THERE ARE NO PROPER TOPOLOGICAL HYPERBOLIC HOMOCLINIC CLASSES FOR AREA-PRESERVING MAPS

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ABSTRACT. We begin by defining a homoclinic class for homeomorphisms. Then we prove that if a topological homoclinic class Λ associated to an area-preserving homeomorphism f on a surface M is topologically hyperbolic (i.e. has the shadowing and expansiveness properties), then $\Lambda = M$ and f is an Anosov homeomorphism.

1. INTRODUCTION

Let $f: M \to M$ be a diffeomorphism defined on a closed and connected Riemannian manifold M. A compact and f-invariant set $\Lambda \subset M$ is called a *hyperbolic set* if the tangent bundle over Λ can be decomposed as a direct sum of two sub-bundles both invariant by the tangent map Df that are uniformly contracted under, respectively, forward and backward iterations (see e.g. [9] for full details). When such a hyperbolic set Λ is given by the orbit of a periodic point, the point is called a *hyperbolic periodic point*. The concept of hyperbolicity played a fundamental role in the development of the stability theory of dynamical systems (see e.g. [9, 10]).

We recall that a *nonwandering point* $x \in M$ is a point such that any neighbourhood $\mathcal{U}$ of x contains points for which some forward iterate is in $\mathcal{U}$. If the set of nonwandering points is the closure of the set of hyperbolic periodic points and furthermore it is a hyperbolic set, then Smale's spectral decomposition theorem ([10]) assures that the set of nonwandering points can be partitioned into a finite number of compact blocks exhibiting a dense orbit, called *basic blocks*. When there is a single piece in this decomposition, f is called an *Anosov map*.

In an attempt to generalize the hyperbolic basic sets which played a central role in Smale's spectral decomposition, Newhouse introduced, in the early seventies, the concept of a homoclinic class (see [5]). A *homoclinic class* is defined as the closure of the set of transversal intersections of the stable and unstable invariant manifolds of a hyperbolic periodic saddle of a diffeomorphism f. Homoclinic classes are f-invariant and display a dense orbit of f in the homoclinic class. Yet the hyperbolicity of the periodic saddle, which gives origin to the homoclinic class, is not enough to spread hyperbolicity to the whole homoclinic class. Indeed, it is well-known that homoclinic classes may fail to be uniformly hyperbolic.

In the particular case when M is a surface and f is area-preserving, Newhouse proved in the mid eighties [6] a simple but elegant result. He proved that f cannot support proper uniformly hyperbolic homoclinic classes. In other words, uniformly hyperbolic homoclinic classes must be the whole manifold, therefore concluding that f is an Anosov map.

In the present paper we intend to reconfigure Newhouse's theorem by allowing one to consider maps that are not diffeomorphisms. Two problems arise: the pointwise hyperbolicity (a homoclinic class foreshadows the need for a hyperbolic periodic saddle point) and set hyperbolicity (the hypothesis that a homoclinic class is hyperbolic).

Clearly, we first need to seek vestiges of hyperbolicity in non differentiable contexts and only then try to conjecture what could be the topological counterpart of Newhouse's theorem. Our proposal is to replace the hyperbolicity by two properties with topological flavor - shadowing and expansiveness. In §2 we fully describe these 'topological hyperbolic sets' in terms of dynamically defined invariant manifolds, canonical coordinates and local product structure which are of utmost importance when we realign Newhouse's strategy. In §3 we define topological homoclinic classes readapted to a type of topological transversality and obtain the Birkhoff-Smale theorem in this non differentiable setting.

For diffeomorphisms it is well-known that periodic points are dense in the whole homoclinic class. However, we do now know how to prove this property for homeomorphisms (see Remark 3.1). Fortunately, we are considering *hyperbolic* topological homoclinic classes which is enough to obtain that periodic points are dense in the whole homoclinic class (cf. Proposition 2.6).

Like Newhouse, who was inspired by a uniformly hyperbolic basic set to consider the closure of transversal intersections of stable/unstable manifolds of a hyperbolic periodic saddle, here we got inspired by a set displaying shadowing and expansiveness (topological set hyperbolicity) to consider the closure of topological transversal intersections of stable/unstable manifolds of a periodic point displaying shadowing and expansiveness (pointwise hyperbolicity).

We hope that our definition of topological homoclinic class can be useful in several aspects of topological dynamics. Finally, in §4 we prove the next:

Theorem 1. *Let M be a surface, $f\colon M \to M$ be an area-preserving homeomorphism and let $\Lambda \subseteq M$ be a topological homoclinic class of f. If f has the shadowing property on Λ and is expansive on Λ, then f is an Anosov homeomorphism (i.e., f has the shadowing property on M and is expansive on M).*

It is interesting to notice that in [4, 3] it was proved that there are no expansive homeomorphisms on the two-dimensional sphere $\mathbb{S}^2$. This result along with Theorem 1 allows us to conclude that $\mathbb{S}^2$ does not support topological homoclinic classes with shadowing and expansiveness associated to an area-preserving homeomorphism.

