GLUING ORBIT PROPERTY AND PARTIAL HYPERBOLICITY

THIAGO BOMFIM, MARIA JOANA TORRES AND PAULO VARANDAS

Abstract. This article is a follow up of our recent works [7, 8], and here we discuss the relation between the gluing orbit property and partial hyperbolicity. First we prove that a partially hyperbolic diffeomorphism with two saddles with different index, and such that the stable manifold of one of these saddles coincides with the strongly stable leaf does not satisfy the gluing orbit property. In particular, the examples of C^1-robustly transitive diffeomorphisms introduced by Mañé [20] do not satisfy the gluing orbit property. We also construct some families of partially hyperbolic skew-products satisfying the gluing orbit property and derive some estimates on their quantitative recurrence.

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1. Introduction

The concept of reconstruction of orbits in topological dynamics and ergodic theory gained substantial importance for its wide range of applications. Among these properties of reconstruction of true orbits from approximate orbits (also called pseudo-orbits) and finite pieces of orbits one can refer the shadowing and the specification properties. On the one hand, in brief terms a dynamical system satisfies the shadowing property whenever any pseudo-orbit can be approximated by true orbits. On the other hand, the specification property, introduced by Bowen [11], roughly means that an arbitrary number of finite pieces of orbits can be "glued together" to obtain a real orbit that shadows the previous ones with a prefixed number of iterates in between. Dynamical systems having either of these properties have a rich dynamical structure (see e.g. [5, 25, 33] and references therein). However, soon it became clear the existence of a strong relation between the previous properties and uniform hyperbolicity in the case of C^1-diffeomorphisms. More precisely, a C^1-generic tame diffeomorphism with the shadowing property is Axiom A without cycles [2] and a C^1-generic diffeomorphism with the specification property is a transitive Anosov diffeomorphism [21]. Related results, on the characterization of the C^1-interior of diffeomorphisms satisfying any of the previous properties, include [23, 24]. However, most dramatically, if one considers the space of C^r-diffeomorphisms, $r \geq 1$, these two properties seldom occur in the absence of uniform hyperbolicity as proved in [9, 27, 28].

The large amount of dynamical consequences which can be obtained for dynamical systems with shadowing or specification soon inspired many authors to pursue

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weakenings of the latter topological invariant (we refer the reader to [19] for an excellent overview of some of these notions). Here we will continue the investigation of the notion of gluing orbit property, introduced in [7, 8]. The underlying notion of the gluing orbit property can be traced also to [13, 29], in the context of differentiable dynamics. The gluing orbit property is much weaker than specification, and it bridges between hyperbolic and completely non-hyperbolic behavior. Indeed, this property is satisfied by both transitive hyperbolic dynamics and by partially hyperbolic diffeomorphisms obtained as time-1 maps of Anosov flows [7], and by zero entropy, minimal and equicontinuous dynamics [8, 30]. Given this wider range of dynamical systems it is quite natural to address the following questions concerning the gluing orbit property:

(1) how common is this property in the space of dynamical systems?
(2) can one characterize dynamical systems with this property?
(3) what is the dynamical richness of such dynamical systems?

As usual, an answer to the first question depends strongly on the topology in the space of dynamical systems. On the one hand, the gluing orbit property holds at every chain recurrent class of a C^0-generic dynamics (cf. [6]). On the other hand, C^1-generic diffeomorphisms with the gluing orbit property are uniformly hyperbolic (see [7] for the precise statement). Given this fact, and inspired by some results on the thermodynamic formalism of partially hyperbolic diffeomorphisms, our aim here is to provide partial answers to questions (1) - (3) in the context of partially hyperbolic dynamics.

As a first step, building over [27], we prove that the existence of hyperbolic periodic points with different index is still an obstruction for the gluing orbit property even for partially hyperbolic diffeomorphisms (we refer the reader to Theorem A for the precise statement). As a consequence, there exist C^1 open and dense subset of the space of robustly transitive and partially hyperbolic diffeomorphisms on a three-dimensional manifold formed by diffeomorphisms that do not satisfy the gluing orbit property (cf. Corollary 2). One other consequence concerns an important class of partially hyperbolic diffeomorphisms on the three-dimensional torus introduced by Mañé [20] known as DA-maps (derived from Anosov). These are C^1-robustly transitive and partially hyperbolic diffeomorphisms, obtained by deformations of Anosov diffeomorphisms by isotopy exhibiting periodic points of different index. In this case we will deduce that the Mañé examples do not satisfy the gluing orbit property. These results provide partial answers to question (1) in the context of partially hyperbolic diffeomorphisms.

Despite the previous results, on the negative, questions (2) and (3) inspired us to pursue the characterization of the three-dimensional partially hyperbolic diffeomorphisms with compact center leaves which satisfy the gluing orbit property (and fail to satisfy specification). This can be made concrete by two different constructions, using skew-products involving either Anosov diffeomorphisms or shift dynamics, and rotations (Theorems B, C and D). While one of such models can be definitely studied using non-autonomous dynamical systems, for the other it seems to exist a dichotomy involving the presence or absence of the gluing orbit property which explores a synchronization on the circle dynamics. This yields some topological and ergodic consequences for the dynamics including quantitative recurrence estimates for some partially hyperbolic dynamics (cf. Corollary 3).

This article is organized as follows. In Section 2 we recall the concepts of partial hyperbolicity and robustly transitive diffeomorphisms and state our main results. Some preliminary results on sequential dynamical systems, gluing orbit properties and partial hyperbolicity are given in Section 3. The proofs of the main results will be given in Subsections 4.1 to 4.4.

2. Statement of the main results

Setting. Throughout, let M be a C^∞ closed manifold with $\dim M \geq 3$, where $\dim E$ denotes the dimension of E, and let $\mathrm{Diff}^r(M)$ denote the space of C^r diffeomorphisms on M endowed with the C^r-topology, $r \geq 1$. Given $f \in \mathrm{Diff}^1(M)$, a Df-invariant splitting $TM = E \oplus F$ is dominated if there is an integer $k \in \mathbb{N}$ such that

$$\frac{\|D_x f^k(u)\|}{\|D_x f^k(w)\|} < \frac{1}{2},$$

for every $x \in M$ and every pair of unitary vectors $u \in E(x)$ and $w \in F(x)$. Generally, a Df-invariant splitting $TM = E_1 \oplus \cdots \oplus E_k$ is *dominated* if for any $1 \leq l \leq k-1$, $(E_1 \oplus \cdots \oplus E_l) \oplus (E_{l+1} \oplus \cdots \oplus E_k)$ is dominated. A Df-invariant bundle E is *uniformly contracting* (resp. *expanding*) if there are $C > 0$ and $0 < \lambda < 1$ such that for every $n > 0$ one has $\|D_x f^n(v)\| \leq C\lambda^n \|v\|$ (resp. $\|D_x f^{-n}(v)\| \leq C\lambda^n\|v\|$) for all $x \in M$ and $v \in E(x)$. We say that a diffeomorphism f is *partially hyperbolic* (resp. *strongly partially hyperbolic*) if there is a Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that E^s and E^u are uniformly contracting and uniformly expanding respectively, and at least one of them is (resp. both of them are) not trivial. A diffeomorphism is hyperbolic if it is strongly partially hyperbolic and E^c is trivial. We say that E^c is the central direction of the splitting. We say that $f \in \mathrm{Diff}^1(M)$ is *transitive* if there is $x \in M$ whose orbit is dense in M. A diffeomorphism f is *robustly transitive* if there is a C^1-neighborhood $\mathcal{U}(f)$ of f in $\mathrm{Diff}^1(M)$ such that any $g \in \mathcal{U}(f)$ is transitive. Denote by $\mathcal{RNT}$ the set of robustly non-hyperbolic and transitive diffeomorphisms in $\mathrm{Diff}^1(M)$, that is, the set of diffeomorphisms f having a C^1-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ is non-hyperbolic and transitive.

Statement of the main results. Our main results concern the relation of partially hyperbolicity with gluing orbit properties and some of its topological and ergodic consequences. The starting point is the following:

Theorem A. *Let $f\colon M \to M$ be a C^1-diffeomorphism admitting a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Assume that there are two hyperbolic periodic points p and q such that either* $\dim E^u = \dim \mathcal{W}^u(p) < \dim \mathcal{W}^u(q)$ *or* $\dim E^s = \dim \mathcal{W}^s(q) < \dim \mathcal{W}^s(p)$. *Then f does not satisfy the gluing orbit property. Moreover, if* $\dim M = 3$ *then there is a C^1-dense open subset $\mathcal{P}$ in $\mathcal{RNT}$ so that every $f \in \mathcal{P}$ does not satisfy the gluing orbit property.*

Here $\mathcal{W}^u(p)$ (resp. $\mathcal{W}^s(p)$) denotes the unstable (resp. stable) manifold of the point p defined as usual. The previous result can be understood as an extension of [27, Theorem A] where, analogously, the holonomy map along the strong unstable foliation plays a key role in the proof. In particular, a counterpart of this result for partially hyperbolic flows is expected to hold, in the spirit of [28].

