or volume-preserving context. We notice that Oxtoby and Ulam proved that generic volume-preserving homeomorphisms are ergodic (see e.g. [1, 31]) and there is a gap in the literature concerning the continuous-time counterpart.

In our approach, the C^0 genericity of the gluing orbit property arises as a spinoff of a stronger characterization for C^0-generic conservative Lipschitz flows. Like in [6] we begin by proving that both the denseness of periodic orbits and the periodic shadowing property are C^0-generic among conservative Lipschitz flows. It is quite interesting that these two properties altogether are enough to assure that these flows satisfy a reparametrized gluing orbit property and, consequently, are strongly transitive: *the shortest hitting time from a ball to any other ball of the same radius is uniformly bounded above by a constant depending only on the radius.*

2. Main definitions and statement of the results

2.1. **Incompressible flows.** Let M be a connected, closed and C^∞ Riemannian manifold of dimension $n \geq 3$. Since along this paper we only deal with divergence-free vector fields we assume that M is also a volume-manifold with a volume-form ν. Furthermore, we equip M with an atlas $\mathcal{A} = \{(\alpha_i, U_i)_i\}$ of M (cf. [28]), such that $(\alpha_i)_*\nu = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, where x_i are the canonical coordinates in the Euclidian space, $\alpha_i \colon U_i \to \mathbb{R}^n$ a local C^∞ diffeomorphism and U_i an open subset of M. The fact that M is compact enables that $\mathcal{A}$ can be taken finite, say $\mathcal{A} = \{(\alpha_i, U_i)\}_{i=1}^k$. We call Lebesgue measure to the measure associated to ν and denote it by μ. More precisely, we let

$$\mu(U) = \mu_\nu(U) := \int_{\alpha(U)} \nu_{\alpha^{-1}(x)}(D\alpha_1^{-1}\, x_1, \dots, D\alpha_n^{-1}\, x_n)\, dx_1 \dots dx_n,$$

for some Borelian $U \subset M$. Let $d(\cdot,\cdot)$ stand for the metric associated to the Riemannian structure.

We say that a function $F \colon M \to \mathbb{R}$ is *Lipschitz* (or Lipschitz continuous) if, there exists $L > 0$ such that $|F(x) - F(y)| < L\, d(x, y)$ for all $x, y \in M$. A vector field X is a map $X \colon M \to TM$ where TM stands for the tangent bundle. Let X be written in the coordinates associated to $\mathcal{A}$ such that $X = \sum_{i=1}^n X_i \frac{\partial}{\partial x_i}$. If, for every $i = 1, \dots, n$, each function X_i is Lipschitz, then X is said to be a *Lipschitz vector field*. The integral family of curves, $X^t \colon M \to M$, associated to X satisfies $X^{t+s}(x) = X^t(X^s(x))$ and $X^0(x) = x$ for all $t, s \in \mathbb{R}$ and $x \in M$ and is called the *flow* associated to X. Picard's theorem ensures that Lipschitz vector fields integrate Lipschitz flows, meaning that $X^t \colon M \to M$ is a lipeomorphism (i.e. a Lipschitz map with Lipschitz inverse) for all $t \in \mathbb{R}$.

Rademacher's theorem ([16, Theorem 3.1.6]) yields that Lipschitz functions admit derivatives μ-almost surely. Thus, the divergence of a Lipschitz vector field X, given by $\nabla \cdot X\colon M \to \mathbb{R}$, where $\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$, is a well defined function on a μ-full measure subset of M.

We say that a Lipschitz vector field X is *divergence-free* (or *incompressible*) if $\nabla \cdot X = 0$ for μ-a.e. $x \in M$. We denote this set by $\mathfrak{X}^{0,1}_\mu(M)$. More generally, $\mathfrak{X}^{k,\alpha}_\mu(M)$ stands for the divergence-free C^k vector fields whose k^{th}-derivative is α-Hölder continuous. The Hamiltonian and geodesic flows are special but meager families of incompressible flows (see [7] and references therein).

2.2. **Topologies.** When dealing with vector fields it is natural to first consider the C^0 Whitney topology. When considering flows one usually uses the compact open topology which basically measures the way points get apart when we evolve the flow in a compact set of time. For continuous flows generated by vector fields, both topologies are related by using Grönwall's inequality (see e.g. [32]) and standard continuity dependence arguments of differential equations: squeezing the distance of vector fields we squeeze the distance of their flows. Hence, in this context the C^0 Whitney topology is stronger than the compact-open topology.

2.3. **Shadowing.** In what follows we introduce the shadowing properties we will consider for continuous flows on compact manifolds. Let us fix real numbers $\delta > 0$ and $T > 1$. We say that a pair of sequences $[x_i, t_i]_{i\in\mathbb{Z}}$, where $x_i \in M$, $t_i \in \mathbb{R}$, $1 \le t_i \le T$, is a (δ, T)-*pseudo-orbit* of $(X^t)_t$ if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$.

For the sequence $(t_i)_{i\in\mathbb{Z}}$ we write, $\sigma(n) = t_0 + t_1 + \dots + t_{n-1}$ if $n > 0$, $\sigma(n) = -(t_n + \dots + t_{-2} + t_{-1})$ if $n < 0$ and $\sigma(0) = 0$. Let $x_0 \star t$ denote a point on a (δ, T)-chain t units time from x_0. More precisely, for $t \in \mathbb{R}$, $x_0 \star t = X^{t-\sigma(i)}(x_i)$ if $\sigma(i) \le t < \sigma(i+1)$. By Rep we denote the set of all increasing homeomorphisms $\tau\colon \mathbb{R} \to \mathbb{R}$, called *reparametrizations*, satisfying $\tau(0) = 0$. Fixing $\varepsilon > 0$, we define the set

$$\mathrm{Rep}(\varepsilon) = \left\{ \tau \in \mathrm{Rep} : \left| \frac{\tau(t) - \tau(s)}{t - s} - 1 \right| < \varepsilon,\ s, t \in \mathbb{R} \right\},$$

of the reparametrizations ε-close to the identity. In rought terms, a reparametrization $\tau : \mathbb{R} \to \mathbb{R}$ belongs to $\mathrm{Rep}(\varepsilon)$ whenever its velocity at all points is ε-close to 1.

Definition 2.1. Let $(X^t)_t$ be a continuous flow on a compact manifold M.

The flow $(X^t)_t$ is said to have the *shadowing property* if, for any $\varepsilon > 0$ and $T > 1$ there exists $\delta = \delta(\varepsilon, T) > 0$ such that for any (δ, T)-pseudo-orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ there is $\tilde{x} \in M$ and a reparametrization $\tau \in \mathrm{Rep}(\varepsilon)$ such that

$$d(X^{\tau(t)}(\tilde{x}), x_0 \star t) < \varepsilon, \text{ for every } t \in \mathbb{R}. \tag{2.1}$$

For simplicity reasons we will just say that the (δ, T)-pseudo-orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ is ε-shadowed by some orbit of X if property (2.1) holds.

If, in addition, any (δ, T)-periodic pseudo-orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ (i.e. there exists $n \in \mathbb{N}$ so that $(x_i, t_i) = (x_{i+n}, t_{i+n})$ for all $i \in \mathbb{Z}$) is ε-shadowed by a periodic point then we say $(X^t)_t$ has the *periodic shadowing property*.

Definition 2.2. We say that a continuous flow $(X^t)_t$ satisfies the *weak shadowing* property if for any $\varepsilon > 0$ and $T > 1$ there exists $\delta > 0$ so that for every (δ, T)-pseudo orbit $[x_i, t_i]_{i\in\mathbb{Z}}$ there exists $\tilde{x} \in M$ so that the pseudo-orbit is contained in the ε-neighborood $\bigcup_{t\in\mathbb{R}} B(X^t(\tilde{x}), \varepsilon)$ of the orbit of $\tilde{x}$.

2.4. **Reparametrized gluing orbit property and strong transitivity.** Given a continuous flow $(X^t)_t$ on M, Poincaré's recurrence theorem ensures that Lebesgue almost every $x \in M$ is recurrent. In particular M is the unique chain recurrent class for a volume-preserving flow $(X^t)_t$ (see e.g. [2] for definitions).

We are interested in much stronger notions of indecomposability. It is well known that C^0-generic volume-preserving homeomorphisms satisfy both the shadowing and the specification properties (see [18]). However, one cannot expect this result to extend for typical continuous flows, due to the existence of the natural foliation by orbits of the flow and reparametrizations of the dynamics. The following notion is more adequate to deal with continuous-time dynamics (see [11] for a discussion).

Definition 2.3. Let $(X^t)_t$ be a continuous flow on M.

We say that $(X^t)_t$ has the *reparametrized gluing orbit property* if for any $\varepsilon > 0$ there exists $K = K(\varepsilon) \in \mathbb{R}^+$ such that for any points $x_0, x_1, \dots, x_k \in M$ and times $t_0, t_1, \dots, t_k \geq 0$ there are $p_0, p_1, \dots, p_{k-1} \leq K(\varepsilon)$, a reparametrization $\tau \in \text{Rep}(\varepsilon)$ and a point $y \in M$ so that $d(X^{\tau(t)}(y)), X^t(x_0)) < \varepsilon$ for every $t \in [0, t_0]$ and also $d(X^{\tau(t+\sum_{j=0}^{i-1} p_j+t_j)}(y), X^t(x_i)) < \varepsilon$ for every $t \in [0, t_i]$ and $1 \leq i \leq k$. If, in addition, the point $y \in M$ can be chosen periodic satisfying $X^{\tau(\sum_{j=0}^{k} p_j+t_j)}(y) = y$ for some $p_k \leq K(\varepsilon)$ then we say that $(X^t)_t$ has the *periodic reparametrized gluing orbit property*.

The previous property implies on a strong form of transitivity for the flow $(X^t)_t$.

Definition 2.4. Let $(X^t)_t$ be a continuous flow on a compact manifold M.

We say that $(X^t)_t$ is *strongly transitive* if for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ so that for any two balls B_1, B_2 of radius ε there exists a point $x \in M$ so that $\{X^t(x) \colon t \in [0, s_\varepsilon]\}$ intersects both B_1 and B_2.

2.5. **Main results.** As mentioned before we will address on topological aspects of the class of volume-preserving flows generated by typical non-singular Lipschitz incompressible vector fields. Let $\mathfrak{X}^{0,1}_{\mu,*}(M) \subset \mathfrak{X}^{0,1}_{\mu}(M)$ denote the subset formed by all vector fields having no singularities. This is clearly a C^0-open subset, although empty for some manifolds (e.g. $M = \mathbb{S}^4$). Our main result says that typical flows are chaotic.

Theorem A. *There exists a C^0-generic subset $\mathscr{R} \subset \mathfrak{X}^{0,1}_{\mu,*}(M)$ such that, if $(X^t)_t$ is the continuous flow generated by a vector field $X \in \mathscr{R}$, then:*

1. *the set of periodic orbits of $(X^t)_t$ is dense in M;*
2. *$(X^t)_t$ satisfies the periodic shadowing property;*
3. *$(X^t)_t$ satisfies the gluing orbit property;*
4. *$(X^t)_t$ is strongly transitive.*

Using that both the general density theorem and the shadowing property are generic (i.e. items (1) and (2) of Theorem A), and that there exists a unique chain recurrent class for volume-preserving dynamics, we conclude that the reparametrized gluing orbit property holds C^0-generically (the argument in [6, Theorem 1] just attends at the topological properties and is independent of the class of flows). In consequence, a C^0-generic volume-preserving flow is strongly transitive. Thus, it is enough to prove the first two items in the theorem. In §3 we prove the general density theorem, and in §4 we prove that the shadowing property is C^0-generic.