Throughout the article we assume that M is a closed, connected Riemannian manifold, d is the distance on M induced by the Riemannian structure and λ is the Lebesgue measure on M associated to a volume form on M. Despite the fact that our main result is about surfaces some results are stated and proved for manifolds on dimension ≥ 2 for eventual future use.

2. TOPOLOGICAL HYPERBOLICITY

2.1. **Hyperbolic homeomorphisms.** Let $f\colon M \to M$ be a homeomorphism. Given $\delta > 0$, a sequence of points $\{x_i\}_{i\in\mathbb{Z}} \subset M$ is called a δ-*pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let $\Lambda \subseteq M$ be a closed f-invariant set (i.e., $f(\Lambda) = \Lambda$). We say that f has the *shadowing property on* Λ if for every $\epsilon > 0$ there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$ of f there is a point $z \in \Lambda$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. When $\Lambda = M$, f is said to have the *shadowing property*.

A homeomorphism f is called *expansive on* Λ (see [9, §7]) if there is $e > 0$, called an *expansive constant*, such that for all $x \in \Lambda$ and $y \in M$ if we have $d(f^n(x), f^n(y)) \le e$ for all $n \in \mathbb{Z}$, then $x = y$. When $\Lambda = M$, f is simply said to be expansive.

A homeomorphism f is called an *Anosov homeomorphism* if it has the shadowing property and is expansive (see [1, §11.3]). We shall say that f is a *hyperbolic homeomorphism on* Λ if it has the shadowing property on Λ and is expansive on Λ.

2.2. **Invariant sets, shadowing and expansiveness.** In this section we obtain some useful local results to be used in the sequel. Let $f \colon M \to M$ be a homeomorphism and let $\Lambda \subseteq M$ be a closed and f-invariant set. Given $x \in M$ and $\epsilon > 0$, the *local stable* and *local unstable set* of x are defined, respectively, by

$$W_\epsilon^s(x) = \{y \in M \colon d(f^n(x), f^n(y)) \le \epsilon, \text{ for } n \ge 0\}$$

and

$$W_\epsilon^u(x) = \{y \in M \colon d(f^{-n}(x), f^{-n}(y)) \le \epsilon, \text{ for } n \ge 0\}.$$

Under the expansiveness hypothesis on Λ, stable and unstable sets of points $x \in \Lambda$ are *dynamically defined*, i.e., the following result holds.

Proposition 2.1. *Let $e > 0$. The homeomorphism f is expansive on Λ with expansive constant e if and only if for all $\tilde{e} > 0$ there exists $N > 0$ such that for all $x \in \Lambda$ and all $n \ge N$ we have*

$$f^n(W_e^s(x)) \subset W_{\tilde{e}}^s(f^n(x)) \quad and \quad f^{-n}(W_e^u(x)) \subset W_{\tilde{e}}^u(f^{-n}(x)). \tag{2.1}$$

Proof. Suppose that f is expansive on Λ with expansive constant e, and suppose, for a contradiction, that there exist sequences $x_n \in \Lambda$ and $y_n \in M$, $n \in \mathbb{N}$, such that $y_n \in W_e^s(x_n)$ and $d(f^n(x_n), f^n(y_n)) > \tilde{e}$. Since $y_n \in W_e^s(x_n)$, we have that $d(f^m \circ f^n(x_n), f^m \circ f^n(y_n)) \le e$ for all $m \ge -n$. Taking subsequences we may assume that there exists $x \in \Lambda$ and $y \in M$ such that $\lim_n f^n(x_n) = x$ and $\lim_n f^n(y_n) = y$. Hence, $d(f^m(x), f^m(y)) \le e$ for all $m \in \mathbb{Z}$. Moreover, $d(x, y) = \lim_n d(f^n(x_n), f^n(y_n)) \ge \tilde{e}$. This contradicts the expansiveness on Λ.

Conversely, let $x \in \Lambda$ and $y \in M$ be such that $d(f^n(x), f^n(y)) \le e$ for all $n \in \mathbb{Z}$. For all $n \ge 0$ we have $f^{-n}(y) \in W_e^s(f^{-n}(x))$. Then, for any $\tilde{e} > 0$ we have $y \in W_{\tilde{e}}^s(x)$ by the first inclusion of (2.1) and hence $x = y$. □

The *stable* and *unstable sets* of $x \in M$ are defined, respectively, by

$$W^s(x) = \left\{y \in M \colon \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0\right\}$$

and

$$W^u(x) = \left\{y \in M \colon \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0\right\}.$$

Proposition 2.2. *If f is expansive on Λ with expansive constant e and $\epsilon \in (0, e)$, then for all $x \in \Lambda$:*

$$W^s(x) = \bigcup_{n \ge 0} f^{-n}(W_\epsilon^s(f^n(x))) \quad and \quad W^u(x) = \bigcup_{n \ge 0} f^n(W_\epsilon^u(f^{-n}(x))). \tag{2.2}$$

Proof. Let $y \in \cup_{n \geq 0} f^{-n} W^s_\epsilon(f^n(x))$. There exists $n \geq 0$ such that $f^n(y) \in W^s_\epsilon(f^n(x))$. By Proposition 2.1, for all $\tilde{e} > 0$ there exists $N > 0$ such that for all $m \geq N$

$$f^{m+n}(y) \in f^m W^s_\epsilon(f^n(x)) \subset W^s_{\tilde{e}}(f^{m+n}(x)).$$

Hence, $d(f^{m+n}(y), f^{m+n}(x)) \leq \tilde{e}$ for all $m \geq N$ and, consequently, $y \in W^s(x)$.