We shall now deduce some implications of Theorem A on the space of robustly transitive diffeomorphisms. Denote by $\mathcal{SPH}_1(M) \subset \mathrm{Diff}^1(M)$ the C^1-open set of strongly partially hyperbolic diffeomorphisms with one-dimensional central direction. In [20], Mañé introduced a class of robustly transitive, non-hyperbolic and strong partially hyperbolic diffeomorphisms, which are nowadays known as *Mañé diffeomorphisms*. In the case that the central direction E^c is one dimensional, any two hyperbolic periodic points with different index verify the assumptions of Theorem A. We obtain the following immediate consequence.

Corollary 1. *If $f \in \mathcal{SPH}_1(M)$ admits two hyperbolic periodic points p, q with different index then f does not satisfy the gluing orbit property. In particular, Mañé diffeomorphisms do not satisfy the gluing orbit property.*

We recall that any hyperbolic measure μ that is invariant by a $C^{1+\alpha}$ diffeomorphism is weak* approximated by invariant probabilities supported on hyperbolic periodic points with the same index of the hyperbolic splitting determined by μ (cf. [16]). Hence, the conclusion of Corollary 1 is satisfied by any strongly partially hyperbolic $C^{1+\alpha}$-diffeomorphism on a three-dimensional compact manifold having two ergodic probabilities with central Lyapunov exponent of different sign.

Theorem A has also consequences on the set of robustly transitive and non-hyperbolic diffeomorphisms. Indeed, [2, Theorem 3.1] ensures that there is an open and dense subset $\mathcal{P}'$ in $\mathcal{RNT}$ such that every diffeomorphism in $\mathcal{P}'$ has two saddles with different index. Thus, we deduce the following:

Corollary 2. *There is a C^1-open and dense set $\mathcal{P} \subset \mathcal{RNT} \cap \mathcal{SPH}_1(M)$ such that every $f \in \mathcal{P}$ does not satisfy the gluing orbit property.*

It is clear from Theorem A and Corollaries 1 and 2, that the set of partially hyperbolic diffeomorphisms satisfying the gluing orbit property is often a meager set. Moreover, in [30, 31], Sun characterized the set of zero-entropy maps with the gluing orbit property (these are minimal and equicontinuous) and proved that any positive entropy map with the gluing orbit property is not minimal. Both results lead us to propose the construction of (non-trivial) partially hyperbolic skew-products (clearly the product map of one transformation with the gluing orbit property and other with the specification property satisfies the gluing orbit property, cf. [8]).

Now, we construct some classes of non-trivial partially hyperbolic skew-products with the gluing orbit property. We endow $\mathbb{S}^1 \times M$ with the distance

$$d((x_1, y_1), (x_2, y_2)) = \max\left\{ |x_1 - x_2| \,,\ d_M(y_1, y_2) \right\} \tag{2.1}$$

where $|\cdot|$ denotes the usual distance in the circle and d_M denotes the distance on M inherited by the Riemann structure.

Theorem B. *Let $f \in Diff^2(M)$ be a transitive Anosov diffeomorphism. There exists a C^2-open neighborhood $\mathcal{V}$ of f such that, for any $\alpha \notin \mathbb{Q}$ and any C^2-smooth family of Anosov diffeomorphisms $(f_x)_{x\in\mathbb{S}^1}$ in $\mathcal{V}$ the map*

$$\begin{array}{rccc} F\colon & \mathbb{S}^1 \times M & \longrightarrow & \mathbb{S}^1 \times M \\ & (x, y) & \mapsto & (\, x + \alpha\,(\,mod 1\,),\ f_x(y)\,) \end{array} \tag{2.2}$$

is a C^2-partially hyperbolic diffeomorphism satisfying the gluing orbit property.

We note that all partially hyperbolic diffeomorphisms in Theorem B do not satisfy the specification property. Indeed, the previous construction enjoys the fact that the central direction is at all points driven by a fixed rotation of irrational angle, hence the resulting diffeomorphism is not topologically mixing.

There are few known results on the quantitative recurrence of partially hyperbolic dynamics. While such a description is nowadays very well known for hyperbolic dynamical systems the situation in very much the opposite beyond the context of uniform hyperbolicity, where even the construction of equilibrium states for partially hyperbolic dynamics faces non-trivial challenges in general.

In what follows we describe some quantitative recurrence results for the partially hyperbolic skew-products described in Theorem B. Given $(x, y) \in \mathbb{S}^1 \times M$, set

$$\tau_n((x, y), \varepsilon) := \inf \left\{ k \geq 1 \colon F^{-k}(B_n((x, y), \varepsilon)) \cap B_n((x, y), \varepsilon) \neq \emptyset \right\}$$

as the return time of the dynamical ball $B_n((x, y), \varepsilon)$ to itself. Analogously, let

$$\tau(B((x, y), \varepsilon)) := \inf \left\{ k \geq 1 \colon F^{-k}(B((x, y), \varepsilon)) \cap B((x, y), \varepsilon) \neq \emptyset \right\}$$

denote the minimal return time of the ball $B((x, y), \varepsilon)$ to itself. Poincaré's recurrence theorem ensures that the latter quantities, usualy known as *short return times* are almost everywhere finite with respect to any F-invariant probability measure (see e.g. [1] and references therein). While some sort of hyperbolicity is generally associated to the fact that the first quantity grows linearly with n and that the second reflects the Lyapunov exponents of the invariant measure (see e.g. [4] and references therein), the situation changes drastically for circle rotations (see e.g. [17]). For that reason, it seemed hopeless to combine these different methods to deal with the previously defined skew-products. Nevertheless we prove the following:

Corollary 3. *The following properties hold:*

(1) *F has super-linear lower asymptotic mixing rates on the family of balls;*
(2) *F has positive lower frequency of visits to balls; and*
(3) *every point $(x, y) \in \mathbb{S}^1 \times M$ is an entropy point for F.*

Moreover, if μ is F-invariant and ergodic then:

(4)
$$\lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{\tau_n(x, \varepsilon)}{n} = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\tau_n(x, \varepsilon)}{n} = 1 \quad \textit{and}$$

(5)
$$0 < \frac{1}{\lambda^+(\mu)} \leq \liminf_{\varepsilon \to 0} \frac{\tau(B(x, \varepsilon))}{-\log \varepsilon}$$

for μ-a.e. x, where $\lambda^+(\mu)$ denotes the largest Lyapunov exponent for μ.

This corollary gives first results on quantitative recurrence for families of partially hyperbolic dynamics. Items (1)-(3) in Corollary 3 can be easily read from the proofs of [8, Theorems A and B] (which consider flows with reparametrized gluing orbit property, and whose proofs become simpler in the case of maps), while the proofs of items (4) and (5) carry almost without change from the proofs of [32, Theorem B and Proposition A] (replacing the role of specification property by the gluing orbit property).

In what follows we consider families of skew-products with partial hyperbolic behavior of a dual nature, modeled by means of iterated function systems on the circle over shift spaces. By some abuse of notation, we still refer to these as partially hyperbolic dynamics. We say that a family $(f_i)_{0 \leq i \leq d-1}$ of circle homeomorphisms is *jointly equicontinuous* if $\{f_i^{\pm 1}\}_{0 \leq i \leq d-1}$ is jointly equicontinuous, meaning that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $|y_1 - y_2| < \delta$ and $\ell \geq 1$ then $|g_{i_\ell} \circ \cdots \circ g_{i_1}(y_1) - g_{i_\ell} \circ \cdots \circ g_{i_1}(y_2)| < \varepsilon$, where $g_{i_j} \in \{f_i^{\pm 1} \colon 0 \leq i \leq d-1\}$ for every

$1 \le j \le \ell$. Set $\Sigma := \{0, 1, \dots, d-1\}^{\mathbb{Z}}$, let $\sigma : \Sigma \to \Sigma$ be the shift map and denote by $\omega = (\dots, \omega_{-1}, \omega_0, \omega_1, \dots)$ the elements in Σ. Consider also the metric

$$d((\omega, y), (\tilde{\omega}, \tilde{y})) = \max\left\{ d_\Sigma(\omega, \tilde{\omega}) \ , \ |y - \tilde{y}| \right\}$$

on $\Sigma \times \mathbb{S}^1$, where $d_\Sigma(\omega, \tilde{\omega}) = \sum_{n\in\mathbb{Z}} 2^{-|n|}|\omega_n - \tilde{\omega}_n|$. The next result shows that joint equicontinuity and a sort of equidistribution of some fiber dynamics are sufficient conditions for locally constant circle extensions to satisfy the gluing orbit property. In other words:

