# Algebras of Singular Integral Operators with Piecewise Continuous Coefficients on Weighted Nakano Spaces

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**Abstract.** We find Fredholm criteria and a formula for the index of an arbitrary operator in the Banach algebra of singular integral operators with piecewise continuous coefficients on Nakano spaces (generalized Lebesgue spaces with variable exponent) with Khvedelidze weights over either Lyapunov curves or Radon curves without cusps. These results "localize" the Gohberg-Krupnik Fredholm theory with respect to the variable exponent.

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## 1. Introduction

The study of one-dimensional singular integral operators (SIOs) with piecewise continuous ($PC$) coefficients on weighted Lebesgue spaces was started by Khvedelidze in the fifties and then was continued in the sixties by Widom, Simonenko, Gohberg, Krupnik, and others. The starting point for those investigations was the sufficient conditions for the boundedness of the Cauchy singular integral operator $S$ on Lebesgue spaces with power weights over Lyapunov curves proved in 1956 by Khvedelidze [27]. Gohberg and Krupnik constructed the Fredholm theory for SIOs with $PC$ coefficients under the assumptions of the Khvedelidze theorem and this theory is the heart of their monograph [16] first published in Russian in 1973 (see also the monographs [6, 20, 33, 35, 36]). In the same year Hunt, Muckenhoupt, and Wheeden proved that for the boundedness of $S$ on $L^p(\mathbb{T}, w)$ it is necessary and sufficient that the weight $w$ belongs to the so-called Muckenhoupt class $A_p(\mathbb{T})$,

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here $\mathbb{T}$ denotes the unit circle. In 1982 David proved that $S$ is bounded on $L^2$ over a rectifiable curve if and only if the curve is a Carleson curve. After some hard analysis one can conclude, finally, that $S$ is bounded on a weighted Lebesgue space over a rectifiable curve if and only the weight belongs to a Carleson curve analog of the Muckenhoupt class (see [11], [2] and also [36]). In 1992 Spitkovsky [43] made the next significant step after Gohberg and Krupnik (20 years later!): he proved Fredholm criteria for an individual SIO with $PC$ coefficients on Lebesgue spaces with Muckenhoupt weights over Lyapunov curves. Finally, Böttcher and Yu. Karlovich extended Spitkovsky's result to the case of arbitrary Carleson curves and Banach algebras of SIOs with $PC$ coefficients. With their work the Fredholm theory of SIOs with $PC$ coefficients is available in the maximal generality (that, is, when the Cauchy singular integral operator $S$ is bounded on weighted Lebesgue spaces). We recommend the nice paper [3] for a first reading about this topic and [2] for a complete and self-contained analysis (see also [4]).

It is quite natural to consider the same problems in other, more general, spaces of measurable functions on which the operator $S$ is bounded. Good candidates for this role are rearrangement-invariant spaces (that is, spaces with the property that norms of equimeasurable functions are equal). These spaces have nice interpolation properties and boundedness results can be extracted from known results for Lebesgue spaces applying interpolation theorems. The author extended (some parts of) the Böttcher-Yu. Karlovich Fredholm theory of SIOs with $PC$ coefficients to the case of rearrangement-invariant spaces with Muckenhoupt weights [22, 24]. Notice that necessary conditions for the Fredholmness of an individual singular integral operator with $PC$ coefficients are obtained in [25] for weighted reflexive Banach function spaces (see [1, Ch. 1]) on which the operator $S$ is bounded.

Nakano spaces $L^{p(\cdot)}$ (generalized Lebesgue spaces with variable exponent) are a nontrivial example of Banach function spaces which are not rearrangement-invariant, in general. Many results about the behavior of some classical operators on these spaces have important applications to fluid dynamics (see [10] and the references therein). Recently Kokilashvili and S. Samko proved [29] that the operator $S$ is bounded on weighted Nakano spaces for the case of nice curves, nice weights, and nice (but variable!) exponents. They also extended the Gohberg-Krupnik Fredholm criteria for an individual SIO with $PC$ coefficients to this situation [30]. So, Nakano spaces are a natural context for the "localization" of the Gohberg-Krupnik theory with respect to the variable exponent. In this paper we proved Fredholm criteria and a formula for the index of an arbitrary operator in the Banach algebra of SIOs with $PC$ coefficients on Nakano spaces (generalized Lebesgue spaces with variable exponent) with Khvedelidze weights over either Lyapunov curves or Radon curves without cusps. These results generalize [30] (see also [25]) to the case of Banach algebras and the results of [15] (see also [14]) to the case of variable exponents (notice also that Radon curves were not considered in [15]). Basically, under the assumptions of the theorem of Kokilashvili and Samko, we can replace the constant exponent $p$ by the value of the variable exponent $p(t)$ at each point $t$ of the contour of integration in the Gohberg-Krupnik Fredholm theory [15].

The paper is organized as follows. In Section 2 we define weighted Nakano spaces and discuss the boundedness of the Cauchy singular integral operator $S$ on weighted Nakano spaces. Section 3 contains Fredholm criteria for an individual SIO with $PC$ coefficients on weighted Nakano spaces. In Section 4 we formulate the Allan-Douglas local principle and the two projections theorem. The results of Section 4 are the main tools allowing us to construct the symbols calculus for the Banach algebra of SIOs with $PC$ coefficients in Section 5. Finally, in Section 6, we prove an index formula for an arbitrary operator in the Banach algebra of SIOs with $PC$ coefficients acting on a Nakano space with a Khvedelidze weight over either a Lyapunov curve or a Radon curve without cusps.

## 2. Preliminaries

### 2.1. The Cauchy singular integral

Let $\Gamma$ be a Jordan (i.e., homeomorphic to a circle) rectifiable curve. We equip $\Gamma$ with the Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. The *Cauchy singular integral* of a measurable function $f : \Gamma \to \mathbb{C}$ is defined by

$$(Sf)(t) := \lim_{R\to 0} \frac{1}{\pi i} \int_{\Gamma\setminus\Gamma(t,R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),$$

where the "portion" $\Gamma(t, R)$ is

$$\Gamma(t, R) := \{\tau \in \Gamma : |\tau - t| < R\} \quad (R > 0).$$

It is well known that $(Sf)(t)$ exists almost everywhere on $\Gamma$ whenever $f$ is integrable (see [11, Theorem 2.22]).