Conversely, let $y \in W^s(x)$. Given $\epsilon > 0$ there exists $N \geq 0$ such that $d(f^n(x), f^n(y)) \leq \epsilon$ for $n \geq N$. Consequently, $d(f^i \circ f^N(x), f^i \circ f^N(y)) \leq \epsilon$ for all $i \geq 0$, i.e., $f^N(y) \in W^s_\epsilon(f^N(x))$. Hence,

$$y \in f^{-N} W^s_\epsilon(f^N(x)) \subset \cup_{n \geq 0} f^{-n} W^s_\epsilon(f^n(x)).$$

The second equality in (2.2) can be proved analogously.

□

We say that f has *canonical coordinates* on Λ if for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y) \cap \Lambda \neq \emptyset$.

Lemma 2.3. *If f has the shadowing property on Λ, then f has canonical coordinates on Λ.*

Proof. Given $\epsilon > 0$, let $\delta > 0$ be given by the shadowing property on Λ. Let us be given any $x, y \in \Lambda$ such that $d(x, y) < \delta$. Take $x_i = f^i(x)$ for $i \geq 0$ and $x_i = f^i(y)$ for $i < 0$. Clearly, $\{x_i\}_{i \in \mathbb{Z}} \subset \Lambda$ is a δ-pseudo orbit of f. By the shadowing property on Λ there exists $z \in \Lambda$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. Thus, $d(f^i(z), f^i(x)) < \epsilon$ for all $i \geq 0$ and $d(f^i(z), f^i(y)) < \epsilon$ for all $i < 0$. Hence, $z \in W^s_\epsilon(x)$ and $z \in W^u_\epsilon(y)$.   □

Lemma 2.4. *If f is an expanding homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \leq e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then there is at most one point of intersection of $W^s_\epsilon(x)$ and $W^u_\epsilon(x)$.*

Proof. Let $z \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$. Suppose, by contradiction, that there exists $w \in M$, $w \neq z$, such that $w \in W^s_\epsilon(x) \cap W^u_\epsilon(y)$. Then, for all $n \geq 0$, $d(f^n(x), f^n(z)) \leq \epsilon$, $d(f^n(x), f^n(w)) \leq \epsilon$, $d(f^{-n}(y), f^{-n}(z)) \leq \epsilon$, and $d(f^{-n}(y), f^{-n}(w)) \leq \epsilon$. This implies that

$$d(f^n(z), f^n(w)) \leq 2\epsilon,$$

for all $n \in \mathbb{Z}$, which contradicts the expansiveness on Λ.   □

As a consequence of Lemmas 2.3 and 2.4 we have:

Corollary 2.5. *If f is a hyperbolic homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \leq e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then*

$$W^s_\epsilon(x) \cap W^u_\epsilon(y) = \{one\ point\} \subset \Lambda.$$

Remark 2.1. *We say that a 1-dimensional set $W \subset M$ is a* topological manifold *if every $x \in W$ has a neighborhood homeomorphic to $\mathbb{R}$. It should be enlightening to say that even with expansiveness we can have several 'prongs' as invariant sets and thus stable (and unstable) sets are not topological manifolds. In Figure 2 we can have expansiveness, nevertheless a set Λ including a ball around p cannot have canonical coordinates. Actually, there exist points $x, y \in \Lambda$ arbitrarily close but still without any intersection between the local stable set of x with the local unstable set*

of y. Shadowing implies no n-prongs structure for n > 2. Expansivity implies no 1-prong structure (cf. Figure 1). Therefore, when we put together shadowing and expansiveness there can only be a 2-prongs structure (Figure 3).
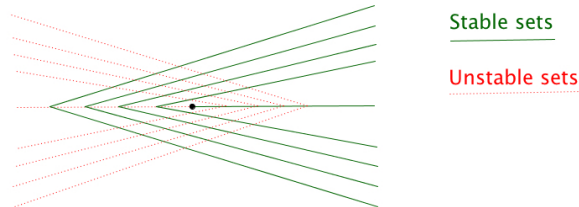

FIGURE 1. Example of a 1-prong structure. In this case local stable and local unstable sets intersect in more than one point contradicting expansivity (cf. Lemma 2.4).
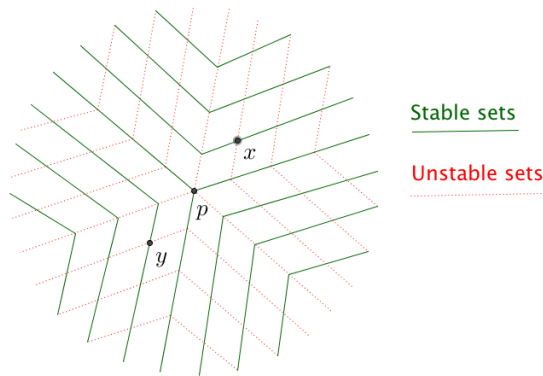

FIGURE 2. Example of a 3-prongs structure. In this case local stable and local unstable sets do not intersect contradicting the shadowing property (cf. Lemma 2.3).
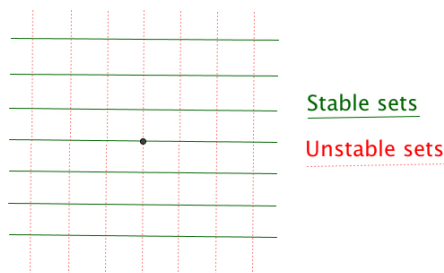

FIGURE 3. Example of 2-prongs structure which is compatible with being a C^0 manifold.

