ALGEBRAS OF SINGULAR INTEGRAL OPERATORS ON NAKANO SPACES WITH KHVEDELIDZE WEIGHTS OVER CARLESON CURVES WITH LOGARITHMIC WHIRL POINTS

A. YU. KARLOVICH

ABSTRACT. We establish a Fredholm criterion for an arbitrary operator in the Banach algebra of singular integral operators with piecewise continuous coefficients on Nakano spaces (generalized Lebesgue spaces with variable exponent) with Khvedelidze weights over Carleson curves with logarithmic whirl points.

1. INTRODUCTION

Fredholm theory of one-dimensional singular integral operators (SIOs) with piecewise continuous (PC) coefficients on weighted Lebesgue spaces was constructed by Gohberg and Krupnik [9] and [7, 8] in the beginning of 70s in the case of Khvedelidze weights and piecewise Lyapunov curves (see also the monographs [5, 12, 26, 27]). Simonenko and Chin Ngok Min [38] suggested another approach to the study of the Banach algebra of singular integral operators with piecewise continuous coefficients on Lebesgue spaces with Khvedelidze weights over piecewise Lyapunov curves. This approach is based on Simonenko's local principle [36]. In 1992 Spitkovsky [39] made a next significant step: he proved a Fredholm criterion for an individual SIO with PC coefficients on Lebesgue spaces with Muckenhoupt weights over Lyapunov curves. Finally, Böttcher and Yu. Karlovich extended Spitkovsky's result to the case of arbitrary Carleson curves and Banach algebras of SIOs with PC coefficients. After their work the Fredholm theory of SIOs with PC coefficients is available in the maximal generality (that, is, when the Cauchy singular integral operator S is bounded on weighted Lebesgue spaces). We recommend the nice paper [3] for a first reading about this topic and the book [2] for a complete and self-contained exposition.

It is quite natural to consider the same problems in other, more general, spaces of measurable functions on which the operator S is bounded. Good candidates for this role are rearrangement-invariant spaces (that is, spaces with the property that norms of equimeasurable functions are equal). These spaces have nice interpolation properties and boundedness results can be extracted from known results for Lebesgue spaces applying interpolation theorems. The author extended (some parts of) the Böttcher-Yu. Karlovich Fredholm theory of SIOs with PC coefficients to the case of rearrangement-invariant spaces with Muckenhoupt weights [14, 16]. Notice that necessary conditions for the Fredholmness of an individual singular integral operator with PC coefficients are obtained in [17] for weighted reflexive Banach function spaces on which the operator S is bounded.

Nakano spaces $L^{p(\cdot)}$ (generalized Lebesgue spaces with variable exponent) are a nontrivial example of Banach function spaces which are not rearrangement-invariant, in general. Many results about the behavior of some classical operators on these spaces have important applications to fluid dynamics (see [11] and the references therein). Kokilashvili and Samko [22] proved that the operator S is bounded on weighted Nakano spaces for the case of nice curves, nice weights, and nice (but variable!) exponents. They also extended the Gohberg-Krupnik Fredholm criterion for an individual SIO with PC coefficients to this situation [23] (see also [33]). The author [18] has found a Fredholm criterion and a formula for the index of an arbitrary operator in the Banach algebra of SIOs with PC coefficients on Nakano spaces with Khvedelidze weights over either Lyapunov curves or Radon curves without cusps.

Very recently Kokilashvili and Samko [24] (see also [21, Theorem 7.1]) have proved a boundedness criterion for the Cauchy singular integral operator S on Nakano spaces with Khvedelidze weights over arbitrary Carleson curves. Combining this boundedness result with the machinery developed in [17], we are able to prove a Fredholm criterion for an individual SIO on a Nakano space with a Khvedelidze weight over a Carleson curve satisfying a "logarithmic whirl condition" (see [1], [2, Ch. 1]) at each point. Further, we extend this result to the case of Banach algebras of SIOs with PC coefficients, using the approach developed in [1, 2, 17, 18].

The paper is organized as follows. In Section 2 we define weighted Nakano spaces and discuss the boundedness of the operator S on these spaces. Section 3 contains a Fredholm criterion for an individual SIO with PC coefficients on weighted Nakano spaces. The proof of this result is based on the local principle of Simonenko type and factorization technique. In Section 4 we formulate the Allan-Douglas local principle and the two projections theorem. The results of Section 4 are the main tools allowing us to construct a symbol calculus for the Banach algebra of SIOs with PC coefficients acting on a Nakano space with a Khvedelidze weight over a Carleson curve with logarithmic whirl points in Section 5.

2. Preliminaries

2.1. Weighted Nakano spaces $L_w^{p(\cdot)}$.

Function spaces $L^{p(\cdot)}$ of Lebesgue type with variable exponent p were studied for the first time by Orlicz [32] in 1931, but notice that another kind of Banach spaces is called after him. Inspired by the successful theory of Orlicz spaces, Nakano defined in the late forties [30, 31] so-called *modular spaces*. He considered the space $L^{p(\cdot)}$ as an example of modular spaces. In 1959, Musielak and Orlicz [29] extended the definition of modular spaces by Nakano. Actually, that paper was the starting point for the theory of Musielak-Orlicz spaces (generalized Orlicz spaces generated by Young functions with a parameter), see [28].

