MODELLING IRREGULARLY SPACED TIME SERIES UNDER PREFERENTIAL SAMPLING

Authors: Andreia Monteiro
– University of Minho & CIDMA, Portugal
andreiaforte50@gmail.com

Raquel Menezes
– University of Minho, Portugal
rmenezes@math.uminho.pt

Maria Eduarda Silva
– Faculty of Economics, University of Porto & CIDMA, Portugal
mesilva@fep.up.pt

Abstract:

- Irregularly spaced time series are commonly encountered in the analysis of time series. A particular case is that in which the collection procedure over time depends also on the observed values. In such situations, there is stochastic dependence between the process being modeled and the times at which the observations are made. Ignoring this dependence can lead to biased estimates and misleading inferences. In this paper, we introduce the concept of preferential sampling in the temporal dimension and we propose a model to make inference and prediction. The methodology is illustrated using artificial data as well a real data set.

Key-Words:

- *Preferential sampling; time series; continuous time autoregressive process; SPDE.*

AMS Subject Classification:

- 62M10, 62M20.

1. INTRODUCTION

Analysis of experimental data that have been observed at different points in time leads to specific problems in statistical modeling and inference. In traditional time series the main emphasis is on the case when a continuous variable is measured at discrete equispaced time points, [24]. There is an extensive body of literature on analyzing equally spaced time series data, see for example [5] and [8]. However, unevenly spaced (also called unequally or irregularly spaced) time series data naturally occurs in many scientific domains. Natural disasters such as earthquakes, floods, or volcanic eruptions typically occur at irregular time intervals. In observational astronomy, for example, measurements of properties such as the spectra of celestial objects are taken at irregularly spaced times determined by seasonal, weather conditions, and availability of observation time slots. In clinical trials (or more generally, longitudinal studies), a patient's state of health may be observed only at irregular time intervals, and different patients are usually observed at different points in time.

It must be noted that sometimes equally spaced time series are treated as irregularly spaced time series, namely time series with missing observations and multivariate data sets that consist of time series with different frequencies, even if the observations of each time series are reported at regular intervals.

There are few methods for the analysis of irregularly spaced series. Some authors, like [15], [16], [4] and [7] have suggested an embedding into continuous diffusion processes. The focus of this literature is mainly on the modeling of univariate autoregressive moving average (ARMA) processes, which oppose the development of a complete set of tools similar to that available for equally spaced data.

Observations with irregularly spaced sampling times are much harder to work with, partly because the established and efficient algorithms developed for equally spaced sampling times are no longer applicable, [19]. A common approach to perform parametric estimation is to construct a log-likelihood function in terms of the unknown parameter [6]. When the sampling times are considered deterministic, the traditional approach is to build the classical Gaussian log-likelihood function. However, because the inversion of the covariance matrix has to be performed, numerical evaluation of this Gaussian log-likelihood function is in general very expensive, [18]. One way to overcome this computational effort is to regulate the sampling scheme, using some form of interpolation, and consider it as being equally spaced. Under the assumption of equally spaced sampling times, the Gaussian log-likelihood function can be approximated, at least for a sufficiently large sample, by the Whittle log-likelihood function [26]. This approach has been successfully applied to irregularity caused by missing values, [21]. While, this methodology, may be reasonable to deal with the minor irregularities in sampling times caused by missing values, the interpolation procedure will typically change the dynamic of the underlying process, leading

to biased estimates for the parameters [13]. Moreover, there is little understanding of which particular interpolation method is the most appropriate on a given data set. Alternatively, a convenient continuous time domain dynamic model may be assumed for the underlying continuous time stationary process such as the Continuous time ARMA (CARMA) model. [24] reviews the application of Kalman recursion techniques to the parametric estimation of CARMA processes. [17] estimate the parameters of an irregularly sampled CARMA process using a Bayesian framework.

A particular case of irregularly spaced data is that in which the collection procedure along time depends also, for practical constraints, on the observed values. For example, a certain health indicator for an individual may be measured at different time points and with different frequencies depending on his health state. In a completely different setting, the times of occurrence of transactions in the financial markets depend largely on the value of the underlying asset. In environmental monitoring applications, or in the context of smart cities if it is decided to monitor more frequently when a value considered critical to human health is exceeded. Therefore, additional information on the phenomena under study is obtained from the frequency or time occurrence of the observations. In such situations, there is stochastic dependence between the process being modeled and times of the observations. In this work, we introduce the concept of preferential sampling, first presented by [11] in the context of spatial statistics, to the temporal dimension, under a model based approach, accounting for the conditional distribution of the time point process on a latent process, with the aim of modeling the unevenly observed process.

