THEORY PROBAB. APPL.
Vol. 64, No. 4, pp. 636–645

TAIL DEPENDENCE UNDER SAMPLE FAILURES*

M. FERREIRA† AND H. FERREIRA‡

Abstract. When collecting samples, sometimes failures of observations occur and consequently missing data. This can have an impact on the analysis and subsequent inference, especially if the study focuses on the extreme values where the data is more scarce. In this work, we analyze the effect of different types of failures on the dependence within the tail of a stationary series. We will also present some examples.

Key words. theory of extreme values, stationary sequence, dependence on extreme values, asymptotic independence of extreme values

DOI. 10.1137/S0040585X97T989751

1. Introduction. The occurrence of extremal events is a topic of concern of analysts in several areas, such as environmental, insurance, and finance. During extremal phenomena, one often encounters increasing dependence on extreme values with a strong impact on risk. For instance, large losses in financial markets influence more losses which may be catastrophic for the stability of whole sectors. The most common procedure for evaluating extremal dependence is the tail dependence coefficient (TDC), a concept introduced in [14] and later formalized in [12]. More precisely,

$$\chi = \lim_{x\to\infty} \frac{\mathbf{P}(X_1 > x,\, X_2 > x)}{1 - F(x)}, \tag{1}$$

where (X_1, X_2) is a random pair with a common marginal distribution function (d.f.) F and an infinite right-end-point. We say that a random pair is tail dependent if $\chi > 0$ and tail independent if $\chi = 0$. In the latter case, Ledford and Tawn [13] noticed that a residual extremal dependence described by the rate of convergence towards zero of $\mathbf{P}(X_1 > x,\, X_2 > x)$ may cause bias in inferences about the tail. Their model assumption,

$$\frac{\mathbf{P}(X_1 > x,\, X_2 > x)}{1 - F(x)} = (1 - F(x))^{1/\eta - 1} L(x), \tag{2}$$

includes a parameter $\eta \in (0,1]$ measuring the degree of residual tail dependence whenever $\chi = 0$ and the function L is slowly varying at ∞ (i.e., $L(tx)/L(x) \to 1$ as $x \to \infty$) and provides a relative dependence strength at each η. We observe that if $\eta = 1$ and $L(x) \to c > 0$ as $x \to \infty$, then $\chi = c > 0$, and thus we have the tail dependence. If $\eta < 1$ or $\eta = 1$ and $L(x) \to 0$, we have tail independence, with positive association if $\eta > 1/2$ and negative if $\eta < 1/2$. The perfect independence leads to $\eta = 1/2$ and $L(x) = 1$. In what follows, we denote by LTC the Ledford and Tawn coefficient η.

*Received by the editors January 3, 2018; revised July 11, 2018. The first author was supported by Portuguese Funds through FCT — Fundação para a Ciência e a Tecnologia within the Projects UID/MAT/00013/2013 of Centre of Mathematics of the University of Minho, UID/Multi/04621/2019 of Center for Computational and Stochastic Mathematics, UID/MAT/00006/2019 of Centre of Statistics and Its Applications, and PTDC/MAT-STA/28243/2017. The second author was partially supported by the research unit UID/MAT/00212/2019 — FCT (Fundação para a Ciência e a Tecnologia). Originally published in the Russian journal *Teoriya Veroyatnostei i ee Primeneniya*, 64 (2019), pp. 798–810.

https://doi.org/10.1137/S0040585X97T989751

†Center of Mathematics, Minho University, Braga, Portugal, and Center for Computational and Stochastic Mathematics and Center of Statistics and Applications, University of Lisbon, Lisbon, Portugal (msferreira@math.uminho.pt).

‡Centro de Matemática e Aplicações (CMA-UBI), Universidade da Beira Interior, Covilhã, Portugal (helena.ferreira@ubi.pt).

In the collection of data, errors in the recordings of the observations (due to failures in the reading devices, technical failures, or even impossibility of collection inherent to the phenomenon) cause missing data. With regard to inference in tails where data is scarce, this can have serious consequences. When a missing value occurs, in the majority of cases either the observation is replaced or the observation is completely lost, and the data sample is actually a subsample with smaller sample size. Hall and Hüsler [10] presented three different models addressing these types of patterns. More precisely, for a stationary sequence $\{X_n\}$ of random variables with marginal d.f. F and an independent and identically distributed (i.i.d.) sequence $\{U_n\}$ independent of $\{X_n\}$, with Bernoulli(p) marginal d.f., $0 < p < 1$, we have the following:

(M1) Model with missing values: $Y_n = U_n X_n$. The marginal d.f. of $\{Y_n\}$ is $F_Y(y) = 1 - p + pF(y)$, $y > 0$.

(M2) Model with subsampling: $\{Z_n\}$ such that $Z_n = X_{i_n}$, where i_n denote the indexes of $\{X_n\}$ for which $U_n = 1$. The marginal d.f. of $\{Z_n\}$ is F.

(M3) Model with replaced missing values: $W_n = U_n X_n + (1 - U_n)X_n^{(1)}$, where $\{X_n^{(1)}\}$ is an independent replica of $\{X_n\}$. The marginal d.f. of $\{W_n\}$ is also F.

For more details and illustrations, see [10] and the references therein.

In the present paper, we cover the impact of missing data under schemes of types (M1), (M2), and (M3) on dependence within the tails of a stationary sequence. In section 2 we present the tail dependence coefficients that we are going to address. The effect of missing data is analyzed in section 3. Illustrative examples are given in section 4.