Theorem C. *Let $(f_i)_{0\le i\le d-1}$ be a jointly equicontinuous family of homeomorphisms on the circle $\mathbb{S}^1$. If there exists $0 \le j \le d-1$ such that f_j is minimal, then the skew-product $F\colon \Sigma \times \mathbb{S}^1 \to \Sigma \times \mathbb{S}^1$ given by $F(\omega, y) = (\sigma(\omega), f_{\omega_0}(y))$ satisfies the gluing orbit property.*

The equicontinuity assumption is natural in the context of non-stationary dynamics and if often necessary in order to recover properties on the entropy and stability of classical dynamical systems (see e.g. [12, 18] and references therein). Since equicontinuity is satisfied by families of circle rotations and irrational rotations are minimal, as a direct consequence of Theorem C, the gluing orbit property is prevalent for circle extensions over the shift, in the following sense:

Corollary 4. *There exists a Baire residual and full Lebesgue measure subset $A \subseteq \mathbb{R}^d$ so that the skew-product $F\colon (\omega, y) \mapsto (\sigma(\omega), y + \alpha_{\omega_0})$ has the gluing orbit property, for every $(\alpha_0, \alpha_1, \dots, \alpha_{d-1}) \in A$.*

The general construction of non-locally constant circle extensions satisfying the gluing orbit property seems to face enormous difficulties, even if the dynamical system is more regular, e.g. Hölder continuous. Despite that, we can build such an example, as described below.

Theorem D. *The skew-product $F\colon \{0, 1\}^{\mathbb{Z}} \times \mathbb{S}^1 \to \{0, 1\}^{\mathbb{Z}} \times \mathbb{S}^1$ given by*

$$F(\omega, y) = (\sigma(\omega), y + \alpha(\omega)),$$

where $\alpha(\omega) = e^{-(\ell(\omega)+1)}$ and $\ell(\omega) = \inf\{k \ge 0 : \omega_k = 1 \text{ or } \omega_{-k} = 1\}$, satisfies the gluing orbit property.

Altogether, the previous results support the conjecture that the majority of the neutral circle extensions with a unique measure of maximal entropy satisfy the gluing orbit property.

3. Preliminaries

3.1. **Sequential dynamics.** In this subsection we collect some necessary definitions and results for sequential dynamics (we refer the reader to [12] and references therein for more details and proofs). A sequential dynamical system is a collection $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ of continuous maps $f_n : M \to M$, where (M, d) is a complete metric space and $n \in \mathbb{Z}$.

Throughout, we will assume that M is a compact Riemannian manifold. Given $r \ge 0$, let $\mathcal{S}^r(M)$ denote the space of sequences $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ of C^r-differentiable maps. Given sequences $\mathcal{F}, \mathcal{G} \in \mathcal{S}^r(M)$, define the normalized distance $|||\mathcal{F} - \mathcal{G}||| := \sup_{n\in\mathbb{Z}} \ d_{C^r}(f_n, g_n)$, where $d_{C^r}(f, g) = \min\{1, \|f - g\|_r\}$ and $\|\cdot\|_r$ denotes the usual C^r-norm between f and g. Given $n \ge 1$ set $F_n = f_{n-1} \circ \dots f_2 \circ f_1 \circ f_0$. As usual the symbol F_0 will stand for the identity on M.

The *(positive) orbit* of $x \in M$ is the set $\mathcal{O}^+_{\mathcal{F}}(x) = \{F_n(x) : n \in \mathbb{Z}_+\}$. If each element of $\mathcal{F}$ is invertible then the *orbit* of $x \in M$ is the set $\mathcal{O}_{\mathcal{F}}(x) = \{F_n(x) : n \in \mathbb{Z}\}$, where $F_{-n} = f_{-n}^{-1} \circ \cdots \circ f_{-2}^{-1} \circ f_{-1}^{-1}$ for every $n \in \mathbb{Z}_+$.

In the sequel, for notational simplicity, we represent all metrics d_n by d. Assume that $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ is a sequence of Anosov diffeomorphisms on M that preserves common stable and unstable cone fields (see e.g. [12] for the definition) and set $\mathcal{F}^{(k)} = \{f_{k+n}\}_{n\in\mathbb{Z}}$, $F^{(k)}_{-n} = f_{k-n}^{-1} \circ \cdots \circ f_{k-2}^{-1} \circ f_{k-1}^{-1}$ and $F^{(k)}_n = f_{n+k-1} \circ \cdots \circ f_{k+1} \circ f_k$, for all $k, n \in \mathbb{Z}$. The existence of invariant cone fields with uniform expansion (λ_n^{-1}) and contraction (λ_n) imply on the following properties:

(1) there exists $a > 0$ such that for every $k \in \mathbb{Z}$ and $x \in M$, the subspaces

$$E^u_{\mathcal{F}^{(k)}}(x) := \bigcap_{n\geq 0} DF_n^{(k-n)}(F_{-n}^{(k)}(x))\, C^u_{a,k-n}(F^{(k)}_{-n}(x)) \subset T_xM$$

and

$$E^s_{\mathcal{F}^{(k)}}(x) := \bigcap_{n\geq 0} DF_{-n}^{(k+n)}(F_{n}^{(k)}(x))\, C^s_{a,k+n}(F^{(k)}_{n}(x)) \subset T_xM$$

satisfy $Df_k(x)E^*_{\mathcal{F}^{(k)}}(x) = E^*_{\mathcal{F}^{(k+1)}}(f_k(x))$ for $* \in \{s, u\}$.

(2) there are constants $C > 0$, $\delta_1 > 0$ and $\tilde{\lambda} \in (\lambda, 1)$, where $\lambda := \sup_{n\in\mathbb{Z}} \lambda_n$, so that, for any $x \in M$, $k \in \mathbb{Z}$ and $* \in \{s, u\}$ there exists a unique smooth submanifold $\mathcal{W}^*_{\mathcal{F}^{(k)},\delta_1}(x)$ of M (of size δ_1) that is tangent to the subbundle $E^*_{\mathcal{F}^{(k)}}(x)$ at x, in such a way that:
   (i) $f_k(\mathcal{W}^*_{\mathcal{F}^{(k)},\delta_1}(x)) = \mathcal{W}^*_{\mathcal{F}^{(k+1)},\delta_1}(f_k(x))$
   (ii) $d_{\mathcal{W}^s}(f_k(y), f_k(z)) \leq \tilde{\lambda}\, d_{\mathcal{W}^s}(y, z)$ for all $y, z \in \mathcal{W}^s_{\mathcal{F}^{(k)},\delta_1}(x)$
   (iii) $d_{\mathcal{W}^u}(f_k^{-1}(y), f_k^{-1}(z)) \leq \tilde{\lambda}\, d_{\mathcal{W}^u}(y, z)$ for all $y, z \in \mathcal{W}^u_{\mathcal{F}^{(k+1)},\delta_1}(x)$
   (iv) the angles between stable and unstable bundles $E^s_{\mathcal{F}^{(k)}}(x)$ and $E^u_{\mathcal{F}^{(k)}}(x)$ is bounded away from zero by some constant $\theta_k > 0$
   (v) for any $0 < \varepsilon < \delta_1$ there exists $\delta_k > 0$ so that for any $x, y \in M$ with $d(x, y) < 2\delta_k$ the transverse intersection $\mathcal{W}^s_{\mathcal{F}^{(k)},\varepsilon}(x) \pitchfork \mathcal{W}^u_{\mathcal{F}^{(k)},\varepsilon}(y)$ consists of a unique point in M
   (vi) $\mathcal{W}^s_{\mathcal{F}^{(k)},\varepsilon}(x) = \{y \in M\colon d(F_n^{(k)}(y), F_n^{(k)}(x)) \leq \varepsilon$ for every $n \geq 0\}$ and $\mathcal{W}^u_{\mathcal{F}^{(k)},\varepsilon}(x) = \{y \in M\colon d(F_{-n}^{(k)}(y), F_{-n}^{(k)}(x)) \leq \varepsilon$ for every $n \geq 0\}$

Here we opted to write the invariant manifolds as $\mathcal{W}^*_{\mathcal{F}^{(k)}}$ ($* \in \{s, u\}$) to specify the shifted sequence $\mathcal{F}^{(k)}$ with respect to which the uniform contracting or expanding behavior holds.