The next result shows that hyperbolic homeomorphisms on Λ behave somehow like basic pieces of classical hyperbolic dynamics theory (cf. [9]).

Proposition 2.6. *Let $f \colon M \to M$ be a volume-preserving homeomorphism which is hyperbolic on Λ. Then, $\overline{Per(f)} \cap \Lambda = \Lambda$.*

Proof. Let $x \in \Lambda$. Since f is volume-preserving, by Poincaré recurrence theorem, almost every point in Λ is recurrent and, therefore, x is a non-wandering point. Hence, it is easy to deduce that for every $\delta > 0$, there exists a δ-chain from x to itself, i.e., there exists a finite δ-pseudo-orbit $\{x_i\}_{i=0}^n \subset \Lambda$ such that $x = x_0 = x_n$, $n > 0$.

Let e be the expansive constant of f, let $0 < \epsilon \leq e/2$ be fixed and let $\delta > 0$ be given by the shadowing property on Λ. Let $\{x_i\}_{i=0}^N \subset \Lambda$ be a δ-chain from x to itself. The sequence $\{\tilde{x}_i\}_{i \in \mathbb{Z}}$ defined by $\tilde{x}_i = x_i$ if $n \equiv i \pmod{N}$ is a δ-pseudo orbit. Therefore, it is ϵ-shadowed by some point $z \in \Lambda$ and, clearly, also by $f^N(z) \in \Lambda$. Applying the triangle inequality, we have that, $d(f^n(z), f^{n+N}(z)) \leq 2\epsilon \leq e$, for all $n \in \mathbb{Z}$. Hence, by the expansive property on Λ, we obtain that $z = f^N(z)$. Thus, $z \in \Lambda$ is a periodic point such that $d(z, x) = d(z, x_0) \leq \epsilon$. $\square$

2.3. **Local product structure.** Given $\epsilon > 0$, let $\Delta(\epsilon) := \{(x, y) : x, y \in \Lambda$ and $d(x, y) \leq \epsilon\}$. We say that a homeomorphism f has a *local product structure* (LPS) on Λ if

(A) there are $\delta_0 > 0$ and a continuous function

$$[\cdot, \cdot] \colon \quad \Delta(\delta_0) \quad \longrightarrow \quad \Lambda$$
$$(x, y) \quad \longmapsto \quad [x, y]$$

such that for all $x, y, z \in \Lambda$ we have

$$[x, x] = x, \ [[x, y], z] = [x, z], \ [x, [y, z]] = [x, z] \text{ and } f([x, y]) = [f(x), f(y)],$$

whenever defined.

(B) There exist $\delta_1 \in (0, \delta_0/2)$ and $\rho \in (0, \delta_1)$ such that for all $x \in \Lambda$ the following three conditions hold:
  (i) denoting $V_{\delta_1}^\sigma(x) := \{y \in W_{\delta_0}^\sigma(x) \cap \Lambda : d(x, y) < \delta_1\}$ ($\sigma \in \{s, u\}$) we have that $[V_{\delta_1}^u(x), V_{\delta_1}^s(x)]$ is an open set of Λ with diameter less than δ_0;
  (ii) $[\cdot, \cdot] \colon V_{\delta_1}^u(x) \times V_{\delta_1}^s(x) \to [V_{\delta_1}^u(x), V_{\delta_1}^s(x)]$ is a homeomorphism and
  (iii) $[V_{\delta_1}^u(x), V_{\delta_1}^s(x)] \supset \{y \in \Lambda : d(x, y) \leq \rho\}$.

When f is a hyperbolic homeomorphism on Λ we shall set $[x, y] := W_\eta^s(x) \cap W_\eta^u(y)$, where $\eta = \frac{e}{4}$ and e is an expansive constant for f (and x is near y).

Lemma 2.7. *If f is a hyperbolic homeomorphism on Λ, then f has a local product structure on Λ.*

Proof. We borrow ideas from [2, Theorem 5.6] and next we adapt it to the local case. Let $e > 0$ be an expansive constant for f and fix $\eta = \frac{e}{4}$. By Corollary 2.5, there exists $0 < \delta_0 < \eta$ such that $W_\eta^s(x) \cap W_\eta^u(y) = \{\text{one point}\} \subset \Lambda$ for $x, y \in \Lambda$ with $d(x, y) \leq \delta_0$. We define the map $[\cdot, \cdot] \colon \Delta(\delta_0) \to \Lambda$ by $[x, y] := W_\eta^s(x) \cap W_\eta^u(y)$, for $(x, y) \in \Delta(\delta_0)$.

