Characterizations of m-EP elements in rings

Honglin Zou*, Jianlong Chen†, Pedro Patrício‡
*School of Mathematics and Statistics, Hubei Normal University, Huangshi 435002, China
†School of Mathematics, Southeast University, Nanjing 210096, China
‡CMAT-Centro de Matemática and Departamento de Matemática e Aplicacões, Universidade do Minho, Braga 4710-057, Portugal

Abstract: Let R be a ring with involution. In this paper, we extend the notions of m-EP matrices and m-EP operators to an arbitrary ring case. A number of new characterizations of m-EP elements in rings are presented. In particular, the existence criteria for 1-EP (i.e. EP) elements are obtained by means of the group inverse, Moore-Penrose inverse, and core inverse. Some properties of 2-EP are also given.

Keywords: m-EP; Drazin inverse; Moore-Penrose inverse; Core inverse; Ring

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1 Introduction

Characterizations of EP matrices, EP linear operators on Banach or Hilbert spaces, and EP elements in rings with involution have been investigated by many authors [1–9]. In 2016, Malik, Rueda and Thome [10] introduced the notion of m-EP matrices, generalizing the notion of EP matrices. The equivalent conditions and the properties of m-EP matrices were obtained. In addition, they took advantage of Hartwig-Spindelböck decomposition to give the representation of Drazin inverse. Later, Wang and Deng [11] gave the definition of m-EP operators. They studied the characteristics of m-EP operators and the properties of the particular case of m-EP operators that are Drazin invertible in terms of the operator matrix decomposition.

The article is motivated by the papers [8, 10, 11]. We give the notion of m-EP elements in the context of rings with involution. Several new characterizations of m-EP elements are obtained. Also, we study the necessary and sufficient conditions for 1-EP applying the group inverse, Moore-Penrose inverse, and core inverse.

*E-mail: honglinzou@163.com
†Corresponding author. E-mail: jlchen@seu.edu.cn
‡E-mail: pedro@math.uminho.pt

2 Preliminary definitions and notations

Throughout this paper, R denotes a unital $*$-ring, that is, a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^* a^*$ for all $a, b \in R$.

For the readers' convenience, we first recall the definitions of some generalized inverses. An element $a \in R$ is said to be Moore-Penrose invertible with respect to the involution $*$ if the following equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax \quad \text{and} \quad (xa)^* = xa$$

have a common solution [12]. Such solution is unique if it exists, and is usually denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of R will be denoted by $R^\dagger$.

The Drazin inverse [13] of $a \in R$ is the element $x \in R$ which satisfies

$$a^k = a^{k+1}x, \quad xax = x \quad \text{and} \quad ax = xa, \quad \text{for some} \quad k \geq 1.$$

The element x above is unique if it exists and is denoted by a^D. The least such k is called the index of a, and denoted by ind(a). In particular, when ind(a)=1, the Drazin inverse a^D is called the group inverse of a and it is denoted by $a^\#$. The set of all Drazin (resp. group) invertible elements of R will be denoted by R^D (resp. $R^\#$).

Baksalary and Trenkler [14] introduced the core inverse for complex matrices and it was extended to the ring case by Rakić, Dinčić and Djordjević [15]. The core inverse [15] of $a \in R$ is the element $x \in R$ which satisfies

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a \quad \text{and} \quad ax^2 = x.$$

The element x above is unique if it exists and is denoted by $a^{\oplus}$. The set of all core invertible elements of R will be denoted by $R^{\oplus}$.

An element $a \in R$ is said to be EP if $a \in R^\# \cap R^\dagger$ and $a^\# = a^\dagger$. An element $a \in R$ satisfying $a^* = a$ is called Hermitian. By the analogy with complex matrices, an element $a \in R^\dagger$ is called bi-EP, bi-dagger, star-dagger if $(aa^\dagger)(a^\dagger a) = (a^\dagger a)(aa^\dagger)$, $(a^2)^\dagger = (a^\dagger)^2$, $a^\dagger a^* = a^* a^\dagger$, respectively.

Let $\mathbb{N}$ denote the set of all positive integers. By R^{nil} we denote the set of all nilpotent elements in R. Also, $M_n(R)$ and $\mathbb{C}^{n\times n}$ stand for the ring of $n \times n$ matrices over R and the algebra of $n \times n$ complex matrices, respectively.

3 m-EP elements

In this section, we will give several characterizations for m-EP elements by means of the Moore-Penrose inverse and Drazin inverse in rings. In [10], Malik, Rueda and Thome introduced the notion of m-EP matrices as follows:

Definition 3.1. [10, Definition 2.1] *A matrix $A \in \mathbb{C}^{n\times n}$ is called m-EP if it satisfies*

$$A^\dagger A^m = A^m A^\dagger,$$

where m is the index of A.

Note that, any $A \in \mathbb{C}^{n\times n}$ has the Moore-Penrose inverse. However, in general, it does not hold in the ring case. Therefore, it is reasonable to add the hypothesis $a \in R^\dagger$ in the following definition. We will see that the Definition 3.2 coincides with the Definition 3.1 in the case $R = \mathbb{C}^{n\times n}$.

Definition 3.2. *An element $a \in R$ with involution is said to be m-EP if $a \in R^\dagger$ and m is the smallest positive integer such that $a^\dagger a^m = a^m a^\dagger$.*

Example 3.3. *Let $\mathbb{Z}_7$ be the ring of integers modulo 7. Take $R = M_5(\mathbb{Z}_7)$ with the transpose of matrices as involution. Setting*

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $a^\dagger = a^*$. We can check that $aa^\dagger \neq a^\dagger a$, $a^2 a^\dagger \neq a^\dagger a^2$ and $a^3 a^\dagger = a^\dagger a^3$. Hence, a is 3-EP.

