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m—step Preconditioners for Nonhermitian Positive Definite Toeplitz Systems

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Abstract It is known that if A is a Toeplitz matrix, then A enjoys a circulant and skew circulant splitting (denoted by CSCS), i. e., $A = C + S$ with C a circulant matrix and S a skew circulant matrix. Based on the CSCS iteration [7], we give m —step preconditioners P_m for certain classes of Toeplitz matrices in this paper. We show that if both C and S are positive definite, then the spectrum of the preconditioned matrix $(P_mA)^* P_mA$ are clustered around one for some moderate size m. Experimental results show that the proposed preconditioners perform slightly better than T. Chan's preconditioners in [3] for some moderate size m.

Key words Irculant — skew circulant splitting m — step polynomial preconditioners Conjugate gradient method Toeplitz matrix

非埃米特正定 Toeplitz 矩阵的 m—步预处理子

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摘 要 众所周知,如果 A 是 *Toeplitz* 矩阵,那么矩阵 A 有一循环与反循环分裂(记为 CSCS)[7],可写为 $A = C + S$,其中 C 为循环矩阵,S 为反循环矩阵.本文针对某类 *Toeplitz* 矩阵,提出了一个 m 步的预处理子 P_m,这个预处理子 P_m 是基于CSCS 迭代方法构建的.本文中证明当 C 和 S 都是正定矩阵时,对于适当的 m,预处理矩阵 $(P_m * A)^* * (P_m * A)$ 的谱半径聚集于 1. 实验结果表明,对于适当的 m,本文提出的预处理子优于 T—*Chan* 预处理子[3].

关键词 循环与反循环分裂 m—步多项式预处理子 共轭梯度方法 *Toeplitz* 矩阵

1 Introduction

Consider the solution to a large linear system of equations

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$$Ax = b, \tag{1}$$

by Preconditioned conjugate gradient (PCG) method, where A is an $n \times n$ nonhermitian positive definite Toeplitz matrix where A is constant along its diagonals.

Toeplitz linear systems arise in a variety of applications in mathematics, scientific computing and engineering, see for instance [3] and references therein. There are two main types of methods for solving Toeplitz systems: direct methods and iterative methods. The direct methods are based on the idea of solving Toeplitz systems recursively. The operational cost of these direct methods is $O(n^2)$ [5]. The second type of method is iterative methods. Conjugate Gradient (CG) method is a popular method for solving Toeplitz systems. An important property of a Toeplitz matrix is that it can be embedded into $2n \times 2n$ circulant matrix. Thus the operational cost for a Toeplitz matrix-vector multiplication is $O(nlogn)$ by using the Fast Fourier Transforms (FFT). One of the main important results of this methodology is that the complexity of solving a large class of Toeplitz systems can be reduced to $O(nlogn)$ operations as compared to the $O(n^2)$ operations required by fast direct Toeplitz solvers, provided that a suitable preconditioner is chosen under certain conditions on Toeplitz solvers. In the context of the preconditioners for Toeplitz matrices, various preconditioners proposed usually correspond to different classes of Toeplitz matrices with certain structures and properties. That is why the theory and algorithms of preconditioners for Toeplitz systems have been intrigued the researchers for decades, see for instance [3] and references therein.

Here we consider the m -step preconditioners (proposed in [1] and denoted by P_m) for such classes of Toeplitz matrices of which each consists of the sum of a positive definite circulant matrix and a positive definite skew circulant matrix. Thepreconditioners are constructed based on CSCS iteration proposed in [7]. Note that the system (1) have the same solution as the following preconditioned system:

$$(P_m A)^* (P_m A)x = (P_m A)^* P_m b. \tag{2}$$

For such classes of Toeplitz matrices, we'll show that the eigenvalues of the coefficient matrix in (2) is clustered around one for some moderate large size$m$. when the conjugate gradient method is applied to solve system (2), we therefore expect fast convergence.

2 m -step preconditioners based on CSCS iteration

In this section, we first review some basic definitions, notation and preliminaries used in the sequel, then introduce the CSCS iteration for Toeplitz system (1) and finally construct the$m$ －step preconditioners based on CSCS.

2.1 Preliminaries

Recall that a matrix $A \in C^{n\times n}$ is said to be positive definite if $x^* Ax > 0$ for all nonzero $x \in C^n$ (x^* denotes the conjugate transpose of a vector x), and that the expression $A = M - N$ is called, respectively, a splitting of A if M is nonsingular and a convergent splitting A if the spectral radius of $M^{-1}N$ is less than one, i. e. , $\rho(M^{-1}N) < 1$.