2. Tail dependence.

Several measures of tail dependence are developed from χ and η or, more precisely, from the respective multivariate formulations

$$\chi^{\mathbf{X}_A} = \lim_{x\to\infty} \frac{\mathbf{P}\big(\bigcap_{i\in A}\{X_i > x\}\big)}{1 - F(x)} \tag{3}$$

and

$$\frac{\mathbf{P}\big(\bigcap_{i\in A}\{X_i > x\}\big)}{1 - F(x)} = (1 - F(x))^{1/\eta^{\mathbf{X}_A} - 1} L^{\mathbf{X}_A}(x), \tag{4}$$

where A is any set of indexes. If $A = \{1, 2\}$, $\chi^{\mathbf{X}_A}$ corresponds to the TDC in (1), and $\eta^{\mathbf{X}_A}$, to the LTC in (2).

In particular, if we consider a stationary sequence $\{X_n\}$, the lag-m TDC and LTC are, respectively, obtained by taking $A = \{1, m\}$, i.e.,

$$\chi_m^{(X)} \equiv \chi^{\mathbf{X}_{\{1,m\}}} = \lim_{x\to\infty} \mathbf{P}(X_m > x \mid X_1 > x) \tag{5}$$

and

$$\mathbf{P}(X_m > x \mid X_1 > x) = (1 - F(x))^{1/\eta_m^{(X)} - 1} L_m^{(X)}(x); \tag{6}$$

here $\eta_m^{(X)} \equiv \eta^{\mathbf{X}_{\{1,m\}}} \in (0, 1]$ with slowly varying function $L_m^{(X)}(x) \equiv L^{\mathbf{X}_{\{1,m\}}}(x)$. The lag-$m$ measures TDC and LTC can be found in, e.g., [11], [4], and references therein.

In [3], for integers s and k such that $1 \leqslant s < d - k + 1 \leqslant d$, the *upper* s,k*-extremal coefficient* of the random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is defined by

$$\chi(X_{s:d} \mid X_{d-k+1:d}) = \lim_{x\to\infty} \mathbf{P}(X_{s:d} > x \mid X_{d-k+1:d} > x),$$

where $X_{1:d} \leqslant \cdots \leqslant X_{d:d}$. This coefficient can be interpreted as the limiting probability that the sth worst performer in a system is attracted by the kth best one, provided the latter has an extremely good performance. Interesting applications are systems in finance, engineering, and insurance, among others (see, e.g., [8], [3]). If $s = k = 1$, we obtain the *upper extremal dependence coefficient*, ϵ^X, considered in [8], i.e.,

$$\epsilon^X = \chi(X_{1:d} \mid X_{d:d}).$$

The *lag-m upper s,k-extremal coefficient*, applied to a stationary sequence $\{X_n\}$ in [7], is obtained by replacing d by m,

$$\chi(X_{s:m} \mid X_{m-k+1:m}) = \lim_{x\to\infty} \mathbf{P}(X_{s:m} > x \mid X_{m-k+1:m} > x), \tag{7}$$

and the *lag-m upper extremal dependence coefficient* is defined by

$$\epsilon_m^X = \chi(X_{1:m} \mid X_{m:m}). \tag{8}$$

The Fragility Index (FI), as introduced in [9], is another measure of extreme risk dependence, concerning the fragility of a system. Given a random vector $\mathbf{X} = (X_1, \ldots, X_d)$, the FI of $\mathbf{X}$ corresponds to the asymptotic expected number of exceedances above a threshold x among $X_1, \ldots, X_d$, $N_x := \sum_{i=1}^d \mathbf{1}_{\{X_i > x\}}$, given that there exists at least one exceedance, i.e., the variables $\mathrm{FI} = \lim_{x\to\infty} \mathbf{E}(N_x \mid N_x > 0)$. The system $\{X_1, \ldots, X_d\}$ is fragile if $\mathrm{FI} > 1$ and weakly fragile whenever $\mathrm{FI} = 1$. This concept was generalized in order to evaluate the stability of a system divided into blocks (see [6]). Denote by $\mathcal{D} = \{I_1, \ldots, I_s\}$ a partition of $D = \{1, \ldots, d\}$, and by $N_{\mathbf{x}}$ the number of blocks where at least one exceedance of $\mathbf{x}$ takes place, i.e.,

$$N_{\mathbf{x}} = \sum_{j=1}^{s} \mathbf{1}_{\{\mathbf{X}_{I_j} \not\leq \mathbf{x}_{I_j}\}};$$

we denote by $\mathbf{X}_{I_j}$ the subvector of $\mathbf{X}$ whose components have indexes in I_j, with $j = 1, \ldots, s$ (i.e., $\mathbf{X}_{I_j}$ is the jth block of a random vector $\mathbf{X}$) and by $\mathbf{x}_{I_j}$ the vector of length $|I_j|$ with components equal to $x \in \mathbf{R}$. The block-FI of a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ relative to partition $\mathcal{D}$ is defined as

$$\mathrm{FI}(\mathbf{X}, \mathcal{D}) = \lim_{x\to\infty} \mathbf{E}(N_{\mathbf{x}} \mid N_{\mathbf{x}} > 0).$$

Taking the partition $\mathcal{D}^* = \{I_j = \{j\} \colon j = 1, \ldots, d\}$, we obtain the FI introduced in [9].

All operations and inequalities on vectors are meant componentwise.

In order to measure the stability of a stochastic process divided into blocks, we consider the lag-m block-FI relative to a partition $\mathcal{D}_m$ of $D_m = \{1, \ldots, m\}$, i.e.,

$$\mathrm{FI}(\mathbf{X}, \mathcal{D}_m) = \lim_{x\to\infty} \mathbf{E}(N_{\mathbf{x}} \mid N_{\mathbf{x}} > 0), \tag{9}$$

where $N_{\mathbf{x}}$ is the number of blocks with at least one occurrence of an exceedance of $\mathbf{x}$ for a horizon of m successive time instants.