Remark 3.4. (1) *Clearly, we have that $a \in R$ is 1-EP if and only if a is EP.*

(2) *Every element $a \in R^\dagger \cap R^{nil}$ with the nilpotence index m is m-EP. Indeed, take any integer $k \leq m-1$, assume that $a^\dagger a^k = a^k a^\dagger$, then $a^k = a^k a^\dagger a = a^\dagger a^{k+1}$, which implies that $a^k = a^\dagger a^k a = a^\dagger(a^\dagger a^{k+1})a = (a^\dagger)^2 a^{k+2} = \cdots = (a^\dagger)^{m-k} a^m = 0$, which contradicts with the nilpotence index m.*

(3) *An element $a \in R$ is m-EP if and only if a^* is m-EP.*

Lemma 3.5. [13, Theorem 4] *Let $a, x, y \in R$ be such that $a^m = xa^{m+1} = a^{m+1}y$, then $a \in R^D$ and $a^D = x^{m+1}a^m = a^m y^{m+1}$. Moreover, $\mathrm{ind}(a) \leq m$.*

Lemma 3.6. *Let $a \in R$. Then the following are equivalent:*

(1) *a is m-EP;*

(2) *$a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$, and $a^\dagger a^m = a^m a^\dagger$.*

In this case, $a^D = a^m (a^\dagger)^{m+1} = (a^\dagger)^{m+1} a^m$.

(1) $\Rightarrow$ (2) Suppose that a is m-EP. Pre-multiplication and post-multiplication of the equation $a^\dagger a^m = a^m a^\dagger$ by a, respectively, we can obtain $a^m = a^{m+1} a^\dagger$ and $a^m = a^\dagger a^{m+1}$, which imply $a \in R^D$ with $\mathrm{ind}(a) \leq m$, and $a^D = a^m (a^\dagger)^{m+1} = (a^\dagger)^{m+1} a^m$ by Lemma 3.5.

Now, assume that $\mathrm{ind}(a) \leq m-1$, then $a^{m-1} = a^m a^D$. Thus, we get the following equation:

$$a^{m-1}a^{\dagger} = a^D a^m a^{\dagger} = a^D a^{\dagger} a^m = (a^D)^2 a a^{\dagger} a^m = (a^D)^2 a^m = a^D a^{m-1}.$$

Similarly, $a^{\dagger}a^{m-1} = a^{m-1}a^D$ holds. Therefore, $a^{\dagger}a^{m-1} = a^{m-1}a^{\dagger}$, which is an contradiction, since m is the smallest positive integer such that $a^{\dagger}a^m = a^m a^{\dagger}$.

(2) $\Rightarrow$ (1) According to Definition 3.2, we only need to prove that m is the smallest positive integer such that $a^{\dagger}a^m = a^m a^{\dagger}$.

Take any integer $k \leq m-1$, assume that $a^{\dagger}a^k = a^k a^{\dagger}$, then $a^k = a^{k+1}a^{\dagger} = a^{\dagger}a^{k+1}$. So, by Lemma 3.5 we have $\mathrm{ind}(a) \leq k \leq m-1$, which contradicts with $\mathrm{ind}(a) = m$. Thus, a is m-EP. □

Remark 3.7. *By Lemma 3.6, we can see that the smallest positive integer m in Definition 3.2 is the index of a. Thus, Definition 3.2 coincides with Definition 3.1 for complex matrices.*

Next, we give some auxiliary lemmas, which we will rely on.

Lemma 3.8. (1) [13, Theorem 1] *Let $a, b \in R$ such that b is group invertible. Then $ab = ba$ if and only if $ab^{\#} = b^{\#}a$.*

(2) [13, Corollary 3, Theorem 2] *Let $a \in R^D$ and $k \geq \mathrm{ind}(a)$. Then $a^k \in R^{\#}$, and $(a^k)^{\#} = (a^k)^D = (a^D)^k$.*

(3) [13, Theorem 3] *Let $a \in R^D$. Then $a^D \in R^{\#}$ and $(a^D)^{\#} = a^2 a^D$.*

(4) *Let $a \in R^{\#}$ and $k \in \mathbb{N}$. Then $a^k \in R^{\#}$ and $(a^k)^{\#} = (a^{\#})^k$.*

(5) *Let $a, b \in R$ such that $aba = a$ and $ab = ba$. Then $a \in R^{\#}$ and $a^{\#} = bab$.*

Proof. (4) and (5) can be obtained by the definition of the group inverse. □

Lemma 3.9. (1) [16, Theorem 7.3] *Let $a \in R$. Then a is EP if and only if $a \in R^{\#}$ and $aa^{\#}$ is Hermitian.*

(2) [9, Lemma 1.1] *Let $a \in R$. Then a is EP if and only if $a \in R^{\dagger}$ and $aa^{\dagger} = a^{\dagger}a$.*

In order to prove our main results, we need the following crucial auxiliary propositions.

Proposition 3.10. *Let $a \in R^D \cap R^{\dagger}$ with $\mathrm{ind}(a) = m$. Then the following are equivalent:*

(1) $a^D = a^m (a^{\dagger})^{m+1}$;

(2) $a^m a^{\dagger} = a^m a^D$;

(3) $a^m = a^{m+1}a^{\dagger}$.

Proof. (1) $\Rightarrow$ (2) Since $a \in R^D$ with ind(a)=m, so, $a^m \in R^{\#}$ and $(a^m)^{\#} = (a^m)^D = (a^D)^m$ by Lemma 3.8(2). Note that the equations $aa^D = a^{m+1}(a^{\dagger})^{m+1}$ and $a^D a = a^m (a^{\dagger})^{m+1} a$, which imply $a^{m+1}(a^\dagger)^{m+1} = a^m(a^\dagger)^{m+1}a$. Thus, we have

$$\begin{aligned}
a^m a^\dagger &= a^m(a^m)^\# a^m a^\dagger = a^m(a^D)^m a^m a^\dagger = a^m(a^D)^{m-1}a^D a^m a^\dagger \\
&= a^m(a^D)^{m-1}(a^m(a^\dagger)^{m+1}a)a^{m-1}a^\dagger \\
&= a^m(a^D)^{m-1}a^{m+1}(a^\dagger)^{m+1}a^{m-1}a^\dagger \\
&= a^m(a^D)^{m-1}a(a^m(a^\dagger)^{m+1}a)a^{m-2}a^\dagger \\
&= a^m(a^D)^{m-1}a^{m+2}(a^\dagger)^{m+1}a^{m-2}a^\dagger \\
&\vdots \\
&= a^m(a^D)^{m-1}a^{2m-1}(a^\dagger)^{m+1}aa^\dagger \\
&= a^m((a^D)^{m-1}a^{m-1})(a^m(a^\dagger)^{m+1}) \\
&= a^m a^D a a^D \\
&= a^m a^D.
\end{aligned}$$

(2) $\Rightarrow$ (3) Pre-multiplying $a^m a^\dagger = a^m a^D$ by a, we have $a^{m+1}a^\dagger = a^{m+1}a^D = a^m$, by the definition of a^D.