We should also highlight that, by [9], the reparametrized gluing orbit property has non-trivial implications on the space of invariant measures and recurrence for the dynamics. In particular, as a direct consequence of [9, Theorem B] (whose proof only explores the gluing orbit property) we obtain the following:

Corollary 1. *There exists a C^0-generic subset $\mathscr{R} \subset \mathfrak{X}^{0,1}_{\mu,*}(M)$ such that, if $(X^t)_t$ is the continuous flow generated by a vector field $X \in \mathscr{R}$, then the set of periodic measures is dense in the space of $(X^t)_t$-invariant probabilities.*

We illustrate another application. Actually, the gluing orbit property can be also used as a key tool in ergodic optimization. Let us recall some basic notions. For each $X \in \mathfrak{X}^{0,1}_{\mu,*}(M)$ denote by $\mathcal{M}_X$ the simplex of $(X^t)_t$-invariant probabilities. For each $\varphi \in C(M,\mathbb{R})$, by continuity of the function $\mathcal{M}_X \ni \mu \mapsto \int \varphi\, d\mu$ (in the weak* topology) one has that

$$M(\varphi) := \sup_{\mu\in\mathcal{M}_X} \int \varphi\, d\mu = \max_{\mu\in\mathcal{M}_X} \int \varphi\, d\mu.$$

Any probability attaining the previous maximum is called a φ-maximizing measure. Some of the fundamental questions in ergodic optimization are to determine wether maximizing measures are unique, full supported, have positive topological entropy or periodic measures, for typical observables (we refer the reader to [23] for a survey on ergodic optimization). Building over [26], Morro, Sant'Anna and the third named author [27] characterized maximizing measures for typical continuous observables of dynamics with the gluing orbit property (cf. Theorem 2.5 in [27]). Together with Theorem A, this has the following consequence.

Corollary 2. *There exists a C^0-generic subset $\mathfrak{R} \subset \mathfrak{X}^{0,1}_{\mu,*}(M)\times C(M,\mathbb{R})$ such that for every $(X,\varphi)\in\mathfrak{R}$ there exists a unique $(X^t)_t$-invariant and φ-maximizing measure. Moreover the unique maximizing measure has full support and zero entropy.*

Let us discuss the shadowing property in the case of vector fields with singularities. We recall that Odani presented in [29] a quite direct strategy to obtain the C^0-genericity of shadowing for dissipative homeomorphisms on nice manifolds, meaning those for which the space of C^1-diffeomorphisms is C^0-dense in the space of homeomorphisms. His proof builds over Shub's C^0-density theorem [37], which says that structurally stable diffeomorphisms are C^0-dense in the space of C^r-diffeomorphisms ($1 \le r \le +\infty$). It turns out that the continuous-time version of Shub's result also holds [30, 42] (see also [41, Section 9]). Moreover, as proved by Hayashi [19], structurally stable flows are precisely those whose nonwandering set is hyperbolic and satisfies the transversality condition. Then, it is not hard to follow the same lines as in [29] and conclude the C^0-genericity of shadowing on $\mathfrak{X}^{0,1}(M)$. We notice this does not contradict the existence of smooth flows displaying hyperbolic singularities (e.g. geometric Lorenz flows or more generally singular-hyperbolic flows) which do not satisfy the shadowing property (see e.g. [40]). Indeed, while these flows are C^1-persistent according to [25], they can be C^0-approximated by structurally stable ones allowing to go ahead with Odani's strategy. A version of the results in [30, 37, 42] for incompressible vector fields is unknown and probably unlikely. For this reason we ask:

Question: Does the shadowing property hold C^0-generically on $\mathfrak{X}^{0,1}_{\mu}(M)$?

While hyperbolicity is clearly a feature of smooth flows, the permanence of singularities and their stable manifolds can be obtained through fixed point index and continuity arguments, respectively. Notwithstanding, while it is unclear wether these obstructions can be made locally C^0-generic (or even C^0-open), here we can prove a technical lemma concerning homogeneity through incompressible vector fields (Lemma 5.1) which will allow to adapt the strategy in [24] to the continuous-time setting and prove the following:

Theorem B. *C^0-generic vector fields in $\mathfrak{X}^{0,1}_{\mu}(M)$ generate incompressible flows with the weak shadowing property, (hence are transitive) and have a dense set of periodic orbits.*

3. The C^0-general density theorem in $\mathfrak{X}^{0,1}_{\mu}(M)$

3.1. **Strategy of the proof.** There is a long story concerning the general density theorem for continuous maps, which culminated with a proof by Hurley (we refer the reader to [21] for the proof and an historical account). Hurley's proof uses that one can create a C^0-stable periodic *sink* by a small C^0-perturbation and Brouwer's fixed point theorem guarantees a fixed point for every C^0-close homeomorphism.

Daalderop and Fokkink overcomed the absence of sinks for volume-preserving homeomorphisms and proved that the general density theorem holds for that systems [14].

In the present paper we mainly follow the arguments in [6], adapting them to the volume-preserving flow framework.

The fundamental steps in the proof are the following:

1st Step We will notice, in §3.2, that any element in $\mathfrak{X}_\mu^{0,1}(M)$ can by C^0-approximated by a vector field in $\mathfrak{X}_\mu^\infty(M)$;

2nd Step Since the proof uses a topological fixed point index argument, in §3.3, and for the sake of completeness we recall some basic definitions;

3rd Step In §3.5 we obtain a result (Lemma 3.3) which gives us a C^0-residual $\mathscr{R} \subset \mathfrak{X}_\mu^{0,1}(M)$ where the periodic orbits are *permanent*. This guarantees that the map $\mathfrak{P}$ defined by $\mathfrak{P}(X) = \overline{\text{Per}(X^t)}$ for $X \in \mathfrak{X}_\mu^{0,1}(M)$ is lower semicontinuous in $\mathscr{R}$. In particular, the continuity points of $\mathfrak{P}|_{\mathscr{R}}$ form a residual subset of $\mathscr{R}$;

4th Step In §3.6 we prove a C^0-closing lemma for Lipschitz vector fields (Corollary 3.10), which will play a key role in the proof of item (1) in Theorem A.

5th Step Finally, the general density theorem will follow by a continuity argument involving the two previous steps and the fact that the divergence-free Kupka-Smale vector fields are contained in the continuity points of $\mathfrak{P}$.

3.2. **A density theorem.** Despite the fact that we are using the C^0 topology which is unsuitable for stability results, certain stability arguments will be used in the sequel. So, C^0-approximating a Lipschtiz vector field by a C^1-smooth one will be a key step toward the proof of several results.

Theorem 3.1. *The set $\mathfrak{X}_\mu^\infty(M)$ is C^0-dense in $\mathfrak{X}_\mu^{0,1}(M)$.*

Proof. The proof of this statement is contained in the proof of [5, Proposition 1]. □

3.3. **Hopf degree and index for periodic orbits for continuous flows.** In [6] we presented with detail the basic concepts related to Hopf degree. Regardless, it is worth to recollect them again to cover every aspect in a self contained way. Given a continuous map f on a compact manifold M, we will recall the fixed point index used in [13]. Let B be an open ball on M whose boundary ∂B is an embedded sphere and assume that either (i) $f(\overline{B}) \cap \overline{B} = \emptyset$, or (ii) $f(\overline{B}) \cup \overline{B}$ is contained in a single coordinate chart. If, in addition, f has no fixed points in ∂B then the *fixed point index* $\iota_f(B)$ is defined as follows:

(1) $\iota_f(B) = 0$, if $f(\overline{B}) \cap \overline{B} = \emptyset$; and
(2) $\iota_f(B) = \deg(\gamma)$, in the case that $f(\overline{B}) \cup \overline{B}$ is contained in a single coordinate chart, where $\deg(\gamma)$ denotes the Hopf degree of the map $\gamma : \partial B \simeq S^{n-1} \to S^{n-1}$ which is defined (after a change of coordinates) by $\gamma(x) = \frac{f(x)-x}{\|f(x)-x\|}$.

This notion is independent of the choice of local coordinates and it is locally constant in a small neighborhood of the continuous mappping f (see e.g. [20]).

In the mid sixties, Fuller [17] introduced a notion of index for periodic orbits for continuous flows in the same vein of the fixed point index for homeomorphisms. As in the discrete time setting, an appropriate notion of index would be homotopy invariant, additive and if it is non-zero then it should guarantee the existence of a periodic orbit. Some of the subtleties that arise in the case of flows concern the fact that the (compact) periodic orbit can escape to infinity or into a singularity by homotopy (see e.g. [17, Section 7]). For that reason, the index of periodic orbits of vector fields or flows is defined for periodic orbits that are isolated in the sense that we describe below. Given a compact boundaryless manifold M and a vector field $X \in \mathfrak{X}^0(M)$, consider the (closed) set

$$\Pi(X) = \{(p,t) \in M \times \mathbb{R}^+ \colon X^t(p) = p\} \subset M \times \mathbb{R}^+.$$

Given an open set $\Omega \subset M$ that does not contain any singularity of X, we say that the open set $\Omega \times]t_1, t_2[\subset M \times \mathbb{R}^+$ is *admissible for* X if $\partial(\Omega \times]t_1, t_2[) \cap \Pi(X) = \emptyset$. In other words, the set $\Omega \times]t_1, t_2[$ is admissible for X if and only if the boundary of Ω contains no periodic orbits for X with period t_1 or t_2.

Following [17], a periodic orbit γ with period π is *isolated* if is has an open neighborhood in $M \times \mathbb{R}^+$ where γ is the unique periodic orbit for X. Clearly, isolated periodic orbits admit admissible neighborhoods. The *Fuller index* of an isolated periodic orbit γ of period $q > 0$ is defined by

$$i(X, \Omega \times]t_1, t_2[) = \frac{\iota(P_X)}{m}$$

where $m = q/p$ is the multiplicity (p is the least period of γ) and $\iota(P_X)$ is the index of any Poincaré return map at some point p of γ. We observe that although in its original paper Fuller [17] was studying C^∞-vector fields, the notion of index that was introduced is clearly independent of the smoothness of the vector field.

First, we observe that since the usual index of fixed points is homotopy invariant then the later is well defined and independent of the admissible neighborhood of γ. Second, up to a factor, the index coincides essentially with the index of a Poincaré map $P_X : \Sigma \to \Sigma$ where Σ is a local cross-section to the flow passing though the point $p \in \gamma$. The most important property for our purposes is that if $i(X, \Omega \times]t_1, t_2[) \neq 0$ then there exists a non-trivial periodic orbit with non-empty intersection with Ω and a period in the range $]t_1, t_2[$. By the later discussion, and when no confusion is possible, we will refer to $\iota(P_X)$ instead of the Fuller index. We refer the reader to [17, 20] for the proofs and other properties of this index.