Let Γ be a Jordan (i.e., homeomorphic to a circle) rectifiable curve. We equip Γ with the Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. Let $p : \Gamma \to (1, \infty)$ be a measurable function. Consider the convex modular (see [28, Ch. 1] for definitions and properties)

$$m(f, p) := \int_\Gamma |f(\tau)|^{p(\tau)} |d\tau|.$$

Denote by $L^{p(\cdot)}$ the set of all measurable complex-valued functions f on Γ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space when equipped with the *Luxemburg-Nakano norm*

$$\|f\|_{L^{p(\cdot)}} := \inf \left\{ \lambda > 0 : \ m(f/\lambda, p) \le 1 \right\}$$

(see, e.g., [28, Ch. 2]). Thus, the spaces $L^{p(\cdot)}$ are a special case of Musielak-Orlicz spaces. Sometimes the spaces $L^{p(\cdot)}$ are referred to as Nakano spaces. We will follow this tradition. Clearly, if $p(\cdot) = p$ is constant, then the Nakano space $L^{p(\cdot)}$ is isometrically isomorphic to the Lebesgue space L^p. Therefore, sometimes the spaces $L^{p(\cdot)}$ are called generalized Lebesgue spaces with variable exponent or, simply, variable L^p spaces.

We shall assume that

(1) $$1 < \operatorname*{ess\,inf}_{t \in \Gamma} p(t), \quad \operatorname*{ess\,sup}_{t \in \Gamma} p(t) < \infty.$$

In this case the conjugate exponent

$$q(t) := \frac{p(t)}{p(t) - 1} \quad (t \in \Gamma)$$

has the same property.

A nonnegative measurable function w on the curve Γ is referred to as a *weight* if $0 < w(t) < \infty$ almost everywhere on Γ. The *weighted Nakano space* is defined by

$$L_w^{p(\cdot)} = \left\{ f \text{ is measurable on } \Gamma \text{ and } fw \in L^{p(\cdot)} \right\}.$$

The norm in $L_w^{p(\cdot)}$ is defined by $\|f\|_{L_w^{p(\cdot)}} = \|fw\|_{L^{p(\cdot)}}$.

2.2. Carleson curves. A rectifiable Jordan curve Γ is said to be a *Carleson* (or *Ahlfors-David regular*) *curve* if

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{|\Gamma(t, R)|}{R} < \infty,$$

where $\Gamma(t, R) := \{ \tau \in \Gamma : |\tau - t| < R \}$ for $R > 0$ and $|\Omega|$ denotes the measure of a measurable set $\Omega \subset \Gamma$. We can write

$$\tau - t = |\tau - t| e^{i \arg(\tau - t)} \quad \text{for} \quad \tau \in \Gamma \setminus \{t\},$$

and the argument can be chosen so that it is continuous on $\Gamma \setminus \{t\}$. Seifullaev [34] showed that for an arbitrary Carleson curve the estimate $\arg(\tau - t) = O(-\log|\tau - t|)$ as $\tau \to t$ holds for every $t \in \Gamma$. A simpler proof of this result can be found in [2, Theorem 1.10]. One says that a Carleson curve Γ satisfies the *logarithmic whirl condition* at $t \in \Gamma$ if

(2) $$\arg(\tau - t) = -\delta(t) \log|\tau - t| + O(1) \quad (\tau \to t)$$

with some $\delta(t) \in \mathbb{R}$. Notice that all piecewise smooth curves satisfy this condition at each point and, moreover, $\delta(t) \equiv 0$. For more information along these lines, see [1], [2, Ch. 1], [3].

2.3. **The Cauchy singular integral operator.** The *Cauchy singular integral* of $f \in L^1$ is defined by

$$(Sf)(t) := \lim_{R \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t,R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

Not so much is known about the boundedness of the Cauchy singular integral operator S on weighted Nakano spaces $L_w^{p(\cdot)}$ for general curves, general weights, and general exponents $p(\cdot)$. From [17, Theorem 6.1] we immediately get the following.

Theorem 2.1. *Let Γ be a rectifiable Jordan curve, let $w : \Gamma \to [0, \infty]$ be a weight, and let $p : \Gamma \to (1, \infty)$ be a measurable function satisfying (1). If the Cauchy singular integral generates a bounded operator S on the weighted Nakano space $L_w^{p(\cdot)}$, then*

$$(3) \qquad \sup_{t \in \Gamma} \sup_{R > 0} \frac{1}{R} \|w \chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \|\chi_{\Gamma(t,R)}/w\|_{L^{q(\cdot)}} < \infty.$$

From the Hölder inequality for Nakano spaces (see, e.g., [28] or [25]) and (3) we deduce that if S is bounded on $L_w^{p(\cdot)}$, then Γ is necessarily a Carleson curve. If the exponent $p(\cdot) = p \in (1, \infty)$ is constant, then (3) is simply the famous Muckenhoupt condition A_p. It is well known that for classical Lebesgue spaces L^p this condition is not only necessary, but also sufficient for the boundedness of the Cauchy singular integral operator S. A detailed proof of this result can be found in [2, Theorem 4.15].

Let $N \in \mathbb{N}$. Consider now a power weight

$$(4) \qquad \varrho(t) := \prod_{k=1}^{N} |t - \tau_k|^{\lambda_k}, \quad \tau_k \in \Gamma, \quad k \in \{1, \dots, N\},$$

where all λ_k are real numbers. Introduce the class $\mathcal{P}$ of exponents $p : \Gamma \to (1, \infty)$ satisfying (1) and

$$(5) \qquad |p(\tau) - p(t)| \leq \frac{A}{-\log|\tau - t|}$$

for some $A \in (0, \infty)$ and all $\tau, t \in \Gamma$ such that $|\tau - t| < 1/2$.