(3) $\Rightarrow$ (1) Since $a^m = a^{m+1}a^\dagger$ and $a^m = a^D a^{m+1}$, applying Lemma 3.5, we conclude $a^D = a^m(a^\dagger)^{m+1}$. $\square$

Remark 3.11. (1) *By Proposition 3.10, we can easily obtain the result: $a \in R^\# \cap R^\dagger$ and $a^\# = a(a^\dagger)^2$ if and only if a is EP.*

(2) *If $a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$, $a^D = a^m(a^\dagger)^{m+1}$, and $a^m a^\dagger$ is Hermitian, then a is m-EP. Indeed, note that $a^m a^\dagger = a^m a^D$ by Proposition 3.10, so $a^m a^D$ is Hermitian, then we get*

$$a^\dagger a^m = a^\dagger a a^m a^D = (a^\dagger a)^*(a^m a^D)^* = (a^m a^D a^\dagger a)^* = a^m a^D = a^m a^\dagger.$$

Dually, we have the following result.

Proposition 3.12. *Let $a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$. Then the following are equivalent:*

(1) $a^D = (a^\dagger)^{m+1}a^m$;

(2) $a^\dagger a^m = a^D a^m$;

(3) $a^m = a^\dagger a^{m+1}$.

Combining Lemma 3.6, Proposition 3.10 and Proposition 3.12, we can directly obtain the following result, which is the first characterization for m-EP elements.

Theorem 3.13. *Let $a \in R$. Then the following are equivalent:*

(1) *a is m-EP;*

(2) *$a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$, and $a^D = a^m(a^\dagger)^{m+1} = (a^\dagger)^{m+1}a^m$.*

Corollary 3.14. *Let $a \in R$ be m-EP, and $a^{m+1} \in R^\dagger$ with $(a^{m+1})^\dagger = (a^\dagger)^{m+1}$. Then*

(1) *a^D is EP;*

(2) *a^k is EP, where $k \geq m$.*

Proof. (1) From Theorem 3.13, we obtain $a \in R^D$ with $\mathrm{ind}(a) = m$, and $a^D = a^m(a^\dagger)^{m+1}$. Note that $a^D \in R^\#$ and $(a^D)^\# = a^2a^D$ by Lemma 3.8(3). Thus, we deduce that $a^D(a^D)^\# = a^D(a^2a^D) = aa^D = a(a^m(a^\dagger)^{m+1}) = a^{m+1}(a^\dagger)^{m+1} = a^{m+1}(a^{m+1})^\dagger$ is Hermitian, which gives that a^D is EP by Lemma 3.9(1).

(2) By Lemma 3.8(2), it follows that $a^k \in R^\#$ and $(a^k)^\# = (a^k)^D = (a^D)^k$. Then, we have that $a^k(a^k)^\# = a^k(a^D)^k = aa^D$. From the proof of (1), we have that aa^D is Hermitian. So, $a^k(a^k)^\#$ is Hermitian. Therefore, a^k is EP by Lemma 3.9(1). □

Now, we present an existence criterion for m-EP elements by means of the commutativity of the Moore-Penrose inverse and Drazin inverse.

Theorem 3.15. *Let $a \in R$. Then the following are equivalent:*

(1) *a is m-EP;*

(2) *$a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$, and $a^\dagger a^D = a^D a^\dagger$.*

Proof. (1) ⇒ (2) By Theorem 3.13, we have

$$aa^Da^\dagger = a^Daa^\dagger = a^m(a^\dagger)^{m+1}aa^\dagger = a^m(a^\dagger)^{m+1} = a^D.$$

We prove similarly that $a^\dagger aa^D = a^D$. Thus, $aa^Da^\dagger = a^\dagger aa^D$, which implies the following two equations:

$$a^Da^\dagger = a^D(aa^Da^\dagger) = a^Da^\dagger aa^D = (a^D)^2(aa^\dagger a)a^D = (a^D)^2$$

and

$$a^\dagger a^D = (a^\dagger aa^D)a^D = aa^Da^\dagger a^D = a^D(aa^\dagger a)(a^D)^2 = (a^D)^2.$$

So, we deduce that $a^Da^\dagger = a^\dagger a^D$.

(2) ⇒ (1) From $a^\dagger a^D = a^Da^\dagger$, by Lemma 3.8 (1) and (3), we can obtain that $a^\dagger(a^D)^\# = (a^D)^\# a^\dagger$, i.e., $a^\dagger a^2a^D = a^2a^Da^\dagger$, which immediately yields that $a^\dagger(a^2a^D)^m = (a^2a^D)^m a^\dagger$ by induction. Note that

$$a^\dagger a^m = a^\dagger a^{m+1}a^D = a^\dagger a^m(aa^D)^m = a^\dagger(a^2a^D)^m$$

and

$$a^ma^\dagger = a^{m+1}a^Da^\dagger = a^m(aa^D)^m a^\dagger = (a^2a^D)^m a^\dagger.$$

Thus, $a^\dagger a^m = a^m a^\dagger$. By Lemma 3.6, we get that a is m-EP. □

Next, we consider the necessary and sufficient conditions for m-EP elements involving powers of elements in rings.

Theorem 3.16. *Let $a \in R$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) *a is m-EP;*

(2) *$a \in R^D \cap R^\dagger$ with $\mathrm{ind}(a) = m$, and $a^na^Da^\dagger = a^\dagger a^na^D$.*

Proof. Suppose that a is m-EP, by Theorem 3.15, we get $a^\dagger a^D = a^D a^\dagger$. From the previous equation, it follows that

$$a^n a^D a^\dagger = a^n a^\dagger a^D = a^n a^\dagger a (a^D)^2 = a^n (a^D)^2 = a^{n-1} a^D.$$

In addition, the equality $a^\dagger a^n a^D = a^D a^{n-1}$ can be obtained, similarly. Therefore, we conclude that $a^n a^D a^\dagger = a^\dagger a^n a^D$.

Conversely, since $a^n a^D a^\dagger = a^\dagger a^n a^D$, we get

$$a^D a^\dagger = (a^D)^n a^n a^D a^\dagger = (a^D)^n a^\dagger a^n a^D = (a^D)^{n+1} a a^\dagger a^n a^D = (a^D)^2.$$

Similarly, we can obtain $a^\dagger a^D = (a^D)^2$. So, $a^\dagger a^D = a^D a^\dagger$. Applying Theorem 3.15 again, we can show that a is m-EP. □

The next equivalent conditions of m-EP elements are stated as follows.