3.4. **The flowbox theorem for Lipschitz incompressible flows.** Given a regular orbit of a C^r flow ($r \geq 1$) it is always possible, using a change of coordinates, to straightening out all orbits in a certain neighborhood of the orbit. This is a very simple, yet important result called *the flowbox theorem* and its proof uses basically the inverse function theorem (see e.g. [32, pp. 40]). This theorem describes completely the local behavior of the orbits in a neighborhood of a regular orbit and shows that, locally, first integrals always exist. Nevertheless, since the change of coordinates is given implicitly we are not sure that it preserves certain geometric invariants of the flow like, for example, preservation of a volume form. Furthermore, when decreasing the smoothness of the vector field some careful is needed to proceed with the proof of the flowbox theorem.

For performing local perturbations of vector fields in $\mathfrak{X}^{0,1}_\mu(M)$ will be of utmost importance to use the flowbox theorem for divergence-free Lipschitz vector fields proved by the first author in [5]. Let us introduce some definitions first. We say that two vector fields $X_1 \colon U_1 \to TU_1$ and $X_2 \colon U_2 \to TU_2$ are *locally topologically conjugate near* $p_1 \in U_1$ and $p_2 \in U_2$ if there exist two open neighborhoods $O_i \ni p_i$ ($i = 1, 2$) and a homeomorphism $\phi \colon O_1 \to O_2$ with $\phi(p_1) = p_2$ such that for any $x \in O_1$ and a small interval I containing 0 the integral curve $\sigma_x \colon I \to O_1$ defined by $\sigma_x(0) = x$ and $\frac{d}{dt}\sigma_x(t) = X_1(\sigma_x(t))$ for all $t \in I$ (i.e. defined by $X_1^t(x)$ for $t \in I$) is a solution associated to X_1 if and only if the integral curve $\phi \circ \sigma_x \colon I \to O_2$ is a solution associated to X_2. The following flowbox theorem corresponds to [5, Theorem 1].

Theorem 3.2. *(Flowbox theorem for Lipschitz divergence-free vector fields)*
Let be given $X \in \mathfrak{X}^{0,1}_\mu(M)$, a non-singular point $p_1 \in M$ and the trivial vector field $T(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n) = (1, 0, \ldots, 0)$ on canonical coordinates $(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ of $\mathbb{R}^n$.

(i) *Then, there exists an open neighborhood $\mathfrak{A}$ of p and a lipeomorphism $\phi : \mathfrak{A} \to \phi(\mathfrak{A})$ onto an open neighborhood of $\hat{0}$ that conjugates the vector fields $X|_{\mathfrak{A}}$ and $T|_{\phi(\mathfrak{A})}$.*

(ii) *Then, X and $T_c = cT$ are locally topologically (volume-preserving) conjugate near p_1 and $p_2 = \hat{0}$ for some $c = c(X, p_1) > 0$. The homeomorphism Φ which gives the conjugacy is a volume-preserving lipeomorphism.*

A neighborhood $\mathfrak{A}$ of p as above is called a flowbox chart *at p.*

3.5. **Permanence of periodic orbits.** We recall the notion of permanence from topological dynamics. Let $X \in \mathfrak{X}^{0,1}_\mu(M)$ and let $(X^t)_t$ be the flow generated by X. We say that a closed orbit γ of a flow $(X^t)_t$ is *permanent* if for any vector field $Y \in \mathfrak{X}^{0,1}_\mu(M)$, C^0-arbitrarily close to X, the flow $(Y^t)_t$ has a periodic orbit $\tilde{\gamma}$ near γ. Let $\mathscr{P}(X^t)$ denote the set of all permanent closed orbits of $(X^t)_t$. We need some instrumental results.

Lemma 3.3. *There exists a residual subset $\mathscr{R}$ of $\mathfrak{X}^{0,1}_\mu(M)$ and with respect to the C^0-Whitney topology such that $Per(X^t) = \mathscr{P}(X^t)$, for any $X \in \mathscr{R}$. Moreover, the residual subset $\mathscr{R}$ contains the space of C^1 Kupka-Smale divergence-free vector fields denoted by $KS^1_\mu(M)$.*

Proof. The proof is similar to the one of [6, Lemma 5.1]. We make use of the denseness of divergence-free Kupka-Smale vector fields as we now describe. Recall that $KS^1_\mu(M)$ is a C^1-residual subset of the Baire space $(\mathfrak{X}^1_\mu(M), \|\cdot\|_{C^1})$ (see [35]), thus it is C^1-dense in $\mathfrak{X}^1_\mu(M)$. In consequence, $KS^1_\mu(M)$ is C^0-dense in $\mathfrak{X}^1_\mu(M)$. Since, by Theorem 3.1, $\mathfrak{X}^1_\mu(M)$ is C^0-dense in $\mathfrak{X}^{0,1}_\mu(M)$ then we conclude that $KS^1_\mu(M)$ is C^0-dense in $\mathfrak{X}^{0,1}_\mu(M)$.

We proceed proving that there exists a C^0-residual $\mathscr{R} \subset \mathfrak{X}^{0,1}_\mu(M)$ such that any closed orbit of a vector field in $\mathscr{R}$ is permanent. We begin by taking a countable base for the topology $\{\mathcal{B}_i\}_{i\in\mathbb{N}}$ of M consisting of open balls whose boundaries are embedded spheres. The index of periodic orbits will play a crucial role along the proof, since the existence of non-zero index on a set (which is a C^0-open condition) assures the existence of a periodic orbit that intersects that set.

We define, for every $i, n \in \mathbb{N}$, the disjoint C^0-subsets $\mathscr{F}_{i,n}, \mathscr{I}_{i,n}$ of $\mathfrak{X}^{0,1}_\mu(M)$ in the following way:

(1) $X \in \mathscr{F}_{i,n}$ if $X^t(x) \neq x$ for all $x \in \overline{\mathcal{B}_i}$ and all $t \in]0, n]$;
(2) $X \in \mathscr{I}_{i,n}$ if there exists $\mathcal{B}_j$ with $diam(\mathcal{B}_j) < diam(\mathcal{B}_i)$ and such that $X^t(x) \neq x$ for all $x \in \partial\mathcal{B}_j$ and all $t \in]0, n]$, and $i(X, \mathcal{B}_i \times]t_1, t_2[\,) \neq 0$ for some $0 < t_1 < t_2 \leq n$.

By Gronwall's inequality the map $\mathfrak{X}^{0,1}_\mu(M) \times]0, t] \ni (X, s) \mapsto X^s \in \text{Homeo}_\mu(M)$ is continuous and, consequently, the sets $\mathscr{F}_{i,n}$ and $\mathscr{I}_{i,n}$ are C^0-open subsets of $\mathfrak{X}^{0,1}_\mu(M)$.

We claim that $KS^1_\mu(M) \subset \mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ and, in particular, $\mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ is a C^0-open and dense subset of $\mathfrak{X}^{0,1}_\mu(M)$ for all $i, n \geq 1$. Indeed, given $i, n \geq 1$ fixed and $X \in KS^1_\mu(M)$ either there are no periodic orbits with period smaller or equal to n in $\overline{\mathcal{B}_i}$ (in which case $X \in \mathscr{F}_{i,n}$) or there are periodic orbits with period smaller or equal to n in $\overline{\mathcal{B}_i}$. In the later case, since the periodic orbits are hyperbolic, hence isolated, we have that $X \in \mathscr{I}_{i,n}$.

Now we claim that the C^0-residual subset $\mathscr{R} := \bigcap_{i,n\geq 1} [\mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}] \subset \mathfrak{X}^{0,1}_\mu(M)$ satisfies the requirements of the lemma. Let us show that $\text{Per}(X^t) = \mathscr{P}(X^t)$. Take $X \in \mathscr{R}$ and $\gamma \in \text{Per}(X^t)$ of period a and any $\mathcal{B}_i$ which intersects γ. Since $X \in \mathscr{F}_{i,n} \cup \mathscr{I}_{i,n}$ and γ intersects $\mathcal{B}_i$ then there exist $n \in \mathbb{N}$ and $\mathcal{B}_j$ with $diam(\mathcal{B}_j) < diam(\mathcal{B}_i)$, such that $X^t(x) \neq x$ for all x in the boundary of $\mathcal{B}_j$ and all $t \in]0, n]$, and the corresponding index is non-zero. Since this property is persistent for small C^0-perturbations of the original vector field we get that γ is permanent. This finishes the proof of the lemma. □

The next result is a kind of C^0-Pasting lemma restricted to a flowbox, and it combines two perturbation results. First we use Theorem 3.2 to put ourselves into a smooth context. Second, [3, Theorem 3.1] implies on the desired perturbation.

Lemma 3.4. *Given $\varepsilon > 0$, there exists $\delta > 0$ such that if $X \in \mathfrak{X}^{0,1}_{\mu}(M)$, K is a compact injective flowbox of M and $Y \in \mathfrak{X}^{0,1}_{\mu}(M)$ is δ-C^0-close to X on a small neighborhood U of K, then there are $Z \in \mathfrak{X}^{0,1}_{\mu}(M)$, sets V and W such that $K \subset V \subset U \subset W$ and V and K are close in the Hausdorff distance, satisfying $Z\mid_V = Y$, $Z\mid_{int(W^c)} = X$ and Z is ε-C^0-close to X.*

Proof. Fix $\varepsilon > 0$. By [3, Theorem 3.1] there exists $\tilde{\delta} > 0$ such that if $X \in \mathfrak{X}^{0,1}_{\mu}(M)$ and ϕ is given by Theorem 3.2, $\tilde{K}$ is a compact subset of M and $\phi(\tilde{K}) \subset Im(\phi)$ (where $Im(\phi)$ is the image of ϕ) is a compact subset of $\mathbb{R}^n$ and $\tilde{Y} \in \mathfrak{X}^2_{\mu}(\mathbb{R}^n)$ is $\tilde{\delta}$-C^1-close to $T = X$ on a small neighborhood $\tilde{U}$ of $\tilde{K}$, then there exist $\tilde{Z} \in \mathfrak{X}^2_{\mu}(\mathbb{R}^n)$ and $\tilde{V}$ and $\tilde{W}$ such that $\tilde{K} \subset \tilde{V} \subset \tilde{U} \subset \tilde{W} \subset Im(\phi)$ satisfying $\tilde{Z}\mid_{\tilde{V}} = \tilde{Y}$, $\tilde{Z}\mid_{\tilde{V}} = \tilde{Y}$, $\tilde{Z}\mid_{int(\tilde{W}^c)} = \tilde{X}$ and $\tilde{Z}$ is ε-C^0-close to $\tilde{X}$. Now define $\delta = \tilde{\delta}$, $V = \phi^{-1}(\tilde{V})$, $W = \phi^{-1}(\tilde{W})$, $\tilde{Y} = \phi_* Y$ and $\tilde{Z} = \phi_* Z$. Clearly, $Z\mid_V = Y$, $Z\mid_{int(W^c)} = X$. If Z is not ε-C^0-close to X decrease $\tilde{\delta}$ accordingly, reboot the proof until Z is ε-C^0-close to X. □

We should notice that [3, Theorem 3.1] provides sharper estimates as we deal with the C^0-topology, instead of the space of C^1-vector fields and C^1-topology considered there.