Preferential sampling in time could be seen as a version of informative follow-up in longitudinal studies, see, for example, [20]. In these studies the follow-up time process is considered dependent on the longitudinal outcome process and it should not be regarded deterministic in the design of the study. The analogous problem in the context of longitudinal clinical trial data has been studied too in the context of issues concerning missing values and dropouts, in the sense that a missing observation conveys partial information about the value that would have been observed. See, for example, [10], [14] and [9].

The paper is organized as follows. Section 2 describes our proposed model for preferential sampling in time dimension, namely to make inference and prediction. In Section 3 we describe the Monte Carlo Maximum Likelihood Estimation. In section 4 we conduct a numerical illustration, in a artificial data set, to analyze the quality of the proposed model and we show the application of the previously described methodology to a real data set relative to lung function of an asthma patient. Section 5 is devoted to make some concluding remarks.

2. A MODEL FOR PREFERENTIAL SAMPLING

In time series, data are obtained by sampling a phenomenon $S(t) : t > 0$ at a discrete set of times t_i, $i = 1, \ldots, n$. Admiting the possibility that the sampling design may be stochastic, $T = (t_1, \ldots, t_n)$ denotes a stochastic process of observation times. In many situations, $S(t)$ cannot be measured without error, hence, if Y_i denotes the measured value at time t_i, a model for the data takes the form:

$$Y(t) = \mu + S(t) + N(0, \tau^2),\ t > 0 \tag{2.1}$$

where μ is a constant mean effect and $S(\cdot)$ is a stationary Gaussian process with $E[S(t)] = 0$. An equivalent formulation is that conditional on $S(\cdot)$, the Y_i are mutually independent, normally distributed with mean $\mu + S(t_i)$ and common variance τ^2.

We consider $S(\cdot)$ as a continuous time autoregressive process of order 1, CAR(1), that satisfies the differential equation $dS(t) + \alpha_0 S(t)dt = dW(t)$ where, α_0 is a constant and $W(t)$ is a Brownian motion with variance parameter σ_w^2. $Y = (Y_1, \ldots, Y_n)$ is multivariate Gaussian with mean $\boldsymbol{\mu}$ and covariance matrix $\Sigma_Y = \frac{\sigma_w^2}{2\alpha_0} R_y(\alpha_0) + \tau^2 I_n$ where I_n is the $n \times n$ identity matrix and $R_y(\alpha_0)$ has elements $r_{ij} = \rho\left(|t_i - t_j|\,; \alpha_0\right)$ defined by

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = e^{-\alpha_0 |h|} \tag{2.2}$$

where $\gamma(\cdot)$ is the covariance function.

Admitting that the sampling times are stochastic, a complete model needs to specify the joint distribution of S, T and Y. Considering the stochastic dependence between S and T, the model to deal with preferential sampling is defined through $[S, T, Y]$ written as:

$$[S][T\,|S][Y\,|S(T)\,] \tag{2.3}$$

where $[\cdot]$ means "the distribution of", $S = \{S(t) : t > 0\}$, $T = (t_1, \ldots, t_n)$ and $S(T)$ represents $\{S(t_1), \ldots, S(t_n)\}$.

We define a specific class of models through the additional assumptions: conditional on S, T is an inhomogeneous Poisson process with intensity $\lambda\left(t\right) = \exp\left\{a + \beta S\left(t\right)\right\}$ and unconditionally T is a log-Gaussian Cox process. The log-Gaussian Cox process is a flexible class of point pattern models that allows conditioning the sampling times to the variable of interest. For example, when $\beta = 2$ the sample times are concentrated, predominantly, near the maximum of the observed values and when $\beta = 0$ it corresponds to the situation of an homogeneous, non-preferential, sampling. Conditional on S and T, Y is a set of mutually independent Gaussian variates with τ^2 being the measurement error variance.