Whenever the system is weakly fragile, i.e., with a unit FI, we might observe an asymptotic independence, i.e., a residual dependence that vanishes at high values. For instance, Gaussian processes are weakly fragile, regardless of the correlation between the variables (see [9]).

The asymptotic independent fragility index (AIFI), introduced in [9], is defined by

$$\eta_A^{\mathbf{X}} = \frac{1}{|A|} \lim_{x\to\infty} \frac{\sum_{i\in A} \ln \mathbf{P}(X_i > x)}{\ln \mathbf{P}\big(\bigcap_{i\in A}\{X_i > x\}\big)} \tag{10}$$

for any set of indexes $A \subseteq D = \{1, \ldots, d\}$. In the case $d = 2$, we find the Ledford and Tawn coefficient of asymptotic independence (see [13]).

The AIFI blocks of $\mathbf{X} = (X_1, \ldots, X_d)$ relative to a partition $\mathcal{D}$ of D are defined in [6] by

$$\eta(\mathbf{X}, \mathcal{D}) = \frac{1}{s} \lim_{x\to\infty} \frac{\sum_{j=1}^{s} \ln \mathbf{P}(\mathbf{X}_{I_j} > \mathbf{x}_{I_j})}{\ln \mathbf{P}(X_1 > x, \ldots, X_d > x)},$$

whenever the limit exists. Similarly, we consider the lag-m AIFI blocks applied to a stationary sequence $\{X_n\}$,

$$\eta(\mathbf{X}, \mathcal{D}_m) = \frac{1}{s} \lim_{x\to\infty} \frac{\sum_{j=1}^{s} \ln \mathbf{P}(\mathbf{X}_{I_j} > \mathbf{x}_{I_j})}{\ln \mathbf{P}(X_1 > x, \ldots, X_m > x)}. \tag{11}$$

Furthermore, we obtain the tail dependence coefficients of a stationary sequence under missing values derived from models (M1), (M2), and (M3) defined in the introduction.

3. Tail dependence under failures. We start by calculating the coefficients (3) and (4) for the sequences $\{Y_n\}$, $\{Z_n\}$, and $\{W_n\}$, respectively, originating from the fault schemes corresponding to the models (M1), (M2), and (M3), that form the basis for the results.

PROPOSITION 3.1. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. Then, for any set of indexes* A,

(i) $\chi^{\mathbf{Y}_A} = \chi^{\mathbf{X}_A} p^{|A|-1}$;

(ii) $\chi^{\mathbf{W}_A} = \chi^{\mathbf{X}_A}(p^{|A|} + (1-p)^{|A|})$, *provided that* $\chi^{\mathbf{X}_A}$ *in* (3) *exists*;

(iii) $\chi^{\mathbf{Z}_A} = \chi^{\mathbf{X}_{\widetilde{A}}}$, *provided that* $\chi^{\mathbf{X}_{\widetilde{A}}}$ *in* (3) *exists for* $\widetilde{A} = \{i_j : U_j = 1,\, j \in A\}$.

Proof. (i) We have that

$$\mathbf{P}\Big(\bigcap_{i\in A}\{Y_i > x\}\Big) = \mathbf{P}\Big(\bigcap_{i\in A}\{X_i > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in A}\{U_i = 1\}\Big), \tag{12}$$

and now the result follows from

$$\chi^{\mathbf{Y}_A} = \lim_{x\to\infty} \frac{\mathbf{P}\big(\bigcap_{i\in A}\{X_i > x\}\big)p^{|A|}}{1-(1-p+pF(x))}.$$

(ii) Observe that

$$\begin{aligned}
\mathbf{P}\Big(\bigcap_{i\in A}\{W_i > x\}\Big) &= \mathbf{P}\Big(\bigcap_{i\in A}\{U_iX_i + (1-U_i)X_i^{(1)} > x\}\Big)\\
&= \sum_{I\subseteq A}\mathbf{P}\Big(\bigcap_{i\in I}\{X_i > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in I}\{U_i = 1\}\Big)\mathbf{P}\Big(\bigcap_{i\in \overline{I}}\{X_i^{(1)} > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in \overline{I}}\{U_i = 0\}\Big),
\end{aligned} \tag{13}$$

and thus we have

$$\begin{aligned}
\chi^{\mathbf{W}_A} &= \lim_{x\to\infty}\frac{\sum_{I\subseteq A}\mathbf{P}\big(\bigcap_{i\in I}\{X_i > x\}\big)p^{|I|}\mathbf{P}\big(\bigcap_{i\in\overline{I}}\{X_i^{(1)} > x\}\big)(1-p)^{|\overline{I}|}}{1-F(x)}\\
&= \lim_{x\to\infty}\bigg(\frac{\mathbf{P}\big(\bigcap_{i\in A}\{X_i > x\}\big)p^{|A|}}{1-F(x)} + \frac{\mathbf{P}\big(\bigcap_{i\in A}\{X_i^{(1)} > x\}\big)(1-p)^{|A|}}{1-F(x)}\bigg).
\end{aligned}$$

(iii) It suffices to observe that

$$\chi^{\mathbf{Z}_A} = \lim_{x\to\infty}\frac{\mathbf{P}\big(\bigcap_{i\in\widetilde{A}}\{X_i > x\}\big)}{1-F(x)}.$$

Proposition 3.1 is proved.