In the case that $f \in \mathrm{Diff}^1(M)$ is an Anosov diffeomorphism there exists a C^1-small open neighborhood $\mathcal{U}$ of f so that the cone fields $C^u_{a_0}$ and $C^s_{a_0}$ (determined by f) are Dg-invariant for all $g \in \mathcal{U}$, and $\lambda := \sup_{g\in\mathcal{U}} \lambda_g < 1$. Reducing $\mathcal{U}$ if necessary we may take constants $\theta, \delta, \varepsilon > 0$ (depending only on f and $\mathcal{U}$) so that $\theta < \theta_n$, $\delta < \delta_n$ and $\varepsilon < \varepsilon_n$ for all n, for any sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ formed by elements of $\mathcal{U}$. In other words, both the angles between stable and unstable subspaces and the sizes given by local product structure are uniformly bounded away from zero, and the sequence $\mathcal{F}$ is δ_1-expansive: if $d(F_n(x), F_n(y)) < \varepsilon$ for all $n \in \mathbb{Z}$ then $x \in \mathcal{W}^s_{\mathcal{F},\delta_1}(y) \cap \mathcal{W}^u_{\mathcal{F},\delta_1}(y) = \{y\}$. We refer the reader e.g. to [12]) for more details. Let us recall the stability of non-autonomous sequences of C^1-Anosov diffeomorphisms:

Theorem 3.1. [12, Theorem 2.4] *Let M be a compact Riemannian manifold and $f \in Diff^1(M)$ be a transitive C^1 Anosov diffeomorphism on M. There exist $K > 0$ and $\varepsilon > 0$ so that if $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ is a sequence with $\|f_n - f\|_{C^1} < \varepsilon$ for all $n \in \mathbb{Z}$ then there exists a unique sequence $(h_n)_{n\in\mathbb{Z}_0^+}$ of homeomorphisms on M so that*

$$f_{n-1} \circ \cdots \circ f_1 \circ f_0 \; = \; h_n^{-1} \circ f^n \circ h_0 \quad \textit{for every } n \in \mathbb{Z}_0^+. \tag{3.1}$$

Moreover, $\|h_n - Id\|_{C^0} \le K \sup_{k\ge n} \|f_k - g_k\|_{C^1}$ for every $n \ge 0$.

Remark 3.2. Although we believe that the sequence of conjugacies $(h_n)_{n\in\mathbb{Z}_0^+}$ may fail to be equicontinuous in general, there are at least two cases where this occurs. First, if the sequence $(f_n)_n$ in Theorem 3.1 is C^1-convergent to some diffeomorphism $f \in \mathcal{U}$ then the sequence of homeomorphisms $(h_n)_n$ is equicontinuous [12, Remark 4.7]. This is also the case whenever the diffeomorphisms are C^2-smooth [3, Theorem 7.6].

3.2. **Gluing orbit properties.** Let X be a compact metric space and let $f : X \to X$ be a continuous map. Given $x \in X$, $\varepsilon > 0$ and an initial time $n_0 \in \mathbb{Z}$, we define the *dynamical ball* of length $n - n_0$, with initial time n_0 and size ε centered at x, by

$$B(x, n, n_0, \varepsilon) = \{y \in X : \, d(f^j(y), f^j(x)) \le \varepsilon, \; \forall \, n_0 \le j \le n\}.$$

When no confusion is possible, set $B_n(x,\varepsilon) = B(x, n, 0, \varepsilon)$ and $B_{-n}(x,\varepsilon) = B(x, 0, -n, \varepsilon)$. In particular, $B(x,\varepsilon) = B_0(x,\varepsilon)$ stands for the closed ball of radius ε around x.

Definition 3.3. We say that a continuous map $f : X \to X$ satisfies the *gluing orbit property* if: given $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon) \ge 1$ such that for any points $x_1, \dots, x_k \in X$ and times $n_1, \dots, n_k \ge 0$ there exist gluing times $0 \le p_1, \dots, p_{k-1} \le N(\varepsilon)$ and a point $z \in X$ so that $d(f^n(z), f^n(x_1)) < \varepsilon$ for every $n = 0 \dots n_1$ and $d(f^{n+\sum_{1\le i<j}(p_i+n_i)}(z), f^n(x_j)) < \varepsilon$ for every $j = 2 \dots k$ and $n = 0 \dots n_j$. Alternatively (to be used in the sequel), using the dynamical balls, there exist gluing times $0 \le p_1, \dots, p_{k-1} \le N(\varepsilon)$ so that

$$\bigcap_{j=1}^{k} f^{-\sum_{1\le i<j}(p_i+n_i)}(B_{n_j}(x_j,\varepsilon)) \neq \emptyset. \tag{3.2}$$

Remark 3.4. In the case that p is a hyperbolic periodic point for a diffeomorphism f, there exists $\varepsilon > 0$ small so that the local unstable set $W^u_\varepsilon(p) = \bigcap_{n\ge0} B_{-n}(p,\varepsilon)$ is the local unstable manifold at p with size ε. Analogously, the local stable set $W^s_\varepsilon(p) = \bigcap_{n\ge0} B_n(p,\varepsilon)$ is the local stable manifold for some $\epsilon > 0$ (see e.g. [26]).

Now, inspired by the concept of orbital specification from [22], we introduce a concept of specification for non-stationary dynamics.

Definition 3.5. Let X be a compact metric space and let $\mathcal{F} = \{f_n\}_{n\in\mathcal{Z}}$ be a sequence of continuous maps $f_n : X \to X$, where $\mathcal{Z} = \mathbb{Z}_+$ or $\mathbb{Z}$. We say that $\mathcal{F}$ satisfies the *specification property* if for any $\varepsilon > 0$ there exists $T(\varepsilon) \ge 1$ such that for any points $x_1, \dots, x_k \in X$, any positive integers $n_1, \dots, n_k$ and any $p_j \ge T(\varepsilon)$ there exists $z \in X$ so that $d(F_\ell(z), F_\ell(x_1)) < \varepsilon$ for every $\ell = 0 \dots n_1$ and

$$d(\; F_{\ell+p_{j-1}+n_{j-1}+\cdots+p_1+n_1}(z), F_\ell^{(p_{j-1}+n_{j-1}+\cdots+p_1+n_1)}(x_j) \;) < \varepsilon$$

for every $j = 2 \dots k$ and $\ell = 0 \dots n_1$.

Although the previous condition seems very rigid, it is satisfied by all sequences of Ruelle expanding maps and certain sequences of Anosov diffeomorphisms (cf. [22] and Proposition 4.5 below).

4. Proofs

4.1. **Proof of Theorem A.** Although the concept of gluing orbit property is much more embracing than specification (recall the discussion at the introduction) from the technical point of view, the proof of Theorem A is a not so hard modification of the arguments in [27]. For that reason we will indicate the key modifications.

Lemma 4.1. *Suppose that $f \in \mathit{Diff}^1(M)$ satisfies the gluing orbit property. Then for every hyperbolic periodic point p both the stable and unstable manifolds $\mathcal{W}^s(p)$ and $\mathcal{W}^u(p)$ are dense in M.*

Proof. We will prove the denseness in M of the unstable manifold $\mathcal{W}^u(p)$ (the proof for the stable manifold $\mathcal{W}^s(p)$ is analogous). For notational simplicity we assume that p is a fixed point (if not just consider f^k where k is the period of p).

Given any point $x \in M$ and $\varepsilon_0 > 0$, we claim that there is a point $y \in M$ such that $y \in \mathcal{W}^u(p)$ and $d(x, y) \le \varepsilon_0$. For some $\varepsilon_1 > 0$, we denote by $\mathcal{W}^u_{\varepsilon_1}(p)$ the local unstable manifold of size ε_1. Set $\varepsilon := \frac{1}{2}\min\{\varepsilon_0, \varepsilon_1\}$ and let $N(\varepsilon) \ge 1$ be given by the gluing orbit property. Then, for any $n \ge 1$ there exists $0 \le k_n \le N(\varepsilon)$ such that

$$f^{\kappa_n}(B_{-n}(p, \varepsilon)) \cap B(x, \varepsilon) \neq \emptyset\,.$$

By the pigeonhole principle, there exists an increasing subsequence $n_i \to \infty$ so that $(\kappa_{n_i})_{i\ge1}$ is constant to $0 \le \kappa \le N(\varepsilon)$. Notice that $(f^\kappa(B_{-n_i}(p, \varepsilon)) \cap B(x, \varepsilon))_{i\ge1}$ is a nested decreasing family of closed non-empty sets. Thus, the compactness of M ensures that there exists a point

$$y \in \bigcap_{i=1}^{\infty} f^\kappa(B_{-n_i}(p, \varepsilon)) \cap B(x, \varepsilon).$$

Reformulating, the point $y \in M$ satisfies

$$d(y, x) \le \varepsilon_0 \quad \text{and} \quad d(f^{-n_i}(f^{-\kappa}(y)), f^{-n_i}(p)) \le \varepsilon_1 \quad \text{for all } i \ge 1.$$

The decreasing nature of the sequence of dynamical balls $(B_{-n}(p, \varepsilon))_n$ implies that $y \in f^\kappa(\mathcal{W}^u_{\varepsilon_1}(p)) = f^\kappa(\mathcal{W}^u_{\varepsilon_1}(f^{-\kappa}(p)))$. Consequently, $y \in \mathcal{W}^u(p)$ and $d(x, y) \le \varepsilon_0$, which shows the claim and completes the proof of the lemma.