The following criterion for the boundedness of the Cauchy singular integral operator on Nakano spaces with power weights (6) has been recently proved by Kokilashvili and Samko [24] (see also [21, Theorem 7.1]).

Theorem 2.2. *Let Γ be a Carleson Jordan curve, let ϱ be a power weight of the form (4), and let $p \in \mathcal{P}$. The Cauchy singular integral operator S is bounded on the weighted Nakano space $L_\varrho^{p(\cdot)}$ if and only if*

$$(6) \qquad 0 < \frac{1}{p(\tau_k)} + \lambda_k < 1 \quad \text{for all} \quad k \in \{1, \dots, N\}.$$

For weighted Lebesgue spaces over Lyapunov curves the above theorem was proved by Khvedelidze [19] (see also the proof in [9, 20, 27]). Therefore the weights of the form (4) are often called *Khvedelidze weights*. We shall follow this tradition.

Notice that if p is constant and Γ is a Carleson curve, then (6) is equivalent to the fact that ϱ is a Muckenhoupt weight (see, e.g., [2, Chapter 2]). Analogously one can prove that if the exponent p belong to the class $\mathcal{P}$ and the curve Γ is Carleson, then the power weight (4) satisfies the condition (3) if and only if (6) is fulfilled. The proof of this fact is based on certain estimates for the norms of power functions

in Nakano spaces with exponents in the class $\mathcal{P}$ (see also [17, Lemmas 5.7 and 5.8] and [22]).

3. Singular integral operators with PC coefficients

3.1. The local principle of Simonenko type. Let I be the identity operator on $L_\varrho^{p(\cdot)}$. Under the conditions of Theorem 2.2, the operators

$$P := (I + S)/2, \quad Q := (I - S)/2$$

are bounded projections on $L_\varrho^{p(\cdot)}$ (see [17, Lemma 6.4]). Let L^∞ denote the space of all measurable essentially bounded functions on Γ. The operators of the form $aP + Q$ with $a \in L^\infty$ are called *singular integral operators* (SIOs). Two functions $a, b \in L^\infty$ are said to be locally equivalent at a point $t \in \Gamma$ if

$$\inf \left\{ \|(a - b)c\|_\infty \; : \; c \in C, \; c(t) = 1 \right\} = 0.$$

Theorem 3.1. *Suppose the conditions of Theorem 2.2 are satisfied and $a \in L^\infty$. Suppose for each $t \in \Gamma$ there is a function $a_t \in L^\infty$ which is locally equivalent to a at t. If the operators $a_t P + Q$ are Fredholm on $L_\varrho^{p(\cdot)}$ for all $t \in \Gamma$, then $aP + Q$ is Fredholm on $L_\varrho^{p(\cdot)}$.*

For weighted Lebesgue spaces this theorem is known as Simonenko's local principle [36]. It follows from [17, Theorem 6.13].

3.2. Simonenko's factorization theorem. The curve Γ divides the complex plane $\mathbb{C}$ into the bounded simply connected domain D^+ and the unbounded domain D^-. Without loss of generality we assume that $0 \in D^+$. We say that a function $a \in L^\infty$ admits a *Wiener-Hopf factorization on $L_\varrho^{p(\cdot)}$* if $1/a \in L^\infty$ and a can be written in the form

$$(7) \qquad\qquad a(t) = a_-(t) t^\kappa a_+(t) \quad \text{a.e. on } \Gamma,$$

where $\kappa \in \mathbb{Z}$, and the factors $a_\pm$ enjoy the following properties:

(i) $a_- \in QL_\varrho^{p(\cdot)} \dot{+} \mathbb{C}, \quad 1/a_- \in QL_{1/\varrho}^{q(\cdot)} \dot{+} \mathbb{C}, \quad a_+ \in PL_{1/\varrho}^{q(\cdot)}, \quad 1/a_+ \in PL_\varrho^{p(\cdot)},$

(ii) the operator $(1/a_+)Sa_+ I$ is bounded on $L_\varrho^{p(\cdot)}$.

One can prove that the number κ is uniquely determined.

Theorem 3.2. *Suppose the conditions of Theorem 2.2 are satisfied. A function $a \in L^\infty$ admits a Wiener-Hopf factorization (7) on $L_\varrho^{p(\cdot)}$ if and only if the operator $aP + Q$ is Fredholm on $L_\varrho^{p(\cdot)}$. If $aP + Q$ is Fredholm, then its index is equal to $-\kappa$.*

This theorem goes back to Simonenko [35, 37]. For more about this topic we refer to [2, Section 6.12], [4, Section 5.5], [9, Section 8.3] and also to [5, 26] in the case of weighted Lebesgue spaces. Theorem 3.2 follows from [17, Theorem 6.14].

3.3. Fredholm criterion for singular integral operators with PC coefficients. We denote by PC the Banach algebra of all piecewise continuous functions on Γ: a function $a \in L^\infty$ belongs to PC if and only if the finite one-sided limits

$$a(t \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)$$

exist for every $t \in \Gamma$.