Theorem 3.17. *Let $a \in R^\dagger$. Then the following are equivalent:*

(1) *a is m-EP;*

(2) *m is the smallest positive integer such that $a^m = a^\dagger a^{m+1} = a^{m+1} a^\dagger$;*

(3) *m is the smallest positive integer such that $a^m \in R^\#$ with $(a^m)^\# = a^\dagger (a^m)^\# a = a (a^m)^\# a^\dagger$.*

Proof. (1) ⇒ (2) Clearly, we have $a^m = a^\dagger a^{m+1} = a^{m+1} a^\dagger$.

Assume that $a^{m-1} = a^\dagger a^m = a^m a^\dagger$, then $\mathrm{ind}(a) \leq m-1$, which is contrary to the fact $\mathrm{ind}(a) = m$ by Theorem 3.13.

(2) ⇒ (1) If $m = 1$, then we obtain $a = a^\dagger a^2 = a^2 a^\dagger$, which implies that $a \in R^\#$. Thus, $aa^\# = a^\dagger a^2 a^\# = a^\dagger a$. So, $aa^\#$ is Hermitian. Therefore, a is EP, i.e., a is 1-EP.

If $m \geq 2$, then $a^m a^\dagger = a^\dagger a^{m+1} a^\dagger$ and $a^\dagger a^m = a^\dagger a^{m+1} a^\dagger$. Thus, $a^m a^\dagger = a^\dagger a^m$. Now, assume that $a^\dagger a^{m-1} = a^{m-1} a^\dagger$. Then, we have $a^\dagger a^m = a^\dagger a^{m-1} a = a^{m-1} a^\dagger a = a^{m-1}$. In the same manner, we can prove that $a^m a^\dagger = a^{m-1}$. So, $a^{m-1} = a^\dagger a^m = a^m a^\dagger$, contrary to the condition (2). Therefore, a is m-EP.

(1) ⇒ (3) From Theorem 3.13, it follows that $a \in R^D$ with $\mathrm{ind}(a) = m$. Thus, m is the smallest positive integer such that $a^m \in R^\#$ by [17, Lemma]. On account of $a^\dagger a^m = a^m a^\dagger$, it is easily seen that $a^\dagger (a^m)^\# = (a^m)^\# a^\dagger$ by Lemma 3.8(1). Then $(a^m)^\# = ((a^m)^\#)^2 a^m a^\dagger a = a^\dagger ((a^m)^\#)^2 a^m a = a^\dagger (a^m)^\# a$. Similarly, we can get $(a^m)^\# = a(a^m)^\# a^\dagger$.

(3) ⇒ (1) According to the condition (3), we have that $(a^m)^\# a^\dagger = a^\dagger (a^m)^\# a a^\dagger$ and $a^\dagger (a^m)^\# = a^\dagger a (a^m)^\# a^\dagger$. Note that $(a^m)^\# a = a(a^m)^\#$. Thus, $a^\dagger (a^m)^\# = (a^m)^\# a^\dagger$, which implies $a^\dagger a^m = a^m a^\dagger$ by Lemma 3.8(1). Assume that $a^{m-1} a^\dagger = a^\dagger a^{m-1}$, then we have that $\mathrm{ind}(a) \leq m-1$, which implies $a^{m-1} \in R^\#$ with $(a^{m-1})^\# = a^\dagger (a^{m-1})^\# a = a(a^{m-1})^\# a^\dagger$. There is a contradiction with the condition (3). Thus, a is m-EP. □

In [18], Patrício and Puystjens gave the notion of $*$-DMP elements in a ring. In addition, the characterizations of $*$-DMP elements were obtained [18]. Next, we will give the relation between $*$-DMP elements and m-EP elements. First, let us recall the following definition.

Definition 3.18. [18, Definition 6] *An element $a \in R$ with involution $*$ is called $*$-DMP (Drazin-Moore-Penrose) of index k if k is the smallest natural number such that $(a^k)^{\#}$ and $(a^k)^{\dagger}$ exist with respect to $*$ and $(a^k)^{\#} = (a^k)^{\dagger}$.*

Theorem 3.19. *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{\dagger}$ and a is $*$-DMP with index m;

(2) There exists an EP element $c \in R$ and an element $q \in R^{\dagger} \cap R^{nil}$ with the nilpotence index m such that $a = c + q$ and $cq = qc = 0$;

(3) a is m-EP and a^m is EP.

Proof. (1) $\Rightarrow$ (2) Since a is $*$-DMP with index m, by [18, Theorem 7], we have that $a \in R^D$ with $\mathrm{ind}(a) = m$. Let $c = a^2 a^D$ and $q = a - a^2 a^D$. Then, $a = c + q$ and $cq = qc = 0$. Also, we have that $q \in R^{nil}$ with the nilpotence index m. From [18, Theorem 10(2)], it follows that c is EP. Applying [18, Theorem 11(2)], we get that $q \in R^{\dagger}$.

(2) $\Rightarrow$ (3) Since c is EP, then c^m is EP. Note that $a^m = c^m + n^m = c^m$, then a^m is EP.

Next, our objective is to show that $a \in R^{\dagger}$ and $a^{\dagger} = c^{\dagger} + q^{\dagger}$. Since c is EP, according to [18, Corollary 3], we have $cR = c^*R$, which implies $c^* = cx = yc$ for $x, y \in R$. Thus, we obtain

$$c^{\dagger}q = c^{\dagger}cc^{\dagger}q = c^{\dagger}(cc^{\dagger})^*q = c^{\dagger}(c^{\dagger})^*c^*q = c^{\dagger}(c^{\dagger})^*ycq = 0.$$

Similarly, we can prove that $qc^{\dagger} = q^{\dagger}c = cq^{\dagger} = 0$. Then, according to the definition of the Moore-Penrose inverse, it is easy to check $a^{\dagger} = c^{\dagger} + q^{\dagger}$.

