3D-Mappings Using Monogenic Functions

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Received 09 July 2006

Key words Clifford Analysis, monogenic functions, 3D-mappings
Subject classification 30G35,41A10

Conformal mappings of plane domains are realized by holomorphic functions with non vanishing derivative. Therefore complex differentiability plays an important role in all questions related to fundamental properties of such mapping. In contrast to the planar case, in higher dimensions the set of conformal mappings consists only of Möbius transformations. But unfortunately Möbius transformations are not monogenic functions and therefore also not hypercomplex differentiable. However the equivalence between both concepts - hypercomplex differentiability in the sense of [9], [11] and monogenicity - suggests the question whether monogenic functions can play or not a special role for other types of 3D-mappings, for instance, for quasi-conformal ones. Our goal is to present a case study of an approach to 3D-mappings by using particularly easy to handle monogenic homogeneous polynomials as basis for approximating the mapping function. Thereby we extend significantly the results obtained in [3]. From the numerical point of view we apply ideas from complex numerical analysis realizing the approximation via polynomials of a small real parameter.

1 Preliminaries

Many questions concerning the general extension of theoretical and practical conformal mapping methods in $\mathbb{C}$ to the higher dimensional case in the setting of Clifford Analysis have not been answered until now (see [11, 3]). But, contrary to the case of several complex variables, in Clifford analysis are no restrictions on the real dimension of being even or odd. This implies that the real 3-dimensional Euclidean space, the most important space for concrete applications, can in principle be subject to a treatment similar to the complex one. For this purpose, let $\{1, e_1, e_2, e_3\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with the (quaternionic) product given according to the multiplication rules $e_1^2 = e_2^2 = e_3^2 = -1,\ \ e_1e_2 = -e_2e_1 = e_3$. (see [2]).

As usual, we identify each element $x = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the *paravector* (sometimes also called *reduced quaternion*) $z = x_0 + x_1e_1 + x_2e_2$.
For $C^1(\Omega, \mathbb{R}^3)$ define the (reduced) quaternionic Cauchy-Riemann operator $D = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2}$.
Solutions of the differential equations $Df = 0$ (resp. $fD = 0$) are called left-monogenic (resp. right-monogenic) functions in the domain Ω. Let us remind that the differential operator D is not only a formal linear combination of the real partial derivatives $\frac{\partial}{\partial x_k}$ but, when applied to a given function $f : \Omega \to \mathbb{H}$, is nothing else than an areolar derivative in the sense of Pompeiu (cf. [13] and [12]). The same is true for the conjugate quaternionic Cauchy-Riemann operator $\overline{D} = \frac{\partial}{\partial x_0} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2}$.
But if f is a function monogenic in Ω, its areolar derivative Df is vanishing and this is equivalent with the fact that the areolar derivative $\frac{1}{2}\overline{D}f$ can be considered as the hypercomplex derivative of the function f. In $\mathbb{C}$ for a complex differentiable function f we have $f' = \frac{df}{dz} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}) = \frac{\partial f}{\partial x}$. The same is true in our case, i.e. $\frac{1}{2}\overline{D}f = \frac{\partial f}{\partial x_0}$. Obviously, this formula guarantees that the (hypercomplex) derivative of a monogenic function is again a monogenic function.
In general we have to assume that a monogenic function f has values in $\mathbb{H}$, i.e., it is of the form $f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2 + f_3(x)e_3$, where $f_k,\ k = 0, 1, 2, 3$ are real valued functions in Ω. But if we are dealing with mappings from one 3-dimensional domain to another 3-dimensional domain we have to restrict f

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to be a quaternion-valued function with one identically zero component. This can be done by different choices. Here we consider f also as a paravector, i.e., as being of the form $f(x) = f_0(x) + f_1(x)e_1 + f_2(x)e_2$.
In this case a monogenic function is monogenic from both sides and its components satisfy the Riesz system

$$\begin{aligned} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} &= 0 \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} &= 0 \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_2}{\partial x_0} &= 0 \\ \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} &= 0. \end{aligned} \tag{1}$$