Theorem 3.3. *Let* Γ *be a Carleson Jordan curve satisfying* (2) *with* $\delta(t) \in \mathbb{R}$ *for every* $t \in \Gamma$. *Suppose* $p \in \mathcal{P}$ *and* ϱ *is a power weight of the form* (4) *which satisfies* (6). *The operator* $aP + Q$, *where* $a \in PC$, *is Fredholm on the weighted Nakano space* $L_\varrho^{p(\cdot)}$ *if and only if* $a(t \pm 0) \neq 0$ *and*

$$(8) \qquad -\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| + \frac{1}{p(t)} + \lambda(t) \notin \mathbb{Z}$$

for all $t \in \Gamma$, *where*

$$\lambda(t) := \left\{ \begin{array}{ll} \lambda_k, & if \quad t = \tau_k, \quad k \in \{1, \ldots, N\}, \\ 0, & if \quad t \notin \Gamma \setminus \{\tau_1, \ldots, \tau_N\}. \end{array} \right.$$

Proof. The *necessity* part follows from [17, Theorem 8.1] because the Böttcher-Yu. Karlovich indicator functions α_t and β_t in that theorem for a Khvedelidze weight ϱ and a Carleson curve Γ satisfying the logarithmic whirl condition (2) at $t \in \Gamma$ are calculated by

$$\alpha_t(x) = \beta_t(x) = \lambda(t) + \delta(t)x \quad \text{for} \quad x \in \mathbb{R}$$

(see [2, Ch. 3] or [16, Lemma 3.9]).

Sufficiency. If $aP + Q$ is Fredholm, then, by [17, Theorem 6.11], $a(t \pm 0) \neq 0$ for all $t \in \Gamma$.

Fix $t \in \Gamma$. For the function a we construct a "canonical" function $g_{t,\gamma}$ which is locally equivalent to a at the point $t \in \Gamma$. The interior and the exterior of the unit circle can be conformally mapped onto D^+ and D^- of Γ, respectively, so that the point 1 is mapped to t, and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let Λ_0 and Λ_∞ denote the images of $[0,1]$ and $[1, \infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins 0 to ∞ and meets Γ at exactly one point, namely t. Let $\arg z$ be a continuous branch of argument in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbb{C}$, define the function $z^\gamma := |z|^\gamma e^{i\gamma \arg z}$, where $z \in \mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, z^γ is an analytic function in $\mathbb{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of z^γ to $\Gamma \setminus \{t\}$ will be denoted by $g_{t,\gamma}$. Obviously, $g_{t,\gamma}$ is continuous and nonzero on $\Gamma \setminus \{t\}$. Since $a(t \pm 0) \neq 0$, we can define $\gamma_t = \gamma \in \mathbb{C}$ by the formulas

$$\operatorname{Re} \gamma_t := \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)}, \quad \operatorname{Im} \gamma_t := -\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right|,$$

where we can take any value of $\arg(a(t-0)/a(t+0))$, which implies that any two choices of $\operatorname{Re} \gamma_t$ differ by an integer only. Clearly, there is a constant $c_t \in \mathbb{C} \setminus \{0\}$ such that $a(t \pm 0) = c_t g_{t,\gamma_t}(t \pm 0)$, which means that a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point $t \in \Gamma$. From (8) it follows that there exists an $m_t \in \mathbb{Z}$ such that

$$0 < m_t - \operatorname{Re} \gamma_t - \delta(t) \operatorname{Im} \gamma_t + \frac{1}{p(t)} + \lambda(t) < 1.$$

By Theorem 2.2, the operator S is bounded on $L_{\widetilde{\varrho}}^{p(\cdot)}$, where

$$\widetilde{\varrho}(\tau) = |\tau - t|^{m_t - \operatorname{Re} \gamma_t - \delta(t) \operatorname{Im} \gamma_t} \varrho(\tau)$$

for $\tau \in \Gamma$. In view of the logarithmic whirl condition (2) we have

$$\begin{aligned}
\left| (\tau - t)^{m_t - \gamma_t} \right| &= |\tau - t|^{m_t - \operatorname{Re} \gamma_t} e^{\operatorname{Im} \gamma_t \arg(\tau - t)} \\
&= |\tau - t|^{m_t - \operatorname{Re} \gamma_t} e^{-\operatorname{Im} \gamma_t (\delta(t) \log |\tau - t| + O(1))} \\
&= |\tau - t|^{m_t - \operatorname{Re} \gamma_t - \delta(t) \operatorname{Im} \gamma_t} e^{-\operatorname{Im} \gamma_t O(1)}
\end{aligned}$$

as $\tau \to t$. Therefore the operator $\varphi_{t,m_t-\gamma_t} S \varphi_{t,\gamma_t-m_t} I$, where

$$\varphi_{t,m_t-\gamma_t}(\tau) = |(\tau - t)^{m_t-\gamma_t}|,$$

is bounded on $L_{\varrho}^{p(\cdot)}$. Then, by [17, Lemma 7.1], the function g_{t,γ_t} admits a Wiener-Hopf factorization on $L_{\varrho}^{p(\cdot)}$. Due to Theorem 3.2, the operator $g_{t,\gamma_t} P + Q$ is Fredholm. Then the operator $c_t g_{t,\gamma_t} P + Q$ is Fredholm, too. Since the function $c_t g_{t,\gamma_t}$ is locally equivalent to the function a at every point $t \in \Gamma$, in view of Theorem 3.1, the operator $aP + Q$ is Fredholm on $L_{\varrho}^{p(\cdot)}$.   □