□

Remark 4.2. The previous lemma quantities the denseness of stable and unstable manifolds. Indeed, if p is a hyperbolic periodic point and $\varepsilon > 0$ then both $f^{-N(\varepsilon)}(\mathcal{W}^s_\varepsilon(p))$ and $f^{N(\varepsilon)}(\mathcal{W}^u_\varepsilon(p))$ are ε-dense.

Proof of Theorem A. Consider $f \in \mathrm{Diff}^1(M)$ admitting a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$. Let p and q be hyperbolic periodic points for f for which $\dim\, E^u = \dim\, \mathcal{W}^u(p) < \dim\, \mathcal{W}^u(q)$ (the case $\dim\, E^s = \dim\, \mathcal{W}^s(q) < \dim\, \mathcal{W}^s(p)$ is analogous).

We assume, by contradiction, that f satisfies the gluing orbit property. Then f is topologically transitive, thus it has neither sinks nor sources. In particular, as $\dim E^u = \dim \mathcal{W}^u(p) > 0$, the sub-bundle E^u is not trivial. We will make use of an intrumental result on the location of the shadowing point in unstable disks proved in [9]. Given $x \in \mathcal{W}^u(p)$ and $\eta > 0$, let $\gamma^u_\eta(x) := \{z \in \mathcal{W}^u(p) : d^u(x, z) \le \eta\}$ be the local unstable disk around x in $\mathcal{W}^u(p)$, where d^u denotes the distance in $\mathcal{W}^u(p)$ induced by the Riemannian metric.

Proposition 4.3. [9, Proposition 3] *There exists a small positive constant ε_0 such that for any $\varepsilon \in (0, \varepsilon_0)$ the following holds: if $x \in \mathcal{W}^u(p)$ and $d(f^{-n}(z), f^{-n}(x)) \le \varepsilon$ for any $n \ge 1$, then $z \in \gamma^u_{4\varepsilon}(x)$.*

The next result, obtained as a modification of [27, Proposition 2.3], reflects the similarities and differences between specification and the more flexible gluing orbit property. In the case of dynamics with the gluing orbit property, the time needed for the shadowing of finite pieces of orbits may be variable. Therefore, creating obstructions for this property is more demanding than for specification. This idea is formalized by equation (4.1) below.

Proposition 4.4. *Let ε_0 be given by Proposition 4.3. There exist $\eta > 0$ and $\varepsilon \in (0, \varepsilon_0)$ with $4\varepsilon < \eta$ such that for all $N \ge 1$ there is $x \in \mathcal{W}^u(p)$ satisfying*

$$\bigcup_{j=1}^{N} f^j(\gamma^u_\eta(x)) \cap W^s_\eta(q) = \emptyset\,. \tag{4.1}$$

Proof. Since the sub-bundle E^u is not trivial, there exists a foliation $\mathcal{F}^u$ which is tangent to E^u (see [15]). We denote by $\mathcal{F}^u(x)$ the leaf of the foliation $\mathcal{F}^u$ containing the point x and given any $\eta > 0$, we define

$$\mathcal{F}^u_\eta(x) := \{w \in \mathcal{F}^u(x) : d^u(x, w) \le \eta\},$$

where d^u is the distance in $\mathcal{F}^u(x)$ induced by the Riemannian metric. By Lemma 4.1, the unstable manifold $\mathcal{W}^u(p)$ is dense in M. Given $r > 0$, we shall consider the following family of unstable disks

$$\mathcal{L}(p) = \{V(w) : w \in B(p, r)\},$$

where $V(w)$ is the connected component of $\mathcal{F}^u(w) \cap B(p, r)$ that contains w. Select a local disk D'_0 and $\eta > 0$ small enough so that D'_0 is transverse to the family $\mathcal{L}(p)$, $p \in D'_0$ and such that $\mathcal{A}(U) := \bigcup_{z \in U} \mathcal{F}^u_\eta(z)$ is homeomorphic to $U \times [-\eta, \eta]^{\dim E^u}$, for any open disk U contained in D'_0.

Set $\varepsilon := \min\{\eta/5, \varepsilon_0/2\}$. Now, we take a compact disk K transverse to E^u and such that $K \supset W^s_\eta(q)$. Let $N \ge 1$ and fix $1 \le j \le N$. Since K is transverse to E^u then $K \cap f^j(\gamma^u_\eta(p))$ is formed by a finite set of points $\{x_{j_1}, x_{j_2}, \cdots, x_{j_{k_j}}\}$. Choose an open subdisk $D_{j,0} \subset D'_0$ containing p such that $K_{j,i} \cap K_{j,i'} = \emptyset$ if $i \ne i'$, where $K_{j,i}$ is a connected component of $K \cap f^j(\mathcal{A}(D_{j,0}))$ containing x_{j_i}, for $1 \le i \le k_j$. For each $1 \le i \le k_j$, let $D_{j,i} = f^{-j}(K_{j,i})$ and consider the holonomy map

$$\begin{array}{rccc} \pi_{j,i}: & D_{j,i} & \to & D_{j,0} \\ & w & \mapsto & v \end{array}$$

where $v = \pi_{j,i}(w)$ is the unique point such that $D_{j,i} \cap \mathcal{F}^u(v) = \{w\}$.

Each holonomy map $\pi_{j,i}$ is a homeomorphism, by our choice of the open disks $D_{j,0}$ and $D_{j,i}$, $1 \le i \le k_j$. Since $W^s_\eta(q)$ is a closed submanifold with $\dim W^s_\eta(q) < \dim K_{j,i}$, $K_{j,i} \setminus W^s_\eta(q)$ is open and dense in $K_{j,i}$. Taking $\gamma^s_{j,i} := \pi_{j,i} \circ f^{-j}(W^s_\eta(q))$, we have that $D_{j,0} \setminus (\bigcup_{i=1}^{k_j} \gamma^s_{j,i})$ is open and dense in $D_{j,0}$. Now let $D_0 := \cup_{j=1}^{N} D_{j,0}$. The latter implies that

$$D_0 \setminus \left(\bigcup_{j=1}^{N} \bigcup_{i=1}^{k_j} \gamma^s_{j,i} \right) \quad \text{is open and dense in } D_0.$$

Consequently, we can choose an open subdisk $U \subset D_0 \setminus (\bigcup_{j=1}^{N} \bigcup_{i=1}^{k_j} \gamma^s_{j,i})$. Since $\mathcal{A}(U)$ is homeomorphic to $U \times [-\eta, \eta]^{\dim E^u}$ and $\mathcal{W}^u(p)$ is dense in M, there exists a point $z' \in \mathcal{A}(U) \cap \mathcal{W}^u(p)$. In other words, there exists a point $x \in U$ such that $z' \in \mathcal{F}^u_\eta(x)$. The choice of ε implies that not only $x \in \mathcal{W}^u(p)$ as $\mathcal{F}^u_\eta(x) = \gamma^u_\eta(x)$. Moreover, the property $\bigcup_{j=1}^{N} f^j(\gamma^u_\eta(x)) \cap W^s_\eta(q) = \emptyset$ is now a consequence of the choice of U. We have proved the proposition. □