PROPOSITION 3.2. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. Then, for any set of indexes* A,

(i) $\eta^{\mathbf{Y}_A} = \eta^{\mathbf{X}_A}$ *and* $L^{\mathbf{Y}_A}(x) = L^{\mathbf{X}_A}(x)p^{|A|-1}$, *provided that* (4) *holds*;

(ii) $\eta^{\mathbf{W}_A} = \eta^{\mathbf{X}_A}$ *and* $L^{\mathbf{W}_A}(x) = L^{\mathbf{X}_A}(x)(p^{|A|} + (1-p)^{|A|})$, *provided that* (4) *holds. Moreover, if* (4) *holds for each* $I \subseteq A$, *then* $1/\eta^{\mathbf{W}_A} = \min\{1/\eta^{\mathbf{X}_I} + 1/\eta^{\mathbf{X}_{\overline{I}}} : I \subseteq A\}$, *where* $1/\eta^{\mathbf{X}_I} = 0$ *if* $I = \varnothing$.

(iii) $\eta^{\mathbf{Z}_A} = \eta^{\mathbf{X}_{\widetilde{A}}}$, *provided that* (4) *holds for* $\widetilde{A} = \{i_j : U_j = 1,\, j \in A\}$.

Proof. The proof follows from Proposition 3.1. For the second part of (ii), see also [5, Proposition 2.9].

The lag-m versions are now straightforward.

COROLLARY 3.1. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. Then*,

(i) $\chi_m^{(Y)} = \chi_m^{(X)} p$;

(ii) $\chi_m^{(W)} = \chi_m^{(X)}(p^2 + (1-p)^2)$, *provided that* $\chi_m^{(X)}$ *in* (5) *exists*;

(iii) $\chi_m^{(Z)} = \chi_{i_m - i_1}^{(X)}$, *provided that* $\chi_{i_m-i_1}^{(X)} \equiv \chi^{\mathbf{X}_{A_m}}$ *in* (3) *exists for some* $A_m = \{i_1, i_m\}$.

COROLLARY 3.2. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. Then the following statements hold*:

(i) $\eta_m^{(Y)} = \eta_m^{(X)}$ *and* $L_m^{(Y)}(x) = L_m^{(X)}(x)p$;

(ii) $\eta_m^{(W)} = \max(\eta_m^{(X)}, 1/2)$ *and* $L_m^{(W)}(x) = (p^2 + (1-p)^2)L_m^{(X)}(x)\mathbf{1}_{\{\eta_m^{(X)} \geqslant 1/2\}} + 2p(1-p)\mathbf{1}_{\{\eta_m^{(X)} \leqslant 1/2\}}$, *provided that* (6) *holds*;

(iii) $\eta_m^{(Z)} = \eta_{i_m-i_1}^{(X)}$ *and* $L_m^{(Z)}(x) = L_{i_m-i_1}^{(X)}(x)$, *provided that* (4) *holds for some* $A_m = \{i_1, i_m\}$.

Proof. (ii) It suffices to observe that (4) holds with $A = \{1, m\}$, and thus it also holds for all $I \subseteq A$. This implies that $\eta_m^{(W)} = \min(1/\eta_m^{(X)}, 2) = \max(\eta_m^{(X)}, 1/2)$.

Remark 3.1. We can see that, compared to the initial sequence $\{X_n\}$, the TDC decreases for sequences $\{Y_n\}$ and $\{W_n\}$ and does not decrease for sequence $\{Z_n\}$. Under asymptotic independence, the LTC of $\{Y_n\}$ and $\{W_n\}$ is given by the LTC of the initial sequence $\{X_n\}$ (although with a slight decrease in the weak dependence that results from the slowly varying function) but may increase with respect to the sequence $\{Z_n\}$. However, for the sequence $\{W_n\}$, if (4) holds for each $I \subseteq A$, the LTC may be greater than or equal to that of $\{X_n\}$. In any case, the LTC does not depend directly on p.

Furthermore, we derive the lag-m upper s,k-extremal coefficients in (7), as well as the particular case of the lag-m upper extremal dependence coefficients defined in (8).

PROPOSITION 3.3. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. If* (3) *holds for all* $A \subseteq D_m = \{1, \dots, m\}$, *then we have successively*

$$\chi(Y_{s:m} \mid Y_{m-k+1:m}) = \frac{\sum_{i=0}^{s-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq I} (-1)^{|J|} \chi^{\mathbf{X}_{\overline{I} \cup J}} p^{|\overline{I} \cup J|-1}}{-\sum_{\varnothing \neq J \subseteq D_m} (-1)^{|J|} \chi^{\mathbf{X}_J} p^{|J|-1} - \sum_{i=1}^{k-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq \overline{I}} (-1)^{|J|} \chi^{\mathbf{X}_{I \cup J}} p^{|I \cup J|-1}},$$

$$\chi(Z_{s:m} \mid Z_{m-k+1:m}) = \frac{\sum_{i=0}^{s-1} \sum_{\widetilde{I} \in \mathcal{F}_i} \sum_{\widetilde{J} \subseteq \widetilde{I}} (-1)^{|\widetilde{J}|} \chi^{\mathbf{X}_{\widetilde{\overline{I}} \cup \widetilde{J}}}}{-\sum_{\varnothing \neq \widetilde{J} \subseteq D_m} (-1)^{|\widetilde{J}|} \chi^{\mathbf{X}_{\widetilde{J}}} - \sum_{i=1}^{k-1} \sum_{\widetilde{I} \in \mathcal{F}_i} \sum_{\widetilde{J} \subseteq \widetilde{\overline{I}}} (-1)^{|\widetilde{J}|} \chi^{\mathbf{X}_{\widetilde{I} \cup \widetilde{J}}}},$$