3.4. Double logarithmic spirals.

Given $z_1, z_2 \in \mathbb{C}$, $\delta \in \mathbb{R}$, and $r \in (0,1)$, put

$$\mathcal{S}(z_1, z_2; \delta, r) := \{z_1, z_2\} \cup \left\{ z \in \mathbb{C} \backslash \{z_1, z_2\} : \arg \frac{z - z_1}{z - z_2} - \delta \log \left| \frac{z - z_1}{z - z_2} \right| \in 2\pi(r + \mathbb{Z}) \right\}.$$

The set $\mathcal{S}(z_1, z_2; \delta, r)$ is a double logarithmic spiral whirling about the points z_1 and z_2. It degenerates to a familiar Widom-Gohberg-Krupnik circular arc whenever $\delta = 0$ (see [2, 9]).

Fix $t \in \Gamma$ and consider a function $\chi_t \in PC$ which is continuous on $\Gamma \backslash \{t\}$ and satisfies $\chi_t(t - 0) = 0$ and $\chi_t(t + 0) = 1$.

From Theorem 3.3 we get the following.

Corollary 3.4. *Let Γ be a Carleson Jordan curve satisfying* (2) *with $\delta(t) \in \mathbb{R}$ for every $t \in \Gamma$. Suppose $p \in \mathcal{P}$ and ϱ is a power weight of the form* (4) *which satisfies* (6). *Then*

$$\left\{ \lambda \in \mathbb{C} : (\chi_t - \lambda)P + Q \text{ is not Fredholm on } L_{\varrho}^{p(\cdot)} \right\} = \mathcal{S}\big(0, 1; \delta(t), 1/p(t) + \lambda(t)\big).$$

4. Tools for the construction of the symbol calculus

4.1. The Allan-Douglas local principle.

Let B be a Banach algebra with identity. A subalgebra Z of B is said to be a central subalgebra if $zb = bz$ for all $z \in Z$ and all $b \in B$.

Theorem 4.1. (see [4, Theorem 1.34(a)]). *Let B be a Banach algebra with unit e and let Z be closed central subalgebra of B containing e. Let $M(Z)$ be the maximal ideal space of Z, and for $\omega \in M(Z)$, let J_ω refer to the smallest closed two-sided ideal of B containing the ideal ω. Then an element b is invertible in B if and only if $b + J_\omega$ is invertible in the quotient algebra B/J_ω for all $\omega \in M(Z)$.*

4.2. The two projections theorem.

The following two projections theorem was obtained by Finck, Roch, Silbermann [6] and Gohberg, Krupnik [10].

Theorem 4.2. *Let F be a Banach algebra with identity e, let $\mathcal{C}$ be a Banach subalgebra of F which contains e and is isomorphic to $\mathbb{C}^{n \times n}$, and let p and q be two projections in F such that $cp = pc$ and $cq = qc$ for all $c \in \mathcal{C}$. Let $W = \mathrm{alg}(\mathcal{C}, p, q)$ be the smallest closed subalgebra of F containing $\mathcal{C}, p, q$. Put*

$$x = pqp + (e - p)(e - q)(e - p),$$

denote by $\mathrm{sp}\, x$ the spectrum of x in F, and suppose the points 0 and 1 are not isolated points of $\mathrm{sp}\, x$. Then

(a) *for each $\mu \in \mathrm{sp}\, x$ the map σ_μ of $\mathcal{C} \cup \{p, q\}$ into the algebra $\mathbb{C}^{2n \times 2n}$ of all complex $2n \times 2n$ matrices defined by*

$$(9) \qquad\qquad \sigma_\mu c = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \sigma_\mu p = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

$$(10) \qquad \sigma_\mu q = \begin{pmatrix} \mu E & \sqrt{\mu(1-\mu)}E \\ \sqrt{\mu(1-\mu)}E & (1-\mu)E \end{pmatrix},$$

where $c \in C$, E denotes the $n \times n$ unit matrix and $\sqrt{\mu(1-\mu)}$ denotes any complex number whose square is $\mu(1-\mu)$, extends to a Banach algebra homomorphism

$$\sigma_\mu : W \to \mathbb{C}^{2n \times 2n};$$

(b) every element a of the algebra W is invertible in the algebra F if and only if

$$\det \sigma_\mu a \neq 0 \quad for \ all \quad \mu \in \mathrm{sp}\, x;$$

(c) the algebra W is inverse closed in F if and only if the spectrum of x in W coincides with the spectrum of x in F.

A further generalization of the above result to the case of N projections is contained in [2].

5. Algebra of singular integral operators with PC coefficients

5.1. The ideal of compact operators. In this section we will suppose that Γ is a Carleson curve satisfying (2) with $\delta(t) \in \mathbb{R}$ for every $t \in \Gamma$, $p \in \mathcal{P}$, and ϱ is a Khvedelidze weight of the form (4) which satisfies (6). Let $X_n := [L_\varrho^{p(\cdot)}]_n$ be the direct sum of n copies of weighted Nakano spaces $X := L_\varrho^{p(\cdot)}$, let $\mathcal{B} := \mathcal{B}(X_n)$ be the Banach algebra of all bounded linear operators on X_n, and let $\mathcal{K} := \mathcal{K}(X_n)$ be the closed two-sided ideal of all compact operators on X_n. We denote by $C^{n \times n}$ (resp. $PC^{n \times n}$) the collection of all continuous (resp. piecewise continuous) $n \times n$ matrix functions, that is, matrix-valued functions with entries in C (resp. PC). Put $I^{(n)} := \mathrm{diag}\{I, \dots, I\}$ and $S^{(n)} := \mathrm{diag}\{S, \dots, S\}$. Our aim is to get a Fredholm criterion for an operator

$$A \in \mathcal{U} := \mathrm{alg}(PC^{n \times n}, S^{(n)}),$$

the smallest Banach subalgebra of $\mathcal{B}$ which contains all operators of multiplication by matrix-valued functions in $PC^{n \times n}$ and the operator $S^{(n)}$.

