Journal of Physics: Conference Series **66** (2007) 012019 doi:10.1088/1742-6596/66/1/012019

Lanczos potentials for linearly perturbed FLRW spacetimes

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Abstract. We study the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes.

1. Introduction

Penrose [17] conjectured that the gravitational entropy should be related to the clumping of matter and therefore associated with the Weyl or conformal curvature. Specifically, Penrose suggested that a measure of the gravitational entropy should involve an integral of a quantity derived from the Weyl tensor, and that the particle number operator for a linear spin-2 massless quantized free-field might provide some clues, since the entropy measure could be taken as an estimate of the 'number of gravitons' [17]. Since then, there have been several attempts to construct gravitational entropy measures using polynomial invariants of the Weyl tensor (see e.g. [8, 4, 16]) as well as density contrast functions [13, 10].

We have used Penrose's conjecture and the particle number from linear theory in flat space to motivate a definition of gravitational entropy in curved space [14]. In order to do that we required a potential for the Weyl tensor which we took to be the Lanczos potential [12]. Illge [11] has shown that any spinor field with the symmetries of the Weyl spinor locally has a Lanczos potential which is determined by its value at a space-like hypersurface. Furthermore, for a vacuum spacetime there exists a potential for the Lanczos potential, i.e. a superpotential for the Weyl spinor [11] (see also [1]).

Apart from Illge's result, which is difficult to apply, there is no general prescription for obtaining a Lanczos potential for a given spacetime. A general expression for a Lanczos potential in the case of perfect fluid spacetimes with zero shear and vorticity was given in [15]. More recently, this result has been extended by Holgersson [9] to Bianchi I perfect-fluid spacetimes. There are also several examples of Lanczos potentials for particular exact solutions, including Gödel, Schwarzschild, Taub and Kerr [3, 15, 5, 6].

In this short note, we consider the problem of deriving the Lanczos potential and superpotential for linearly perturbed Friedman-Lemaitre-Robertson-Walker (FLRW) spacetimes, which we then use to propose a new measure of the gravitational entropy in [14].

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2. The perturbed FLRW model

We consider a spacetime with a distinguished time-like direction given by the velocity vector field u^a of the fluid, and use the formalism of [7, 18], with the projected metric

$$h_{ab} = g_{ab} + u_a u_b,$$

which is orthogonal to u^a. The covariant derivative of u_a can be written as

$$\nabla_b u_a = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} - \dot{u}_a u_b$$

where

$$\sigma_{ab} = \sigma_{(ab)}; \quad \sigma^a_a = 0; \quad \sigma_{ab}u^b = 0; \quad \omega_{ab} = \omega_{[ab]}; \quad \omega_{ab}u^b = 0.$$

Then $\dot{u}^a$ is the acceleration (so that the overdot is $u^a\nabla_a$), ω_{ab} is the vorticity tensor, σ_{ab} the shear, and θ the expansion. The stress–energy tensor for perfect fluids is $T_{ab} = \rho u_a u_b + p h_{ab}$, where ρ is the energy density and p the isotropic pressure of the fluid.

The Weyl tensor can be decomposed into its electric and magnetic parts, E_{ab} and H_{ab} relative to the velocity vector u^a as

$$E_{ab} = C_{acbd}u^c u^d, \quad H_{ab} = C^*_{acbd}u^c u^d,$$

where $C^*_{acbd} = \frac{1}{2}\eta_{ac}{}^{st}C_{stbd}$. An FLRW background is conformally-flat with the fluid-flow being geodesic, shear-free and twist-free so that $\dot{u}_a = \omega_{ab} = \sigma_{ab} = 0 = E_{ab} = H_{ab}$.

We shall now consider the FLRW metric g_{ab} with linear perturbations $\delta g_{ab} = \Phi_{ab}$ such that

$$\Phi_{ab}u^b = \Phi^a{}_a = \nabla^a\Phi_{ab} = 0. \tag{1}$$

The perturbation is characterised as purely gravitational by requiring:

$$\delta R^b_a = 0. \tag{2}$$

This implies that $\delta\rho = \delta p = 0$, and with the gauge conditions (1) also $\delta u^a = \delta u_a = 0$, so that $\delta T^b_a = 0$ and $\delta\theta = 0 = \delta\omega_{ab} = \delta\dot{u}_a$, while for the shear we introduce the notation:

$$\Sigma_{ab} := \delta\sigma_{ab} = \frac{1}{2}\dot{\Phi}_{ab}. \tag{3}$$

For the Weyl tensor, which is zero in the background, we find

$$E^{ab} = -\dot{\Sigma}^{ab} - \frac{2}{3}\theta\,\Sigma^{ab}, \tag{4}$$

$$H^{ab} = \text{curl}\,\Sigma^{ab}, \tag{5}$$

with

$$\text{curl}\,X^{ab} \equiv (\text{curl}\,X)^{ab} := \eta^{cd(a}D_c X^{b)}{}_d,$$

where D_c is the covariant derivative on hypersurfaces orthogonal to u^a, $\eta_{abc} = \eta_{abcd}u^d$ is the hypersurface volume form and η_{abcd} the space-time volume form. Now, the field equation (2) reduces to

$$\Box\Phi_{ab} = \frac{2}{3}\rho\,\Phi_{ab}$$

and from (3) and (2) we get

$$\Box\Sigma_{ab} = \frac{2}{3}\theta\,\dot{\Sigma}_{ab} + (\frac{1}{6}\rho - \frac{3}{2}p + \frac{1}{3}\theta^2)\Sigma_{ab}. \tag{6}$$