Lemma 3.5. *Assume that $X \in \mathfrak{X}^{0,1}_{\mu}(M)$ and that γ is a periodic orbit. There exists an arbitrarily small C^0-perturbation $Y \in \mathfrak{X}^{0,1}_{\mu}(M)$ of X such that γ is an isolated periodic orbit for $(Y_t)_t$, hence permanent.*

Proof. Let γ be a periodic orbit with prime period $T(\gamma) > 0$ for $X \in X^{0,1}_{\mu}(M)$ and L be its Lipschitz constant. Fix $p \in \gamma$ and an arbitrary $\varepsilon > 0$. Covering each compact piece of orbit by trivializing charts given by Theorem 3.2 we obtain a long tubular floxbox chart: for any $\xi > 0$ there exists $\zeta > 0$, an open neighborhood U_ζ of p and a volume-preserving lipeomorphism $\varphi : \overline{U_\zeta} \to [0, T(\gamma) - \xi] \times \overline{B(\vec{0}, \zeta)}$ so that $T = \varphi_* X \equiv (1, 0, \ldots, 0)$.

Let $C > 0$ be a Lipschitz constant for both φ and φ^{-1}.

Consider the vector field $Y_\delta \in \mathfrak{X}^\infty([0, T(\gamma) - \xi] \times \overline{B(\vec{0}, \zeta)})$ given by

$$Y_\delta(x_1, x_2, \ldots, x_{n-1}, x_n) = (1, \delta x_2, \ldots, \delta x_{n-1}, -\delta(n-1)x_n),$$

for some $0 < \delta \leq \frac{L}{C}$.

This is a divergence-free vector field with Lipschitz constant bounded above by $\frac{L(n-1)}{C}$. Moreover, it generates an incompressible flow $(Y^t_\delta)_t$, and the piece of orbit

$$Y^t_\delta(0, 0, \ldots, 0) = (t, 0, \ldots, 0), \qquad t \in [0, \xi]$$

has hyperbolic behavior. By the pasting lemma (see Lemma 3.4), there exists $0 < \delta \ll \frac{L}{C}$ and $Z \in X^{0,1}_{\mu}(M)$ so that

$$Z\mid_{\overline{[0,T(\gamma)-\xi]\times B(\vec{0},\zeta/4)}} \equiv Y \quad \text{and} \quad Z\mid_{\overline{[0,T(\gamma)-\xi]\times B(\vec{0},\zeta)\setminus B(\vec{0},\zeta/2)}} \equiv (1, 0, \ldots, 0)$$

is C^1-close to the constant vector field. The resulting vector field $W = (\varphi^{-1})_* Z \in \mathfrak{X}^{0,1}_{\mu}(M)$ has Lipschitz constant bounded by $L(n-1)$.

By construction, taking $\xi > 0$ small enough, we obtain that the orbit γ is preserved by $(Z_t)_t$ and it is an isolated periodic orbit.

□

3.6. **The C^0-closing lemma on $\mathfrak{X}^{0,1}_{\mu}(M)$.** If p is a non-singular point for $X \in \mathfrak{X}^{0,1}_{\mu}(M)$, $(X^t)_t$ is the flow generated by X and $(T^t)_t$ is the trivial flow on $\mathbb{R} \times \mathbb{R}^{n-1}$ generated by T defined by $T^t(s, z) = (s + t, z)$, then Theorem 3.2 implies that there exist $\delta > 0$, an open neighborhood U_δ of p and a lipeomorphism $\varphi : \overline{U_\delta} \to [0, \delta] \times \overline{B(\vec{0}, \delta)}$ (here $B(\vec{0}, \delta)$ denotes the usual ball in $\mathbb{R}^{n-1}$) verifying $\varphi \circ X^t(x) = T^t \circ \varphi(x)$ for every $x \in U_\delta$ and $t \in \mathbb{R}$ so that $X^t(x) \in \overline{U_\delta}$. Hence, given $0 < r < \delta$ we will consider also the flowbow 'cylinders' $\mathcal{F}_{X,\delta}(B) := \varphi^{-1}((0,\delta)\times B)$, for $B \subset B(\vec{0},\delta)$), and the local cross-section $\Sigma_p := \varphi^{-1}(\{0\}\times B(\vec{0},\delta))$ to the flow $(X^t)_t$ at p.

The proof of the closing lemma in our context (Corollary 3.10) will be crucial in the sequel. This is the volume-preserving version of [6, Lemma 5.3] but its proof is completely different due to the fact that our perturbation must be divergence-free. The proof will be performed in three steps: Our aim will be to put ourselves in the setup of Pugh and Robinson perturbation arguments in [34, Section 8 (c)] allowing to 'lift' points. So firstly, considering Theorem 3.2 we change from $X \in \mathfrak{X}^{0,1}_\mu(M)$ to a C^∞ trivial vector field T displaying a trivial C^∞ flow $T^t(s,z) = (s+t,z)$ (here C^1 would be enough). Secondly, we change again the coordinates in a conservative way using Lemma 3.7 to transform a tubular neighborhood of a path connecting two points into a ball neighborhood. Finally, we use the techniques from [34, Section 8 (c)] which construct the volume form preserved by a flow performing our desired perturbation instead of constructing the volume-preserving flow which preserves a *given* volume-form. Estimates here are simpler than in [34] because we are only interested in C^0 approximations. In overall, our work will be to prove that vector fields in $\mathfrak{X}^{0,1}_\mu(M)$ satisfy the *Lift Axiom* (see [34, Figure 2]) which is the main content of Lemma 3.9.

Next result, which is one of the main tools to perform conservative change of coordinates, will be instrumental to prove Lemma 3.7.

Theorem 3.6. *(Dacorogna and Moser [15, Theorem 1]) Let $\Omega = B(x,r)$ and $f,g \in C^1(\overline{\Omega})$ two positive functions. There exists a diffeomorphism φ with $\varphi,\varphi^{-1} \in C^2(\overline{\Omega},\mathbb{R}^n)$, where $\alpha < 1$, satisfying*

$$g(\varphi(x))\det D\varphi_x = \lambda f(x), \tag{3.1}$$

for all $x \in \Omega$ where $\lambda = \int g/\int f$. We also have $\varphi = Id$ at $\partial\Omega$.

Theorem 3.2 trivialize flowboxes and the next result trivialize supports of perturbations. The main goal will be to put us in the Pugh and Robinson hypothesis of liftability in the conservative setting and so will play a fundamental part to obtain the C^0-closing lemma for $X \in \mathfrak{X}^{0,1}_\mu(M)$.

Lemma 3.7. *Let T be a trivial vector field with transversal section the hyperplane $\mathcal{H}$ defined by $x_1 = 0$, let $p,q \in \mathcal{T} \subset \mathcal{B} \subset \mathcal{H}$ where $\mathcal{B}$ is a ball and $\mathcal{T}$ a neighborhood of the points p and q with smooth boundary. There exist $r > 0$, a volume-preserving map $\psi\colon \mathcal{T} \to B(\vec{0},r)$ such that $B(\vec{0},r) \subset \mathcal{H}$ and, consequently, a volume-preserving smooth change of coordinates Ψ from the flowbox with base $\mathcal{T}$ into the flowbox with base $B(\vec{0},r)$.*

Proof. First choose $r > 0$ such that $\mathcal{T}$ and $B(\vec{0},r)$ have the same volume. Take any smooth map $\psi_0\colon \mathcal{T} \to B(\vec{0},r)$. We feed Theorem 3.6 with $\Omega = B(\vec{0},r)$, $g = 1$ and $f = \det\nabla\psi_0^{-1}$. Clearly, $\lambda = 1$. Now, Theorem 3.6 gives φ satisfying (3.1), i.e. $\det D\varphi = \det\nabla\psi_0^{-1}$. Defining $\psi = \varphi\circ\psi_0$ we obtain that: $\det\psi = \det(\varphi\circ\psi_0) = \det\varphi\det\psi_0 = 1$ and that $\psi\big|_{\partial\mathcal{T}} = \varphi\circ\psi_0\big|_{\partial\mathcal{T}} = \psi_0\big|_{\partial\mathcal{T}}$ (see Figure 1). Finally, the map $\Psi(t,z) = (t,\psi(z))$ defined in flowbox coordinates $z \in \mathcal{T}$ (base coordinate) and $t > 0$ (heigh coordinate) is volume-preserving and can easily be extended in a neighborhood $\mathcal{B}$ of $\mathcal{T}$. Indeed, taking a ball $\mathcal{B}_1 \supset B(\vec{0},r)$ such that $\mathcal{B}_1$ and $\mathcal{B}$ have the same volume we can repeat the same proceedure above this time constructing a volume-preserving diffeomorphism $\tilde{\psi}$ from $\mathcal{B}\setminus\mathcal{T}$ into $\mathcal{B}_1\setminus B(\vec{0},r)$. Since $\psi = \tilde{\psi} = Id$ in the common boundary both diffeomorphisms glue continuously. □

Remark 3.8. If, for $i = 1,\dots,n$, we have paths γ_i connecting p_i and q_i with tubular neighborhoods $\mathcal{T}_i$ pairwise disjoint and inside $\mathcal{B}$ we can construct ψ_i and $\tilde{\psi}$ such that the union of these $i+1$ diffeomorphisms transform conservatively $\mathcal{T}_i$ into $B(s_i,r_i)$ where $B(s_i,r_i) \subset \mathcal{B}_1$.

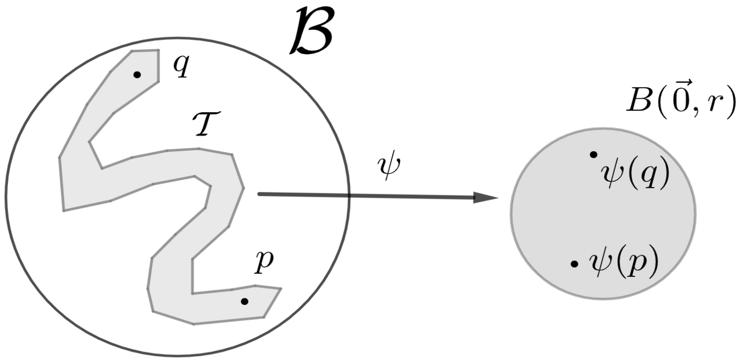

FIGURE 1. The regularization map ψ given by Dacorogna-Moser theorem. The ball $B(\vec{0}, r)$ and the set $\mathcal{T}$ have the same volume.