Lemma 5.1. *The ideal $\mathcal{K}$ is contained in the algebra $\mathrm{alg}(C^{n \times n}, S^{(n)})$, the smallest closed subalgebra of $\mathcal{B}$ which contains the operators of multiplication by continuous matrix-valued functions and the operator $S^{(n)}$.*

The proof of this statement is standard and can be developed as in [13, Lemma 9.1] or [18, Lemma 5.1].

5.2. Operators of local type. We shall denote by $\mathcal{B}^\pi$ the Calkin algebra $\mathcal{B}/\mathcal{K}$ and by A^π the coset $A + \mathcal{K}$ for any operator $A \in \mathcal{B}$. An operator $A \in \mathcal{B}$ is said to be of *local type* if $AcI^{(n)} - cA$ is compact for all $c \in C$, where $cI^{(n)}$ denotes the operator of multiplication by the diagonal matrix-valued function $\mathrm{diag}\{c, \dots, c\}$. This notion goes back to Simonenko [36] (see also the presentation of Simonenko's local theory in his joint monograph with Chin Ngok Min [38]). It easy to see that the set $\mathcal{L}$ of all operators of local type is a closed subalgebra of $\mathcal{B}$.

Proposition 5.2.   (a) *We have $\mathcal{K} \subset \mathcal{U} \subset \mathcal{L}$.*

(b) *An operator $A \in \mathcal{L}$ is Fredholm if and only if the coset A^π is invertible in the quotient algebra $\mathcal{L}^\pi := \mathcal{L}/\mathcal{K}$.*

Proof. (a) The embedding $\mathcal{K} \subset \mathcal{U}$ follows from Lemma 5.1, the embedding $\mathcal{U} \subset \mathcal{L}$ follows from the fact that $cS - ScI$ is a compact operator on $L_{\varrho}^{p(\cdot)}$ for $c \in C$ (see, e.g., [17, Lemma 6.5]).

(b) The proof of this fact is straightforward. $\square$

5.3. Localization.
From Proposition 5.2(a) we deduce that the quotient algebras $\mathcal{U}^{\pi} := \mathcal{U}/\mathcal{K}$ and $\mathcal{L}^{\pi} := \mathcal{L}/\mathcal{K}$ are well defined. We shall study the invertibility of an element A^{π} of $\mathcal{U}^{\pi}$ in the larger algebra $\mathcal{L}^{\pi}$ by using a localization techniques (more precisely, Theorem 4.1). To this end, consider

$$\mathcal{Z}^{\pi} := \{(cI^{(n)})^{\pi} : c \in C\}.$$

From the definition of $\mathcal{L}$ it follows that $\mathcal{Z}^{\pi}$ is a central subalgebra of $\mathcal{L}^{\pi}$. The maximal ideal space $M(\mathcal{Z}^{\pi})$ of $\mathcal{Z}^{\pi}$ may be identified with the curve Γ via the Gelfand map $\mathcal{G}$ given by

$$\mathcal{G} : \mathcal{Z}^{\pi} \to C, \quad (\mathcal{G}(cI^{(n)})^{\pi})(t) = c(t) \quad (t \in \Gamma).$$

In accordance with Theorem 4.1, for every $t \in \Gamma$ we define $\mathcal{J}_t \subset \mathcal{L}^{\pi}$ as the smallest closed two-sided ideal of $\mathcal{L}^{\pi}$ containing the set

$$\{(cI^{(n)})^{\pi} : c \in C, \ c(t) = 0\}.$$

Consider a function $\chi_t \in PC$ which is continuous on $\Gamma \setminus \{t\}$ and satisfies $\chi_t(t-0) = 0$ and $\chi_t(t+0) = 1$. For $a \in PC^{n \times n}$ define the function $a_t \in PC^{n \times n}$ by

$$(11) \qquad a_t := a(t-0)(1-\chi_t) + a(t+0)\chi_t.$$

Clearly $(aI^{(n)})^{\pi} - (a_t I^{(n)})^{\pi} \in \mathcal{J}_t$. Hence, for any operator $A \in \mathcal{U}$, the coset $A^{\pi} + \mathcal{J}_t$ belongs to the smallest closed subalgebra $\mathcal{W}_t$ of $\mathcal{L}^{\pi}/\mathcal{J}_t$ containing the cosets

$$(12) \qquad p := ((I^{(n)} + S^{(n)})/2)^{\pi} + \mathcal{J}_t, \ q := (\chi_t I^{(n)})^{\pi} + \mathcal{J}_t,$$

where $\chi_t I^{(n)}$ denotes the operator of multiplication by the diagonal matrix-valued function $\mathrm{diag}\{\chi_t, \dots, \chi_t\}$ and the algebra

$$(13) \qquad \mathcal{C} := \{(cI^{(n)})^{\pi} + \mathcal{J}_t : c \in \mathbb{C}^{n \times n}\}.$$

The latter algebra is obviously isomorphic to $\mathbb{C}^{n \times n}$, so $\mathcal{C}$ and $\mathbb{C}^{n \times n}$ can be identified with each other.