2 Polynomial approximation of monogenic 3D-mappings

For our purpose we use monogenic polynomials in terms of two hypercomplex monogenic variables $z_k = x_k - x_0 e_k = -\frac{z e_k + e_k z}{2}$, $k = 1, 2$, following the approach in [10], [12]; for other approaches and notations see eg. [2, 5, 14]. This leads to *generalized powers* of degree n that are by convention symbolically written as $z_1^{n-k} \times z_2^k$ and defined as an *n-nary* symmetric product by

$$\begin{aligned} z_1^{n-k} \times z_2^k &= \underbrace{z_1 \times z_1 \times \cdots \times z_1}_{n-k\ \ times} \times \underbrace{z_2 \times z_2 \times \cdots \times z_2}_{k\ \ times} \\ &= \frac{1}{n!} \sum_{\pi(i_1,\ldots,i_n)} z_{i_1} \cdots z_{i_n}\,, \end{aligned}$$

where the sum is taken over *all* permutations of $(i_1, \ldots, i_n)$. Generalized powers form a paravector valued basis for the Taylor series of a monogenic function. Given a paravector-valued function f it is possible to prove (see [3]) a particular form for its general power series development ([10]):

Theorem 2.1 *Let $f = f(z) = f_0 + f_1 e_1 + f_2 e_2$ be a paravector-valued monogenic function of the paravector $z = x_0 + x_1 e_1 + x_2 e_2$. The Taylor series of $f(z)$ in terms of z_k in a neighborhood of the origin is given by*

$$f(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \, {z_1}^{n-k} \times {z_2}^{k} \;\; \alpha_{(n-k,\,k)}$$

with

$$\alpha_{(n-k,\,k)} = \frac{1}{n!} \frac{\partial^n f(0)}{\partial x_1^{n-k} \partial x_2^k} \quad \textit{and} \quad [\alpha_{(n-k,\,k)}]_2 = [\alpha_{(n-k-1,\,k+1)}]_1\;, k = 0, \ldots, n, \tag{2}$$

where

$$\alpha_{(n-k,\,k)} = [\alpha_{(n-k,\,k)}]_0 + [\alpha_{(n-k,\,k)}]_1\, e_1 + [\alpha_{(n-k,\,k)}]_2\, e_2, \quad n = 1, 2, \ldots, \quad k = 0, \ldots, n.$$

Our main goal is to realize approximations of 3D-mappings by partial sums of this type of series. Inspired by the corresponding complex approach (see [8]) we normalize the series by the conditions given below, restricting ourselves only to a situation where the interior of a domain should be transformed into the interior of a sphere. Hence, the considered domain should contain the origin and the origin should be an invariant point under the considered mapping f. Furthermore,

$$\overline{D} f(\mathbf{0}) = (\frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}) f(\mathbf{0}) = 2 \;\text{ and }\; \tilde{D} f(\mathbf{0}) := (\frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2}) f(\mathbf{0}) = 1. \tag{3}$$

It is worth noticing that the first condition in (3) means that the hypercomplex derivative at the origin should be equal to one. All together implies that the general form of the series to be studied is

$$f(z_1, z_2) = \frac{1}{2}(z_1 e_1 + z_2 e_2) + \sum_{n=2}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \;\; {z_1}^{n-k} \times {z_2}^{k} \alpha_{(n-k,\,k)} \tag{4}$$

and that the coefficients satisfy the compatibility conditions (2).