### 2.2. Weighted Nakano spaces $L^{p(\cdot)}$

Function spaces $L^{p(\cdot)}$ of Lebesgue type with variable exponent $p$ were studied for the first time by Orlicz [42] in 1931, but notice that other kind of Banach spaces are named after him. Inspired by the successful theory of Orlicz spaces, Nakano defined in the late forties [40, 41] so-called *modular spaces*. He considered the space $L^{p(\cdot)}$ as an example of modular spaces. Musielak and Orlicz [38] in 1959 extended the definition of modular spaces by Nakano. Actually, that paper was the starting point for the theory of Musielak-Orlicz spaces (generalized Orlicz spaces generated by Young functions with a parameter), see [37].

Let $p : \Gamma \to [1, \infty)$ be a measurable function. Consider the convex modular (see [37, Ch. 1] for definitions and properties)

$$m(f, p) := \int_\Gamma |f(\tau)|^{p(\tau)} |d\tau|.$$

Denote by $L^{p(\cdot)}$ the set of all measurable complex-valued functions $f$ on $\Gamma$ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space with respect to the *Luxemburg-Nakano norm*

$$\|f\|_{L^{p(\cdot)}} := \inf\Big\{\lambda > 0 : \ m(f/\lambda, p) \le 1\Big\}$$

(see, e.g., [37, Ch. 2]). So, the spaces $L^{p(\cdot)}$ are a special case of Musielak-Orlicz spaces. Sometimes the spaces $L^{p(\cdot)}$ are referred to as Nakano spaces (see, e.g., [13, p. 151], [19, p. 179]). We will follow this tradition. Clearly, if $p(\cdot) = p$ is constant, then the Nakano space $L^{p(\cdot)}$ is isometrically isomorphic to the Lebesgue space $L^p$. Therefore, sometimes $L^{p(\cdot)}$ are called generalized Lebesgue spaces with variable exponent or, simply, variable $L^p$ spaces.

A nonnegative measurable function $w$ on the curve $\Gamma$ is referred to as a *weight* if $0 < w(t) < \infty$ almost everywhere on $\Gamma$. The *weighted Nakano space* is defined by

$$L^{p(\cdot)}_w = \Big\{f \text{ is measurable on } \Gamma \text{ and } fw \in L^{p(\cdot)}\Big\}.$$

The norm in this space is defined as usual by $\|f\|_{L^{p(\cdot)}_w} = \|fw\|_{L^{p(\cdot)}}$.

### 2.3. Carleson, Lyapunov, and Radon curves

A rectifiable Jordan curve $\Gamma$ is said to be a *Carleson* (or *Ahlfors-David regular*) *curve* if

$$\sup_{t\in\Gamma}\sup_{R>0}\frac{|\Gamma(t,R)|}{R} < \infty,$$

where $|\Omega|$ denotes the measure of a measurable set $\Omega \subset \Gamma$. Much information about Carleson curves can be found in [2].

On a rectifiable Jordan curve we have $d\tau = e^{i\theta_\Gamma(\tau)}|d\tau|$ where $\theta_\Gamma(\tau)$ is the angle between the positively oriented real axis and the naturally oriented tangent of $\Gamma$ at $\tau$ (which exists almost everywhere). A rectifiable Jordan curve $\Gamma$ is said to be a *Lyapunov curve* if

$$|\theta_\Gamma(\tau) - \theta_\Gamma(t)| \le c|\tau - t|^\mu$$

for some constants $c > 0, \mu \in (0,1)$ and for all $\tau, t \in \Gamma$. If $\theta_\Gamma$ is a function of bounded variation on $\Gamma$, then the curve gamma $\Gamma$ is called a *Radon curve* (or a *curve of bounded rotation*). It is well known that Lyapunov curves are smooth, while Radon curves may have at most countable set of corner points or cusps. All Lyapunov curves and Radon curves without cusps are Carleson curves (see, e.g., [28, Section 2.3]).

### 2.4. Boundedness of the Cauchy singular integral operator

We shall assume that

$$1 < \operatorname*{ess\,inf}_{t\in\Gamma} p(t), \quad \operatorname*{ess\,sup}_{t\in\Gamma} p(t) < \infty. \tag{1}$$

In this case the conjugate exponent

$$q(t) := \frac{p(t)}{p(t)-1} \quad (t \in \Gamma)$$

has the same property.

Not so much is known about the boundedness of the Cauchy singular integral operator $S$ on weighted Nakano spaces $L^{p(\cdot)}_w$ for general curves, general weights, and general exponents $p(\cdot)$. From [25, Theorem 6.1] we immediately get the following.

**Theorem 2.1.** *Let $\Gamma$ be a rectifiable Jordan curve, let $w : \Gamma \to [0, \infty]$ be a weight, and let $p : \Gamma \to [0, \infty)$ be a measurable function satisfying* (1). *If the Cauchy singular integral generates a bounded operator $S$ on the weighted Nakano space $L^{p(\cdot)}_w$, then*

$$\sup_{t\in\Gamma} \sup_{R>0} \frac{1}{R} \|w\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \|\chi_{\Gamma(t,R)}/w\|_{L^{q(\cdot)}} < \infty. \tag{2}$$

From the Hölder inequality for Nakano spaces (see, e.g., [37] or [32]) and (2) we deduce that if $S$ is bounded on $L^{p(\cdot)}_w$, then $\Gamma$ is necessarily a Carleson curve. If the exponent $p(\cdot) = p \in (1, \infty)$ is constant, then (2) is simply the famous Muckenhoupt condition $A_p$ (written in the symmetric form):

$$\sup_{t\in\Gamma} \sup_{R>0} \frac{1}{R} \left( \int_{\Gamma(t,R)} w^p(\tau)|d\tau| \right)^{1/p} \left( \int_{\Gamma(t,R)} w^{-q}(\tau)|d\tau| \right)^{-1/q} < \infty,$$

where $1/p + 1/q = 1$. It is well known that for classical Lebesgue spaces $L^p$ this condition is not only necessary, but also sufficient for the boundedness of the Cauchy singular integral operator $S$. A detailed proof of this result can be found in [2, Theorem 4.15].

Consider now a power weight of the form

$$\varrho(t) := \prod_{k=1}^{N} |t - \tau_k|^{\lambda_k}, \quad \tau_k \in \Gamma, \quad k \in \{1, \dots, N\}, \quad N \in \mathbb{N}, \tag{3}$$

where all $\lambda_k$ are real numbers. Introduce the class $\mathcal{P}$ of exponents $p : \Gamma \to [1, \infty)$ satisfying (1) and

$$|p(\tau) - p(t)| \le \frac{A}{-\log|\tau - t|} \tag{4}$$

for some $A \in (0, \infty)$ and all $\tau, t \in \Gamma$ such that $|\tau - t| < 1/2$.