We notice that $cc^{\dagger} = c^{\dagger}c$, which implies $c^m c^{\dagger} = c^{\dagger}c^m$ by induction. In addition, we have $a^{\dagger}a^m = (c^{\dagger} + q^{\dagger})c^m = c^{\dagger}c^m$ and $a^m a^{\dagger} = c^m c^{\dagger}$. So, $a^{\dagger}a^m = a^m a^{\dagger}$.

Take any integer $k \leq m - 1$. Assume that $a^k a^{\dagger} = a^{\dagger}a^k$, then we can get $\mathrm{ind}(a) \leq k$. Note that $a^D = c^{\#}$. Thus, on one hand, $a^k = a^{k+1}a^D = (c^{k+1} + q^{k+1})c^{\#} = c^{k+1}c^{\#} = c^k$. On the other hand, $a^k = c^k + q^k$. So, $q^k = 0$, which contradicts with the nilpotence index m. Therefore, a is m-EP.

(3) $\Rightarrow$ (1) Assume that a^{m-1} is EP, then $a^{m-1} \in R^{\#}$, which implies $\mathrm{ind}(a) \leq m-1$. Since a is m-EP, then $a \in R^D \cap R^{\dagger}$ with $\mathrm{ind}(a) = m$. There exists a contradiction. So, a^{m-1} is not EP. Similarly, $a, a^2, \cdots, a^{m-2}$ are not EP. Hence, a is $*$-DMP with index m. □

Remark 3.20. *(1) In general, the condition that $a \in R$ is $*$-DMP with index m can not imply that a is m-EP. For example, let $R = \mathbb{C}^{2\times 2}$ and the involution be the transpose. Take $a = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$. Then $a^2 = 0$. So, a^2 is EP. In addition, since $a \notin aa^*R$, then a is not Moore-Penrose invertible, which yields that a is not EP. Therefore, a is $*$-DMP with index 2. However, a is not 2-EP.*

(2) Let R be a $*$-regular ring R (i.e. every element in R is Moore-Penrose invertible). If $a \in R$ is $*$-DMP with index m, then a is m-EP by Theorem 3.19 (1) and (3).

4 Special cases of m-EP elements

In this section, the special cases of m-EP elements will be considered in a ring. We mainly investigate the characterizations of EP elements by means of the group inverse, Moore-Penrose inverse, and core inverse. We also give some properties of 2-EP elements.

The following lemmas will be very useful in proving our main results.

Lemma 4.1. *Let $a \in R^{\textcircled{\#}}$. Then*
(1) [15, Theorem 2.19] $a^{\#} = (a^{\textcircled{\#}})^2 a$.
(2) [15, Theorem 2.18] $a^{\textcircled{\#}}$ *is EP and* $(a^{\textcircled{\#}})^{\#} = (a^{\textcircled{\#}})^{\dagger} = (a^{\textcircled{\#}})^{\textcircled{\#}} = a^2 a^{\textcircled{\#}}$.
(3) [15, Theorem 2.19(v)] *If $a \in R^{\dagger}$, then* $a^{\textcircled{\#}} = a^{\#} a a^{\dagger}$.

Lemma 4.2. [15, Theorem 3.1] *Let $a \in R$. Then the following are equivalent:*
(1) *a is EP;*
(2) $a \in R^{\textcircled{\#}}$ *and* $a^{\#} = a^{\textcircled{\#}}$;
(3) $a \in R^{\textcircled{\#}}$ *and* $a a^{\textcircled{\#}} = a^{\textcircled{\#}} a$;
(4) $a \in R^{\#} \cap R^{\dagger}$ *and* $a^{\dagger} = a^{\textcircled{\#}}$.

Lemma 4.3. [3, Lemma 2.1] *Let $a \in R^{\dagger}$. Then a is EP if and only if $a^{\dagger}$ is EP.*

In [8], Mosić and Djordjević characterized EP elements by certain conditions involving powers of the group inverse and Moore-Penrose inverse. Motivated by them, we will state some new equivalent conditions for EP elements using the powers of the group inverse, Moore-Penrose inverse, and core inverse.

Theorem 4.4. *Let $a \in R$ and $n \in \mathbb{N}$. Then the following are equivalent:*
(1) *a is EP;*
(2) $a \in R^{\textcircled{\#}}$ *and* $a^n a^{\textcircled{\#}} = a^{\textcircled{\#}} a^n$;
(3) $a \in R^{\textcircled{\#}}$ *and* $(a^{\textcircled{\#}})^n = (a^{\#})^n$;
(4) $a \in R^{\textcircled{\#}}$ *and* $a(a^{\textcircled{\#}})^n = (a^{\textcircled{\#}})^n a$;
(5) $a \in R^{\textcircled{\#}}$ *and* $a a^{\textcircled{\#}} (a^*)^n = (a^*)^n a a^{\textcircled{\#}}$;
(6) $a \in R^{\#} \cap R^{\dagger}$ *and* $(a^{\textcircled{\#}})^n = (a^{\dagger})^n$;
(7) $a \in R^{\#} \cap R^{\dagger}$ *and* $a^{\dagger}(a^{\textcircled{\#}})^n = (a^{\textcircled{\#}})^n a^{\dagger}$;
(8) $a \in R^{\#} \cap R^{\dagger}$ *and* $a^{\dagger}(a^{\textcircled{\#}})^n = (a^{\#})^n a^{\dagger}$;
(9) $a \in R^{\#} \cap R^{\dagger}$, $(a^{\dagger})^{n+1}(a^{\textcircled{\#}})^n = (a^{\textcircled{\#}})^n (a^{\dagger})^{n+1}$ *and* $a^{\dagger} a = (a^{\dagger})^2 a^2$;
(10) $a \in R^{\#} \cap R^{\dagger}$ *and* $a(a^{\textcircled{\#}})^n = (a^{\dagger})^n a$;
(11) $a \in R^{\#} \cap R^{\dagger}$ *and* $a(a^{\dagger})^n = (a^{\dagger})^n a$;
(12) $a \in R^{\#} \cap R^{\dagger}$ *and* $a a^{\dagger} = (a^{\dagger})^n a^n$ *(or* $a^{\dagger} a = a^n (a^{\dagger})^n$*)*;
(13) $a \in R^{\#} \cap R^{\dagger}$ *and* $(a^{\dagger})^n = a(a^{\dagger})^{n+1}$.

Proof. If a is EP, then we have $a^{\#} = a^{\dagger} = a^{\textcircled{\#}}$ by Lemma 4.2, which implies that the conditions (2)-(13) hold.