To obtain the predicted value of $S(\cdot)$ at an unsampled time point $t_{n_i} < t_0 < t_{n_j}$, $S(t_0|T)$, we use the fact that the process $CAR(1)$ is Markovian, [8] and thus

(2.4)
$$S(t_0|T) = \mathrm{E}\big[S(t_0)|Y(T)\big] = exp\left(-\alpha_0(t_0 - t_{n_i})\right) Y(t) + \mu\left(1 - exp\left(-\alpha_0(t_0 - t_{n_i})\right)\right)$$

The variance of the prediction is

$$\sigma^2(t_0) = Var\big[S(t_0)|Y(T)\big] = \frac{\sigma_w^2}{2\alpha_0}\left(1 - exp\left(-2\alpha_0(t_0 - t_{n_i})\right)\right) \tag{2.5}$$

3. MONTE CARLO MAXIMUM LIKELIHOOD ESTIMATION

We consider a discretization of the S process with N points and a partition of S into $S = \{S_0, S_1\}$ where S_0 denotes the values of S at each of n times $t_i \in T$, and S_1 are the values of S at the remaining $(N - n)$.

The likelihood function for data T and Y can be expressed as

(3.1)
$$L(\boldsymbol{\theta}) = [T, Y] = \int_S [S][T, Y|S]dS = \int_S [S][T|S][Y|T, S]dS = \int_S [S][T|S][Y|T, S]dS$$

where $\boldsymbol{\theta} = (\mu, \sigma_w, \alpha_0, \tau, \beta)$ represents all the model parameters.
As $[Y|T, S]$ can be approximated by $[Y|S_0]$, we can rewrite the integral as

$$L(\boldsymbol{\theta}) = \int_S [S][T|S][Y|S_0]\frac{[S|Y]}{[S|Y]}dS \tag{3.2}$$

Considering that $[S] = [S_1, S_0] = [S_1|S_0][S_0]$ and replacing the term $[S|Y]$ in the denominator of expression (3.2) by $[S|Y] = [S_0, S_1|Y] = [S_1|S_0, Y][S_0|Y] = [S_1|S_0][S_0|Y]$, equation (3.2) becomes

$$\begin{aligned} L(\theta) &= \int_S [S_1|S_0][S_0][T|S][Y|S_0]\frac{[S|Y]}{[S_1|S_0][S_0|Y]}dS \\ &= \int_S [T|S]\frac{[Y|S_0]}{[S_0|Y]}[S_0][S|Y]dS \\ &= E_{S|Y}\left[[T|S]\frac{[Y|S_0]}{[S_0|Y]}[S_0]\right] \end{aligned} \tag{3.3}$$

Taking into account that the above conditional expectation can be approximated by Monte Carlo, MLE's are obtained by maximizing the Monte Carlo likelihood

$$L_{MC}(\boldsymbol{\theta}) = m^{-1}\sum_{j=1}^{m}[T|S_j]\frac{[Y|S_{0j}]}{[S_{0j}|Y]}[S_{0j}] \tag{3.4}$$

where S_j are simulated from $[S|Y]$ and m is the number of Monte Carlo replicas. With this purpose, we use a technique known as conditioning by kriging [23] and we use the following construction. The new sample $S_j = U + \Sigma_S A^T \left(A\Sigma_S A^T + \tau^2 I_n\right)^{-1} (V - AU)$ where A is the $n \times N$ matrix whose ith row consists of $N-1$ 0s and a single 1 to identify the position of t_i within $T = (t_1, \ldots, t_n)$; $U = \Sigma_S^{1/2} u \sim MVN(0, \Sigma_S)$ with $u \sim N(0,1)$ and $\Sigma_S^{1/2}$ is obtained from the Cholesky decomposition and $V \sim MVN(y, \Sigma_Y)$. Then S_j has the required multivariate Gaussian distribution of S given $Y = y$. In practice, we use antithetic pairs of realizations to reduce Monte Carlo variance [11].

$T|S_j$ in (3.4) is an inhomogeneous Poisson process with intensity

$$\lambda(t) = \exp(a + \beta S_j(t)) \tag{3.5}$$

For computational reasons, we work with logarithm and thus,

$$log([T|S_j]) = \sum_{i=1}^{n} (a + \beta S_j(t_i)) - nlog\left(\int_0^T \exp(a + \beta S_j(t))dt\right) \tag{3.6}$$

As the S_j replica is not known in $[0, T]$ domain, we can not calculate the integral presented in expression (3.6), so, we approximate the integral using the composed trapezium formula for unequally spaced data.