Proceeding with the proof of the theorem, for each $\varepsilon > 0$ let $N = N(\varepsilon) \geq 1$ be given by the gluing orbit property. Proposition 4.4 assures that there are $\eta > 0$, $\varepsilon \in (0, \varepsilon_0)$ with $4\varepsilon < \eta$ and a point $x \in \mathcal{W}^u(p)$ satisfying

$$\bigcup_{j=1}^{N} f^j(\gamma^u_\eta(x)) \cap W^s_\eta(q) = \emptyset.$$

On the other hand, the gluing orbit property implies that for any $n \geq 1$ there is $0 \leq k_n \leq N$ such that $f^{k_n}(B_{-n}(x, \varepsilon)) \cap B_n(q, \varepsilon) \neq \emptyset$. Pick an increasing subsequence $n_i \to \infty$ so that $(k_{n_i})_{i \geq 1}$ is constant to κ ($\leq N$). As before, the compactness of M ensures that there exists a point $z \in M$ such that

$$z \in \bigcap_{i=1}^{\infty} f^\kappa(B_{-n_i}(x, \varepsilon)) \cap B_{n_i}(q, \varepsilon).$$

Once again, because the sequences of dynamical balls $(B_{-n}(x, \varepsilon))_n$ and $(B_n(q, \varepsilon))_n$ are decreasing, the latter implies that $d(f^{-n}(f^{-\kappa}(z)), f^{-n}(x)) \leq \varepsilon$ and $d(f^n(z), f^n(q)) \leq \varepsilon$ for any $n \geq 0$. Thus, it follows from Proposition 4.3 that $z \in f^\kappa(\gamma^u_\eta(x)) \cap W^s_\eta(q)$ ($\kappa \leq N$). Hence, $z \in \bigcup_{j=1}^{N} f^j(\gamma^u_\eta(x)) \cap W^s_\eta(q)$, which is a contradiction. Thus, f does not satisfy the gluing orbit property.

Now suppose that $\dim M = 3$. Then it follows from [2, Theorem 3.1] that there is a C^1-dense open subset $\mathcal{P}$ in $\mathcal{RNT}$ such that any diffeomorphism $f \in \mathcal{P}$ has two saddles with different index.

Since $f \in \mathcal{P}$ is robustly transitive, it follows from [14] that f has a partially hyperbolic splitting $E^u \oplus E^c \oplus E^s$. Thus, the existence of two saddles with different index, imply that $f \in \mathcal{P}$ does not satisfy the gluing orbit property. This completes the proof of Theorem A.

4.2. **Proof of Theorem B.** Let $f \in \mathrm{Diff}^2(M)$ be a transitive Anosov diffeomorphism, let $\mathcal{V}$ be a C^2-small open neighborhood of f, let $\alpha \notin \mathbb{Q}$ and consider the C^2 skew-product

$$\begin{array}{rccc} F: & \mathbb{S}^1 \times M & \longrightarrow & \mathbb{S}^1 \times M \\ & (x, y) & \mapsto & (\, x + \alpha \,(mod 1\,),\ f_x(y)\,) \end{array} \tag{4.2}$$

where $f_x \in \mathcal{V}$ for all $x \in \mathbb{S}^1$. Since f is a transitive Anosov diffeomorphism, by spectral decomposition it is topologically mixing and, by Bowen [10], is satisfies the specification property.

The proof of the theorem has two steps. First we explore the non-autonomous dynamics generated by C^2-close Anosov diffeomorphisms and prove that this satisfies the gluing orbit property (independently of the base dynamics of the skew-product). This is only possible after [3, 12]. Second, we combine the latter with the fact that an irrational rotation on the circle satisfies the gluing orbit property. The first ingredient can be read as follows:

Proposition 4.5. *If f is a C^2 Anosov diffeomorphism then there exists a C^2-open neighborhood $\mathcal{V}$ of f such that, for any sequence $\mathcal{F} = \{f_n\}_{n\in\mathbb{Z}}$ with $f_n \in \mathcal{V}$ and any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ so that $\mathcal{W}^s_{\mathcal{F},\varepsilon}(z) \cap \mathcal{W}^u_{\mathcal{F},T(\varepsilon)}(y) \neq \emptyset$ for every $y, z \in M$. In particular, the sequence $\mathcal{F}$ satisfies the specification property.*

Proof. Since f is a topologically mixing Anosov diffeomorphism then $\mathcal{W}^u_f$ is minimal (i.e. all unstable leaves are dense). Using f^{-1} instead of f we have that the stable foliation $\mathcal{W}^s_f$ is also minimal. Thus, by compactness of M and continuity of unstable leaves on compact parts, for every $\varepsilon > 0$ there exists $L(\varepsilon) > 0$ so that $\mathcal{W}^u_{f,L(\varepsilon)}(x)$ is ε-dense in M for every $x \in M$. Taking $\mathcal{F} = \{f_n\}_n$ and $\mathcal{G} = \{f\}_n$, Theorem 3.1 and Remark 3.2 ensures that there is a equicontinuous sequence of homeomorphisms $(h_n)_n$ where $h_n = h_{\mathcal{F}^{(n)},\mathcal{G}^{(n)}}$ satisfy $\|h_{\mathcal{F}^{(n)},\mathcal{G}^{(n)}} - Id\|_{C^0} \leq K \sup_{\ell\geq n} \|f_{k+\ell} - g_{k+\ell}\|_{C^1}$ for every $n \geq 0$, and

$$F_n = h^{-1}_{\mathcal{F}^{(n)},\mathcal{G}^{(n)}} \circ f^n \circ h_{\mathcal{F},\mathcal{G}} \quad \forall n \in \mathbb{Z}_+. \tag{4.3}$$

Equation (4.3) implies that for any $k \geq 0$, the shifted sequences $\mathcal{G}^{(k)}$ and $\mathcal{F}^{(k)}$ are such that

$$F^{(k)}_n = h^{-1}_{\mathcal{F}^{(k+n)},\mathcal{G}^{(k+n)}} \circ f^n \circ h_{\mathcal{F}^{(k)},\mathcal{G}^{(k)}} \quad \forall n \in \mathbb{Z}_+ \tag{4.4}$$

and, consequently,

$$h_0(\mathcal{W}^*_{\mathcal{F}}(z)) = \mathcal{W}^*_f(h_0(z)) \quad \text{and} \quad h_n(\mathcal{W}^*_{\mathcal{F}^{(n)}}(z)) = \mathcal{W}^*_f(h_n(z))$$

for every $z \in M$, for every n and $* \in \{s, u\}$. This, together with the fact that $\{h_n : n \in \mathbb{Z}\}$ is equicontinuous, guarantees that for every $\varepsilon > 0$ there exists $T(\varepsilon) \geq L(\varepsilon)$ such that $\mathcal{W}^*_{\mathcal{F},T(\varepsilon)}(z)$ is ε-dense for every $z \in M$ and $* \in \{s, u\}$. The transversality statement in Subsection 3.1 implies that

$$\mathcal{W}^s_{\mathcal{F},\varepsilon}(z) \cap \mathcal{W}^u_{\mathcal{F},T(\varepsilon)}(y) \neq \emptyset \quad \text{for every } y, z \in M. \tag{4.5}$$

This proves the first assertion in the proposition.

We now proceed to prove that $\mathcal{F}$ has the specification property. Using (4.4) and the fact that the sequence $(h_n^{-1})_n$ is equicontinuous, given $\varepsilon > 0$ take $0 < \hat{\varepsilon} \leq \varepsilon$ so that $\|h_n^{-1}(x) - h_n^{-1}(y)\| < \varepsilon$ for all $x \in B(y, \hat{\varepsilon})$ and $n \in \mathbb{Z}_+$. Let $N(\hat{\varepsilon}) \geq 1$ be given by the specification property associated to f. Fix $x_1, \ldots, x_k \in M$ and $n_1, \ldots, n_k \in \mathbb{Z}_+$ and time lags $p_1, \ldots, p_{k-1} \geq N(\hat{\varepsilon})$. Applying the specification of f to the points $\hat{x}_1 = h_0(x_1), \ldots, \hat{x}_k = h_{\sum_{1\leq j<k}(n_j+p_j)}(x_k)$, times $n_1, \ldots, n_k$ and the scale $\hat{\varepsilon}$, there exists $y \in M$ so that

$$y \in \bigcap_{j=1}^{k} f^{-\sum_{1\leq i<j}(p_i+n_i)}(B_{n_j}(\hat{x}_j, \hat{\varepsilon})) \neq \emptyset, \tag{4.6}$$

where the dynamical balls are with respect the map f. Take $z := h_0^{-1}(y)$. By the choice of $\hat{\varepsilon}$ we have that

$$\begin{aligned} d_M(F_\ell(z), F_\ell(x_1)) &= d_M(h_\ell^{-1} \circ f^\ell \circ h_0 \circ h_0^{-1}(y), h_\ell^{-1} \circ f^\ell \circ h_0(x_1)) \\ &= d_M(h_\ell^{-1} \circ f^\ell(y), h_\ell^{-1} \circ f^\ell \circ h_0(x_1)) < \varepsilon \end{aligned} \tag{4.7}$$