5.4. The spectrum of $pqp + (e-p)(e-q)(e-p)$.
Since $P^2 = P$ on $L_{\varrho}^{p(\cdot)}$ (see, e.g., [17, Lemma 6.4]) and $\chi_t^2 - \chi_t \in C$, $(\chi_t^2 - \chi_t)(t) = 0$, it is easy to see that

$$(14) \qquad p^2 = p, \quad q^2 = q, \quad pc = cp, \quad qc = cq$$

for every $c \in \mathcal{C}$, where p, q and $\mathcal{C}$ are given by (12) and (13). To apply Theorem 4.2 to the algebras $F = \mathcal{L}^{\pi}/\mathcal{J}_t$ and $W = \mathcal{W}_t = \mathrm{alg}(\mathcal{C}, p, q)$, we have to identify the spectrum of

$$(15) \qquad pqp + (e-p)(e-q)(e-p) = (P^{(n)}\chi_t P^{(n)} + Q^{(n)}(1-\chi_t)Q^{(n)})^{\pi} + \mathcal{J}_t$$

in the algebra $F = \mathcal{L}^{\pi}/\mathcal{J}_{\tau}$; here $P^{(n)} := (I^{(n)} + S^{(n)})/2$ and $Q^{(n)} := (I^{(n)} - S^{(n)})/2$.

Lemma 5.3. *Let $\chi_t \in PC$ be a continuous function on $\Gamma \setminus \{t\}$ such that*

$$\chi_t(t-0) = 0, \quad \chi_t(\tau+0) = 1,$$

and

$$\chi_t(\Gamma \setminus \{t\}) \cap \mathcal{S}(0, 1; \delta(t), 1/p(t) + \lambda(t)) = \emptyset.$$

Then the spectrum of (15) *in* $\mathcal{L}^\pi / \mathcal{J}_t$ *coincides with* $\mathcal{S}\big(0, 1; \delta(t), 1/p(t) + \lambda(t)\big)$.

Proof. Once we have Corollary 3.4 at hand, the proof of this lemma can be developed by a literal repetition of the proof of [13, Lemma 9.4]. □

5.5. **Symbol calculus.** Now we are in a position to prove the main result of this paper.

Theorem 5.4. *Define the "double logarithmic spirals bundle"*

$$\mathcal{M} := \bigcup_{t \in \Gamma} \Big(\{t\} \times \mathcal{S}\big(0, 1; \delta(t), 1/p(t) + \lambda(t)\big) \Big).$$

(a) *For each point* $(t, \mu) \in \mathcal{M}$, *the map*

$$\sigma_{t,\mu} : \{S^{(n)}\} \cup \{aI^{(n)} : a \in PC^{n \times n}\} \to \mathbb{C}^{2n \times 2n}$$

given by

$$\sigma_{t,\mu}(S^{(n)}) = \begin{pmatrix} E & O \\ O & -E \end{pmatrix}, \quad \sigma_{t,\mu}(aI^{(n)}) = \begin{pmatrix} a_{11}(t, \mu) & a_{12}(t, \mu) \\ a_{21}(t, \mu) & a_{22}(t, \mu) \end{pmatrix},$$

where

$$\begin{aligned}
a_{11}(t, \mu) &:= a(t+0)\mu + a(t-0)(1 - \mu), \\
a_{12}(t, \mu) &= a_{21}(t, \mu) := (a(t+0) - a(t-0))\sqrt{\mu(1 - \mu)}, \\
a_{22}(t, \mu) &:= a(t+0)(1 - \mu) + a(t-0)\mu,
\end{aligned}$$

and O *and* E *are the zero and identity* $n \times n$ *matrices, respectively, extends to a Banach algebra homomorphism*

$$\sigma_{t,\mu} : \mathcal{U} \to \mathbb{C}^{2n \times 2n}$$

with the property that $\sigma_{t,\mu}(K)$ *is the zero matrix for every compact operator* K *on* X_n;

(b) *an operator* $A \in \mathcal{U}$ *is Fredholm on* X_n *if and only if*

$$\det \sigma_{t,\mu}(A) \neq 0 \quad \text{for all} \quad (t, \mu) \in \mathcal{M};$$

(c) *the quotient algebra* $\mathcal{U}^\pi$ *is inverse closed in the Calkin algebra* $\mathcal{B}^\pi$, *that is, if a coset* $A^\pi \in \mathcal{U}^\pi$ *is invertible in* $\mathcal{B}^\pi$, *then* $(A^\pi)^{-1} \in \mathcal{U}^\pi$.

Proof. The idea of the proof of this theorem is borrowed from [2] and is based on the Allan-Douglas local principle (Theorem 4.1) and the two projections theorem (Theorem 4.2).