(2) $\Rightarrow$ (1) If $n = 1$, then a is EP by Lemma 4.2(3).

If $n \geq 2$, then

$$a^{\#}a = (a^{\#})^{n-1}aa^{n-2} = (a^{\#})^{n-1}a^{\oplus}a^2a^{n-2} = (a^{\#})^{n-1}a^{\oplus}a^n = (a^{\#})^{n-1}a^na^{\oplus} = aa^{\oplus}.$$

Since $aa^{\oplus}$ is Hermitian, we get that $aa^{\#}$ is also Hermitian. Thus, a is EP by Lemma 3.9(1).

(3) $\Rightarrow$ (1) The condition $a \in R^{\oplus}$ and $(a^{\oplus})^n{=}(a^{\#})^n$ ensures that

$$aa^{\#} = a^n(a^{\#})^n = a^n(a^{\oplus})^n = a^n(a^{\oplus})^naa^{\oplus} = a^n(a^{\#})^naa^{\oplus} = aa^{\#}aa^{\oplus} = aa^{\oplus}$$

is Hermitian. Consequently, a is EP.

(4) $\Rightarrow$ (1) Suppose that $n = 1$. Then, a is EP by Lemma 4.2(3).

Suppose that $n \geq 2$. From $a \in R^{\oplus}$ and Lemma 4.1(1), it follows that $a(a^{\oplus})^2 = a^{\oplus}$ and $(a^{\oplus})^2a = a^{\#}$, which yield $a(a^{\oplus})^n = (a^{\oplus})^{n-1}$ and $(a^{\oplus})^na = (a^{\#})^{n-1}$ by induction. Therefore, $(a^{\oplus})^{n-1} = (a^{\#})^{n-1}$, which implies that a satisfies the condition (3). So, a is EP.

(5) $\Rightarrow$ (1) From the condition $a \in R^{\oplus}$ and $aa^{\oplus}(a^*)^n = (a^*)^naa^{\oplus}$, we obtain

$$\begin{aligned} a^{\#}a &= (a^{\#})^na^n = (a^{\#})^n((a^n)^*)^* = (a^{\#})^n((aa^{\oplus}a^n)^*)^* = (a^{\#})^n((a^n)^*aa^{\oplus})^* \\ &= (a^{\#})^n(aa^{\oplus}(a^*)^n)^* = (a^{\#})^na^naa^{\oplus} = aa^{\oplus}. \end{aligned}$$

Thus, $a^{\#}a$ is Hermitian, which implies that a is EP.

(6) $\Rightarrow$ (1) If $n = 1$, then a is EP by Lemma 4.2(4).

If $n \geq 2$, in the proof of (4) $\Rightarrow$ (1), we have shown that $(a^{\#})^{n-1} = (a^{\oplus})^na$. Thus,

$$\begin{aligned} a^{\#}a &= (a^{\#})^{n-1}a^{n-1} = (a^{\oplus})^naa^{n-1} = (a^{\dagger})^na^n = a^{\dagger}a(a^{\dagger})^na^n = a^{\dagger}a((a^{\oplus})^na)a^{n-1} \\ &= a^{\dagger}a(a^{\#})^{n-1}a^{n-1} = a^{\dagger}aa^{\#}a = a^{\dagger}a, \end{aligned}$$

which implies that $a^{\#}a$ is Hermitian. Thus, a is EP.

(7) $\Rightarrow$ (1) Assume that $n = 1$. Observe that $a^{\oplus} \in R^{\#}$ and $(a^{\oplus})^{\#} = a^2a^{\oplus}$ by Lemma 4.1(2). From the condition $a^{\dagger}a^{\oplus} = a^{\oplus}a^{\dagger}$, we have that $a^{\dagger}(a^{\oplus})^{\#} = (a^{\oplus})^{\#}a^{\dagger}$ by Lemma 3.8(1), which yields $a^2a^{\oplus}a^{\dagger} = a^{\dagger}a^2a^{\oplus}$. In addition, according to Lemma 4.1(3), we have $a^{\oplus} = a^{\#}aa^{\oplus}$. Note that $a^{\#} = (a^{\oplus})^2a$. Then

$$\begin{aligned} a^{\dagger}a &= a^{\dagger}a^{\#}a^2 = a^{\dagger}((a^{\oplus})^2a)a^2 = a^{\dagger}a^{\oplus}a^{\oplus}a^3 = a^{\oplus}a^{\dagger}a^{\oplus}a^3 = a^{\#}aa^{\oplus}a^{\dagger}a^{\oplus}a^3 \\ &= (a^{\#})^2(a^2a^{\oplus}a^{\dagger})a^{\oplus}a^3 = (a^{\#})^2(a^{\dagger}a^2a^{\oplus})a^{\oplus}a^3 = (a^{\#})^3a(a^{\dagger}a^2a^{\oplus})a^{\oplus}a^3 \\ &= (a^{\#})^3a^2(a^{\oplus})^2a^3 = a^{\#}((a^{\oplus})^2a)a^2 = a^{\#}a^{\#}a^2 = a^{\#}a. \end{aligned}$$

Therefore, $a^{\#}a$ is Hermitian, which implies that a is EP.

Assume that $n \geq 2$. Since $a^{\oplus} \in R^{\#}$, be Lemma 3.8(4) we have $(a^{\oplus})^n \in R^{\#}$. From $a^{\dagger}(a^{\oplus})^n = (a^{\oplus})^na^{\dagger}$, it follows that $a^{\dagger}((a^{\oplus})^n)^{\#} = ((a^{\oplus})^n)^{\#}a^{\dagger}$. Note that $((a^{\oplus})^n)^{\#} = ((a^{\oplus})^{\#})^n = (a^2a^{\oplus})^n = a^{n+1}a^{\oplus}$ by induction. So, we get $a^{\dagger}a^{n+1}a^{\oplus} = a^{n+1}a^{\oplus}a^{\dagger}$. Also, $(a^{\#})^{n-1} = (a^{\oplus})^na$, $a^{n-1}(a^{\oplus})^n = a^{\oplus}$, and $a^{\oplus}a^{n+1} = a^n$ by induction. Hence,

$$\begin{aligned} a^{\dagger}a &= a^{\dagger}(a^{\#})^{n-1}a^n = a^{\dagger}((a^{\oplus})^na)a^n = (a^{\oplus})^na^{\dagger}a^{n+1} = a^{\#}a(a^{\oplus})^na^{\dagger}a^{n+1} \\ &= (a^{\#})^{n-1}(a^{n-1}(a^{\oplus})^n)a^{\dagger}a^{n+1} = (a^{\#})^{n-1}a^{\oplus}a^{\dagger}a^{n+1} \\ &= (a^{\#})^{2n}(a^{n+1}a^{\oplus}a^{\dagger})a^{n+1} = ((a^{\#})^{2n+1}a)(a^{\dagger}a^{n+1}a^{\oplus})a^{n+1} \\ &= ((a^{\#})^{2n+1}a^{n+1})(a^{\oplus}a^{n+1}) = (a^{\#})^na^n = a^{\#}a. \end{aligned}$$

Consequently, a is EP.