Lemma 3.9 (Lifting Lemma). *Let $X \in \mathfrak{X}_\mu^{0,1}(M)$, let p be a non-singular point, Σ_p be a local cross-section through p and let Φ be the lipeomorphism given by Theorem 3.2 and Ψ be the diffeomorphism given by Lemma 3.7. Assume there exists a C^1-path $\gamma\colon [0, \delta] \to \Sigma_p$ without self-intersections and with constant speed v (hence of length $v\delta$) such that $\gamma(0) = p$ and $\gamma(\delta) = q$. Then there exists $C > 0$ so that for any $\eta > v\delta$, any open tubular neighborhood $\mathcal{T} \subset \Sigma_p$ forming a narrow band around the curve and of diameter smaller than η, and any $\varepsilon > 0$ there exist $\zeta = \zeta(C, \eta, \varepsilon) > 0$ (so that $\zeta \to 0$ when $\eta \to 0$) and a vector field Z such that: (a) $Z \in \mathfrak{X}_\mu^{0,1}(M)$; (b) Z is ζ-C^0-close to X; (c) $Z^\delta(p) = X^\delta(q)$ and (d) $Z = X$ outside $\mathcal{F}_{X,\delta}(W)$ where W is arbitrarilly close to $[-\delta, \delta] \times \mathcal{T}$ (w.r.t. the Hausdorff distance).*

Proof. Assume that $q \neq p$ otherwise the proof is trivial. The volume-preserving lipeomorphism Φ conjugates $(X^t)_t$ with the trivial flow $(T^t)_t$. The volume-preserving diffeomorphism Ψ makes a change of coordinates from the flowbox with base $\mathcal{T}$ into the flowbox with base $B(\vec{0}, r)$. Hence, we may assume that $(X^t)_t$ is the trivial flow on $[-\delta, \delta] \times \overline{B(\vec{0}, \delta)}$ in the flowbox coordinates $(t, z) \in \mathbb{R} \times \mathbb{R}^{n-1}$, given by $X^s(t, z) = (t + s, z)$ and $\mathcal{T}$ is a ball $B(\vec{0}, r)$ for a certain $r \in (0, \delta)$. Thus the result will follow immediately if we prove the lemma on $[-\delta, \delta] \times \overline{B(\vec{0}, \delta)}$. Take $C > 0$ given by uniform continuity of the lipeomorphism $(\Psi \circ \Phi)^{-1}$.

Take an arbitrary $\eta > v\delta$ and $\mathcal{T} = B(\vec{0}, r) \subset \Sigma_p$ such that $r < \frac{\eta}{2}$. Fix an open ball $B(\vec{0}, \hat{r}) \subsetneq \overline{B(\vec{0}, \hat{r})} \subsetneq \mathcal{T}$ containing γ and $\varepsilon > 0$. Now we are in conditions to apply the proficient volume-preserving perturbative arguments from [34, Section 8 (c)] to find $Y \in \mathfrak{X}_\mu^{0,1}(M)$ such that $Y^\delta(p) = X^\delta(q)$ where the perturbation is performed in $U = [-\delta, \delta] \times B(\vec{0}, \hat{r})$ (cf. Figure 2). By Lemma 3.4 given $\varepsilon > 0$, there exists $\Delta > 0$ such that if $X \in \mathfrak{X}_\mu^{0,1}(M)$, K is a compact injective flowbox of M and $Y \in \mathfrak{X}_\mu^{0,1}(M)$ is Δ-C^0-close to X on a small neighborhood U of K, then there exist $Z \in \mathfrak{X}_\mu^{0,1}(M)$ and V and W such that $K \subset V \subset U \subset W$ satisfying $Z|_V = Y$, $Z|_{int(W^c)} = X$ and Z is ε-C^0-close to X.

The items (a), (c) and (d) are trivially satisfied by construction. Moreover, we saw above that Z is ε-C^0-close to X but we are considering 'good' coordinates which are affected of a distortion factor bounded by C. Furthermore, the size of the flowbox W depends on its height and width which is given by δ and the speed of perturbation is given by v. Hence, the constant η is also relevant when we estimate the C^0-distance from Z to X and (b) also holds. □

The C^0-closing lemma for flows generated by Lipschitz divergence-free vector fields (Corollary 3.10) will play a crucial role in the proof of the general density theorem.

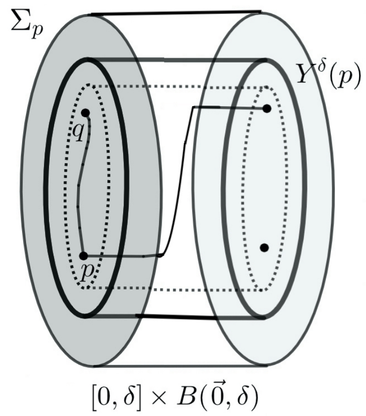

FIGURE 2. The volume-preserving lifting lemma.

Corollary 3.10. *(C^0-closing lemma for Lipschitz divergence-free vector fields) Let $X \in \mathfrak{X}^{0,1}_\mu(M)$ and $p \in M$ be a non-singular point. Then, for every $\varepsilon > 0$ there exists a vector field $X \in \mathfrak{X}^{0,1}_\mu(M)$ which is ε-C^0-close to X, coincides with X outside of a small open neighborhood of p, and such that the flow generated by Y has a periodic point $\tilde{p}$ so that $d(p, \tilde{p}) < \varepsilon$.*

Proof. If p is periodic for $(X^t)_t$ we are done. Otherwise, since $p \in M$ is a non-singular point, for any $0 < r < \delta$ and corresponding flowbow $\mathcal{F}_{X,\delta}(B(\vec{0}, r)) = \varphi^{-1}((0,\delta) \times B(\vec{0}, r))$ associated to p there exists $T > 0$ such that $X^T(\Sigma_p) \cap \Sigma_p \neq \emptyset$ where $\Sigma_p = \varphi^{-1}(\{0\} \times B(\vec{0}, r))$. Consequently, just pick $0 < r < \delta$ small (it is enough that $4Cr < \varepsilon$) so that the perturbated vector field $Y \in \mathfrak{X}^{0,1}_\mu(M)$ obtained by Lemma 3.9 is ε-C^0-close to the vector field X and it admits a periodic orbit that intersects $\mathcal{F}_{X,\delta}(B(\vec{0}, r))$. This proves the corollary. □

3.7. **Proof of the General Density Theorem.** Now we prove item (1) in Theorem A. Let $M^\star$ be the set of compact subsets of M endowed with the Hausdorff topology. We need the following semicontinuity result.

Lemma 3.11. *Let $\mathfrak{X}^{0,1}_\mu(M)$ be endowed with the C^0-topology. Then the map*

$$\begin{array}{rccc} \mathfrak{P}: & \mathfrak{X}^{0,1}_\mu(M) & \to & M^\star \\ & X & \mapsto & \overline{Per(X^t)} \end{array}$$

is lower semicontinuous on the residual $\mathscr{R}$ given by Lemma 3.3.

Proof. We must prove that for any $X \in \mathscr{R}$, and any $\varepsilon > 0$ there exists a neighborhood V of X such that $\mathfrak{P}(Y) \subseteq \mathcal{B}_\varepsilon(\mathfrak{P}(X))$ for all $Y \in V$, or in other words there are no *implosions* of the set of closed orbits when we C^0-perturb X. But Lemma 3.3 says that $\mathrm{Per}(X^t) = \mathscr{P}(X^t)$ and the proof follows immediately from the definition of permanent closed orbits. □

We are now in a position to prove item (1) of Theorem A, that is the General Density Theorem for Lipschitz divergence-free vector fields equipped with the C^0-Whitney topology. Since the map $\mathfrak{P}\colon \mathfrak{X}^{0,1}_\mu(M) \to M^\star$ defined by $\mathfrak{P}(X) = \overline{\mathrm{Per}(X^t)}$ is lower semicontinuous on $\mathscr{R}$ (by Lemma 3.11) then the continuity points of $\mathfrak{P}|_{\mathscr{R}}$ form a residual subset $\mathscr{R}_1 \subset \mathscr{R}$.

Recall that by the Poincaré recurrence theorem the non-wandering set of a incompressible flow is the whole manifold.

Thus, to prove the theorem it is enough to show that $M = \overline{\mathrm{Per}(X^t)}$ for every vector field $X \in \mathscr{R}_1$. Assume, by contradiction, that there exists a vector field $X \in \mathscr{R}_1$ such that $M \setminus \overline{\mathrm{Per}(X^t)} \neq \emptyset$ and take $p \in M \setminus \overline{\mathrm{Per}(X^t)}$. Now, the closing lemma (Corollary 3.10) implies that X can be C^0-approximated by a vector field $Y_1 \in \mathfrak{X}^{0,1}_\mu(M)$ so that $p \in \mathrm{Per}(Y_1^t)$. By Lemma 3.5, one can perform a C^0-small perturbation of Y_1 so that p becomes a permanent periodic point for the resulting vector field $Y_2 \in \mathfrak{X}^{0,1}_\mu(M)$. Since the set of C^1 Kupka-Smale divergence-free vector fields is contained in $\mathscr{R}$ (cf. Lemma 3.11) and is also C^0-dense in $\mathfrak{X}^{0,1}_\mu(M)$ (cf. Theorem 3.1), we conclude that the permanence of the periodic point p guarantees that Y_2 can be arbitrarily C^0-approximated by some C^1 Kupka-Smale divergence-free vector field $Y_3 \in \mathscr{R}$ with a periodic point $\tilde{p}$ arbitrarily close to p. Altogether, we conclude that X is C^0-approximated by vector fields in $\mathscr{R}$ with periodic points arbitrarily close to p, which is in contradiction to the fact that X is a continuity point of $\mathfrak{P} \mid_{\mathscr{R}}$. This proves that $M = \overline{\mathrm{Per}(X^t)}$ and completes the proof of (1) of Theorem A.

4. Periodic shadowing is C^0-generic in $\mathfrak{X}^{0,1}_{\mu,*}(M)$

Unless stated to the contrary we let M be a compact Riemmanian manifold of dimension ≥ 3.

4.1. **Realization of covering relations**. We begin by considering a very useful simple version of a result by Zgliczynski and Gidea [43] which we apply to Poincaré maps of a Lipschitz continuous flow. Let us first present some notation. We say that $N = (|N|, N^\ell, N^r)$ is a h-set in $\mathbb{R}^{n-1}$ if $|N| = [a,b] \times D^{n-2}_r$, $N^\ell = (-\infty, a] \times \mathbb{R}^{n-2}$ and $N^r = [b, \infty) \times \mathbb{R}^{n-2}$ for some $a, b \in \mathbb{R}$. We consider the left and right edges $N^{\ell e} = |N| \cap N^\ell$ and $N^{re} = |N| \cap N^r$, respectivelly. Given two h-sets N, M and a continuous map $h\colon |N| \to |M| \subset \mathbb{R}^{n-1}$, we say that N *h-covers* M if:

- (A) $h(|N|) \subset \mathrm{Int}|M| \cup M^\ell \cup M^r$, and either $h(N^{\ell e}) \subset \mathrm{Int}M^\ell$ and $h(N^{\ell r}) \subset \mathrm{Int}M^r$ or else $h(N^{\ell e}) \subset \mathrm{Int}M^r$ and $h(N^{\ell r}) \subset \mathrm{Int}M^\ell$; or
- (B) $h(|N|) \subset \mathrm{Int}|M|$.

The covering relation of type (A) is denoted by $N \overset{h}{\Longrightarrow} M$ and the covering relation of type (B) is denoted by $N \overset{h,0}{\Longrightarrow} M$. Notice that if h is a homeomorphism obtained as Poincaré first hitting time map from a local smooth cross section N onto a cross-section M and $|N|$ and $|M|$ have the same volume then h may display only the covering relation of type (A).