3 Numerical experiments with monogenic 3D-mappings

As a concrete example, our case study is concerned with the mapping of the interior of the oblate ellipsoid $\mathcal{E}_\lambda,\ (0 \leq \lambda < 1)$, defined by $z(s,t,\lambda) = x_0(s,t,\lambda) + x_1(s,t,\lambda)e_1 + x_2(s,t,\lambda)e_2$ with $x_0 = (1+\lambda)\cos s,\ x_1 = 2(1-\lambda)\sin s \cos t,\ x_2 = 2(1-\lambda)\sin s \sin t$, where $0 \leq s \leq \pi$ and $0 \leq t < 2\pi$, into the interior of a ball $\mathcal{B}$. In fact, $\mathcal{E}_\lambda$ is a small perturbation of the *canonical oblate spheroid* $\mathcal{O} := \{(x_0, x_1, x_2) : {x_0}^2 + \frac{1}{4}{x_1}^2 + \frac{1}{4}{x_2}^2 = 1\}$ which is mapped into the unit sphere ($w\overline{w} = w_0^2 + w_1^2 + w_2^2 = 1$) by the linear monogenic function $w = \frac{1}{2}(z_1 e_1 + z_2 e_2) = x_0 + \frac{1}{2}(x_1\ e_1 + x_2\ e_2)$ that appears as the linear part of the normalized series. Notice that $\mathcal{E}_\lambda$ is described by the hypercomplex equation $(1+\lambda^2)w\,\overline{w} - \lambda(w\,w + \overline{w}\,\overline{w}) = (1-\lambda^2)^2$.

By taking into account several symmetry properties of $\mathcal{E}_\lambda$, which imply certain invariance properties of the mapping function f, we could show in [3] that the number of coefficients of the mapping function can be substantially reduced. Indeed, the Taylor series contains only generalized powers of odd order with real coefficients and therefore the adequate polynomial approximation of f up to a certain degree m can be realized by a polynomial of the form

$$\begin{aligned}\varphi_m(z_1,z_2) \quad &= \quad \frac{1}{2}(z_1e_1 + z_2e_2) + \\ &+a_{(3,0)}\ z_1^3 e_1 + a_{(2,1)}\ z_1^2 \times z_2^1 e_2 + a_{(1,2)}\ z_1^1 \times z_2^2 e_1 + a_{(0,3)}\ z_2^3 e_2 + \\ &+a_{(5,0)}\ z_1^5\ e_1 + a_{(4,1)}\ z_1^4 \times z_2\ e_2 + a_{(3,2)}\ z_1^3 \times z_2^2\ e_1 + a_{(2,3)}\ z_1^2 \times z_2^3\ e_2 + \\ &+a_{(1,4)}\ z_1 \times z_2^4\ e_1 + a_{(0,5)}\ z_2^5\ e_2 + \\ &+ \cdots . \end{aligned} \tag{5}$$

Due to the fact that on the sphere (i.e. on the boundary of $\mathcal{B}$) the polynomial (5) is a function of the parameter λ and the value of $|\varphi(z_1,z_2,\lambda))|^2$ should be constant and equal to some ϱ^2, the corresponding development of $|\varphi(z(s,t,\lambda))|^2$ as a polynomial with respect to λ results in a nonlinear system of algebraic equations, with $a(\lambda)_{(n-k,k)}$ as unknowns. The corresponding numerical procedures have been executed by using the powerful Maple-Quatpackage from [7].

Of course, the relatively high number of indeterminate coefficients together with the non-linearity of the system causes problems for executing the numerical procedures in a reasonable time and with high accuracy. It is not our aim here to discuss these problems in detail. Instead of this we shall pay attention to an essential simplification of the mentioned general approach. This simplification consists in the use of a special set of monogenic basis functions defined and studied to some extend with respect to its algebraic properties in [4], namely functions of the form

$$\mathcal{P}_k(x) = \sum_{s=0}^{k} T_s^k\, z^{k-s}\, \bar{z}^s, \quad \text{with} \quad T_s^k = \frac{1}{k+1}\frac{(\frac{3}{2})_{(k-s)}(\frac{1}{2})_{(s)}}{(k-s)!s!}, \tag{6}$$