(8) $\Rightarrow$ (1) Using the equality $a^{\oplus}a^2 = a$, we can easily get $a^{\oplus}a = (a^{\oplus})^m a^m$ for any $m \in \mathbb{N}$ by induction. Note that $(a^{\#})^n = (a^{\oplus})^{n+1}a$. In addition, from the condition $a^{\dagger}(a^{\oplus})^n = (a^{\#})^n a^{\dagger}$, we deduce that

$$\begin{aligned} a^{\oplus}a &= (a^{\oplus})^{n+1}a^{n+1} = ((a^{\oplus})^{n+1}a)a^{\dagger}aa^n = ((a^{\#})^n a^{\dagger})a^{n+1} = a^{\dagger}((a^{\oplus})^n a^n)a \\ &= a^{\dagger}a^{\oplus}aa = a^{\dagger}a. \end{aligned}$$

Note that $a^{\dagger}a$ is Hermitian, then $a^{\oplus}a$ is Hermitian. According to the definitions of the core inverse and Moore-Penrose inverse, we obtain $a^{\dagger} = a^{\oplus}$. Applying Lemma 4.2(4), we conclude that a is EP.

(9) $\Rightarrow$ (1) We can get $(a^{\oplus})^n \in R^{\#}$ and $((a^{\oplus})^n)^{\#} = a^{n+1}a^{\oplus}$ by the proof of (7) $\Rightarrow$ (1). Since $(a^{\dagger})^{n+1}(a^{\oplus})^n = (a^{\oplus})^n(a^{\dagger})^{n+1}$, then we have $(a^{\dagger})^{n+1}((a^{\oplus})^n)^{\#} = ((a^{\oplus})^n)^{\#}(a^{\dagger})^{n+1}$, which becomes to $(a^{\dagger})^{n+1}a^{n+1}a^{\oplus} = a^{n+1}a^{\oplus}(a^{\dagger})^{n+1}$. Since $a^{\dagger}a = (a^{\dagger})^2a^2$, it is easy to verify $(a^{\dagger})^{n+1}a^{n+1} = a^{\dagger}a$ by induction. Thus, we have $a^{\dagger}aa^{\oplus} = a^{n+1}a^{\oplus}(a^{\dagger})^{n+1}$. Note that $a^{\oplus} = a^{\#}aa^{\dagger}$, then

$$a^{\dagger} = a^{\dagger}aa^{\oplus} = a^{n+1}a^{\oplus}(a^{\dagger})^{n+1} = a^{n+1}a^{\#}aa^{\dagger}(a^{\dagger})^{n+1} = a^{n+1}(a^{\dagger})^{n+2}.$$

Thus,

$$\begin{aligned} aa^{\#} &= a^{n+1}(a^{\#})^{n+1} = a^{n+1}(a^{\dagger}a)a^{\#}(a^{\#})^n = a^{n+1}(a^{\dagger})^2a^2a^{\#}(a^{\#})^n \\ &= a^{n+1}a^{\dagger}(a^{\dagger}a)(a^{\#})^n = a^{n+1}a^{\dagger}(a^{\dagger})^{n+1}a^{n+1}(a^{\#})^n \\ &= a^{n+1}(a^{\dagger})^{n+2}a = a^{\dagger}a. \end{aligned}.$$

So, a is EP.

(10) $\Rightarrow$ (1) For the case $n = 1$. Note that $a^{\oplus} = a^{\#}aa^{\dagger}$, then $aa^{\oplus} = aa^{\#}aa^{\dagger} = aa^{\dagger}$. Thus, $aa^{\dagger} = a^{\dagger}a$, which implies that a is EP.

For the other case $n \geq 2$. Observe that $a(a^{\oplus})^n = (a^{\oplus})^{n-1}$ and $a^{\oplus}a = (a^{\oplus})^{n-1}a^{n-1}$. Then, we get

$$\begin{aligned} a^{\oplus}a &= (a^{\oplus})^{n-1}a^{n-1} = a(a^{\oplus})^n a^{n-1} = (a^{\dagger})^n aa^{n-1} = a^{\dagger}a(a^{\dagger})^n aa^{n-1} \\ &= a^{\dagger}aa(a^{\oplus})^n a^{n-1} = a^{\dagger}a^2(a^{\oplus})^2a = a^{\dagger}a. \end{aligned}$$

Thus, $a^{\dagger} = a^{\oplus}$. So, a is EP.

(11) $\Rightarrow$ (1) Suppose that $n = 1$. It is clear that a is EP by Lemma 3.9(2).

Suppose that $n \geq 2$. Applying [15, Theorem 2.18(ii)], we have that $a^{\dagger} \in R^{\#}$. Then, the following equations hold:

$$\begin{aligned} aa^{\dagger} &= a(a^{\dagger})^n((a^{\dagger})^{\#})^{n-1} = (a^{\dagger})^n a((a^{\dagger})^{\#})^{n-1} = (a^{\dagger})^n aa^{\dagger}((a^{\dagger})^{\#})^n \\ &= (a^{\dagger})^n((a^{\dagger})^{\#})^n = a^{\dagger}(a^{\dagger})^{\#}, \end{aligned}$$

which implies that $a^{\dagger}(a^{\dagger})^{\#}$ is Hermitian. Thus, $a^{\dagger}$ is EP, i.e., a is EP by Lemma 4.3.

(12) $\Rightarrow$ (1) Note that $a^{\dagger} \in R^{\#}$. Then we can get

$$(a^{\dagger})^{\#}a^{\dagger} = (a^{\dagger})^{\#}a^{\dagger}aa^{\dagger} = (a^{\dagger})^{\#}a^{\dagger}(a^{\dagger})^n a^n = (a^{\dagger})^n a^n = aa^{\dagger}.$$

Thus, $a^\dagger$ is EP, so that a is EP.