Theorem 4.1. *(Zgliczynski and Gidea,* [43, Theorem 4]*) If*

$$N_0 \overset{h_1}{\Longrightarrow} N_1 \overset{h_2}{\Longrightarrow} N_2 \overset{h_3}{\Longrightarrow} \dots \overset{h_k}{\Longrightarrow} N_k$$

then there exists $x \in \mathrm{Int}|N_0|$ such that $(h_i \circ \cdots \circ h_2 \circ h_1)(x) \in \mathrm{Int}|N_i|$ for all $i \in \{1, \dots, k\}$.

We are going to modify the strategy in [6, 18] to prove the C^0-genericity of periodic shadowing for volume-preserving Lipschitz flows.

The idea in [18] is to consider suitable partitions and to perform small C^0-perturbations of a given volume-preserving homeomorphism so that the resulting homeomorphism satisfies the covering relations of Theorem 4.1 thus obtaining the C^0-denseness of the setup of Theorem 4.1.

In the argument we will make C^0 perturbations of the original Lipschitz vector field yielding a smooth vector field with the desired covering relations (to be defined below). For that purpose, the next result is a key step.

Lemma 4.2 (C^0-Realization of smooth Poincaré maps on flowboxes). *Let $X \in \mathfrak{X}^1_\mu(M)$, let $\mathfrak{U} \subset M$ be an open set and let $\phi : \mathfrak{U} \to [0,1] \times B(\vec{0}, \Delta_0)$ be a flowbox chart that conjugates the vector fields $X \mid_{\mathfrak{U}}$ and $T \mid_{[0,1]\times B(\vec{0},\Delta_0)}$, for some $\Delta_0 > 0$. If $\Sigma_0 := \phi^{-1}(\{0\} \times B(\vec{0}, \Delta_0))$ and $\Sigma_1 := \phi^{-1}(\{1\} \times B(\vec{0}, \Delta_0))$ there* *exists $C>0$ so that for any volume-preserving diffeomorphism isotopic to the identity $h:\Sigma_1\to\Sigma_1$ which coincides with the identity in the complement of a ball $B_\Delta\subset\Sigma_1$ of radius $0<\Delta<\Delta_0$ then there exists a vector field $Y\in\mathfrak{X}^1_\mu(M)$ such that:*

(a) *Y is $C\Delta$-C^0-close to X;*
(b) *$Y=X$ outside the tubular set $\phi^{-1}([0,1]\times[B(\vec{0},\Delta_0)\setminus B_\Delta])$;*
(c) *$\mathcal{P}_Y(z)=h\circ\mathcal{P}_X(z)$ for all $z\in\Sigma_0$, where $\mathcal{P}_Z=Z^1\big|_{\Sigma_0}$ is the time-1 Poincaré map of $Z\in\mathfrak{X}^1_\mu(M)$.*

Proof. Using the flowbox theorem for smooth volume preserving flows ([8, Lemma 2.1]), we use trivial coordinates to make our computations clearer. Observe that such volume-preserving change of coordinates keeps the assumption of being isotopic to the identity unaltered for the diffeomorphism in the arrival cross-section. In overall, we let $T^s(t,z)=(t+s,z)$ be the trivialized flow and $\mathcal{P}^t_T=Id$ its Poincaré map (after the identification between the fibered disks in the cross sections $\Sigma_0:=\phi^{-1}(\{0\}\times B(\vec{0},\Delta))$ and Σ_1). Consider also Σ_s to be the hyperplane $t=s$.

As h is a volume-preserving diffeomorphism isotopic to the identity, and all fibers are identified through a projection Π onto Σ_1, there exists a smooth family $h_t\colon\Sigma_0\to\Sigma_t$ so that $\Pi\circ h_t$ is an isotopy between Id and h in Σ_1. We claim that the vector field defined, for $s\in[0,1]$ and $z\in\Sigma_s$, by

$$Y(s,z)=(1,\dot{h}_s\circ h_s^{-1}(z)) \tag{4.1}$$

satisfies our demands. Observe that the map $h_s^{-1}(z)$ in (4.1) should be seen as a map $h_s^{-1}\colon\Sigma_s\to\Sigma_s$ taking into account the identifications we did. After that, the infinitesimal generator of the isotopy $\dot{h}_s$ is applied accordingly. Note that, for allowed t, we get

$$Y^t(s,z)=\left(t+s,\mathcal{P}^t_Y(z)\right):=\left(t+s,z+\int_0^t\dot{h}_r\circ h_r^{-1}\circ\mathcal{P}^r_Y(z)\,dr\right). \tag{4.2}$$

Item (a) just depends on the thickness of W and so is easily controlled. Item (b) follows by pasting considerations of Lemma 3.4. We need to prove that (c) holds i.e. $\mathcal{P}_Y(z)=h\circ\mathcal{P}_T(z)$. Notice that $h\circ\mathcal{P}_T(z)=h$ hence we will show that $\mathcal{P}_Y(z)=h$ or, more generally, $\mathcal{P}^t_Y(z)=h_t$. But this is a direct consequence of (4.2) since

$$\mathcal{P}^t_Y(z)=z+\int_0^t\dot{h}_r\circ h_r^{-1}\circ\mathcal{P}^r_Y(z)\,dr=z+\int_0^t\dot{h}_r(z)\,dr=z+h_t(z)-h_0(z)=h_t(z).$$

We are left to check that Y is divergence-free. Observe that $P^t_Y=D\mathcal{P}^t_Y(z)=Dh_t$ and so $\det P^t_Y=\det Dh_t=1$. In overall, we have

$$\det DY^t(s,z)=\det\begin{pmatrix}1 & 0\\ \dot{P}^t_Y & P^t_Y\end{pmatrix}=\det P^t_Y=\det Dh_t=1,$$

which by Abel-Jacobi-Liouville formula allow us to conclude that Y is divergence-free.

□

Remark 4.3. Although Lemma 4.2 is stated for a time-1 flowbox chart, it can be obtained for any flowbox chart defined in time $[0,s]$. Moreover, the constant $C>0$ is not affected by time-reparameterization because we are dealing with the C^0-topology instead of the C^1-topology.

Hence, the previous lemma ensures that if a finite number of cross-sections is fixed there exists a constant $C_X>0$ so that any smooth perturbation of the Poincaré map supported in a Δ-ball can be realized by a $C_X\Delta$-C^0-small perturbation of the smooth vector field.

4.2. **Fixed point free flows.** We now complete the proof of item (2) of Theorem A. Let $\mathfrak{X}^{0,1}_{\mu,*}(M)$ be the set of Lipschitz continuous incompressible vector fields having no singularities.

Given $\varepsilon > 0$, let $\mathcal{R}_{\varepsilon,*} \subset \mathfrak{X}^{0,1}_{\mu,*}(M)$ be the space of vector fields for which there exists a finite set $\mathcal{S} = (\Sigma_i)_i$ formed by smooth local cross-sections to X, each of these with diameter smaller than $\varepsilon > 0$ and there is $r \in (0,1)$ such that the following properties hold:

(P_0) $\Sigma := \bigcup_{1\le i\le n} \Sigma_i$ is a global cross-section for the flow $(X^t)_t$, with Poincaré first return map
$$P_X \colon \Sigma \to \Sigma \quad \text{given by} \quad P_X(x) = X^{\tau(x)}(x),$$
where $\tau : \Sigma \to (0,+\infty)$ is given by $\tau(x) = \inf\{t > 0 \colon X^t(x) \in \Sigma\}$;

(P_1) for all $\Sigma_i, \Sigma_j \in \mathcal{S}$ either $P_X(\Sigma_i) \cap \Sigma_j = \emptyset$ or $P_X(\mathrm{Int}(\Sigma_i)) \cap \mathrm{Int}(\Sigma_j) \neq \emptyset$;

(P_2) for every $\Sigma_i, \Sigma_j \in \mathcal{S}$ so that the intersection $P_X(\Sigma_i) \cap \Sigma_j$ has non-empty interior there exists a sectional rectangle $V = [a,b] \times D^{n-2}_r \subset \mathrm{Int}(\Sigma_i)$, for some $[a,b] \subset [-r,r]$, so that
$$V \overset{P_X}{\Longrightarrow} [-r,r] \times D^{n-2}_r \subset \mathrm{Int}(\Sigma_j) \quad \text{or} \quad V \overset{P_X,0}{\Longrightarrow} [-r,r] \times D^{n-2}_r \subset \mathrm{Int}(\Sigma_j)$$
where D^d_r denotes the d-dimensional closed ball with radius r.

As it will be clear from the proof, the choice of Σ depends on the vector field and its diameter is usually much smaller than ε, with no uniform lower bound. We also observe that a direct application of Gronwall's inequality implies that the time-1 map of a flow varies continuously with respect to the underlying generating vector field. In particular, it is easy to check that for any $\varepsilon > 0$ the set $\mathcal{R}_{\varepsilon,*}$ is C^0-open in $\mathfrak{X}^{0,1}_{\mu,*}(M)$. We now prove its denseness.

We will make use of the following consequence of Lemma 4.2 and Remark 4.3.

Lemma 4.4. *(Realization of covering relations) Given $X \in \mathfrak{X}^1_\mu(M)$, two smooth local cross-sections $\hat\Sigma_1, \hat\Sigma_2 \subset M$ to the flow and the Poincaré map $\mathcal{P}_X : D \subset \hat\Sigma_1 \to \mathcal{P}(\hat\Sigma_1) \cap \hat\Sigma_2$, let $\Delta > 0$ be so that the Δ-sectional rectangles around the point $x \in \hat\Sigma_1$ and its image $P_X(x) \in \hat\Sigma_2$ are contained in the corresponding cross-sections. There exists a constant $L_X > 0$ so that the following holds: if $0 < r \ll \Delta$ and $V = [a,b] \times D^{n-2}_r \subset D$ is a sectional rectangle centered at x and $W = [-r,r] \times D^{n-2}_r \subset \hat\Sigma_2$ is a sectional rectangle centered at $P_X(x)$ then there exists a vector field $Y \in \mathfrak{X}^1_\mu(M)$ satisfying*

(a) *Y is $L_X r$-C^0-close to X;*
(b) *$Y = X$ outside an r-tubular neighborhood of a compact piece of orbit of x;*
(c) *$V \overset{P_Y}{\Longrightarrow} [-r,r] \times D^{n-2}_r \subset \hat\Sigma_2$.*

In particular, Y satisfies the covering relation (P_2) between the cross-sections $\hat\Sigma_1$ and $\hat\Sigma_2$.

Proof. The argument is a simple consequence of Lemma 4.2. We give details for completeness. Let $C_X > 0$ be the constant determined by Lemma 4.2 and Remark 4.3 for X, and consider the cross sections $\hat\Sigma_1$ and $\hat\Sigma_2$. Since the covering relation can be adapted to any cross section in the flowbox chart by displacement along the flow direction (observe the cross-sections $\tilde\Sigma_2$ and $\Psi(\hat\Sigma_2)$ in Figure 3) we shall assume that the hitting time between the cross sections $D \subset \hat\Sigma_1$ and $P_X(D) \subset \hat\Sigma_2$ is constant to $s > 0$, and that these are initial and final cross-sections of a flowbox chart.