for every $\ell = 1 \dots n_1$ and, using (4.4) once more,

$$\begin{aligned} & d_M(F_{\ell+\sum_{1\le i<j}(p_i+n_i)}(z), F_\ell^{(\sum_{1\le i<j}(p_i+n_i))}(x_j)\ ) \\ & \quad = d_M(h^{-1}_{\ell+\sum_{1\le i<j}(p_i+n_i)} \circ f^{\ell+\sum_{1\le i<j}(p_i+n_i)} \circ h_0 \circ h_0^{-1}(y), h^{-1}_{\ell+\sum_{1\le i<j}(p_i+n_i)} \circ f^\ell \circ h_{\sum_{1\le i<j}(p_i+n_i)}(x_j)) \\ & \quad = d_M(h^{-1}_{\ell+\sum_{1\le i<j}(p_i+n_i)} \circ f^{\ell+\sum_{1\le i<j}(p_i+n_i)}(y), h^{-1}_{\ell+\sum_{1\le i<j}(p_i+n_i)} \circ f^\ell \circ h_{\sum_{1\le i<j}(p_i+n_i)}(x_j)) < \varepsilon \end{aligned} \tag{4.8}$$

for every $j = 2 \dots k$ and $\ell = 1 \dots n_j$. This proves the specification property for $\mathcal{F}$ and completes the proof of the proposition. □

We are now in a position to complete the proof of Theorem B. Observe first that for any $x \in \mathbb{S}^1$, the sequence of Anosov diffeomorphisms $\mathcal{F}_x = \{f_n\}_{n\in\mathbb{Z}}$ determined by $f_n := f_{x+n\alpha(mod 1)}$, for every $n \in \mathbb{Z}$, satisfies the requirements of Proposition 4.5.

Now, note that the constant $T(\varepsilon) > 0$ given by the specification property obtained for $\mathcal{F}_x$ in Proposition 4.5 can be taken uniform in a small open neighborhood of f. Thus, given $\varepsilon > 0$, let $T_1(\varepsilon) \ge 1$ be given by the gluing orbit property for the irrational rotation $x \mapsto x + \alpha(mod 1)$ (cf. [8]) and let $T_2(\varepsilon) \ge 1$ be given by the specification property for the sequence $\mathcal{F}_x$ given in the statement of Proposition 4.5 (this is independent of x).

Set $T(\varepsilon) = T_1(\varepsilon) + T_2(\varepsilon) \ge 1$ and take arbitrary $(x_1, y_1), \dots, (x_k, y_k) \in \mathbb{S}^1 \times M$ and integers $n_1, \dots, n_k \ge 0$. Since the irrational rotation R_α has the gluing orbit property, there exists $z \in \mathbb{S}^1$ and $0 \le p_1, \dots, p_{k-1} \le T_1(\varepsilon)$ so that

$$|R^n_\alpha(z)) - R^n_\alpha(x_1)| < \varepsilon \quad \text{for every } 0 \le n \le n_1 + T_2(\varepsilon)$$

and

$$|R_\alpha^{n+\sum_{1\le i<j}(p_i+n_i)}(z) - R^n_\alpha(x_j)| < \varepsilon \quad \text{for every } 0 \le n \le n_j + T_2(\varepsilon)$$

for every $2 \le j \le k$ (in other words, z shadows the piece of orbit of x_j during $n_j + T_2(\varepsilon)$ iterates).

Fix $z \in \mathbb{S}^1$ as above and let $\mathcal{F} \equiv \mathcal{F}_z$. The specification property for the family $\mathcal{F}$ (choosing the points $y_1, y_2, \dots, y_k \in M$, positive integers n_j and time lags $\tilde{p}_j = p_j + T_2(\varepsilon) \ge T_2(\varepsilon)$) ensures that there exists $y \in M$ so that $d_M(F_\ell(y), F_\ell(y_1)) < \varepsilon$ for every $\ell = 1 \dots n_1$ and

$$d_M(\ F_{\ell+\tilde{p}_{j-1}+n_{j-1}+\dots+\tilde{p}_1+n_1}(y), F_\ell^{(\tilde{p}_{j-1}+n_{j-1}+\dots+\tilde{p}_1+n_1)}(y_j)\ ) < \varepsilon$$

for every $j = 2 \dots k$ and $\ell = 0 \dots n_j$. Finally, since the skew-product F in (4.2) satisfies $F^k(x, y) = (\ R_{k\alpha(x)},\ \ F^k_x(y)\ )$, the gluing orbit property follows from the previously specified shadowing of orbits expressed in both coordinates and the choice of the metric d in (2.1). This proves the theorem.

4.3. **Proof of Theorem C.** Let $\varepsilon > 0$ and take points $(\omega_1, y_1), \dots, (\omega_k, y_k) \in \Sigma \times \mathbb{S}^1$ and times $n_1, \dots, n_k \ge 0$. Since the shift map $\sigma \colon \Sigma \to \Sigma$ satisfies the specification property, the choice of the metric d implies that we only need to deal with the fiber direction. We assume $k = 2$ (the general case is similar). Let $\omega_1 = (\dots, \omega^1_0, \omega^1_1, \omega^1_2, \dots)$ and $\omega_2 = (\dots, \omega^2_0, \omega^2_1, \omega^2_2, \dots)$. We claim there exists $\delta = \delta_\varepsilon > 0$ such that

$$B_{n_i}((\omega_i, y_i), \varepsilon) \supseteq B_{n_i}(\omega_i, \varepsilon) \times B(y_i, \delta_\varepsilon) \tag{4.9}$$

for $i = 1, 2$. Indeed, since F is locally constant on fibers, the dynamic ball $B_{n_i}((\omega_i, y_i), \varepsilon)$ coincides with

$$B_{n_i}(\omega_i, \varepsilon) \times \{y \in \mathbb{S}^1 : \left| f_{\omega^i_{n_i-1}} \circ \cdots \circ f_{\omega^i_1} \circ f_{\omega^i_0}(y) - f_{\omega^i_{n_i-1}} \circ \cdots \circ f_{\omega^i_1} \circ f_{\omega^i_0}(y_i) \right| < \varepsilon \}.$$

Now, using that $\{f_i^{\pm 1}\}_{0 \le i \le d-1}$ is jointly equicontinuous, we obtain that there exists $\delta_\varepsilon > 0$ such that $B_{n_i}((\omega_i, y_i), \varepsilon) \supseteq B_{n_i}(\omega_i, \varepsilon) \times B(y_i, \delta_\varepsilon)$, as claimed. The theorem will follow once we prove there exists $N(\varepsilon) \ge 1$ and $0 \le \ell \le N(\varepsilon)$ and $z_\ell \in B_{n_1}(\omega_1, \varepsilon)$ such that

$$F^{n_1+\ell}_{z_\ell} (B(y_1, \delta_\varepsilon)) \cap B(y_2, \delta_\varepsilon) \neq \emptyset,$$

where $F_\omega(\cdot) = F(\omega, \cdot)\colon \mathbb{S}^1 \to \mathbb{S}^1$ and $F^k_\omega = F_{\sigma^{k-1}(\omega)} \circ \cdots \circ F_{\sigma(\omega)} \circ F_\omega$. Observe first that there exists $K(\varepsilon) \ge 1$ such that

$$B_s(\omega, \varepsilon) \subseteq [\omega_{-s-K(\varepsilon)} \cdots \omega_{-1} \underline{\omega}_0\, \omega_1 \cdots \omega_{s+K(\varepsilon)}] \subseteq B_s(\omega, 2\,\varepsilon),$$

for every $\omega \in \Sigma$ and all $s \ge 1$, where $[\omega_{-m_1} \cdots \omega_{-1} \underline{\omega}_0\, \omega_1 \cdots \omega_{m_2}]$ denotes the cylinder set $\{(x_n) \in \Sigma : x_i = \omega_i \text{ for all } -m_1 \le i \le m_2\}$ with 0^{th} position ω_0. On the other hand, given $\omega = (\omega_n) \in \Sigma, y_1 \in \mathbb{S}^1, k \in \mathbb{N}$ and $\varepsilon > 0$ denote the set $\{y \in \mathbb{S}^1 : \left| f_{\omega_{k-1}} \circ \cdots f_{\omega_1} \circ f_{\omega_0}(y) - f_{\omega_{k-1}} \circ \cdots \circ f_{\omega_1} \circ f_{\omega_0}(y_1) \right| < \varepsilon\}$ by $B_\omega(y_1, k, \varepsilon)$. Then

$$F^k_\omega (B_\omega(y_1, k, \varepsilon)) \supseteq B\left(F^k_\omega(y_1), \delta_\varepsilon\right)$$

for every $\omega \in \Sigma$ and every $k \ge 1$ (otherwise, the inverse dynamics $F^{-n_1}_{\omega_1}$ would not be equicontinuous). In particular we deduce that

$$F^{n_1+K(\varepsilon)}_\omega (B_\omega(y_1, n_1 + K(\varepsilon), \varepsilon)) \supseteq B\left(F^{n_1+K(\varepsilon)}_\omega(y_1), \delta_\varepsilon\right)$$

for every $\omega \in [\omega^1_{-n_1-K(\varepsilon)} \cdots \omega^1_{-1} \underline{\omega}^1_0\, \omega^1_1 \cdots \omega^1_{n_1+K(\varepsilon)}]$. Given any $\ell \ge 1$, we pick z_ℓ in the cylinder set

$$[\omega^1_{-n_1-K(\varepsilon)} \cdots \underline{\omega}^1_0 \cdots \omega^1_{n_1} \cdots \omega^1_{n_1+K(\varepsilon)} \underbrace{j \cdots j}_{\ell} \omega^2_{-n_2-K(\varepsilon)} \cdots \omega^2_0 \cdots \omega^2_{n_2} \cdots \omega^2_{n_2+K(\varepsilon)}].$$