$[S_{0j}]$ in (3.4) is multivariate Gaussian with mean 0 and covariance matrix $\Sigma_{S_{0j}} = \frac{\sigma_w^2}{2\alpha_0} R_{S_{0j}}(\alpha_0)$, where $R_{S_{0j}}(\alpha_0)$ is the $n \times n$ correlation matrix with elements $r_{ij} = \rho\left(|t_i - t_j|\,; \alpha_0\right)$ defined by (2.2).

$[S_{0j}|Y]$ in (3.4) is multivariate Gaussian with mean $\mu_{S_{0j}|Y} = \Sigma_{S_{0j}} \Sigma_Y^{-1} (y - \mu 1)$ and covariance matrix $\Sigma_{S_{0j}|Y} = \Sigma_{S_{0j}} - \Sigma_{S_{0j}} \Sigma_Y^{-1} \Sigma_{S_{0j}}^t$. For more details about conditional distribution see for e.g. [3].

Obtained the Maximum Likelihood Estimates (MLE's) we can plug them into (2.4) and (2.5), treating them as known. We are in position of doing the so-called plug-in predictions.

4. NUMERICAL ILLUSTRATION

4.1. ARTIFICIAL DATA SET

In this section we assess the performance of estimation procedure proposed in Section 3 and compare the model based approach proposed with the traditional Kalman filter approach (used in cts package [25]). We generate 400 equally

spaced time points from model (2.1), with $\alpha_0 = 0.2$, $\sigma_w^2 = 1$, $\mu = 0$, $\sigma = \sqrt{\frac{\sigma_w^2}{2\alpha_0}} = 1.581$, $\phi = \frac{1}{\alpha_0} = 5$ and $\tau = 0.1$ and conditional on the realization of S we obtained $n = 70$ sampling times T following an inhomogeneous Poisson process with intensity function defined in (3.5). We simulated 250 realizations of S and conditional on each realization of S we obtained $n = 70$ sampling times T following an inhomogeneous Poisson process with intensity function defined in (3.5). We conducted three separate sampling procedures over each realization of S:

- preferential sampling where the observed times are determined with a value of $\beta = 2$ in the Poisson process intensity function;
- irregular sampling with $\beta = 0$, illustrating the situation without preferential sampling;
- regular sampling with equidistant observations.

The parameters μ, σ, ϕ, τ and β are the target of estimation. The estimates are obtained under (3.4), henceforward denoted by MCMLE's and from the Kalman filter, denoted by MLE's. For the maximization of our Monte Carlo log-likelihood function we considered a total of grid points $N = 400$ and a total number of replicas $m = 1000$.

Figure 1 shows a realization of one of these simulations, on a single realization of the process S. We have 70 sampling times (black points), considering $\beta = 2$ in the process intensity function, in which the preferential nature of the sampling process results in sample times falling predominantly near the maxima. For 70 sampling times (white points), we consider $\beta = 0$, the situation without preferential sampling and with irregularly sampling points. For the remaining 70 points (star points) we have the situation of regular spaced sampling times.

The results of the mean and standard errors of each parameter, obtained from a total of 250 independent samples are summarized in Table 1.

		PS Data set ($\beta = 2$)		Irregularly Sampling ($\beta = 0$)		Regular sampling	
	True	PS model	CTS	PS Model	CTS	PS Model	CTS
$\widehat{\mu}$	**0**	**0.13** (0.18)	**0.38** (0.31)	**0.04** (0.12)	**0.26** (0.34)	**0.02** (0.22)	**0.71** (0.62)
$\widehat{\sigma}$	**1.58**	**1.53** (0.21)	**0.99** (0.18)	**1.64** (0.11)	**1.52** (0.21)	**1.60** (0.13)	**1.45** (0.24)
$\widehat{\phi}$	**5**	**5.71** (1.01)	**3.17** (2.55)	**5.20** (0.48)	**5.52** (1.96)	**5.12** (0.89)	**6.78** (2.93)
$\widehat{\tau}$	**0.1**	**0.12** (0.04)	**0.27** (0.13)	**0.11** (0.01)	**0.30** (0.18)	**0.11** (0.02)	**0.55** (0.28)
$\widehat{\beta}$	**2 or 0**	**1.76** (0.39)		**0.00** (0.07)		**0.00** (0.02)	

Table 1: Maximum likelihood estimates, under PS model (MCMLE's) and by cts package (MLE's), mean (standard errors) obtained from a total of 250 independent samples.