(13) $\Rightarrow$ (1) From the equality $(a^\dagger)^n = a(a^\dagger)^{n+1}$ and $a^\dagger \in R^\#$, it follows that

$$a^\dagger(a^\dagger)^\# = (a^\dagger)^n((a^\dagger)^\#)^n = a(a^\dagger)^{n+1}((a^\dagger)^\#)^n = aa^\dagger.$$

Therefore, $a^\dagger$ is EP, which implies that a is EP. □

In [10], Malik, Rueda and Thome used Hartwig-Spindelböck decomposition to prove that if $A \in \mathbb{C}^{n\times n}$ is 2-EP, then A^2 is EP and A is bi-dagger (bi-EP). In [11], Wang and Deng extended this result to the operator matrix by the operator matrix decomposition. Next, we will use new method to further generalize this result to the ring case, depending on algebraic properties of $*$-ring.

Theorem 4.5. *Let $a \in R$ be 2-EP. Then*

(1) *a^2 is EP;*

(2) *a is bi-EP;*

(3) *a is bi-dagger;*

(4) *a^D is EP;*

(5) *$a^D = a(a^\dagger)^2 = (a^\dagger)^2 a$;*

(6) *If a is star-dagger, then $a^D a^* = a^* a^D$.*

Proof. Suppose that a is 2-EP, by Theorem 3.13, we obtain $a \in R^D$ with $\mathrm{ind}(a) = 2$. Moreover, $a^D = a^2(a^\dagger)^3 = (a^\dagger)^3 a^2$. Note that $a^2 \in R^\#$, $(a^2)^\# = (a^2)^D = a^4(a^\dagger)^6$ and $a^2 a^\dagger = a^\dagger a^2$. Then, we get

$$a^2(a^2)^\# = a^6(a^\dagger)^6 = a^4a^2a^\dagger(a^\dagger)^5 = a^4a^\dagger a^2(a^\dagger)^5 = a^5(a^\dagger)^5 = \cdots = a^2(a^\dagger)^2 = a^\dagger a^2 a^\dagger.$$

(1) First, we prove $(a^2)^\# = (a^\dagger)^2 a^2 (a^\dagger)^2$. Let $x = (a^\dagger)^2$. Since

$$a^2xa^2 = a^2(a^\dagger)^2a^2 = a^2a^\dagger(a^\dagger a^2) = (a^2a^\dagger a^2)a^\dagger = a^3a^\dagger = a(a^2a^\dagger) = aa^\dagger a^2 = a^2$$

and

$$a^2x = a^2(a^\dagger)^2 = (a^\dagger)^2a^2 = xa^2,$$

by Lemma 3.8(5), we have $(a^2)^\# = xa^2x = (a^\dagger)^2a^2(a^\dagger)^2$.

Next, we prove that $a^2(a^2)^\#$ is Hermitian. Note that

$$\begin{aligned} a^2(a^2)^\# &= a^3(a^\dagger)^3 = a(a^2(a^\dagger)^3)aa^\dagger = aa^Daa^\dagger = a(a^D)^2a^2a^\dagger \\ &= a(a^2)^\# a^2a^\dagger = a(a^\dagger)^2a^2(a^\dagger)^2a^2a^\dagger \\ &= (aa^\dagger)(a^\dagger a)(aa^\dagger)(a^\dagger a)(aa^\dagger) = yzyzy, \end{aligned}$$

where $y = aa^\dagger$, $z = a^\dagger a$. Then, we get

$$(a^2(a^2)^\#)^* = (yzyzy)^* = y^*z^*y^*z^*y^* = yzyzy = a^2(a^2)^\#.$$

Applying Lemma 3.9(1), we deduce that a^2 is EP.

(2) From the proof of (1), we know that $a^2(a^2)^\# = a^\dagger a^2 a^\dagger$ is Hermitian. which implies $(a^\dagger a)(aa^\dagger) = ((a^\dagger a)(aa^\dagger))^* = (aa^\dagger)(a^\dagger a)$. Therefore, a is bi-EP.

(3) We will show that $a^2 \in R^\dagger$ and $(a^2)^\dagger = (a^\dagger)^2$ by the definition of the Moore-Penrose inverse. In fact, in the proof of (1), we see that $a^2(a^\dagger)^2a^2 = a^2$, and $a^2(a^\dagger)^2 = (a^\dagger)^2a^2$ is Hermitian. In addition, we have

$$(a^\dagger)^2a^2(a^\dagger)^2 = a^\dagger(a^\dagger a^2 a^\dagger)a^\dagger = a^\dagger a(a^\dagger)^2 aa^\dagger = (a^\dagger)^2.$$

(4) Note that $a^D \in R^\#$ and $a^D(a^D)^\# = aa^D = a^3(a^\dagger)^3$ is Hermitian, Then a^D is EP.

(5) From (2), we obtain $a(a^\dagger)^2a = a^\dagger a^2 a^\dagger$. Multiplying the previous equality by $a^\dagger$ from the right side and left side, respectively, then we get

$$a(a^\dagger)^2 = a^\dagger a^2(a^\dagger)^2 = a^2(a^\dagger)^3 = a^D \quad \text{and} \quad (a^\dagger)^2a = (a^\dagger)^2a^2a^\dagger = (a^\dagger)^3a^2 = a^D.$$

Therefore, $a^D = a(a^\dagger)^2 = (a^\dagger)^2a$.

(6) By (5), we have $a^D = a(a^\dagger)^2 = (a^\dagger)^2a$, which immediately yields

$$a^Da^* = (a^\dagger)^2aa^* = a^\dagger a^* = a^*a^\dagger = a^*a(a^\dagger)^2 = a^*a^D. \quad \square$$

Remark 4.6. *(1) In general, any item of (1)-(5) in Theorem 4.5 can not imply that a is 2-EP. In fact, take an EP element $a \in R$. Then, a satisfies any item of (1)-(5). However, a is not 2-EP, since* $\mathrm{ind}(a) = 1 \neq 2$.

(2) In general, under the condition that $a \in R$ is m-EP, where $m \geq 3$, we can not obtain that a^m is EP. This can be illustrated by Example 2.17 of [10].

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