The map $[\cdot, \cdot]$ is continuous. Indeed, let $\{(x_n, y_n)\} \subset \Delta(\delta_0)$ be a sequence that converges to $(x, y) \in \Delta(\delta_0)$. Let $z_n = [x_n, y_n]$. Since Λ is compact, taking subsequences, we may assume that $(z_n)_n$ converges to $z \in \Lambda$. Since $z_n \in W_\eta^s(x_n)$, we have that $d(f^i(x_n), f^i(z_n)) \leq \eta$ ($i \geq 0$). Hence, $d(f^i(x), f^i(z)) \leq \eta$ and, therefore, $z \in W_\eta^s(x)$. Analogously, $z \in W_\eta^u(y)$. Thus $z = [x, y]$ which shows that $(z_n)_n = ([x_n, y_n])_n$ converges to $[x, y]$.

Clearly, $[x, x] = x$ for all $x \in \Lambda$. Since $[x, y] \in W_\eta^s(x)$, we have that $[[x, y], z] \in W_{2\eta}^s(x) \cap W_\eta^u(z)$. Hence, by expansivity on Λ, $[[x, y], z] = [x, z]$. Analogously, $[x, [y, z]] = [x, z]$. By uniform

continuity of f on Λ, it is easy to conclude that $f[x,y] = [f(x), f(y)]$. All together this proves (A) in the definition of LPS on Λ.

We shall now prove (B) (i)-(iii) in the definition of LPS on Λ. We define a map $g_1 \colon \Lambda \times \Delta(\delta_0) \to \mathbb{R}$ by $g_1(x, (y, z)) := d(x, [y, z])$, for $x \in \Lambda$ and $(y, z) \in \Delta(\delta_0)$. Clearly, g_1 is continuous and $g_1(x, (x, x)) = 0$. Since g_1 is uniformly continuous, there exists $\delta_1 \in (0, \delta_0/2)$ such that $\mathrm{diam}\{x, y, z\} < 2\,\delta_1$ implies $d(x, [y, z]) < \delta_0/3$. Therefore, given $(y, z) \in V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$, we have that $d(x, [y, z]) < \delta_0/3$. Take $w_1, w_2 \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$. Then there exists $y_1, y_2 \in V^u_{\delta_1}(x)$ and $z_1, z_2 \in V^s_{\delta_1}(x)$ such that $w_1 = [y_1, z_1]$ and $w_2 = [y_2, z_2]$. We have that $d(w_1, w_2) = d([y_1, z_1], [y_2, z_2]) \le d(x, [y_1, z_1]) + d(x, [y_2, z_2]) \le \delta_0/3 + \delta_0/3 < \delta_0$. Hence, the diameter of $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is smaller than δ_0. To show that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is open in Λ, let $w \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$. Then there exists $y \in V^u_{\delta_1}(x)$ and $z \in V^s_{\delta_1}(x)$ such that $w = [y, z]$. Since $\mathrm{diam}\{x, y, z\} < 2\,\delta_1$ we have that $d(x, w) < \delta_0/3$. Thus, we can define the maps $P_u \colon \mathcal{B}_{\delta_0/3}(w) \cap \Lambda \to W^u_\eta(x) \cap \Lambda$ and $P_s \colon \mathcal{B}_{\delta_0/3}(w) \cap \Lambda \to W^s_\eta(x) \cap \Lambda$ by $P_u(v) := [v, x]$ and $P_s(v) := [x, v]$, for $v \in \mathcal{B}_{\delta_0/3}(w)$. These maps are clearly continuous. Observe that, given $v \in \mathcal{B}_{\delta_0/3}(w) \cap \Lambda$, we have that $d(x, v) < d(x, w) + d(w, v) < \delta_0/3 + \delta_0/3 < \delta_0$. Given that $w = [y, z]$, we have $P_u(w) = y$ and $P_s(w) = z$, by expansivity on Λ. Hence, there is a neighborhood $U \subset \mathcal{B}_{\delta_0/3}(w) \cap \Lambda$ of w in Λ such that $P_u(U) \subset V^u_{\delta_1}(x)$ and $P_s(U) \subset V^s_{\delta_1}(x)$. Take $v \in U$. By expansivity on Λ, we have that $v = [[v, x], [x, v]]$. Therefore, $v \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$, which proves that $[V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$ is an open set of Λ. Thus, B(i) is proved.

To prove B(ii), define a map $h \colon [V^u_{\delta_1}(x), V^s_{\delta_1}(x)] \to V^u_{\delta_1}(x) \times V^s_{\delta_1}(x)$ by $h(w) := ([w, x], [x, w])$, for $w \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$. Clearly, h is continuous and h itself is the inverse map of $[\cdot, \cdot]$.