Criteria for the boundedness of the Cauchy singular integral operator on Nakano spaces with power weights (3) were recently proved by Kokilashvili and Samko [29] under the condition that the curve $\Gamma$ and the variable exponent $p(\cdot)$ are sufficiently nice.

**Theorem 2.2.** (see [29, Theorem 2]). *Let $\Gamma$ be either a Lyapunov Jordan curve or a Radon Jordan curve without cusps, let $\varrho$ be a power weight of the form* (3), *and let $p \in \mathcal{P}$. The Cauchy singular integral operator $S$ is bounded on the weighted Nakano space $L^{p(\cdot)}_\varrho$ if and only if*

$$0 < \frac{1}{p(\tau_k)} + \lambda_k < 1 \quad \textit{for all} \quad k \in \{1, \dots, N\}. \tag{5}$$

For weighted Lebesgue spaces this result is classic, for Lyapunov curves it was proved by Khvedelidze [27]. Therefore the weights of the form (3) are often called *Khvedelidze weights*. We shall follow this tradition. For Lebesgue spaces over Radon curves without cusps the above result was proved by Danilyuk and Shelepov [8, Theorem 2]. The proofs and history can be found in [7, 16, 28, 36].

Notice that if $p$ is constant and $\Gamma$ is a Carleson curve, then (5) is equivalent to the fact that $\varrho$ is a Muckenhoupt weight (see, e.g., [2, Chapter 2]). Analogously one can prove that if the exponent $p$ belong to the class $\mathcal{P}$ and the curve $\Gamma$ is Carleson, then the power weight (3) satisfies the condition (2) if and only if (5) is fulfilled. The proof of this fact is essentially based on the possibility of estimation of the norms of power functions in Nakano spaces with exponents in the class $\mathcal{P}$ (see also [25, Lemmas 5.7 and 5.8] and [29], [31]).

### 2.5. Is the condition $p \in \mathcal{P}$ necessary for the boundedness?

What can be said about the necessity of the condition $p \in \mathcal{P}$ in Theorem 2.2? We conjecture that this condition is not necessary, that is, the Cauchy singular integral operator can be bounded on $L^{p(\cdot)}_{\varrho}$, but $p$ does not satisfy (4). This conjecture is supported by the following observation made by Andrei Lerner [34].

It is well known that, roughly speaking, singular integrals can be controlled by maximal functions. Denote by $\mathcal{M}(\mathbb{R}^n)$ the class of exponents $p : \mathbb{R}^n \to [1, \infty)$ which are essentially bounded and bounded away from 1 and such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. Diening and Růžička [10, Theorem 4.8] proved that if $p \in \mathcal{M}(\mathbb{R}^n)$ and there exists $s \in (0, 1)$ such that $s/p(t) + 1/\widetilde{q}(t) = 1$ and $\widetilde{q} \in \mathcal{M}(\mathbb{R}^n)$, then the Calderón-Zygmund singular integral operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. A weighted analog of this theorem was used by Kokilashvili and Samko (see [29] and also [31]) to prove Theorem 2.2. Notice also that the author and Lerner [26, Theorem 2.7] proved that if $p, q \in \mathcal{M}(\mathbb{R}^n)$, then the Calderón-Zygmund singular integral operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. On the other hand, Diening [9] showed that the following conditions are equivalent:

(i) $p \in \mathcal{M}(\mathbb{R}^n)$;
(ii) $q \in \mathcal{M}(\mathbb{R}^n)$;
(iii) there exists $s \in (0, 1)$ such that $s/p(t) + 1/\widetilde{q}(t) = 1$ and $\widetilde{q} \in \mathcal{M}(\mathbb{R}^n)$.

So, $p \in \mathcal{M}(\mathbb{R}^n)$ implies the boundedness of the Calderón-Zygmund singular integral operator on $L^{p(\cdot)}(\mathbb{R}^n)$.

Lerner [34], among other things, observed that

$$p(x) = \alpha + \sin(\log\log(1/|x|)\chi_E(x)),$$

where $\alpha > 2$ is some constant and $\chi_E$ is the characteristic function of the ball $E := \{x \in \mathbb{R}^n : |x| \le 1/e\}$, belongs to $\mathcal{M}(\mathbb{R}^n)$. Clearly, the exponent $p$ in this example is discontinuous at the origin, so it does not satisfy (an $\mathbb{R}^n$ analog of) the condition (4). This exponent belongs to the class of pointwise multipliers for $BMO$ (the space of functions of bounded mean oscillation). For descriptions of pointwise multipliers for $BMO$, see Stegenga [44], Janson [18] (local case) and Nakai, Yabuta [39] (global case). So, we strongly believe that necessary and sufficient conditions for the boundedness of the Cauchy singular integral operator (and other singular integrals and maximal functions) on Nakano spaces $L^{p(\cdot)}$ should be formulated in terms of integral means of the exponent $p$ (i.e., in $BMO$ terms), but not in pointwise terms like (4).

## 3. Fredholm criteria

### 3.1. Fredholm operators

A bounded linear operator $A$ on a Banach space is said to be Fredholm if its image is closed and both so-called defect numbers

$$n(A) := \dim\ker A, \quad d(A) := \dim\ker A^*$$

are finite. In this case the difference $n(A) - d(A)$ is referred to as the index of the operator $A$ and is denoted by $\operatorname{Ind} A$. Basic properties of Fredholm operators are discussed in [5, 16, 20, 35, 36] and in many other monographs.

### 3.2. Singular integral operators with piecewise continuous coefficients

In the following we shall suppose that $\Gamma$ is either a Lyapunov Jordan curve or a Radon Jordan curve without cusps, the variable exponent $p$ belongs to the class $\mathcal{P}$, and the Khvedelidze weight (3) satisfies the conditions (5). Then, by Theorem 2.2, the operator $S$ is bounded on the weighted Nakano space $L^{p(\cdot)}_\varrho$. Let $I$ be the identity operator on $L^{p(\cdot)}_\varrho$. Put

$$P := (I + S)/2, \quad Q := (I - S)/2.$$

Let $L^\infty$ denote the space of all measurable essentially bounded functions on $\Gamma$. We denote by $PC$ the Banach algebra of all piecewise continuous functions on $\Gamma$: a function $a \in L^\infty$ belongs to $PC$ if and only if the finite one-sided limits

$$a(t \pm 0) := \lim_{\tau \to t \pm 0} a(\tau)$$

exist for every $t \in \Gamma$.