and, denoting here $q = 1 - p$,

$$\chi(W_{s:m} \mid W_{m-k+1:m}) = \frac{\sum_{i=0}^{s-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq I} (-1)^{|J|} \chi^{\mathbf{X}_{\overline{I} \cup J}} (p^{|\overline{I} \cup J|} + q^{|\overline{I} \cup J|})}{-\sum_{\varnothing \neq J \subseteq D_m} (-1)^{|J|} \chi^{\mathbf{X}_J} (p^{|J|} + q^{|J|}) - \sum_{i=1}^{k-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq \overline{I}} (-1)^{|J|} \chi^{\mathbf{X}_{I\ cupJ}} (p^{|I \cup J|} + q^{|I \cup J|})},$$

provided the ratios are defined, where $\mathcal{F}_i$ *denotes the family of all subsets of* D_m *with cardinality equal to* i, $\overline{I}$ *is the complement set of* $I \in \mathcal{F}_i$ *in* D_m, *and* $\widetilde{A} = \{i_j : U_j = 1,\ j \in A\}$ *for each* $A \subseteq D_m$.

Proof. According to [3, Proposition 2.9], we have

$$\chi(X_{s:m} \mid X_{m-k+1:m}) = \frac{\sum_{i=0}^{s-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq I} (-1)^{|J|} \chi^{\mathbf{X}_{\overline{I} \cup J}}}{-\sum_{\varnothing \neq J \subseteq D_m} (-1)^{|J|} \chi^{\mathbf{X}_J} - \sum_{i=1}^{k-1} \sum_{I \in \mathcal{F}_i} \sum_{J \subseteq \overline{I}} (-1)^{|J|} \chi^{\mathbf{X}_{I \cup J}}},$$

provided the ratio is defined. Now the results follow from Proposition 3.1 above. Proposition 3.3 is proved.

COROLLARY 3.3. *Under the conditions of Proposition* 3.3, *we have successively*

$$\epsilon_m^Y = \frac{\chi^{\mathbf{X}_{D_m}} p^{m-1}}{-\sum_{\varnothing\neq J\subseteq D_m}(-1)^{|J|}\chi^{\mathbf{X}_J}p^{|J|-1}},$$
$$\epsilon_m^Z = \frac{\chi^{\mathbf{X}_{\widetilde{D_m}}}}{-\sum_{\varnothing\neq \widetilde{J}\subseteq D_m}(-1)^{|\widetilde{J}|}\chi^{\mathbf{X}_{\widetilde{J}}}},$$
$$\epsilon_m^W = \frac{\chi^{\mathbf{X}_{D_m}}(p^m+(1-p)^m)}{-\sum_{\varnothing\neq J\subseteq D_m}(-1)^{|J|}\chi^{\mathbf{X}_J}(p^{|J|}+(1-p)^{|J|})}.$$

Finally, we address the concept of tail dependence through fragility and compute the lag-m FI in (9) and the lag-m AIFI in (11).

PROPOSITION 3.4. *Under the conditions of Proposition* 3.3, *we have successively*

$$\mathrm{FI}(\mathbf{Y},\mathcal{D}_m) = \lim_{x\to\infty}\frac{\sum_{j=1}^s\sum_{k\in I_j}(-1)^{k-1}\sum_{J\subset I_j;\,|J|=k}\chi^{\mathbf{X}_J}p^{|J|-1}}{\sum_{k=1}^m(-1)^{k-1}\sum_{J\subset D_m;\,|J|=k}\chi^{\mathbf{X}_J}p^{|J|-1}},$$
$$\mathrm{FI}(\mathbf{Z},\mathcal{D}_m) = \lim_{x\to\infty}\frac{\sum_{j=1}^s\sum_{k\in I_j}(-1)^{k-1}\sum_{\widetilde{J}\subset I_j;\,|\widetilde{J}|=k}\chi^{\mathbf{X}_{\widetilde{J}}}}{\sum_{k=1}^m(-1)^{k-1}\sum_{\widetilde{J}\subset D_m;\,|\widetilde{J}|=k}\chi^{\mathbf{X}_{\widetilde{J}}}}$$
$$\mathrm{FI}(\mathbf{W},\mathcal{D}_m) = \lim_{x\to\infty}\frac{\sum_{j=1}^s\sum_{k\in I_j}(-1)^{k-1}\sum_{J\subset I_j;\,|J|=k}\chi^{\mathbf{X}_J}(p^{|J|}+(1-p)^{|J|})}{\sum_{k=1}^m(-1)^{k-1}\sum_{J\subset D_m;\,|J|=k}\chi^{\mathbf{X}_J}(p^{|J|}+(1-p)^{|J|})}.$$

Proof. By definition (9), we have

$$\begin{aligned}\mathrm{FI}(\mathbf{X},\mathcal{D}_m) &= \lim_{x\to\infty}\frac{\sum_{j=1}^s\mathbf{P}(\bigcup_{i\in I_j}\{X_i>x\})}{1-\mathbf{P}(\bigcap_{i\in D_m}\{X_i<x\})}\\ &= \lim_{x\to\infty}\frac{\sum_{j=1}^s\sum_{k\in I_j}(-1)^{k-1}\sum_{J\subset I_j;\,|J|=k}\mathbf{P}(\bigcap_{i\in J}\{X_i>x\})}{\sum_{k=1}^m(-1)^{k-1}\sum_{J\subset D_m;\,|J|=k}\mathbf{P}(\bigcap_{i\in J}\{X_i>x\})},\end{aligned}$$

provided the ratios are defined. For more details, see [6] and [7]. The results are now straightforward from Proposition 3.1. Proposition 3.4 is proved.