We proceed as follows. If $r > 0$ is small then $P_X(V)$ is contained in $\hat\Sigma_1$. Moreover, Gronwall's inequality ensures that the diameter of $P_X(V)$ is bounded above by $e^{s\|X\|_\infty} r$. Moreover, the image $P_X(V)$ is isotopic to a subset W_0 which intersects W according to the covering relation (A) (see Figure 3). Thus, using the perturbative Lemma 4.2 on the tubular neighborhood of radius $e^{s\|X\|_\infty} r$ we conclude that there exists a vector field $Y \in \mathfrak{X}^1_\mu(M)$ which is $C_X e^{s\|X\|_\infty} r$-C^0-close to X and so that $P_Y(V) = W_0$, hence satisfying the covering relation. □

The previous result put us in a position to prove the following:

Lemma 4.5. *For any $\varepsilon \in \mathbb{Q}_+$ the set $\mathcal{R}_{\varepsilon,*}$ is C^0-dense in $\mathfrak{X}^{0,1}_{\mu,*}(M)$.*

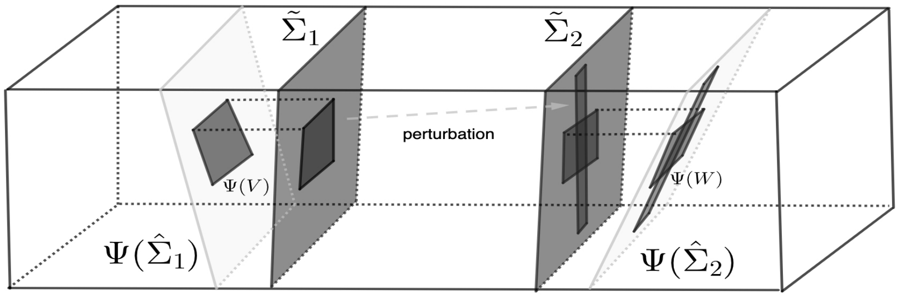

FIGURE 3. Perturbation supported in the flowbox between $\widetilde{\Sigma}_1$ and $\widetilde{\Sigma}_2$, leading to covering relations on $\widetilde{\Sigma}_2$ and $\psi(\widehat{\Sigma}_2)$.

Proof. Fix $X \in \mathfrak{X}^{0,1}_{\mu,*}(M)$ and $\varepsilon \in \mathbb{Q}_+$. Up to an arbitrary small C^0-perturbation we may assume that X is C^∞ smooth (cf. Theorem 3.1). For any $0 < \zeta \ll \varepsilon$ we proceed to obtain $Z \in \mathcal{R}_{\varepsilon,*}$ so that $\|X - Z\|_{C^0} < \zeta$. Since $0 < \zeta \ll \varepsilon$, the vector field X has no singularities and M is compact, one can cover M by a finite number of Lipschitz flowbox charts $(\mathfrak{U}_i)_{1\le i\le n}$, given by Theorem 3.2. Moreover, we can choose these flowbox charts in such a way that:

(i) all pieces of orbits in $\mathfrak{U}_i$ have length one for $(X^t)_t$; and
(ii) on each flowbox chart $\mathfrak{U}_i$ there exists a smooth global cross-section $\Sigma_i \subset \mathfrak{U}_i$ for the vector field $X\mid_{\mathfrak{U}_i}$ so that the Poincaré first return time to $\Sigma := \bigcup_{i=1}^n \Sigma_i$ lies in the interval $[\frac{1}{2}, 2]$ and $\operatorname{diam}\Sigma_i < \frac{\zeta}{100}$ for every $1 \le i \le n$.

Let $\mathcal{S}$ denote the previous family of cross-sections. Changing the flowbox charts slightly, if necessary, we may assume that property (P_1) above holds for X. Moreover, since Σ is a global cross-section to any vector field $Y \in \mathfrak{X}^{0,1}_{\mu,*}(M)$ that is C^0-close to X and the intersections of the image of the sections persist by C^0-perturbation of the vector field, then property (P_1) holds for the Poincaré map $P_Y : \Sigma \to \Sigma$ associated to any vector field Y that is C^0-close to X. It suffices to prove that (P_2) can also be attained by a small C^0-perturbation of X.

While the Poincaré maps are piecewise continuous (except in the case that $\Sigma = \Sigma_1$ is a smooth global cross-section to X, where the Poincaré map is continuous), the perturbations will be performed in the space of Lipschitz continuous vector fields. Given any pair $\Sigma_i, \Sigma_j \in \mathcal{S}$ of local cross-sections for X so that $P_X(\Sigma_i) \cap \Sigma_j$ has non-empty interior, a perturbation identical to the one used in the proof of Lemma 3.1 in [18] ensures that there exist a segment $[a, b] \subset [-r, r]$, a sectional cube $V = [a, b] \times D^{n-2}_r \subset \operatorname{Int}(\Sigma_i)$ and a volume-preserving homeomorphism $h : \Sigma_j \to \Sigma_j$, homotopic to the identity, so that $d_{C^0}(h, id) \le \operatorname{diam}(\Sigma_j) < \zeta$ and

$$V \overset{h\circ P_X}{\Longrightarrow} [-r, r] \times D^{n-2}_r. \tag{4.3}$$

The Realization of covering relations on Lemma 4.4 ensures that there exists a vector field $Y \in \mathfrak{X}^1_{\mu,*}(M)$ that is C^0-close to X and such that $P_Y = h \circ P_X$, hence the covering relation

$$V \overset{P_Y}{\Longrightarrow} [-r, r] \times D^{n-2}_r$$

holds. Since there are finitely many pairs of elements $\Sigma_i, \Sigma_j \in \mathcal{S}$, we conclude that property (2) holds for a vector field $Z \in \mathfrak{X}^1_{\mu,*}(M)$ obtained from X by a finite number of arbitrary small C^0-perturbations with disjoint supports. This proves the lemma. ◻

Consider the C^0-generic subset

$$\mathcal{R}_* = \bigcap_{\varepsilon\in\mathbb{Q}_+} \mathcal{R}_{\varepsilon,*} \subset \mathfrak{X}^{0,1}_{\mu,*}(M). \tag{4.4}$$

We are now in a position to prove that the periodic shadowing property is C^0-generic on $\mathfrak{X}^{0,1}_{\mu,*}(M)$.

Lemma 4.6. *Every vector field $X \in \mathcal{R}_*$ satisfies the periodic shadowing property.*

Proof. Fix $X \in \mathcal{R}_*$ and let $\varepsilon > 0$, $T > 1$ be arbitrary. By compactness of the ambient space it is enough to prove finite shadowing and we shall do so. Given $j \geq 1$ we will prove that the shadowing property holds for all pseudo-orbits with j pieces of orbits.

As $X \in \mathcal{R}_{\varepsilon,*}$, there exists a finite set $\mathcal{S} = (\Sigma_i)_i$ formed by smooth local cross-sections to X, each of these with diameter smaller than $\varepsilon > 0$ and there is $r \in (0,1)$ so that properties (P_0), (P_1) and (P_2) above hold.

Let $\delta = \delta(\varepsilon, \mathcal{S}) > 0$ be smaller than the minimum of the inner diameter of the sets $P_X(\Sigma_i) \cap \Sigma_j$ (here the minimum is taken over all pairs of local cross sections $\Sigma_i, \Sigma_j \in \mathcal{S}$ $(1 \leq i, j \leq n)$ so that the previous intersection is non-empty). By construction, the constant δ is independent of the size of the pieces of orbits t_i and the number of jumps in pseudo-orbits. Furthermore, by construction

$$M = \bigcup_{1\leq i\leq n} \Big[\bigcup_{t\in[0,1]} X^t(\Sigma_i) \Big],$$

is a union of flowboxes. Since we consider subsets of flowbox charts, it is important to estimate how much the time fluctuates on these sets. If $L > 0$ denotes the Lipschitz constant of X then any two points $x, y \in \Sigma_i$ so that $P_X(x)$ and $P_X(y)$ belong to the same local cross-section in $\mathcal{S}$ satisfy $\|X^t(x) - X^t(y)\| \leq e^{Lt}\|x - y\|$ for all $t \in \left[\frac{1}{2}, 2\right]$ (using a lipeomorphism, local coordinates in $\mathbb{R}^d$ and Gronwall's inequality). Thus, as $P_X(\cdot) = X^{\tau(\cdot)}(\cdot)$,

$$\|P_X(x) - X^{\tau(x)-\tau(y)}(P_X(y))\| \leq e^{L\tau(x)}\|x - y\| \leq e^L\varepsilon$$

and, by triangular inequality,

$$\|P_X(y) - X^{\tau(x)-\tau(y)}(P_X(y))\| \leq e^L\varepsilon + \|P_X(y) - P_X(x)\| \leq (e^L + 1)\,\varepsilon.$$

Since the vector field X has no singularities, this together with Gronwall inequality implies that $|\tau(x) - \tau(y)| \lesssim C\varepsilon$.

Reduce $\delta > 0$ if necessary, so that it becomes smaller than the Lebesgue number of the finite covering $\{\bigcup_{t\in(0,1)} X_t(\Sigma_i)\colon\ 1 \leq i \leq n\}$ of M by flowbox charts. With this choice of δ, the jumps in any (δ, T)-pseudo orbit will lie in the same flowbox chart. Now we claim that every (δ, T)-pseudo orbit is ε-shadowed by a true orbit. Consider a (δ, T)-pseudo orbit $[x_\ell, t_\ell]_{\ell=1}^j$. The choice of $\delta > 0$ and uniform continuity of the flow on the time interval $[0, 1]$ implies that the sequence

$$\Big\{y_\ell := P_X(x_\ell)\colon 1 \leq \ell \leq j\Big\} \subset \Sigma \tag{4.5}$$

is a δ-pseudo-orbit for the Poincaré map P_X. In (4.5), by some abuse of notation, we keep denoting by $P_X : M \to \Sigma$ the first hitting time map, and let

$$s_i = \kappa(x_i) := \inf\Big\{s \geq 0\colon X^s(x_i) \in \Sigma\Big\} \tag{4.6}$$

denote the corresponding first hitting times.

By the choice of δ, the points $P_X(y_\ell)$ and $y_{\ell+1}$ belong to the same local cross-section $\Sigma_{i_\ell} \in \mathcal{S}$, for every $1 \leq \ell \leq j$, on which there are the covering relations defined by item (P_2) above. Then, Theorem 4.1 guarantees that there exists $y \in \Sigma_{i_1}$ so that

$$d(P^\ell_X(y), y_{\ell+1}) < \varepsilon \quad \text{for every } 0 \leq \ell \leq j - 1.$$

If, in addition, the pseudo-orbit is periodic then the point y can be chosen periodic for the composition of the Poincaré maps.

Take $x = X^{-s_1}(y)$. We claim that there exists a reparametrization $\tau \in \mathrm{Rep}(C\varepsilon)$ such that

$$d(X^{\tau(t)}(x), x_0 \star t) < 2\varepsilon, \ \text{ for every } t \in [0, \sigma(j)].$$

Remark 4.7. Using that $1 \le t_i \le T$ for all $1 \le i \le n$ and the uniform continuity of $[0,T] \times M \ni (t,x) \mapsto X^t(x)$ we can reduce $\delta = \delta(\varepsilon, T) > 0$, if necessary, to ensure that the trajectory of the point $X^{\sum_{j=0}^{\ell-1} \kappa(P_X^j(y))}(y)$ and $y_{\ell+1}$ remain 2ε-close for time $[0,T]$. Even though we are considering flowbox charts, where orbits travel at constant speed, this is not immediate from the discrete-time shadowing on local cross-sections because of an eventual slide overlaping (see Figure 4 below).