Finally we shall prove B(iii). Define the map $g_2 \colon \Delta(\delta_0) \to \mathbb{R}$ by $g_2(x, y) := \mathrm{diam}\{x, [y, x], [x, y]\}$, for $(x, y) \in \Delta(\delta_0)$. Then g_2 is (uniformly) continuous and, therefore, there exists $\rho \in (0, \delta_1)$ such that $d(x, y) < \rho$ implies $g_2(x, y) < \delta_1$. Hence, $[y, x] \in V^u_{\delta_1}(x)$ and $[x, y] \in V^s_{\delta_1}(x)$. Therefore, given $y \in \Lambda$ with $d(x, y) < \rho$, we have that $y = [[y, x], [x, y]] \in [V^u_{\delta_1}(x), V^s_{\delta_1}(x)]$, which proves B(iii).

$\square$

3. Topological homoclinic classes

3.1. **Topological transversality.** The intersection between two curves in the plane can be quite odd. Indeed, given any compact set $K \subset \mathbb{R}$, let us define the continuous function $f \colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \min\{|x - k| \colon k \in K\}$. The intersection between the 1-dimensional manifolds defined by the graph of f and the x-axis is the set $(k, 0) \in \mathbb{R}^2$ where $k \in K$. Let $B(0, r) \subset \mathbb{R}^2$ stands for the open ball centered in $(0, 0)$ and with radius r. Given 1-dimensional manifolds $N_1, N_2 \subset \mathbb{R}^2$, we say that $q \in \mathbb{R}^2$ is a *transversal intersection* between N_1 and N_2 if there exist an open ball B containing q and a homeomorphism $h \colon B(0, r) \to B$ where $r > 0$ such that $h(x, 0) = \hat{N}_1$ and $h(0, y) = \hat{N}_2$, where $\hat{N}_i$ are the connected components of $N_i \cap B$ containing q.

In the sequel we apply the previous concept of transversality to stable and unstable topological manifolds. Clearly, if $q \in W^s(p) \cap W^u(p)$ is a transversal intersection between the stable and unstable manifold of p, then this intersection is persistent in the sense that the intersection between these two 1-dimensional manifolds cannot disappear under an arbitrarily small C^0-perturbation of the map. We denote that $q \in W^s(p) \cap W^u(p)$ is a transversal intersection by writing $q \in W^s(p) \pitchfork W^u(p)$.

3.2. **Definition of topological homoclinic classes.** Let $f: M \to M$ be an area-preserving home-omorphism and $p \in M$ a periodic point of period n.

We say that p is *topologically hyperbolic* if f is a hyperbolic homeomorphism on $\Lambda = O(p)$, the orbit of p.

By Proposition 2.1, under the expansiveness hypothesis on Λ, stable and unstable sets of points $x \in \Lambda$ are dynamically defined. Clearly, by Remark 2.1 any stable/unstable structure at topological hyperbolic periodic points must be a 2-prongs structure.

We define the *topological homoclinic class* of a topological hyperbolic periodic point p by the closure of the transversal intersections of the stable and unstable manifolds of $f^i(p)$, for $i = 0, \ldots, n-1$, that is by $\overline{W^s(f^i(p)) \pitchfork W^u(f^i(p))}$, for $i = 0, \ldots, n-1$. Clearly, a homoclinic class is compact and f-invariant. The compactness follows from the fact that it is a closed subset of a compact manifold. To check that $f(\Lambda) = \Lambda$, observe that if we have $x \in \Lambda$, then $x_n \to x$ for a sequence $\{x_n\}_n \subset W^s(p) \cap W^u(p)$. By the f-invariance of the sets $W^s(p)$ and $W^u(p)$ we get that $\{f(x_n)\}_n \subset W^s(p) \cap W^u(p)$. Finally, the continuity of f ensures that $f(x) \in \Lambda$.

3.3. **The Birkhoff-Smale theorem.** In Theorem 3.2 we will obtain a slightly different version of the well-known Birkhoff-Smale theorem but with a topological flavor. The fixed point index will play a crucial role along the proof since, in rough terms, the existence of non-null index on a set assures a fixed point in that set. Let us recall the definition of fixed point index: take an open ball $B \subset M$ such that $\text{Fix}(f, \partial B) = \emptyset$ (meaning that f has no fixed points in the boundary of B denoted by ∂B) and $\overline{B} \cap \overline{f(B)} \neq \emptyset$ and ∂B, $\overline{B} \cup \overline{f(B)}$ are labelled by the same chart. In this case we say that the index of f in B is the degree of the map defined in chart coordinates by the vector field:

$$X_f: \quad \partial B \simeq \mathbb{S}^1 \quad \longrightarrow \quad \mathbb{S}^1$$
$$x \quad \longmapsto \quad \frac{f(x)-x}{\|f(x)-x\|}$$

If the index is $\neq 0$, then f has a fixed point in B. When $\overline{B} \cap \overline{f(B)} = \emptyset$ we say that the index is zero.

In the proof of Theorem 3.2 we will need a weak version of the well-known Lambda-lemma ([7]). This lemma tells us that if Σ is a section transversal to the stable manifold, then $f^n(\Sigma)$ $(n > 0)$ becomes arbitrarily C^1 close to some compact subset contained in the unstable manifold. The difficult part in the proof of the classical Lambda-Lemma is to obtain the C^1 closeness of the two sets. Indeed, we can obtain easily the next:

Lemma 3.1. *(Topological Lambda-Lemma) Let Σ and $W^s(p)$ be transversal topological manifolds and $\Gamma \subset W^u(p)$ be a compact set. Then, given any $\epsilon > 0$, there exist $\Sigma' \subset \Sigma$ and $n_0 \in \mathbb{N}$ such that $d_H(f^{n_0}(\Sigma'), \Gamma) < \epsilon$ (where d_H stands for the Hausdorff distance between sets).*

Now we are ready to state the main result in this section.