PROPOSITION 3.5. *Consider a stationary sequence* $\{X_n\}$ *and the sequences* $\{Y_n\}$, $\{Z_n\}$, *and* $\{W_n\}$ *derived according to models* (M1), (M2), *and* (M3), *respectively. Then the following results hold*:

(i) *If* (10) *holds for all* $I_j\in\mathcal{D}_m$, $j=1,\dots,s$, *and also for* D_m, *then*

$$\eta(\mathbf{Y},\mathcal{D}_m)=\eta(\mathbf{W},\mathcal{D}_m)=\eta(\mathbf{X},\mathcal{D}_m).$$

Moreover, if (10) *holds for all* $A\subseteq D_m$, *with the limit given by* $\eta_A^{\mathbf{X}}$, *then*

$$\eta(\mathbf{W},\mathcal{D}_m)=\eta_{D_m}^{\mathbf{W}}\frac{1}{s}\sum_{j=1}^s\frac{1}{\eta_{I_j}^{\mathbf{W}}},$$

with $1/\eta_A^{\mathbf{W}}=\min\{1/\eta^{\mathbf{X}_I}+1/\eta^{\mathbf{X}_{\overline{I}}}\colon I\subseteq A\}$, *where* $1/\eta^{\mathbf{X}_I}=0$ *if* $I=\varnothing$, *respectively.*

(ii) *If* (10) *holds for all* $\widetilde{I}_j$, $j=1,\dots,s$, *and also for* $\widetilde{D_m}$, *with notation* $\widetilde{A}=\{i_j\colon U_j=1,\ j\in A\}$ *for each* $A\subseteq D_m$, *then*

$$\eta(\mathbf{Z},\mathcal{D}_m)=\eta_{\widetilde{D_m}}^{\mathbf{X}}\frac{1}{s}\sum_{j=1}^s\frac{1}{\eta_{\widetilde{I}_j}^{\mathbf{X}}}.$$

Proof. According to Proposition 6.1 in [6], if (10) holds for all $I_j \in \mathcal{D}_m$, $j = 1, \ldots, s$, and also for D_m, with the limit given by $\eta^{\mathbf{X}}_{I_j}$ and $\eta^{\mathbf{X}}_{D_m}$, respectively, then we have

$$\eta(\mathbf{X}, \mathcal{D}_m) = \eta^{\mathbf{X}}_{D_m} \frac{1}{s} \sum_{j=1}^{s} \frac{1}{\eta^{\mathbf{X}}_{I_j}}. \tag{14}$$

The required result now follows from Proposition 3.2. Proposition 3.5 is proved.

4. Examples. Here we present two examples: the first is a process presenting tail dependence, and the second is an asymptotic tail independent process.

Example 4.1. We consider the Yeh–Arnold–Robertson Pareto (III) (see [15]), abbreviated YARP (III)(1), given by

$$X_n = \min\biggl(b^{-1/\alpha} X_{n-1},\, \frac{1}{1 - B_n} \varepsilon_n\biggr),$$

where innovations $\{\varepsilon_n\}_{n \geqslant 1}$ are i.i.d. with the distribution Pareto (III)(0,σ,α), i.e.,

$$1 - F_\varepsilon(x) = \left[1 + \left(\frac{x - \mu}{\sigma}\right)^{\alpha}\right]^{-1}, \qquad x > \mu,$$

with $\sigma, \alpha > 0$. The sequence $\{B_n\}_{n \geqslant 1}$ has i.i.d. with Bernoulli distribution (b) (independent of the innovations). We interpret 1/0 as $+\infty$.

Define the m-step transition probability function (t.p.f.), $Q^m(x,]1, y]) = \mathbf{P}(X_{n+m} < y \mid X_n = x)$. The m-step t.p.f. of YARP (III)(1) was deduced in [2] as

$$Q^m(x,]0, y]) = \begin{cases} 1 - \prod_{j=0}^{m-1} [\overline{F}_\varepsilon(b^{j/\alpha} y)(1 - b) + b], & x > y b^{m/\alpha}, \\ 1, & x \leqslant y b^{m/\alpha}. \end{cases}$$

For $i_1 < i_2 < i_3$, we have

$$\begin{aligned} &\mathbf{P}(X_{i_1} > x,\, X_{i_2} > x,\, X_{i_3} > x) \\ &\quad = \int_x^{\infty} \int_x^{\infty} [1 - Q^{i_3 - i_2}(u_2,]0, x])] Q^{i_2 - i_1}(u_1, du_2)\, dF_X(u_1), \end{aligned} \tag{15}$$

and thus

$$\begin{aligned} &\mathbf{P}(X_{i_1} > x,\, X_{i_2} > x,\, X_{i_3} > x) \\ &\quad = [1 - F(x) + b^{i_3 - i_2} F(x)][1 - F(x) + b^{i_2 - i_1} F(x)](1 - F(x)). \end{aligned}$$

For more details, see [7]. Similarly, for $i_1 < i_2 < \cdots < i_k$,

$$\begin{aligned} \mathbf{P}(X_{i_1} > x, \ldots, X_{i_k} > x) &= \int_x^{\infty} \cdots \int_x^{\infty} \bigl(1 - Q^{i_k - i_{k-1}}(u_{i_{k-1}},]0, x])\bigr) \\ &\qquad \times \prod_{j=2}^{k-1} Q^{i_{k-j} - i_{k-j+1}}(u_{i_{k-j}}, du_{i_{k-j+1}})\, dF_X(u_{i_1}) \\ &= \prod_{j=2}^{k} (1 - F(x) + b^{i_j - i_{j-1}} F(x))(1 - F(x)). \end{aligned}$$