Analysing table 1 we conclude that the model for Temporal Preferential Sampling presents estimates for the parameters less biased, even when the prefer-

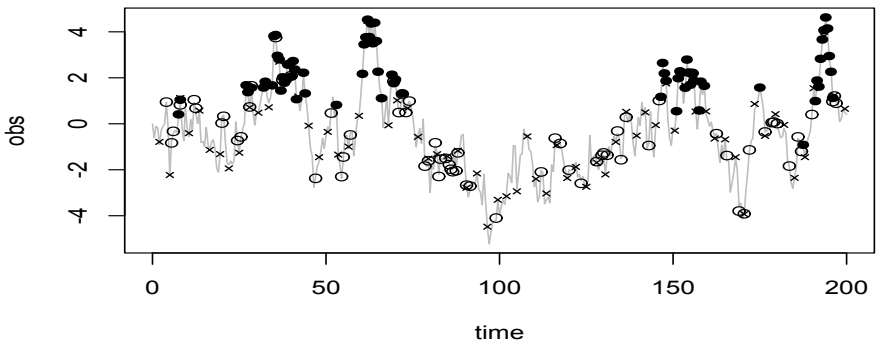

Figure 1: Sample times with preferential sampling nature (black points), without preferential sampling and irregularly spaced time points (white points), regular spaced time points (star points) and underlying process S (gray line).

ability degree is null, with regular and irregularly sampling.

To analyse the impact of ignoring preferential sampling on the quality of predictions, we conducted a second simulation study. We simulated 250 realizations of S and for each we constructed a preferential sampling data set. Then, the proposed MCMLE's and the MLE's from the Kalman filter approach were obtained and plugged-in equation (2.4) to predict $S(t)$ at 50 equally spaced time points. These together with the corresponding standard errors, in (2.5), allowed us to calculate prediction 95% confidence intervals and estimate their coverage.

Figure 2 represents one simulation of S(t) (black line), the corresponding preferential sampling data (black points) and the predictions acquired from MCMLE's (white points) and MLE's (gray points). The results indicate that the latter overestimate the observations and underestimate the variability of the underlying process. In fact, in the overall simulation results confidence intervals from MCMLE's present an estimated coverage of 88% while the MLE's provide an estimated coverage of just 73%.

4.2. LUNG FUNCTION OF AN ASTHMA PATIENT

[4] analyzed 209 measurements of the lung function of an asthma patient. The time series is measured mostly at 2 hour time intervals but with irregular gaps as demonstrated by the unequal space of tick marks in Figure 3. This data is available in the package cts [25] with the name of "asth".

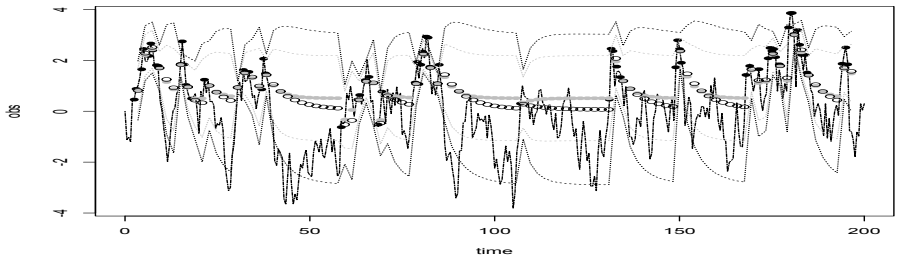

Figure 2: Predictions acquired from MCMLE's (white points) and MLE's (gray points), dashed line are confidence bands, black points are the preferential sampling data and black line is the underlying process S .