For $a \in PC$ denote by $aI$ the operator of multiplication by $a$. Obviously, it is bounded on $L^{p(\cdot)}_\varrho$. If $B$ is a bounded operator, then we will simply write $aB$ for the product $aI \cdot B$. The operators of the form $aP + bQ$ with $a, b \in PC$ are called *singular integral operators (SIOs) with piecewise continuous (PC) coefficients.*

**Theorem 3.1.** *The operator $aP+bQ$, where $a, b \in PC$, is Fredholm on the weighted Nakano space $L^{p(\cdot)}_\varrho$ if and only if*

$$a(t \pm 0) \neq 0, \quad b(t \pm 0) \neq 0, \quad -\frac{1}{2\pi}\arg\frac{g(t-0)}{g(t+0)} + \frac{1}{p(t)} + \lambda(t) \notin \mathbb{Z}$$

*for all $t \in \Gamma$, where $g = a/b$ and*

$$\lambda(t) := \begin{cases} \lambda_k, & \text{if} \quad t = \tau_k, \quad k \in \{1, \dots, N\}, \\ 0, & \text{if} \quad t \notin \Gamma \setminus \{\tau_1, \dots, \tau_N\}. \end{cases}$$

If $a, b$ have only finite numbers of jumps and $\varrho = 1$, this result was obtained in [30, Theorem A] (as well as a formula for the index of the operator $aP + bQ$). In the present form this result is contained in [25, Theorem 8.3]. For Lebesgue spaces with Khvedelidze weights over Lyapunov curves the corresponding result was obtained in the late sixties by Gohberg and Krupnik [16, Ch. 9].

### 3.3. Widom-Gohberg-Krupnik arcs

Given $z_1, z_2 \in \mathbb{C}$ and $r \in (0, 1)$, put

$$\mathcal{A}(z_1, z_2; r) := \{z_1, z_2\} \cup \left\{ z \in \mathbb{C} \setminus \{z_1, z_2\} : \arg \frac{z - z_1}{z - z_2} \in 2\pi r + 2\pi\mathbb{Z} \right\}.$$

This is a circular arc between $z_1$ and $z_2$ (which contains its endpoints $z_1$ and $z_2$). Clearly, $\mathcal{A}(z, z; \nu)$ degenerates to the point $\{z\}$ and $\mathcal{A}(z_1, z_2; 1/2)$ is the line segment between $z_1$ and $z_2$. A connection of these arcs to Fredholm properties of singular integral operators with piecewise continuous coefficients on $L^p(\mathbb{R})$ was first observed by Widom in 1960. Gohberg and Krupnik expressed their Fredholm theory of SIOs with $PC$ coefficients on Lebesgue spaces with Khvedelidze weights over piecewise Lyapunov curves in terms of these arcs. For more about this topic we refer to the books [5, 16, 20, 36], where the Gohberg-Krupnik Fredholm theory is presented; see also more recent monographs [2, 4], where generalizations of Widom-Gohberg-Krupnik arcs play an essential role in the Fredholm theory of Toeplitz operators with $PC$ symbols on Hardy spaces with Muckenhoupt weights.

Fix $t \in \Gamma$ and consider a function $\chi_t \in PC$ which is continuous on $\Gamma \setminus \{t\}$ and satisfies $\chi_t(t - 0) = 0$ and $\chi_t(t + 0) = 1$.

From Theorem 3.1 we immediately get the following.

**Corollary 3.2.** *We have*

$$\big\{\lambda \in \mathbb{C} : (\chi_t - \lambda)P + Q \text{ is not Fredholm on } L^{p(\cdot)}_\varrho\big\} = \mathcal{A}\big(0, 1; 1/p(t) + \lambda(t)\big).$$

## 4. Tools for the construction of the symbol calculus

### 4.1. The Allan-Douglas local principle

Let $B$ be a Banach algebra with identity. A subalgebra $Z$ of $B$ is said to be a central subalgebra if $zb = bz$ for all $z \in Z$ and all $b \in B$.

**Theorem 4.1.** (see [5, Theorem 1.34(a)]). *Let $B$ be a Banach algebra with unit $e$ and let $Z$ be closed central subalgebra of $B$ containing $e$. Let $M(Z)$ be the maximal ideal space of $Z$, and for $\omega \in M(Z)$, let $J_\omega$ refer to the smallest closed two-sided ideal of $B$ containing the ideal $\omega$. Then an element $b$ is invertible in $B$ if and only if $b + J_\omega$ is invertible in the quotient algebra $B/J_\omega$ for all $\omega \in M(Z)$.*

### 4.2. The two projections theorem

The following two projections theorem was obtained by Finck, Roch, Silbermann [12] and Gohberg, Krupnik [17].

**Theorem 4.2.** *Let $F$ be a Banach algebra with identity $e$, let $\mathcal{C} = \mathbb{C}^{n\times n}$ be a Banach subalgebra of $F$ which contains $e$, and let $p$ and $q$ be two projections in $F$ such that $cp = pc$ and $cq = qc$ for all $c \in \mathcal{C}$. Let $W = \operatorname{alg}(\mathcal{C}, p, q)$ be the smallest closed subalgebra of $F$ containing $\mathcal{C}, p, q$. Put*

$$x = pqp + (e-p)(e-q)(e-p),$$

*denote by $\operatorname{sp} x$ the spectrum of $x$ in $F$, and suppose the points $0$ and $1$ are not isolated points of $\operatorname{sp} x$. Then*

(a) *for each $\mu \in \operatorname{sp} x$ the map $\sigma_\mu$ of $\mathcal{C} \cup \{p, q\}$ into the algebra $\mathbb{C}^{2n\times 2n}$ of all complex $2n \times 2n$ matrices defined by*

$$\sigma_\mu c = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, \quad \sigma_\mu p = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \tag{6}$$

$$\sigma_\mu q = \begin{pmatrix} \mu E & \sqrt{\mu(1-\mu)}E \\ \sqrt{\mu(1-\mu)}E & (1-\mu)E \end{pmatrix}, \tag{7}$$

*where $c \in \mathcal{C}, E$ denotes the $n \times n$ unit matrix and $\sqrt{\mu(1-\mu)}$ denotes any complex number whose square is $\mu(1-\mu)$, extends to a Banach algebra homomorphism $\sigma_\mu : W \to \mathbb{C}^{2n\times 2n}$;*

(b) *an element $a \in W$ is invertible in $F$ if and only if $\det \sigma_\mu a \neq 0$ for all $\mu \in \operatorname{sp} x$;*

(c) *the algebra $W$ is inverse closed in $F$ if and only if the spectrum of $x$ in $W$ coincides with the spectrum of $x$ in $F$.*

A further generalization of the above result to the case of $N$ projections is contained in [2].