3. The Lanczos potential

The Lanczos potential is a tensor $L_{abc} = -L_{bac}$ such that:

$$C_{ab}{}^{cd} = -\nabla^{[c} L_{ab}{}^{d]} - \nabla_{[a} L^{cd}{}_{b]} - 2\delta^{[}_{[a}{}^{c} \nabla^e L_{b]e}{}^{d]},$$

in the Lanczos gauge:

$$L_{ab}{}^{a} = 0 = \eta^{abcd} L_{abc} \; ; \qquad \nabla_c L_{ab}{}^{c} = 0.$$

Holgersson [9] gave a useful decomposition of the Lanczos potential into irreducible parts as:

$$L_{abc} = 2u_{[a} A_{b]} u_c - A_{[a} h_{b]c} - 2u_{[a} C_{b]c} + \eta_{ab}{}^{d} S_{dc} + u_{[a} \eta_{b]cd} P^d - u_c \eta_{abd} P^d \;, \tag{7}$$

where A_a and P_a are orthogonal to u^a and S_{ab} and C_{ab} are trace-free, symmetric and orthogonal to u^a.

Since the FLRW perturbation is trace-free, symmetric and orthogonal to u^a, we seek a Lanczos potential as in (7) with $A_a = P_a = 0$. Then from (7) and (3) we find the following expressions for E_{ab} and H_{ab}:

$$E_{ab} = \frac{1}{2}(\text{curl } S_{ab} - \dot{C}_{ab}), \tag{8}$$

$$H_{ab} = \frac{1}{2}(\text{curl } C_{ab} + \dot{S}_{ab}). \tag{9}$$

which equated to (4) and (5) give the expressions for C_{ab} and S_{ab}.

Now, suppose a superpotential ϕ_{ab} existed for all times with

$$L_{abc} = \nabla_{[a} \phi_{b]c}, \tag{10}$$

then from (7), we get expressions for C_{ab} and S_{ab} as:

$$C_{ab} = \frac{1}{2}(\dot{\phi}_{ab} + \frac{\theta}{3}\phi_{ab}),$$

$$S_{ab} = \frac{1}{2}\text{curl } \phi_{ab},$$

which turn out to be incompatible with the Bianchi identities [14] (as is to be expected, since the superpotential should not exist for non-vacuum). However, this procedure suggests the ansatz:

$$C_{ab} = \frac{1}{2}(\psi_{ab} + \frac{\theta}{3}\phi_{ab})$$

$$S_{ab} = \frac{1}{2}\text{curl } \phi_{ab} \tag{11}$$

in terms of another unknown tensor ψ_{ab}. So, we find from (9) and (11)

$$H_{ab} = \frac{1}{4}\text{curl } (\dot{\phi} + \psi)_{ab}.$$

Comparing this equation with (5) we can choose

$$\Sigma_{ab} = \frac{1}{4}(\dot{\phi}_{ab} + \psi_{ab}), \tag{12}$$

so that ψ_{ab} is known once ϕ_{ab} has been found. Then, from (8)

$$E_{ab} = \frac{1}{4}(-\dot{\psi}_{ab} - \frac{\dot{\theta}}{3}\phi_{ab} - \frac{\theta}{3}\dot{\phi}_{ab} + \text{curl curl } \phi_{ab}),$$

and combining this with (4) and (12) we get

$$\Box\phi_{ab} + \frac{4}{3}\theta\dot{\phi}_{ab} + (\frac{\dot{\theta}}{3} + \frac{\theta^2}{9} - \rho)\phi_{ab} = \frac{8}{3}\theta\Sigma_{ab}, \tag{13}$$

which is a wave equation for ϕ_{ab}. We therefore have a complete prescription to determine a unique L_{abc} for linearly perturbed FLRW, subject to choice of initial data. We can achieve (10), at a given instant t_0 by choosing the data for (13) to be

$$\begin{aligned} \phi_{ab}(\mathbf{x}, t_0) &= \Phi_{ab}(\mathbf{x}, t_0), \\ \dot{\phi}_{ab}(\mathbf{x}, t_0) &= \dot{\Phi}_{ab}(\mathbf{x}, t_0). \end{aligned} \tag{14}$$

since then, by (12), $\psi_{ab}(\mathbf{x}, t_0) = \dot{\phi}_{ab}(\mathbf{x}, t_0)$.

We summarize our results in the following proposition:

Proposition *Given a perturbed FLRW spacetime and a choice of time t_0, a Lanczos potential L_{abc}, in the Lanczos gauge, may be uniquely specified by (7) with (11), (12) and (13), subject to the data (14). We may define a superpotential ϕ_{ab} such that (10) holds at t_0 but this will not hold at other times.*

Acknowledgments
FCM thanks FCT (Portugal) for grant SFRH/BPD/12137/2003 and Centre of Mathematics (CMAT), University of Minho, for support.

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