We are reduced to prove that there exists $N(\varepsilon) \ge 1$ such that for any $J_1, I_1 \subseteq \mathbb{S}^1$ with $|J_1| > \delta_\varepsilon$ there is $0 \le \ell \le N(\varepsilon)$ such that $F^\ell_{\sigma^{n_1+K(\varepsilon)}(z_\ell)}(J_1) \cap I_1 \neq \emptyset$. By construction,

$$F^\ell_{\sigma^{n_1+K(\varepsilon)}(z_\ell)} = \underbrace{f_j \circ \cdots f_j \circ f_j}_{\ell} = f^\ell_j.$$

Since f_j is minimal, there exists $N(\varepsilon) \ge 1$ so that $\bigcup_{\ell=0}^{N(\varepsilon)} f^\ell_j(J_1) = \mathbb{S}^1$. Then just choose $\ell \in [0, N(\varepsilon)]$ such that $f^\ell_j(J_1) \cap I_1 \neq \emptyset$. This completes the proof.

4.4. **Proof of Theorem D.** We first observe that $F^n(\omega, y) = (\sigma^n(\omega), y + S_n\alpha(\omega))$, where $S_n\alpha(\omega) := \sum_{j=0}^{n-1} \alpha(\sigma^j(\omega))$. Fix $0 < \varepsilon < 1$ and choose $N \in \mathbb{N}$ so that $2^{-(N-1)} < \varepsilon \le 2^{-(N-2)}$. Take arbitrary points $(\omega_1, y_1), \ldots, (\omega_k, y_k) \in \Sigma \times \mathbb{S}^1$, where $\omega_i = (\ldots, \omega^i_0, \omega^i_1, \omega^i_2, \ldots)$ for every $1 \le i \le k$, and positive integers $n_1, \ldots, n_k \ge 0$. We assume $k = 2$ (the general case is similar). Now, for every $1 \le i \le 2$ define

$$\bar{\omega}_i = (\ldots, 1, 1, \omega^i_{-N-2}, \ldots, \underline{\omega}^i_0, \omega^i_1, \ldots, \omega^i_{n_i+N+2}, 1, 1, \ldots).$$

We will need the following estimate from [8] (using that the irrational rotation $R_{\frac{1}{e}} : \mathbb{S}^1 \to \mathbb{S}^1$ has the gluing orbit property and is a minimal isometry):

Claim: *There exists $N(\varepsilon) \ge 1$ such that for any $b_1, b_2 \in \mathbb{S}^1$ there exists a gluing time $0 \le m \le N(\varepsilon)$ so that $\left| b_1 + \frac{m}{e} - b_2 \right| < \frac{1}{2^{N+1}}$.*

Taking $b_1 = y_1 + S_{n_1+N+2}\,\alpha(\bar\omega_1)$ and $b_2 = y_2 - S_{N+2}\alpha(\sigma^{-N-2}(\bar\omega_1))$ in the claim, there exists $0 \le m \le N(\varepsilon)$ such that

$$\Big| y_1 + S_{n_1+N+2}\,\alpha(\bar\omega_1) + \frac{m}{e} + S_{N+2}\alpha(\sigma^{-N-2}(\bar\omega_2)) - y_2 \Big| < \frac{1}{2^{N+1}}\,. \tag{4.10}$$

The positive integer m will be used below to define the gluing time. Choose

$$\begin{aligned} z =&(\dots,1,1,\omega^1_{-N-2},\dots,\underline{\omega^1_0},\omega^1_1,\dots,\omega^1_{n_1+N+2},\\ &\underbrace{1,\cdots,1}_{m},\omega^2_{-N-2},\dots,\omega^2_0,\omega^2_1,\dots,\omega^2_{n_2+N+2},1,1,\dots). \end{aligned}$$

By construction, z and $\bar\omega_1$ belong to $[\omega^1_{-N-2},\dots,\underline{\omega^1_0},\omega^1_1,\dots,\omega^1_{n_1+N+2}]$. Therefore $S_n\alpha(z) = S_n\alpha(\bar\omega_1)$ for every $0 \le n \le n_1 + N + 2 + m$. Moreover, it is not hard to check that

$$|\alpha(\sigma^n(\omega_1)) - \alpha(\sigma^n(\bar\omega_1))| \le e^{-(N+2+n)} \quad \text{for every } 0 \le n \le n_1.$$

In consequence,

$$\begin{aligned} d(F^n(\omega_1,y_1),F^n(z,y_1)) &= \max\Big\{ d_\Sigma(\sigma^n(\omega_1),\sigma^n(z))\ ,\ |S_n\alpha(\omega_1) - S_n\alpha(z)| \Big\} \\ &\le \max\Big\{ \frac{1}{2^{N+2}}\ ,\ |S_n\alpha(\omega_1) - S_n\alpha(\bar\omega_1)| \Big\} < \varepsilon \end{aligned}$$

for every $0 \le n \le n_1$. Take the gluing time $p := m + 2(N+2) \le N(\varepsilon) + 2(4 - \frac{\ln\varepsilon}{\ln 2})$. If $0 \le n \le n_2$. Observe also that $\sigma^{n_1+2(N+2)+m}(z)$ and $\bar\omega_2$ coincide in $2(N+2)+1$ coordinates and

$$\begin{aligned} S_{n+n_1+p}\alpha(z) &= S_{n_1+N+2}\,\alpha(z) + \frac{m}{e} \\ &\quad + S_{N+2}\alpha(\sigma^{n_1+N+2+m}(z)) + S_n\alpha(\sigma^{n_1+2(N+2)+m}(z)) \\ &= S_{n_1+N+2}\alpha(\bar\omega_1) + \frac{m}{e} + S_{N+2}\alpha(\sigma^{n_1+N+2+m}(z)) + S_n\alpha(\bar\omega_2). \end{aligned} \tag{4.11}$$

The equalities (4.11) and $S_{N+2}\alpha(\sigma^{n_1+N+2+m}(z)) = S_{N+2}\alpha(\sigma^{-N-2}(\bar\omega_2))$, together with equation (4.10) and the triangular inequality yield that

$$\Big| y_1 + S_{n+n_1+p}\alpha(z) - y_2 - S_n\alpha(\omega_2) \Big| \le \frac{1}{2^{N+1}} + |S_n\alpha(\bar\omega_2) - S_n\alpha(\omega_2)|.$$

As before, $|S_n\alpha(\bar\omega_2) - S_n\alpha(\omega_2)| \le \frac{1}{2^{N+1}}$ and, in conclusion,

$$\begin{aligned} d(F^n(\omega_2,y_2),F^{n+n_1+p}(z,y_1)) \le \max\{ & d_\Sigma(\sigma^n(\omega_2),\sigma^{n+n_1+p}(z))\ , \\ & \Big| y_2 + S_n\alpha(\omega_2)\ -\ y_1 - S_{n+n_1+p}\alpha(z) \Big| \Big\} \end{aligned}$$

which is bounded above by $\frac{1}{2^N} \le \varepsilon$. This proves that F satisfies the gluing orbit property.

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Thiago Bomfim, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil
Email address: tbnunes@ufba.br

Maria Joana Torres, CMAT and Departamento de Matemática, Universidade do Minho, Campus de Gualtar, 4700-057 Braga, Portugal
Email address: jtorres@math.uminho.pt

Paulo Varandas, CMUP and Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil
Email address: paulo.varandas@ufba.br