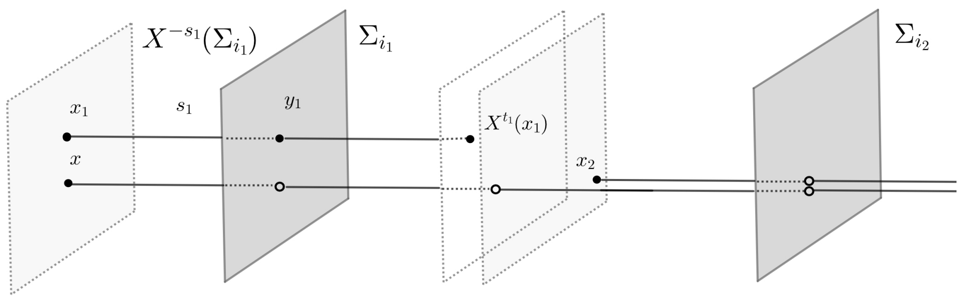

FIGURE 4. Allowing some reparametrization ε-close to identity implied by an eventual δ-slide overlaping.

Recall that, by construction, the jumps in the (δ, T)-pseudo orbit $[x_\ell, t_\ell]_{\ell=1}^{j}$ lie in the same flowbox chart. Hence, the reparametrization τ is built recursively using the flowbox charts. More precisely, by (4.5), $y_1 = P_X(x_1) \in \Sigma_{i_1}$ and

$$\begin{cases} y_2 = P_X(x_2) \in \Sigma_{i_2} \\ P_X(X^{t_1}(x_1)) = X^{t_1+s_1}(x_1) \in \Sigma_{i_2}. \end{cases}$$

Since any two points are in the same flowbox chart, up to a conjugation lipeomorphism φ, these can be identified with points in a rectangle $[0,1] \times D_{i_1}$ for some disk $D_{i_1} \subset \mathbb{R}^{n-1}$, and up to this change of coordinates, we say that $p \prec q$ if $\pi_1(\varphi(p)) < \pi_1(\varphi(q))$. While there are several cases to consider, according to the relative position of initial and end points of pseudo-orbits, assume that $j = 2$ and $X^{t_1}(x_1) \prec x_2$ (the other cases are analogous). In this case, if $0 < \delta_1 := \pi_1(\varphi(x_2)) - \pi_1(\varphi(X^{t_1}(x_1))) < \delta$ the reparametrization τ is given by

$$\tau(t) = \begin{cases} t & , \ t \in [0, t_1] \\ \frac{t_2+\delta_1}{t_2}(t - t_1) + t_1 & , \ t \in (t_1, t_1 + t_2]. \end{cases}$$

It is clear that $\tau(t) - \tau(s)$ is maximized whenever $s < t_1 < t$ hence

$$\tau(t) - \tau(s) \le \frac{t_2 + \delta_1}{t_2}(t - t_1) + t_1 - s = \left(\frac{t_2 + \delta_1}{t_2} - 1\right)(t - t_1) + t - s \le \frac{t_2 + \delta_1}{t_2}(t - s) \le (1 + \delta)(t - s)$$

or, equivalently,

$$\frac{\tau(t) - \tau(s)}{t - s} - 1 \le \delta < \varepsilon \quad \forall t, s \in [0, t_1 + t_2].$$

The reasoning for periodic (δ, T)-pseudo orbits is entirely analogous. Finally, as ε can be chosen arbitrary, the latter completes the proof of the lemma. □

5. Incompressible vector fields with singularities

This section is devoted to the proof of Theorem B. The argument for the general density theorem is completely analogous to the one in Section 3.7. Hence it remains to prove that C^0-generic incompressible vector fields satisfy the weak shadowing property. The *homogeneity property* on a given compact boundaryless manifold M endowed with a metric d of dimension $n \geq 2$ was caracterized in [1, Proposition 2] on which was proved that given any $\eta > 0$, there exists $\delta > 0$ such that if $\{(x_i, y_i)\}_{i=1}^n$ are n pairs of distinct points such that $d(x_i, y_i) < \delta$ for every $i = 1, ..., n$, then there exists a family of n pairwise disjoint topological balls $\{B_i\}_{i=1}^n$ in the interior of M, with $diam(B_i) < \eta$ and so that $x_i, y_i \in B_i$, for every $i = 1, \dots, n$. The following result proves the much stronger result, that in dimension larger than two such homotopies can be performed using incompressible flows.

Lemma 5.1 (Homogeneity property by incompressible Lipschitz vector fields). *Let M be a compact boundaryless manifold of dimension $n > 2$. Given $\beta > 0$ there exists $\alpha > 0$ so that for any given $n \geq 1$ and pairs of distinct points $\{(x_i, y_i)\}_{i=1}^n$ satisfying $d(x_i, y_i) < \alpha$ for all $i = 1, \dots, n$ there exists a vector field $Z \in \mathfrak{X}_\mu^{0,1}(M)$ avoiding any finite number of curves such that $\|Z\|_0 < \beta$ and $Z^\alpha(x_i) = y_i$, for every $i = 1, \dots, n$.*

Proof. By homogeneity, given $\varepsilon > 0$ there exists $\alpha > 0$ so that for any given $n \geq 1$ and pairs of distinct points $\{(x_i, y_i)\}_{i=1}^n$ satisfying $d(x_i, y_i) < \alpha$ for all $i = 1, \dots, n$ there exist n pairwise disjoint paths γ_i of length bounded above by α and connecting the points x_i to y_i. These can be chosen to avoid any finite number of curves in M as it has dimension at least 3. In particular one can choose n narrow bands $\mathcal{T}_i$ $(1 \leq i \leq n)$ with pairwise disjoint closures and smooth boundary, diffeomorphic to disjoint balls. The lemma follows by applying Lemma 3.9 in finitely many disjoint domains. □

Now, given the main technical Lemma 5.1, the proof of Theorem B can now be obtained by a rather simple modification of the argument used by Mazur [24, Theorem 1] for dissipative homeomorphisms. For each $\varepsilon > 0$ let $(U_i^\varepsilon)_{1 \leq i \leq k_\varepsilon}$ be a finite open covering of M by sets with diameter smaller than ε. Given a vector field $X \in \mathfrak{X}_\mu^{0,1}$ consider the set

$$\mathfrak{I}_X^\varepsilon = \left\{ \{U_{j_1}, U_{j_2}, \dots, U_{j_t}\} \in 2^{\{1,2,\dots,k_\varepsilon\}} \colon \exists x \in M \text{ s.t. } \overline{\{X^t(x) \colon t \in \mathcal{R}\}} \cap U_{j_\ell} \neq \emptyset,\ \forall 1 \leq \ell \leq t \right\}$$

of configurations that are realized by a true orbit of the flow, independently of the order. By Grownall's inequality it is clear that the itinerary map

$$\mathfrak{X}_\mu^{0,1} \ni X \mapsto \mathfrak{I}_X^\varepsilon$$

is strongly lower-semicontinuous: for every $X \in \mathfrak{X}_\mu^{0,1}(M)$ there exists a C^0-open neighborhood $\mathcal{V}$ of X such that $\mathfrak{I}_X^\varepsilon \subset \mathfrak{I}_Y^\varepsilon$ for every $Y \in \mathcal{V}$.

Consider the set $\mathcal{S}_\varepsilon \subset \mathfrak{X}_\mu^{0,1}(M)$ of vector fields with stable ε-itinerary, meaning that for each $X \in \mathcal{S}$ there exists a C^0-open neighborhood $\mathcal{V}_X$ of X so that $\mathfrak{I}_X^\varepsilon = \mathfrak{I}_Y^\varepsilon$ for every $Y \in \mathcal{V}_X$. This is also a C^0-dense set because of the previous lower-semicontinuity and the fact that, by finitude of set $2^{\{1,2,\dots,k_\varepsilon\}}$, in any C^0-open set in $\mathfrak{X}_\mu^{0,1}(M)$ there exists a vector field X whose itinerary $\mathfrak{I}_X^\varepsilon$ is a maximal element (with respect to inclusion). Therefore, $\mathfrak{R} := \bigcap_{n \geq 1} \mathcal{S}_{\frac{1}{n}}$ is a C^0-generic subset of $\mathfrak{X}_\mu^{0,1}(M)$.

We claim that every vector field $X \in \mathfrak{R}$ generates a flow $(X^t)_t$ with the weak shadowing property. Fix $\varepsilon > 0$ and $n > \frac{1}{\varepsilon}$. As $X \in \mathcal{S}_{\frac{1}{n}}$, there exists $\delta_0 > 0$ so that $\mathfrak{I}_X^{\frac{1}{n}} = \mathfrak{I}_Y^{\frac{1}{n}}$ for every $Y \in \mathfrak{X}_\mu^{0,1}(M)$ such that $\|Y - X\|_{C^0} < \delta_0$. Let $\delta = \frac{\alpha}{2} > 0$ where α is given by Lemma 5.1 when $\beta = \delta_0 > 0$. Fix $T > 1$ and consider any (δ, T)-pseudo orbit $[(x_i, t_i)]_{i \in \mathbb{Z}}$. By compactness of M, there exists $\ell \geq 1$ such that the latter is ε-shadowed by the finite (δ, T)-pseudo orbit $[(x_i, t_i)]_{-\ell \le i \le \ell}$. Moreover, since M is a manifold, changing the points x_i slightly, if necessary, we produce a $(2\delta, T)$-pseudo orbit $[(x_i, t_i)]_{-\ell \le i \le \ell}$ such that all pieces of orbits $\{X^{[0,t_i]}(x_i)\colon -\ell \le i \le \ell\}$ are pairwise disjoint. By Lemma 5.1 there exists a vector field $Y \in \mathfrak{X}^{0,1}_{\mu}(M)$ that is $\frac{1}{n}$-C^0-close to X so that $X^{[0,t_i]}(x_i) = Y^{[0,t_i]}(x_i)$, $\forall -\ell \le i \le \ell$ and $Y^{\alpha}(X^{t_i}(x_i)) = x_{i+1}$ for all i. In particular, as $\Im_X^{\frac{1}{n}} = \Im_Y^{\frac{1}{n}}$, there exists $x \in M$ so that the orbit $\{X^t(x)\colon t \in \mathbb{R}\}$ intersects all elements in $\Im_X^{\frac{1}{n}}$. Altogether this ensures that every (δ, T)-pseudo orbit $[(x_i, t_i)]_{i\in\mathbb{Z}}$ is 3ε-weakly shadowed by a true orbit of the flow $(X^t)_t$. Since ε was chosen arbitrary, this proves that $(X^t)_t$ satisfies the weak shadowing property. Finally, it is clear that in the case of incompressible vector fields the weak shadowing property implies on transitivity. This completes the proof of the theorem.

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