## 5. Algebra of singular integral operators

### 5.1. The ideal of compact operators

The curve $\Gamma$ divides the complex plane $\mathbb{C}$ into the bounded simply connected domain $D^+$ and the unbounded domain $D^-$. Without loss of generality we assume that $0 \in D^+$. Let $X_n := [L^{p(\cdot)}_\varrho]_n$ be a direct sum of $n$ copies of weighted Nakano spaces $X := L^{p(\cdot)}_\varrho$, let $\mathcal{B} := \mathcal{B}(X_n)$ be the Banach algebra of all bounded linear operators on $X_n$, and let $\mathcal{K} := \mathcal{K}(X_n)$ be the closed two-sided ideal of all compact operators on $X_n$. We denote by $C^{n\times n}$ (resp. $PC^{n\times n}$) the collection of all continuous (resp. piecewise continuous) $n \times n$ matrix functions, that is, matrix-valued functions with entries in $C$ (resp. $PC$). Put $I^{(n)} := \operatorname{diag}\{I, \dots, I\}$ and $S^{(n)} := \operatorname{diag}\{S, \dots, S\}$. Our aim is to get Fredholm criteria for an operator $A \in \mathcal{U} := \operatorname{alg}(PC^{n\times n}, S^{(n)})$, the smallest Banach subalgebra of $\mathcal{B}$ which contains all operators of multiplication by matrix-valued functions in $PC^{n\times n}$ and the operator $S^{(n)}$.

**Lemma 5.1.** *$\mathcal{K}$ is contained in* $\mathrm{alg}(C^{n\times n}, S^{(n)})$*, the smallest closed subalgebra of $\mathcal{B}$ which contains the operators of multiplication by continuous matrix-valued functions and the operator $S^{(n)}$.*

*Proof.* The proof of this statement is standard, here we follow the presentation in [21, Lemma 9.1]. First, notice that it is sufficient to prove the statement for $n = 1$. By [32, Theorem 2.3 and Corollary 2.7] (see also [37]), (1) is equivalent to the reflexivity of the Nakano space $L^{p(\cdot)}$. Then, in view of [25, Proposition 2.11], the set of all rational functions without poles on $\Gamma$ is dense in both weighted spaces $L^{p(\cdot)}_{\varrho}$ and $L^{q(\cdot)}_{1/\varrho}$. Hence $\{t^k\}_{k=-\infty}^{\infty}$ is a basis in $L^{p(\cdot)}_{\varrho}$ (we assumed that $0 \in D^+$), whence $L^{p(\cdot)}_{\varrho}$ has the approximating property: each compact operator on $L^{p(\cdot)}_{\varrho}$ can be approximated in the operator norm by finite-rank operators as closely as desired. So, it is sufficient to show that a finite-rank operator on $L^{p(\cdot)}_{\varrho}$ belongs to $\mathrm{alg}(C, S)$. Since $[L^{p(\cdot)}_{\varrho}]^* = L^{q(\cdot)}_{1/\varrho}$ (again see [32] or [37]), a finite-rank operator on $L^{p(\cdot)}_{\varrho}$ is of the form

$$(Kf)(t) = \sum_{j=1}^{m} a_j(t) \int_{\Gamma} b_j(\tau) f(\tau)\, d\tau, \quad t \in \Gamma, \tag{8}$$

where $a_j \in L^{p(\cdot)}_{\varrho}$ and $b_j \in L^{q(\cdot)}_{1/\varrho}$. Since $C$ is dense in $L^{p(\cdot)}_{\varrho}$ and in $L^{q(\cdot)}_{1/\varrho}$, one can approximate in the operator norm every operator of the form (8) by operators of the same form but with $a_j, b_j \in C$. Therefore it is sufficient to prove that the operator (8) with $a_j, b_j \in C$ belongs to $\mathrm{alg}(C, S)$. But the latter fact is obvious because

$$K = \sum_{j=1}^{m} a_j (S\chi I - \chi S) b_j I,$$

where $\chi(\tau) = \tau$ for $\tau \in \Gamma$. □

### 5.2. Operators of local type

We shall denote by $\mathcal{B}^\pi$ the Calkin algebra $\mathcal{B}/\mathcal{K}$ and by $A^\pi$ the coset $A + \mathcal{K}$ for any operator $A \in \mathcal{B}$. An operator $A \in \mathcal{B}$ is said to be of local type if $AcI^{(n)} - cA$ is compact for all $c \in C$, where $cI^{(n)}$ denotes the operator of multiplication by the diagonal matrix-valued function $\mathrm{diag}\{c, \dots, c\}$. It easy to see that the set $\mathcal{L}$ of all operators of local type is a closed subalgebra of $\mathcal{B}$.

**Proposition 5.2.** (a) *We have $\mathcal{K} \subset \mathcal{U} \subset \mathcal{L}$.*

(b) *An operator $A \in \mathcal{L}$ is Fredholm if and only if the coset $A^\pi$ is invertible in the quotient algebra $\mathcal{L}^\pi := \mathcal{L}/\mathcal{K}$.*

*Proof.* (a) The embedding $\mathcal{K} \subset \mathcal{U}$ follows from Lemma 5.1, the embedding $\mathcal{U} \subset \mathcal{L}$ follows from the fact that $cS - ScI$ is a compact operator on $L^{p(\cdot)}_{\varrho}$ for $c \in C$ (see, e.g., [25, Lemma 6.5]).

(b) Straightforward. □

### 5.3. Localization

From Proposition 5.2(a) we deduce that the quotient algebras $\mathcal{U}^\pi := \mathcal{U}/\mathcal{K}$ and $\mathcal{L}^\pi := \mathcal{L}/\mathcal{K}$ are well defined. We shall study the invertibility of an element $A^\pi$ of $\mathcal{U}^\pi$ in the larger algebra $\mathcal{L}^\pi$ by using the localization techniques (more precisely, Theorem 4.1). To this end, consider

$$\mathcal{Z}^\pi := \big\{(cI^{(n)})^\pi : c \in C\big\}.$$

From the definition of $\mathcal{L}$ it follows that $\mathcal{Z}^\pi$ is a central subalgebra of $\mathcal{L}^\pi$. The maximal ideal space $M(\mathcal{Z}^\pi)$ of $\mathcal{Z}^\pi$ may be identified with the curve $\Gamma$ via the Gelfand map $\mathcal{G}$ given by

$$\mathcal{G} : \mathcal{Z}^\pi \to C, \quad \big(\mathcal{G}(cI^{(n)})^\pi\big)(t) = c(t) \quad (t \in \Gamma).$$