where $a_{(r)}$ denotes the Pochhammer symbol, i.e. $a_{(r)} := \frac{\Gamma(a+r)}{\Gamma(a)}$, for any integer $r > 1$, and $a_{(0)} := 1$. In terms of generalized powers these polynomials are of the form

$$\mathcal{P}_k(x) = \mathcal{P}_k(z_1,z_2) = c_k \sum_{k=0}^{n} z_1^{n-k} \times z_2^k \binom{n}{k} e_1^{n-k} \times e_2^k \quad \text{with} \quad c_k = \begin{cases} \frac{(k-1)!!}{k!!}, \text{ if } k \text{ is even,} \\ \\ \frac{k!!}{(k+1)!!}, \text{ if } k \text{ is odd.} \end{cases} . \tag{7}$$

The first polynomials are given by

$$\begin{aligned} \mathcal{P}_0 \quad &= \quad 1 \\ \mathcal{P}_1(z) \quad &= \quad \frac{3}{4}z + \frac{1}{4}\bar{z}; \quad \mathcal{P}_1(z_1,z_2) = \frac{1}{2}(z_1e_1 + z_2e_2) \\ \mathcal{P}_2(z) \quad &= \quad \frac{5}{8}z^2 + \frac{1}{4}z\bar{z} + \frac{1}{8}\bar{z}^2; \quad \mathcal{P}_2(z_1,z_2) = -\frac{1}{2}(z_1^2 + z_2^2) \\ \mathcal{P}_3(z) \quad &= \quad \frac{35}{64}z^3 + \frac{15}{64}z^2\bar{z} + \frac{9}{64}z\bar{z}^2 + \frac{5}{64}\bar{z}^3 \\ \mathcal{P}_3(z_1,z_2) \quad &= \quad -\frac{3}{8}(z_1^3e_1 + z_1^2 \times z_2e_2 + z_1 \times z_2^2e_1 + z_2^3e_2) \end{aligned}$$

It is easy to prove, in the general case, that a linear combination with real coefficients of these homogeneous polynomials up to degree m has exactly the deduced structure (5) and hence can be used as a monogenic polynomial for the mapping problem discussed in the beginning of this section, although only one and the same coefficient appears in every homogeneous degree. Nevertheless, the corresponding numerical experiments have shown, not only an enormous reduction of numerical costs but also an increasing rate of convergence. Several conjectures about the efficiency of such special polynomials in mapping problems will be presented in our talk.

In the following figures we present approximations of degree 3 for the image of the ellipsoid $\mathcal{E}_{0.01}$, obtained by considering the described general approach and the simplified approach corresponding to the use of (6).

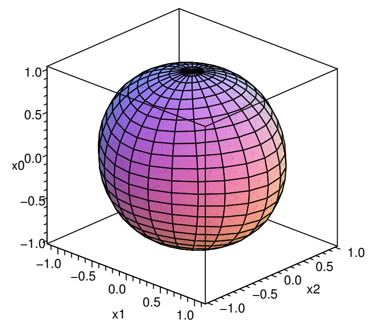

Figure 1: Image of $\mathcal{E}_{0.01}$: general case

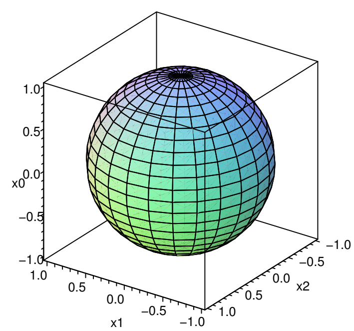

Figure 2: Image of $\mathcal{E}_{0.01}$: simplified case

Acknowledgments: The research of the first author was partially supported by the R&D unit *Matemática e Aplicações* (UIMA) of the University of Aveiro, through the Portuguese Foundation for Science and Technology (FCT). The research of the second author was partially supported by the Portuguese Foundation for Science and Technology (FCT) through the research program POCTI.

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