Fix $t \in \Gamma$ and choose a function $\chi_t \in PC$ such that χ_t is continuous on $\Gamma \setminus \{t\}$, $\chi_t(t-0) = 0$, $\chi_t(t+0) = 1$, and $\chi_t(\Gamma \setminus \{t\}) \cap \mathcal{S}(0, 1; \delta(t), 1/p(t) + \lambda(t)) = \emptyset$. From (14) and Lemma 5.3 we deduce that the algebras $\mathcal{L}^\pi / \mathcal{J}_t$ and $\mathcal{W}_t = \mathrm{alg}(\mathcal{C}, p, q)$, where p, q and $\mathcal{C}$ are given by (12) and (13), respectively, satisfy all the conditions of the two projections theorem (Theorem 4.2).

(a) In view of Theorem 4.2(a), for every $\mu \in \mathcal{S}(0, 1; \delta(t), 1/p(t) + \lambda(t))$, the map $\sigma_\mu : \mathbb{C}^{n \times n} \cup \{p, q\} \to \mathbb{C}^{2n \times 2n}$ given by (9)–(10) extends to a Banach algebra homomorphism $\sigma_\mu : \mathcal{W}_t \to \mathbb{C}^{2n \times 2n}$. Then the map

$$\sigma_{t,\mu} = \sigma_\mu \circ \pi_t : \mathcal{U} \to \mathbb{C}^{2n \times 2n},$$

where $\pi_t : \mathcal{U} \to \mathcal{W}_t = \mathcal{U}^\pi / \mathcal{J}_t$ is acting by the rule $A \mapsto A^\pi + \mathcal{J}_t$, is a well defined Banach algebra homomorphism and

$$\sigma_{t,\mu}(S^{(n)}) = 2\sigma_\mu p - \sigma_\mu e = \begin{pmatrix} E & O \\ O & -E \end{pmatrix}.$$

If $a \in PC^{n \times n}$, then in view of (11) and $(aI^{(n)})^\pi - (a_t I^{(n)})^\pi \in \mathcal{J}_t$ it follows that

$$\begin{aligned}
\sigma_{t,\mu}(aI^{(n)}) &= \sigma_{t,\mu}(a_t I^{(n)}) = \sigma_\mu(a(t-0))\sigma_\mu(e-q) + \sigma_\mu(a(t+0))\sigma_\mu q \\
&= \begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{pmatrix}.
\end{aligned}$$

From Proposition 5.2(a) it follows that $\pi_t(K) = K^\pi + \mathcal{J}_t = \mathcal{J}_t$ for every $K \in \mathcal{K}$ and every $t \in \Gamma$. Hence,

$$\sigma_{t,\mu}(K) = \sigma_\mu(0) = \begin{pmatrix} O & O \\ O & O \end{pmatrix}.$$

Part (a) is proved.

(b) From Proposition 5.2 it follows that the Fredholmness of $A \in \mathcal{U}$ is equivalent to the invertibility of $A^\pi \in \mathcal{L}^\pi$. By Theorem 4.1, the former is equivalent to the invertibility of $\pi_t(A) = A^\pi + \mathcal{J}_t$ in $\mathcal{L}^\pi / \mathcal{J}_t$ for every $t \in \Gamma$. By Theorem 4.2(b), this is equivalent to

(16) $$\det \sigma_{t,\mu}(A) = \det \sigma_\mu \pi_t(A) \neq 0 \text{ for all } (t,\mu) \in \mathcal{M}.$$

Part (b) is proved.

(c) Since $\mathcal{S}(0, 1; \delta(t), 1/p(t) + \lambda(t))$ does not separate the complex plane $\mathbb{C}$, it follows that the spectra of (15) in the algebras $\mathcal{L}^\pi / \mathcal{J}_t$ and $\mathcal{W}_t = \mathcal{U}^\pi / \mathcal{J}_t$ coincide, so we can apply Theorem 4.2(c). If A^π, where $A \in \mathcal{U}$, is invertible in $\mathcal{B}^\pi$, then (16) holds. Consequently, by Theorem 4.2(b), (c), $\pi_t(A) = A^\pi + \mathcal{J}_t$ is invertible in $\mathcal{W}_t = \mathcal{U}^\pi / \mathcal{J}_t$ for every $t \in \Gamma$. Applying Theorem 4.1 to $\mathcal{U}^\pi$, its central subalgebra $\mathcal{Z}^\pi$, and the ideals $\mathcal{J}_t$, we obtain that A^π is invertible in $\mathcal{U}^\pi$, that is, $\mathcal{U}^\pi$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi$. $\square$

Note that the approach to the study of Banach algebras of SIOs based on the Allan-Douglas local principle and the two projections theorem is nowadays standard. It was successfully applied in many situations (see, e.g., [1, 2, 6, 13, 14, 16, 18]). However, it does not allow to get formulas for the index of an arbitrary operator in the Banach algebra of SIOs with PC coefficients. These formulas can be obtained similarly to the classical situation considered by Gohberg and Krupnik [7, 8] (see also [2, Ch. 10]). For reflexive Orlicz spaces over Carleson curves with logarithmic whirl points this was done by the author [15]. In the case of Nakano spaces with Khvedelidze weights over Carleson curves with logarithmic whirl points the index formulas are almost the same as in [15]. It is only necessary to replace the both Boyd indices α_M and β_M of an Orlicz space L^M by the numbers $1/p(t) + \lambda(t)$ in corresponding formulas.

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Universidade do Minho, Centro de Matemática, Escola de Ciências, Campus de Gualtar, 4710-057 Braga, Portugal

E-mail address: oleksiy@math.uminho.pt