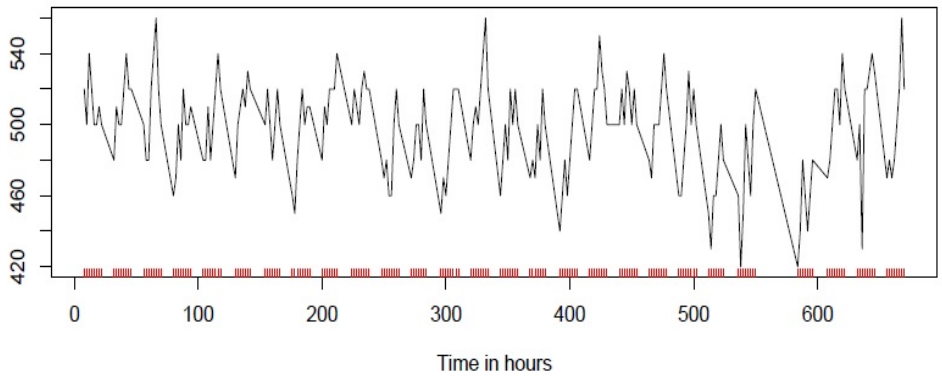

Figure 3: Measurements of the lung function.

To assess the performance and the utility of the proposed model we select the last 50 observations of "asth" data, corresponding to the period with more missing observations. We considered a log-transformation in the data, which leads to more symmetric distribution of measured values and we make predictions, within the period of these observations, aiming to "complete" the data set. Figure 4 shows predictions of (log of), the variable of interest for that patient at regular time points. The MCMLE's for model parameters are $\hat{\mu} = 6.18$, $\hat{\phi} = 2.83$, $\hat{\sigma} = 0.06$, $\hat{\tau} = 0.03$ and $\hat{\beta} = 0.62$. The positive value for $\hat{\beta}$ can be justified by observed points being closer to maxima. This kind of study is important, for example, to analyse when a new measurement of the patient's health indicator should be taken.

Besides the previous analysis, we also conducted three separate analysis each one with 50 time points selected from the 209 measurements considering: two preferential samples, where the sampling times are determined with a value of $\beta = -2$ and $\beta = 2$ in (3.5), as an illustration of a case when the patient

was observed in a state of poor health and in a healthy state respectively; and a non-preferential sample with $\beta = 0$. Estimated coverage from 95% nominal CI obtained from MCMLE's and MLE's calculated in the preferential sample, as described before, are 92% in both preferential samples ($\beta = \pm 2$) and 97% in the case of the random sample. These results help to justify the acceptable behavior of proposed model even under a preferential sampling design.

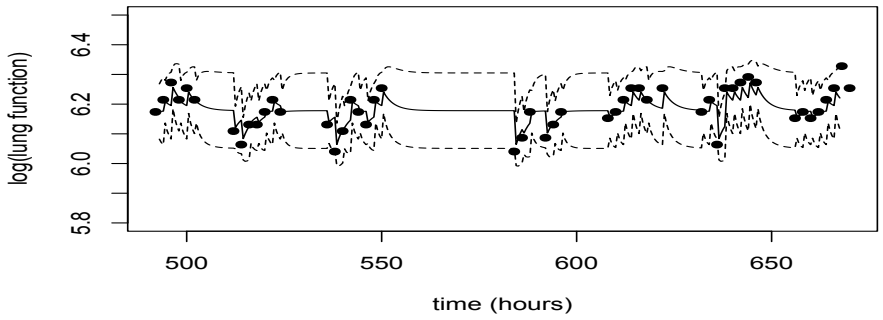

Figure 4: Predictions of (log of) the variable of interest (black line) and Confidence Intervals (dashed line). Black points are observations for the logarithm of lung function of an asthma patient.

5. Concluding Remarks and Future Work

In this work we propose a methodology to deal with irregularly spaced time series but also a methodology that take into account the frequency or time occurrence of the observations. The proposed model not only provides good estimates for model parameters but also reveals quite satisfactory results for prediction. A key aspect of this methodology is that it provides a tool, for example in the context of clinical trials, supporting a better knowledge of the underlying stochastic process, goal of study.

[12] affirm that the use of a single parameter in (3.5) to capture both the strength of the non-preferentiality and the amount of non-uniformity in sampling locations is somewhat inflexible. Alternatively, a more flexible and computational more efficient class of models, based on the proposal of [22], is discussed. These authors suggest an extension to the model proposed by [11], by adding a second Gaussian process and use of stochastic partial differential equation models. For future investigation we intend to adapt those suggestions to the time dimension.

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