In accordance with Theorem 4.1, for every $t \in \Gamma$ we define $\mathcal{J}_t \subset \mathcal{L}^\pi$ as the smallest closed two-sided ideal of $\mathcal{L}^\pi$ containing the set

$$\big\{(cI^{(n)})^\pi \ : \quad c \in C, \quad c(t) = 0\big\}.$$

Consider a function $\chi_t \in PC$ which is continuous on $\Gamma \setminus \{t\}$ and satisfies $\chi_t(t-0) = 0$ and $\chi_t(t+0) = 1$. For $a \in PC^{n\times n}$ define the function $a_t \in PC^{n\times n}$ by

$$a_t := a(t-0)(1-\chi_t) + a(t+0)\chi_t. \tag{9}$$

Clearly $(aI^{(n)})^\pi - (a_tI^{(n)})^\pi \in \mathcal{J}_t$. Hence, for any operator $A \in \mathcal{U}$, the coset $A^\pi + \mathcal{J}_t$ belongs to the smallest closed subalgebra $\mathcal{W}_t$ of $\mathcal{L}^\pi/\mathcal{J}_t$ containing the cosets

$$p := \big((I^{(n)} + S^{(n)})/2\big)^\pi + \mathcal{J}_t, \quad q := (\chi_t I^{(n)})^\pi + \mathcal{J}_t, \tag{10}$$

where $\chi_t I^{(n)}$ denotes the operator of multiplication by the diagonal matrix-valued function $\operatorname{diag}\{\chi_t, \dots, \chi_t\}$ and the algebra

$$\mathcal{C} := \big\{(cI^{(n)})^\pi + \mathcal{J}_t \ : \quad c \in \mathbb{C}^{n\times n}\big\}. \tag{11}$$

The latter algebra is obviously isomorphic to $\mathbb{C}^{n\times n}$, so $\mathcal{C}$ and $\mathbb{C}^{n\times n}$ can be identified to each other.

### 5.4. The spectrum of $pqp + (e-p)(e-q)(e-p)$

Since $P^2 = P$ on $L^{p(\cdot)}_\varrho$ (see, e.g., [25, Lemma 6.4]) and $\chi_t^2 - \chi_t \in C$, $(\chi_t^2 - \chi_t)(t) = 0$, it is easy to see that

$$p^2 = p, \quad q^2 = q, \quad pc = cp, \quad qc = cq \tag{12}$$

for every $c \in \mathcal{C}$, where $p, q$ and $\mathcal{C}$ are given by (10) and (11). To apply Theorem 4.2 to the algebras $F = \mathcal{L}^\pi/\mathcal{J}_t$ and $W = \mathcal{W}_t = \operatorname{alg}(\mathcal{C}, p, q)$, we have to identify the spectrum of

$$pqp + (e-p)(e-q)(e-p) = \big(P^{(n)}\chi_t P^{(n)} + Q^{(n)}(1-\chi_t)Q^{(n)}\big)^\pi + \mathcal{J}_t \tag{13}$$

in the algebra $F = \mathcal{L}^\pi/\mathcal{J}_\tau$, here $P^{(n)} := (I^{(n)} + S^{(n)})/2$ and $Q^{(n)} := (I^{(n)} - S^{(n)})/2$.

**Lemma 5.3.** *Let $\chi_t \in PC$ be a continuous function on $\Gamma \setminus \{t\}$ such that $\chi_t(t-0) = 0$, $\chi_t(\tau + 0) = 1$ and $\chi_t(\Gamma \setminus \{t\}) \cap \mathcal{A}(0,1;1/p(t)+\lambda(t)) = \emptyset$. Then the spectrum of* (13) *in the algebra $\mathcal{L}^\pi/\mathcal{J}_t$ coincides with $\mathcal{A}(0,1;1/p(t)+\lambda(t))$.*

*Proof.* Once we have at hand Corollary 3.2, the proof of this lemma can be developed by a literal repetition of the proof of [21, Lemma 9.4]. It is only necessary to replace the spiralic horn $\mathcal{S}(0,1;\delta_t;\alpha_M,\beta_M)$ in that proof by the Widom-Gohberg-Krupnik circular arc $\mathcal{A}(0,1;1/p(t)+\lambda(t))$. A nice discussion of the relations between (spiralic) horns and circular arcs and their role in the Fredholm theory of SIOs can be found in [2] and [3]. □

### 5.5. Symbol calculus

Now we are in a position to prove the main result of this paper.

**Theorem 5.4.** *Define the "arcs bundle"*

$$\mathcal{M} := \bigcup_{t\in\Gamma} \Big( \{t\} \times \mathcal{A}\big(0,1;1/p(t)+\lambda(t)\big)\Big).$$

(a) *for each point $(t,\mu) \in \mathcal{M}$, the map*

$$\sigma_{t,\mu} : \{S^{(n)}\} \cup \{aI^{(n)} : a \in PC^{n\times n}\} \to \mathbb{C}^{2n\times 2n},$$

*given by*

$$\sigma_{t,\mu}(S^{(n)}) = \begin{pmatrix} E & O \\ O & -E \end{pmatrix}, \quad \sigma_{t,\mu}(aI^{(n)}) = \begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{pmatrix},$$

*where*

$$\begin{aligned} a_{11}(t,\mu) &:= a(t+0)\mu + a(t-0)(1-\mu), \\ a_{12}(t,\mu) &= a_{21}(t,\mu) := (a(t+0)-a(t-0))\sqrt{\mu(1-\mu)}, \\ a_{22}(t,\mu) &:= a(t+0)(1-\mu) + a(t-0)\mu, \end{aligned}$$

*and $O$ and $E$ are the zero and identity $n\times n$ matrices, respectively, extends to a Banach algebra homomorphism*

$$\sigma_{t,\mu} : \mathcal{U} \to \mathbb{C}^{2n\times 2n}$$

*with the property that*

$$\sigma_{t,\mu}(K) = \begin{pmatrix} O & O \\ O & O \end{pmatrix}$$

*for every compact operator $K$ on $X_n$;*

(b) *an operator $A \in \mathcal{U}$ is Fredholm on $X_n$ if and only if*

$$\det \sigma_{t,\mu}(A) \neq 0 \quad \textit{for all} \quad (t,\mu) \in \mathcal{M};$$

(c) *the quotient algebra $\mathcal{U}^\pi$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi$, that is, if an arbitrary coset $A^\pi \in \mathcal{U}^\pi$ is invertible in $\mathcal{B}^\pi$, then $(A^\pi)^{-1} \in \mathcal{U}^\pi$.*

*Proof.* The idea of the proof of this theorem based on the Allan-Douglas local principle and the two projections theorem is borrowed from [2].