Theorem 3.2. *(Topological Birkhoff-Smale theorem) Let M be a surface, $f: M \to M$ a homeomorphism, $p \in M$ a periodic topological hyperbolic point of period n and $q \in W^s(p) \pitchfork W^u(p)$. Then f has a periodic point in any neighborhood of q.*

Proof. Let $q \in W^s(p) \pitchfork W^u(p)$ and let $\Gamma \subset W^u(p)$ be the compact arc with extremes p and q. By hypothesis there exists a section Σ which is transversal to $W^s(p)$ and containing q. Then, by Lemma 3.1, given any $\epsilon > 0$, there exist $\Sigma' \subset \Sigma$ and $n_0 \in \mathbb{N}$ such that $d_H(f^{n_0}(\Sigma'), \Gamma) < \epsilon$. Take a rectangle R like in Figure 4 sufficiently thin and sufficiently close to $W^u(p)$ in order to contain Σ'. We obtain $R \cap f^{n_0}(R) \neq \emptyset$. Moreover, we can even assure that $\text{Fix}(f^{n_0}, \partial R) = \emptyset$. Indeed, the
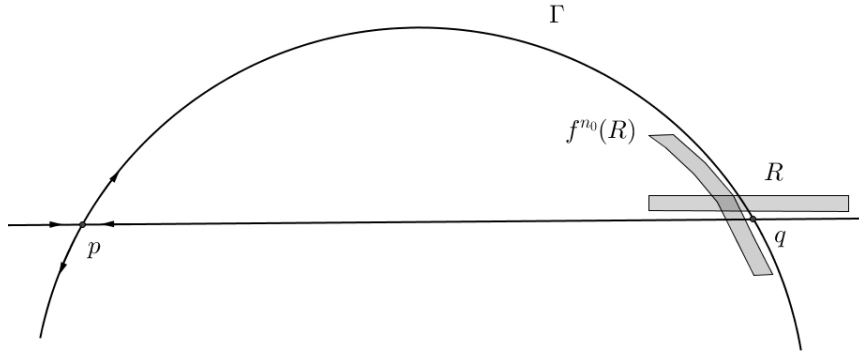
FIGURE 4. Support to the proof of Theorem 3.2.

top and the bottom of R, in a configuration like in Figure 4, turn into the bottom and the top of $f^{n_0}(R)$ respectively. Of course that a configuration like in Figure 4 can be achieved by allowing the change of R a little bit. Since ∂R is homeomorphic to $\mathbb{S}^1$ we let $X_{f^{n_0}}$ be a vector field defined in ∂R as above. The angular variation of $X_{f^{n_0}}$ as x moves once around ∂R is not zero, then f^{n_0} has a fixed point in R. Consequently, f has a periodic point in any neighborhood of q.

$\square$

Remark 3.1. *We point out that the classic Birkhoff-Smale theorem ([11]) ensures that the periodic points in any neighborhood of q are homoclinic related with p. This allows us to conclude that the periodic orbits in the homoclinic class are dense in the homoclinic class. However, the arguments used in [11] are clearly not valid outside the differentiable context. An alternative approach using tubular family theorems (cf. [8]) is also not suitable under C^0 hypothesis.*

4. PROOF OF THEOREM 1

Let M be a surface, $f \colon M \to M$ be an area-preserving homeomorphism and let $\Lambda \subseteq M$ be a topological homoclinic class of f. We already know by Corollary 2.5 that, if f is a hyperbolic homeomorphism on Λ with expansive constant e, then for each $0 < \epsilon \le e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then

$$W_\epsilon^s(x) \cap W_\epsilon^u(y) = \{\text{one point}\} \subset \Lambda. \tag{4.1}$$

So, if Λ is a topological homoclinic class displaying shadowing and expansiveness with expansive constant e, then for each $0 < \epsilon \le e/2$ there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then (4.1) holds. Since Λ is a homoclinic class it is closed by definition. As M is connected it is sufficient to prove that Λ is open in M.

The next result will be central in the proof of Theorem 1.

Lemma 4.1. *If $x \in \Lambda$ then $\Lambda \cap W^u(x)$ is dense in $W^u(x)$ and $\Lambda \cap W^s(x)$ is dense in $W^s(x)$.*

Proof. The proof is by contradiction. We assume that $\Lambda \cap W^u(x)$ is not dense in $W^u(x)$. Denote by $I \subset W^u(x)$ a *gap interval* in $W^u(x) \setminus \Lambda$ that is, its extremes are in Λ when I is bounded (from one side, from the other or both).