The following two lemmas are classical results in matrix analysis[4]. A matrix $I - H \in C^{n\times n}$ is invertible if there is a matrix norm $||\cdot||$ such that $||H|| < 1$. If this condition is satisfied, then $(I-H)^{-1} = \sum_{k=0}^{\infty} H^k$.

Let $A \in C^{n\times n}$, and $\varepsilon > 0$ be given. There is a matrix norm $||\cdot||_\varepsilon$ such that $\rho(A) \leqslant ||A||_\varepsilon \leqslant \rho(A) + \varepsilon$.

The following conclusion is very often used in iterative methods[6]. Given a nonsingular matrix A and H such that $(I-H)^{-1}$ exists, there exists a unique pair of matrices M_H , N_H , such that $H = M_H^{-1}N_H$ and $A = M_H - N_H$, where M_H is nonsingular.

It is said that $A = M_H - N_H$ is an induced splitting of A by H .

2.2 The CSCS iteration

Recall that any Toeplitz matrix$A$ enjoys a circulant and skew－circulant splitting [7]. If $A = C + S$ is a CSCS of A , then the CSCS iteration for solving (1) in [7] can be described as follows.

Algorithm. (CSCS iteration) Given an initial guess x^0 , for $k = 0,1,2,\cdots$ until converges, compute

$$\begin{cases} (\alpha I + C)x^{k+\frac{1}{2}} = (\alpha I - S)x^k + b \\ (\alpha I + S)x^{k+1} = (\alpha I - C)x^{k+\frac{1}{2}} + b \end{cases}, \tag{3}$$

where α is positive constants.

It is shown in [7] that the CSCS iteration is convergent unconditionally if both$C$ and S are positive definite, and that the parameter α has an optimal choice.

2.3 m －step preconditioners

For a general nonsingular matrix A , if $A = M - N$ is a convergent splitting, then an m －step approximate inverse preconditioner of A can be defined as

$$P_m = (I + H + H^2 + \cdots + H^{m-1})M^{-1}, \tag{4}$$

where $H = M^{-1}N$, see [1].

Before giving the preconditioners of this paper, we first establish the following theorem.

Let $A=C+S$ be the circulant and skew-circulant splitting of A. If both C and S are positive definite, then the CSCS iteration induces a convergent splitting $A = M - N$ with $M = \frac{(\alpha I + C)(\alpha I + S)}{2\alpha}$ and $M^{-1}N = (\alpha I + S)^{-1}(\alpha I - C)(\alpha I + C)^{-1}(\alpha I - S)$.

Proof In fact, one CSCS iteration (3) consists of two half step iterations, which can be thought of as the following single step iteration

$$x^{k+1} = Hx^k + Gb, \tag{5}$$

where $H = (\alpha I + S)^{-1}(\alpha I - C)(\alpha I + C)^{-1}(\alpha I - S)$ and $G = 2\alpha(\alpha I + S)^{-1}(\alpha I + C)^{-1}$. From the assumption, $\alpha > 0$, which implies that G is invertible. By Lemma 2.1, we then have the following induced splitting by H,

$$A = G^{-1} - G^{-1}H, \tag{6}$$

By setting $M = G^{-1}$ and $N = G^{-1}H$, we thus complete the proof.

For system (1), if $A = C + S$ is a CSCS with positive definite C and S, by Theorem 4, we then have that $A = G^{-1} - G^{-1}H$ is a convergent splitting, where G and H are defined as in (6). Thus the m-step preconditioner P_m in (4) for A can be defined.

3 Analysis of the convergence

The following lemma concerns an error bound which describes the convergence rate of the preconditioned system. ([2]) Let x^k be the kth iteration of the CG method applied to the symmetric positive definite system $Bx = b$ and x be the exact solution of the system. If the eigenvalues λ_j of B are ordered such that $0 < \alpha \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \beta$, where α and β are two constants, then $|||x-x^k||| \leqslant 2\left(\frac{\gamma-1}{\gamma+1}\right)^k |||x-x^0|||$, where $|||\cdot|||$ is the energy norm given by $|||v|||^2 = v^TBv$ and $\gamma = (\beta/\alpha)^{\frac{1}{2}} \geqslant 1$.

This Lemma tell us that the more clustered the eigenvalues are, the faster the convergence rate will be. The following theorem is the main result of this paper, in which the bounds of eigenvalues of the preconditioned matrix in (2) is given.