Fix $t \in \Gamma$ and choose a function $\chi_t \in PC$ such that $\chi_t$ is continuous on $\Gamma \setminus \{t\}$, $\chi_t(t-0) = 0$, $\chi_t(t+0) = 1$, and $\chi_t(\Gamma \setminus \{t\}) \cap \mathcal{A}(0,1;1/p(t)+\lambda(t)) = \emptyset$. From (12) and Lemma 5.3 we deduce that the algebras $\mathcal{L}^\pi/\mathcal{J}_t$ and $\mathcal{W}_t = \operatorname{alg}(\mathcal{C}, p, q)$, where $p, q$ and $\mathcal{C}$ are given by (10) and (11), respectively, satisfy all the conditions of the two projections theorem (Theorem 4.2).

(a) In view of Theorem 4.2(a), for every $\mu \in \mathcal{A}(0,1;1/p(t)+\lambda(t))$, the map $\sigma_\mu : \mathbb{C}^{n\times n} \cup \{p,q\} \to \mathbb{C}^{2n\times 2n}$ given by (6)–(7) extends to a Banach algebra homomorphism $\sigma_\mu : \mathcal{W}_t \to \mathbb{C}^{2n\times 2n}$. Then the map

$$\sigma_{t,\mu} = \sigma_\mu \circ \pi_t : \mathcal{U} \to \mathbb{C}^{2n\times 2n},$$

where $\pi_t : \mathcal{U} \to \mathcal{W}_t = \mathcal{U}^\pi/\mathcal{J}_t$ is acting by the rule $A \mapsto A^\pi + \mathcal{J}_t$, is a well defined Banach algebra homomorphism and

$$\sigma_{t,\mu}(S^{(n)}) = 2\sigma_\mu p - \sigma_\mu e = \begin{pmatrix} E & O \\ O & -E \end{pmatrix}.$$

If $a \in PC^{n\times n}$, then in view of (9) and $(aI^{(n)})^\pi - (a_t I^{(n)})^\pi \in \mathcal{J}_t$ it follows that

$$\begin{aligned} \sigma_{t,\mu}(aI^{(n)}) &= \sigma_{t,\mu}(a_t I^{(n)}) = \sigma_\mu(a(t-0))\sigma_\mu(e-q) + \sigma_\mu(a(t+0))\sigma_\mu q \\ &= \begin{pmatrix} a_{11}(t,\mu) & a_{12}(t,\mu) \\ a_{21}(t,\mu) & a_{22}(t,\mu) \end{pmatrix}. \end{aligned}$$

From Proposition 5.2(a) it follows that $\pi_t(K) = K^\pi + \mathcal{J}_t = \mathcal{J}_t$ for every $K \in \mathcal{K}$ and every $t \in \Gamma$. Hence,

$$\sigma_{t,\mu}(K) = \sigma_\mu(0) = \begin{pmatrix} O & O \\ O & O \end{pmatrix}.$$

Part (a) is proved.

(b) From Proposition 5.2 it follows that the Fredholmness of $A \in \mathcal{U}$ is equivalent to the invertibility of $A^\pi \in \mathcal{L}^\pi$. By Theorem 4.1, the former is equivalent to the invertibility of $\pi_t(A) = A^\pi + \mathcal{J}_t$ in $\mathcal{L}^\pi/\mathcal{J}_t$ for every $t \in \Gamma$. By Theorem 4.2(b), this is equivalent to

$$\det \sigma_{t,\mu}(A) = \det \sigma_\mu \pi_t(A) \neq 0 \quad \text{for all} \quad (t,\mu) \in \mathcal{M}. \tag{14}$$

Part (b) is proved.

(c) Since $\mathcal{A}(0,1;1/p(t)+\lambda(t))$ does not separate the complex plane $\mathbb{C}$, it follows that the spectra of (13) in the algebras $\mathcal{L}^\pi/\mathcal{J}_t$ and $\mathcal{W}_t = \mathcal{U}^\pi/\mathcal{J}_t$ coincide, so we can apply Theorem 4.2(c). If $A^\pi$, where $A \in \mathcal{U}$, is invertible in $\mathcal{B}^\pi$, then (14) holds. Consequently, by Theorem 4.2(b), (c), $\pi_t(A) = A^\pi + \mathcal{J}_t$ is invertible in $\mathcal{W}_t = \mathcal{U}^\pi/\mathcal{J}_t$ for every $t \in \Gamma$. Applying Theorem 4.1 to $\mathcal{U}^\pi$, its central subalgebra $\mathcal{Z}^\pi$, and the ideals $\mathcal{J}_t$, we obtain that $A^\pi$ is invertible in $\mathcal{U}^\pi$, that is, $\mathcal{U}^\pi$ is inverse closed in the Calkin algebra $\mathcal{B}^\pi$. □

## 6. Index of a Fredholm SIO

### 6.1. Functions on the cylinder $\Gamma \times [0,1]$ with an exotic topology

Let us consider the cylinder $\mathfrak{M} := \Gamma \times [0,1]$. Following [14, 15], we equip it with an exotic topology, where a neighborhood base is given as follows:

$$\begin{aligned}\Omega(t,0) &:= \{(t,x) \in \mathfrak{M} : |\tau - t| < \delta, \tau \prec t, x \in [0,1]\} \cup \{(t,x) \in \mathfrak{M} : x \in [0,\varepsilon)\},\\ \Omega(t,1) &:= \{(t,x) \in \mathfrak{M} : |\tau - t| < \delta, t \prec \tau, x \in [0,1]\} \cup \{(t,x) \in \mathfrak{M} : x \in (\varepsilon,1]\},\\ \Omega(t,x_0) &:= \{(t,x) \in \mathfrak{M} : x \in (x_0 - \delta_1, x_0 + \delta_2)\},\end{aligned}$$

where $x_0 \neq 0, 0 < \delta_1 < x_0, 0 < \delta_2 < 1 - x_0$, and $0 < \varepsilon < 1$.