Case 1: We begin by considering that $I = [y_1, y_2]$ with $y_1, y_2 \in \Lambda$. In §2.3 we considered small constants like $\epsilon, \delta_0, \delta_1$ and ρ because the arguments involving the LPS hold only in a microscopic level. Therefore, we iterate backward I to be sufficiently close to x and so be under the hypotheses of §2.3. Clearly, an iterate of I will still be a gap interval. Using Proposition 2.6, we consider a periodic point $z \in \Lambda$ arbitrarily close to x and such that $W^u(O(x)) \cap W^u(O(z)) = \emptyset$. Since $y_1, y_2 \in \Lambda$, by Corollary 2.5, we obtain that $W_\epsilon^s(y_1)$ intersects $W_\epsilon^u(z)$ in a single point $x_1 \in \Lambda$ and $W_\epsilon^s(y_2)$ intersects $W_\epsilon^u(z)$ in a single point $x_2 \in \Lambda$. Let R be the rectangle with sides I, the segment I_1 with extremes y_1 and x_1, the segment I_2 with extremes y_2 and x_2 and the segment I_3 with extremes x_1 and x_2. To assure that $R \cap \Lambda = \emptyset$ we can go up in the proof and pick other periodic point z closer to x building another rectangle R satisfying $R \cap \Lambda = \emptyset$. Indeed, if such z do not exists, then there would a point $w \in R \cap \Lambda$ such that $W_\epsilon^s(w) \pitchfork W_\epsilon^u(x) \in I$ which contradicts the fact that I is a gap interval. From Poincaré's recurrence theorem λ-almost every point in R is recurrent and so $f^n(R) \cap R \neq \emptyset$ for infinitely many choices of n. Moreover, since f is an area-preserving map, we obtain $\partial R \cap \partial f^n(R) \neq \emptyset$. In fact, preservation of area avoids $R \subset f^n(R)$ or $R \supset f^n(R)$ and so the intersection of boundaries is inevitable. Furthermore, from stable/unstable sets arguments we get $f^n(I) \cap I = \emptyset$, $f^n(I) \cap I_3 = \emptyset$, $f^n(I_3) \cap I = \emptyset$, $f^n(I_3) \cap I_3 = \emptyset$, $f^n(I_1) \cap I_1 = \emptyset$, $f^n(I_1) \cap I_2 = \emptyset$, $f^n(I_2) \cap I_2 = \emptyset$ and $f^n(I_2) \cap I_1 = \emptyset$. Therefore, we must have an intersection between f^n iterates of the segments I, I_3 with the segments I_1, I_2, i.e., $f^n(I \cup I_3) \cap (I_1 \cup I_2) \neq \emptyset$, contradicting the fact that I was a gap interval.

Case 2: Finally, we consider the case of having an unbounded interval $I = [y_1, +\infty) \subset W^u(x)$ with $y_1 \in \Lambda$. Once again we pull I near x. Clearly, an iterate of I will still be an unbounded interval. We take a periodic point $z \in \Lambda$ arbitrarily close to x. Since $z \in \Lambda$, by Corollary 2.5, we get that $W_\epsilon^u(z) \cap W_\epsilon^s(x) \neq \emptyset$ and $W_\epsilon^u(z) \cap W_\epsilon^s(y_1) \neq \emptyset$ and, moreover, each of these intersections is a single point in the topological homoclinic class Λ.

Therefore, in $W^u(z)$ and since z is periodic, if exists, a gap interval cannot be unbounded and we can apply Case 1 and conclude that $W^u(z) \cap \Lambda$ is dense in $W^u(z)$. Since, by Proposition 2.6, we can take periodic points $z_n \in \Lambda$ such that $W^u(z_n) \cap \Lambda$ is dense in $W^u(z_n)$, we obtain that $I \cap \Lambda \neq \emptyset$ which is a contradiction.

$\square$

Finally we prove that Λ is open in M. By Lemma 2.7, f has a local product structure on Λ. Consider $\delta_0 > 0$ and $\delta_1 \in (0, \delta_0/2)$ given by the local product structure on Λ. We have that $[V_{\delta_1}^u(x), V_{\delta_1}^s(x)]$ is an open set of Λ with diameter less than δ_0 and also $[\cdot, \cdot]: V_{\delta_1}^u(x) \times V_{\delta_1}^s(x) \to [V_{\delta_1}^u(x), V_{\delta_1}^s(x)]$ is a homeomorphism.

By Lemma 4.1 the set $V_{\delta_1}^u(x)$ is dense in $W_{\delta_1}^u(x)$ and the set $V_{\delta_1}^s(x)$ is dense in $W_{\delta_1}^s(x)$. Clearly, the closure of the Cartesian product $V_{\delta_1}^u(x) \times V_{\delta_1}^s(x)$ is a topological disk and a neighborhood of x. Since $[\cdot, \cdot]: V_{\delta_1}^u(x) \times V_{\delta_1}^s(x) \to [V_{\delta_1}^u(x), V_{\delta_1}^s(x)]$ is a homeomorphism, then $[\cdot, \cdot](\overline{V_{\delta_1}^u(x) \times V_{\delta_1}^s(x)}) = \overline{[V_{\delta_1}^u(x), V_{\delta_1}^s(x)]}$. Hence, x is in the interior of Λ in M. Thus, Λ is open in M.

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