Hence, we have, for $A = \{i_1, \ldots, i_k\}$,

$$\chi^{\mathbf{X}_A} = \lim_{x \to \infty} \prod_{j=2}^{k} (1 - F(x) + b^{i_j - i_{j-1}} F(x)) = b^{i_k - i_1}.$$

Denoting by $\zeta(A)$ and $\varsigma(A)$ the maximum and minimum of A, respectively, we obtain

$$\chi^{\mathbf{X}_A} = b^{\zeta(A)-\varsigma(A)}.$$

In particular, $\chi_m^{(X)} = b^{m-1}$, and therefore $\chi_m^{(Y)} = b^{m-1}p$, $\chi_m^{(Z)} = b^{i_m-i_1}$, and $\chi_m^{(W)} = b^{m-1}(p^2+(1-p)^2)$.

The lag-m upper s,k-extremal coefficients, upper extremal dependence coefficients, and block-FI can be readily deduced for $\{Y_n\}$, $\{Z_n\}$, and $\{W_n\}$ from, respectively, Proposition 3.3, Corollary 3.3, and Proposition 3.4.

Example 4.2. Consider the ARMAX$_p$ model (see [1])

$$X_n = \max(X_{n-1}^c, \varepsilon_n) \tag{16}$$

with univariate marginals having Pareto (α) d.f., i.e., $1-F(x) = x^{-\alpha}$, $x > 1$, $\alpha > 0$, and innovations $\{\varepsilon_n\}_{n\geqslant 1}$ with marginal d.f.

$$F_\varepsilon(x) = \frac{1-x^{-\alpha}}{1-x^{-\alpha/c}}.$$

We can observe that, $1-F_\varepsilon(x) \sim x^{-\alpha}$ as $x\to\infty$, where $a_n \sim b_n$ denote that $\lim_{n\to\infty} a_n/b_n = k \neq 0$.

The m-step t.p.f. of the ARMAX$_p$ is

$$Q^m(x,]1,y]) = \begin{cases} \dfrac{F(y)}{F(y^{1/c^m})}, & x \leqslant y^{1/c^m}, \\ 0, & x > y^{1/c^m}. \end{cases}$$

By the same argument as in (15), we obtain

$$\begin{aligned} \mathbf{P}(X_{i_1} > x, X_{i_2} > x, X_{i_3} > x) &= 1 - 3F(x) + \frac{F^2(x)}{F(x^{1/c^{i_2-i_1}})} + \frac{F^2(x)}{F(x^{1/c^{i_3-i_1}})} \\ &\quad + \frac{F^2(x)}{F(x^{1/c^{i_3-i_2}})} - \frac{F^3(x)}{F(x^{1/c^{i_3-i_2}})F(x^{1/c^{i_2-i_1}})} \\ &\sim x^{-3\alpha} + x^{-\alpha/c^{i_3-i_1}}. \end{aligned}$$

Analogously, for $i_1 < i_2 < \cdots < i_k$, we have

$$\begin{aligned} \mathbf{P}(X_{i_1} > x, \dots, X_{i_k} > x) &= \int_x^\infty \cdots \int_x^\infty \left(1 - Q^{i_k - i_{k-1}}(u_{i_{k-1}},]0,x])\right) \\ &\quad \times \prod_{j=2}^{k-1} Q^{i_{k-j}-i_{k-j+1}}(u_{i_{k-j}}, du_{i_{k-j+1}})\, dF_X(u_{i_1}) \\ &\sim x^{-k\alpha} + x^{-\alpha/c^{i_k-i_1}}, \end{aligned}$$

and thus $\eta^{\mathbf{X}_{\{i_1,\dots,i_k\}}} = \max(1/k, c^{i_k-i_1})$. We also have $\eta_m^{(X)} = \max(1/2, c^{m-1})$, already obtained in [1]. More generally, for any $A \subseteq D_m$,

$$\eta^{\mathbf{X}_A} = \max\left(\frac{1}{|A|}, c^{\zeta(A)-\varsigma(A)}\right),$$

with $\zeta(A)$ and $\varsigma(A)$ denoting the maximum and minimum of A, respectively. Therefore, by (14), the lag-m blocks AIFI of $\{X_n\}$ is

$$\eta(\mathbf{X}, \mathcal{D}_m) = \frac{\max(1/m, c^{m-1})}{s} \sum_{j=1}^s \min\left(|I_j|, \frac{1}{c^{\zeta(I_j)-\varsigma(I_j)}}\right). \tag{17}$$

The lag-m blocks AIFI of $\{Y_n\}$, $\{Z_n\}$, and $\{W_n\}$ are now straightforward from Proposition 3.5.