Note that $\mathcal{A}(z_1, z_2; r)$ has the following parametric representation

$$z(x) = z_1 + (z_2 - z_1)\omega(x,r), \quad 0 \le x \le 1,$$

where $\omega(x,r) = x$ for $r = 1/2$ and

$$\omega(x,r) := \frac{\sin(\theta x)\exp(i\theta x)}{\sin\theta \exp(i\theta)}, \quad \theta := \pi(1-2r), \quad r \neq 1/2.$$

Let $\Lambda$ be the set of all piecewise continuous scalar functions having only finitely many jumps. For $a \in \Lambda$, put

$$U_a(t,x) := a(t+0)\omega(x, 1/p(t)+\lambda(t)) + a(t-0)\big(1-\omega(x,1/p(t)+\lambda(t))\big), \quad (t,x) \in \mathfrak{M}.$$

Let us consider the function

$$F(t,x) := \prod_{j=1}^{k} U_{a_j}(t,x), \quad (t,x) \in \mathfrak{M}, \tag{15}$$

where $a_j \in \Lambda$, $1 \le j \le k$, and $k \ge 1$. If $F(t,x) \neq 0$ for all $(t,x) \in \mathfrak{M}$, then $F$ is continuous on $\mathfrak{M}$ and the image of this function is a continuous closed curve that does not pass through the origin and can be oriented in a natural way. Namely, at the points where the functions $a_j$ are continuous, the orientation of the curve is defined correspondingly to the orientation of $\Gamma$. Along the complementary arcs connecting the one-sided limits at jumps the orientation is defined by the variation of $x$ from 0 to 1. The index $\operatorname{ind}_{\mathfrak{M}} F$ of $F$ is defined as the winding number of the above defined curve about the origin.

By $\mathfrak{F}(\mathfrak{M})$ we denote the class of functions $H : \mathfrak{M} \to \mathbb{C}$ satisfying the following two conditions:

(i) $H(t,x) \neq 0$ for all $(t,x) \in \mathfrak{M}$;

(ii) $H$ can be represented as the uniform limit with respect to $(t,x) \in \mathfrak{M}$ of a sequence of functions $F_s$ of the form (15).

The numbers $\operatorname{ind}_{\mathfrak{M}} F_s$ are independent of $s$ starting from some number $s_0$. The number

$$\operatorname{ind}_{\mathfrak{M}} H := \lim_{s\to\infty} \operatorname{ind}_{\mathfrak{M}} F_s$$

will be called the index of $H \in \mathfrak{F}(\mathfrak{M})$. One can see that the index just defined is independent of the choice of a sequence $F_s$ of the form (15).

### 6.2. Index formula

The matrix function

$$\mathfrak{A}(t,x)=\sigma_{t,\omega(x,1/p(t)+\lambda(t))}(A),\quad (t,x)\in\mathfrak{M},$$

is said to be the symbol of the operator $A\in\mathcal{U}$. We can write the symbol in the block form

$$\mathfrak{A}(t,x)=\begin{pmatrix}\mathfrak{A}_{11}(t,x) & \mathfrak{A}_{12}(t,x)\\ \mathfrak{A}_{21}(t,x) & \mathfrak{A}_{22}(t,x)\end{pmatrix},\quad (t,x)\in\mathfrak{M},$$

where $\mathfrak{A}_{ij}(t,x)$ are $n\times n$ matrix functions.

**Theorem 6.1.** *If an operator $A\in\mathcal{U}$ is Fredholm on $X_n$, then the function*

$$Q_A(t,x):=\frac{\det\mathfrak{A}(t,x)}{\det\mathfrak{A}_{22}(t,0)\det\mathfrak{A}_{22}(t,1)},\quad (t,x)\in\mathfrak{M},$$

*belongs to $\mathfrak{F}(\mathfrak{M})$ and*

$$\operatorname{Ind}A=-\operatorname{ind}_{\mathfrak{M}}Q_A.$$

*Proof.* The proof of this theorem is developed as in the classical situation [14, 15] (see also [22, 23] and [2]) in several steps. We do not present all details here, although we mention the main steps.

1) The index formula for the scalar Fredholm operator $aP+Q$ with $a\in\Lambda$:

$$\operatorname{Ind}(aP+Q)=-\operatorname{ind}_{\mathfrak{M}}U_a.$$

In a slightly different form (and in the non-weighted case) this formula was proved by Kokilashvili and Samko [30].

2) The index formula for $aP^{(n)}+Q^{(n)}$, where $a\in C^{n\times n}$:

$$\operatorname{Ind}(aP^{(n)}+Q^{(n)})=-\frac{1}{2\pi}\{\operatorname{Arg}\det a(t)\}_\Gamma,$$

where the latter denotes the Cauchy index of the continuous function $\det a$. This formula can be proved by using standard homotopic arguments.

3) The index formula for $aP^{(n)}+Q^{(n)}$, where $a$ is a function in $\Lambda^{n\times n}$, the set of $n\times n$ matrices with entries in $\Lambda$:

$$\operatorname{Ind}(aP^{(n)}+Q^{(n)})=-\operatorname{ind}_{\mathfrak{M}}\det U_a.$$

A proof of this fact is based on the possibility of a representation of $a\in\Lambda^{n\times n}$ as the product $c_1Yc_2$, where $c_1$ and $c_2$ are nonsingular continuous matrix functions and $Y$ is an invertible upper-triangular matrix function in $\Lambda^{n\times n}$. A proof of this representation can be found, e.g., in [6, Ch. VIII, Lemma 2.2].

4) An index formula for the operators of the form

$$\sum_{j=1}^{k}(a_{j1}P^{(n)}+b_{j1}Q^{(n)})\times\cdots\times(a_{jr}P^{(n)}+b_{jr}Q^{(n)}),\tag{16}$$

where $a_{jl},b_{jl}\in\Lambda^{n\times n}$, $1\le l\le r$, $k\ge 1$, can be proved by using the previous step an a procedure of linear dilation as in [14, Theorem 7.1] or [15, Theorem 3.1].

5) Every operator $A \in \mathcal{U}$ can be represented as a limit (in the operator topology) of operators of the form (16). So, the index formula in the general case follows from the fourth step by passing to the limits. Notice that if a sequence of operators $A_s \in \mathcal{U}$ converges to $A$, then

$$\det \mathfrak{A}^{(s)} \to \det \mathfrak{A}, \quad \det \mathfrak{A}_{11}^{(s)} \to \det \mathfrak{A}_{11}, \quad \det \mathfrak{A}_{22}^{(s)} \to \det \mathfrak{A}_{22}$$

uniformly on $\mathfrak{M}$, where $\mathfrak{A}$ and $\mathfrak{A}^{(s)}$ are the symbols of $A$ and $A_s$ (see [23, Theorem 3]), so passage to the limits is legitimate. □

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