Let $A = C+S$ be the circulant and skew-circulant splitting of A, let the preconditioner P_m be defined as in (4), and the eigenvalues of the preconditioned matrix $\hat{A} = (P_mA)^*(P_mA)$ in (2) be ordered such that $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. If both C and S are positive definite, then we have the following bounds

$$(1-\|\hat{H}\|_c)^2 \leqslant \lambda_i(\hat{A}) \leqslant (1+\|\hat{H}\|_c)^2, \tag{7}$$

where $\dot{H} = H^m$, $\dot{A} = (P_mA)^*(P_mA)$, $||\cdot||_\varepsilon$ is the matrix norm in Lemma

Table 1 No. of iters. with P_m and C_T in Example 4.1-4.2

n	Precons							
	Example 4.1				Example 4.2			
	P_0	C_T	P_1	P_2	P_0	C_T	P_1	P_2
128	18	7	5	3	23	7	5	3
256	18	7	5	3	23	7	5	3
512	18	7	5	3	23	7	5	3
1024	18	7	5	3	23	7	5	3

Proof Now we consider the preconditioned linear system (2), $P_mA = I - \dot{H}$, since $\rho(H) < 1$, we get $\rho(\dot{H}) < 1$. By lemma 2.1, for any sufficient small $\varepsilon > 0$, there exists a norm $||\cdot||_\varepsilon > 0$, such that $\rho(\dot{H}) \leqslant ||\dot{H}||_\varepsilon \leqslant \rho(\dot{H}) + \varepsilon < 1$. It suffices to show that $\lambda_n(\dot{A}) \geqslant (1 - ||\dot{H}||_\varepsilon)^2$ and $\lambda_1(\dot{A}) \leqslant (1 + ||\dot{H}||_\varepsilon)^2$. We first prove the left inequality (7): By lemma 2.1, we get

$$\lambda_n(\dot{A}) = \frac{1}{\rho(\dot{A}^{-1})} \geqslant \frac{1}{||((I-\dot{H})^*)^{-1}(I-\dot{H})^{-1}||_\varepsilon} \geqslant \frac{1}{||(I-\dot{H})^{-1}||_\varepsilon^2}$$

$$\geqslant \frac{1}{||(I+\dot{H}+\dot{H}^2+\cdots||_\varepsilon^2} \geqslant \frac{1}{(1+||\dot{H}||_\varepsilon+||\dot{H}||_\varepsilon^2+\cdots)^2} = (1-||\dot{H}||_\varepsilon)^2.$$

For the right inequality of (7), we have

$$\lambda_1(\dot{A}) \leqslant \rho((I-\dot{H})^*)(I-\dot{H})) \leqslant ||((I-\dot{H})^*)(I-\dot{H}))||_\varepsilon \leqslant (1+||\dot{H}||_\varepsilon)^2.$$

Thus we complete the proof.

4 Numerical examples

All the numerical tests were done on a Founder desktop PC with Pentium dual-core E6700 CPU 3.20 GHz with Matlab 7.4.0.287 (R2007a). When CG method is applied to the preconditioned system (2), the initial guess x^0 is chosen to be zero vector. The stopping criteria is $\tau = \frac{||r^k||_2}{||r^0||_2} \leqslant 10^{-7}$, where r^k is the residual vector at kth iteration.

To verify the effectiveness of our preconditioners, three kinds of generating functions were tested and listed as follows

Example 4.1 $a_0 = 5$, $a_1 = -1$, $a_{-1} = 1$, $a_{-2} = -2$, and $a_j = 0$, elsewhere; $b = e = (1,1,\cdots,1)^T$.

Example 4.2 $a_0 = 10$, $a_1 = -1 - 2 * i$, $a_2 = -1 - 3 * i$, $a_{-1} = 2 * i$, $a_{-2} = 3 * i$, $b = 5e$.

Example 4.3 $a_j = (0.1 + |j|)^{-\mu}, j \geq 0$; and $a_j = i(0.1 + |j|)^{-\mu}, j < 0$, $b = Ae$.

Table 2 No. of iters with P_m and C_T in Example 4.3

n	Precons											
	C_T			P_0			P_1			P_2		
	0.9	1	1.1	0.9	1	1.1	0.9	1	1.1	0.9	1	1.1
128	7	6	6	12	10	9	4	4	3	3	2	2
256	7	6	6	13	11	9	5	4	3	3	3	2
512	7	6	6	14	11	10	5	4	3	3	3	2
1024	7	6	6	15	12	10	5	4	3	3	3	2

For comparison, we also test T. Chan's circulant preconditioner C_T in [3]. The numerical results are illustrated in Tables 1−2, where n is the order of coefficient matrix A, C_T is Chan's preconditioner and P_m, $m=0,1,2$ are our new preconditioners. In the numerical test of Example , the scalar μ is taken to be 0.9, 1.0 and 1.1, respectively. Numerical experiments show that our preconditioners P_m, $m=1,2$, perform slightly better than Chan's one.

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