Remark 4.1. Let $\{U_n\}$ be a stationary sequence presenting some dependence, e.g., of Markov type, such as

$$\mathbf{P}(U_n = 1 \mid U_{n-1} = 1) = \phi \quad \text{and} \quad \mathbf{P}(U_n = 1 \mid U_{n-1} = 0) = \varphi,$$

and thus rendering a process, where the probability of failure depends uniquely on whether a failure has just occurred. Only the formulas of the TDC in (3), and consequently of the s, k-extremal coefficients and the fragility indexes, are affected for sequences $\{Y_n\}$ and $\{W_n\}$. See Remark 3.1. Indeed, denoting the transition matrix by

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1-\varphi & \varphi \\ 1-\phi & \phi \end{bmatrix},$$

and taking without loss of generality $A = \{i_1, \ldots, i_k\}$, by (12), we have

$$\begin{aligned} \mathbf{P}\Big(\bigcap_{i\in A}\{Y_i > x\}\Big) &= \mathbf{P}\Big(\bigcap_{i\in A}\{X_i > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in A}\{U_i = 1\}\Big) \\ &= \mathbf{P}\Big(\bigcap_{i\in A}\{X_i > x\}\Big)\prod_{j=2}^{k} p_{11}^{(i_j - i_{j-1})} p_1, \end{aligned}$$

where $p_{11}^{(i_j-i_{j-1})}$ is the respective p_{11} element of the transition matrix $P^{i_j-i_{j-1}}$, for each $j = 2, \ldots, k$, and $p_1 = \mathbf{P}(U_n = 1) = \varphi/(1-\phi+\varphi)$.

Applying (13) and stating $I = \{i_1^{(I)}, \ldots, i_l^{(I)}\}$ with $l \leqslant k$, we have

$$\begin{aligned} \mathbf{P}\Big(\bigcap_{i\in A}\{W_i > x\}\Big) &= \sum_{I\subseteq A}\mathbf{P}\Big(\bigcap_{i\in I}\{X_i > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in I}\{U_i = 1\}\Big) \\ &\qquad \times \mathbf{P}\Big(\bigcap_{i\in \overline{I}}\{X_i^{(1)} > x\}\Big)\mathbf{P}\Big(\bigcap_{i\in \overline{I}}\{U_i = 0\}\Big) \\ &= \sum_{I\subseteq A}\mathbf{P}\Big(\bigcap_{i\in I}\{X_i > x\}\Big)\prod_{j=2}^{l} p_{11}^{(i_j^{(I)} - i_{j-1}^{(I)})} p_1 \\ &\qquad \times \mathbf{P}\Big(\bigcap_{i\in \overline{I}}\{X_i^{(1)} > x\}\Big)\prod_{j=2}^{k-l} p_{00}^{(i_j^{(\overline{I})} - i_{j-1}^{(\overline{I})})}(1-p_1), \end{aligned}$$

where $p_{11}^{(i_j-i_{j-1})}$ and $p_{00}^{(i_j^{(\overline{I})}-i_{j-1}^{(\overline{I})})}$ are the respective p_{11} element of the transition matrix $P^{i_j^{(I)}-i_{j-1}^{(I)}}$, for each $j = 2, \ldots, l$, and the p_{00} element of the transition matrix $P^{i_j^{(\overline{I})}-i_{j-1}^{(\overline{I})}}$, for each $j = 2, \ldots, k-l$.

Other types of dependence within $\{U_n\}$ and between $\{X_n\}$ and $\{U_n\}$ also may be interesting to consider. This will be addressed in a future work.

REFERENCES

[1] M. Ferreira and L. Canto e Castro, *Tail and dependence behavior of levels that persist for a fixed period of time*, Extremes, 11 (2008), pp. 113–133.
[2] M. Ferreira, *On the extremal behavior of a Pareto process: An alternative for ARMAX modeling*, Kybernetika (Prague), 48 (2012), pp. 31–49.
[3] H. Ferreira and M. Ferreira, *Tail dependence between order statistics*, J. Multivariate Anal., 105 (2012), pp. 176–192.
[4] M. Ferreira and H. Ferreira, *On extremal dependence: Some contributions*, TEST, 21 (2012), pp. 566–583.
[5] H. Ferreira and M. Ferreira, *On extremal dependence of block vectors*, Kybernetika (Prague), 48 (2012), pp. 988–1006.

[6] H. Ferreira and M. Ferreira, *Fragility index of block tailed vectors*, J. Statist. Plann. Inference, 142 (2012), pp. 1837–1848.
[7] M. Ferreira, *Tail dependence of a Pareto process*, in New Advances in Statistical Modeling and Applications, Stud. Theor. Appl. Stat. Sel. Papers Stat. Soc., Springer, Cham, 2014, pp. 177–185.
[8] G. Frahm, *On the extremal dependence coefficient of multivariate distributions*, Statist. Probab. Lett., 76 (2006), pp. 1470–1481.
[9] J. L. Geluk, L. De Haan, and C. G. De Vries, *Weak and Strong Financial Fragility*, Tinbergen Institute Discussion Paper 07-023/2, 2007, https://www.tinbergen.nl/discussion-paper/2279/07-023-2-weak-amp-strong-financial-fragility.
[10] A. Hall and J. Hüsler, *Extremes of stationary sequences with failures*, Stoch. Models, 22 (2006), pp. 537–557.
[11] J. E. Heffernan, J. A. Tawn, and Z. Zhang, *Asymptotically (in)dependent multivariate maxima of moving maxima processes*, Extremes, 10 (2007), pp. 57–82.
[12] H. Joe, *Multivariate Models and Dependence Concepts*, Monogr. Statist. Appl. Probab. 73, Chapman & Hall, London, 1997.
[13] A. W. Ledford and J. A. Tawn, *Statistics for near independence in multivariate extreme values*, Biometrika, 83 (1996), pp. 169–187.
[14] M. Sibuya, *Bivariate extreme statistics.* I, Ann. Inst. Statist. Math. Tokyo, 11 (1960), pp. 195–210.
[15] H.-C. Yeh, B. C. Arnold, and C. A. Robertson, *Pareto processes*, J. Appl. Probab., 25 (1988), pp. 291–301.