Characterizations and representations of core and dual core inverses in rings

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Abstract

In this paper, double commutativity and reverse order law for core inverse are considered. Then, new characterizations of Moore-Penrose inverse are given by one-sided invertibilities in a ring. Also, we characterize core inverse and dual core inverse of a regular element by units in a ring R. Moreover, their expressions are shown.

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1. Introduction

In this paper, R means an associative ring with unity 1. We say that $a \in R$ is (von Neumann) regular if there exists $x \in R$ such that $axa = a$. Such x is called an inner inverse of a, and is denoted by a^-. Let $a\{1\}$ be the set of all inner inverses of a. Recall that an element $a \in R$ is said to be group invertible if there exists $x \in R$ such that $axa = a$, $xax = x$ and $ax = xa$. The element x satisfying the conditions above is called a group inverse of a. The group inverse of a is unique if it exists, and is denoted by $a^\#$.

An involution in R is an anti-isomorphism of degree 2, which satisfies $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all a, $b \in R$. An element

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$a \in R$ is called Moore-Penrose invertible (see [8]) if there exists $x \in R$ satisfying the following equations

$$\text{(i) } axa = a \quad \text{(ii) } xax = x \quad \text{(iii) } (ax)^* = ax \quad \text{(iv) } (xa)^* = xa.$$

Any element x satisfying the equations (i)-(iv) is called a Moore-Penrose inverse of a. If such x exists, it is unique and is denoted by $a^\dagger$. If x satisfies the conditions (i) and (iii), then x is called a $\{1,3\}$-inverse of a, and is denoted by $a^{(1,3)}$. If x satisfies the conditions (i) and (iv), then x is called a $\{1,4\}$-inverse of a, and is denoted by $a^{(1,4)}$. The symbols R^{-1}, $R^{\#}$, $R^\dagger$, $R^{(1,3)}$ and $R^{(1,4)}$ denote the sets of all invertible, group invertible, Moore-Penrose invertible, $\{1,3\}$-invertible and $\{1,4\}$-invertible elements in R, respectively.

The concept of core inverse of a complex matrix was first introduced by Baksalary and Trenkler in [2]. Recently, Rakić et al. [10] gave an equivalent definition of core inverse in rings. An element $a \in R$ is core invertible (see [10, Definition 2.3]) if there exists $x \in R$ such that $axa = a$, $xR = aR$ and $Rx = Ra^*$. It is known that the core inverse x of a is unique if it exists, and is denoted by $a^{\oplus}$. The dual core inverse of a when exists is defined as the unique $a_{\oplus}$ such that $aa_{\oplus}a = a$, $a_{\oplus}R = a^*R$ and $Ra_{\oplus} = Ra$. By $R^{\oplus}$ and $R_{\oplus}$ we denote the sets of all core invertible and dual core invertible elements in R, respectively.

In this paper, double commutativity and reverse order law for core inverse proposed in [1] are considered. Also, we characterize the Moore-Penrose inverse of a regular element by one-sided invertibilities in a ring R. Further, new existence criteria of core inverse and dual core inverse of a regular element are given by units. Moreover, their expressions are shown.

2. Main results

In what follows, R always denotes an associative unital ring with involution. We first give the representation of (dual) core inverse of a in R.

Proposition 2.1. *Let $a \in R$. Then*

(i) *$a \in R^{\oplus}$ if and only if $a \in R^{\#} \cap R^{(1,3)}$. In this case, $a^{\oplus} = a^{\#}aa^{(1,3)}$.*

(ii) *$a \in R_{\oplus}$ if and only if $a \in R^{\#} \cap R^{(1,4)}$. In this case, $a_{\oplus} = a^{(1,4)}aa^{\#}$.*

Proof. (i) " $\Rightarrow$ " By [10, Theorem 2.14], we have $a \in R^{(1,3)}$. Also, $a = aa^{\oplus}a = aa(a^{\oplus})^2a = a^2(a^{\oplus})^2a \in a^2R$, which combines with $a = a^{\oplus}a^2 \in Ra^2$ yield $a \in R^{\#}$. Hence, $a \in R^{\#} \cap R^{(1,3)}$.

" $\Leftarrow$ " Let $x = a^{\#}aa^{(1,3)}$. We next show that x is the core inverse of a.

(1) It is direct to check that $axa = a$.

(2) We have $xR = a^{\#}aa^{(1,3)}R = aa^{\#}a^{(1,3)}R \subseteq aR$ and $aR = a^{\#}a^2R = a^{\#}aa^{(1,3)}a^2R \subseteq xR$.

(3) From $x = a^{\#}aa^{(1,3)} = a^{\#}(a^{(1,3)})^*a^*$ and $a^* = a^*aa^{(1,3)} = a^*ax$, it follows that $Rx = Ra^*$.

Hence, $a^{\oplus} = a^{\#}aa^{(1,3)}$.

(ii) By a similar proof of (i). $\square$

It is known that $a \in R^{\dagger}$ if and only if $a \in R^{(1,3)} \cap R^{(1,4)}$. By Proposition 2.1, we obtain $a \in R^{\oplus} \cap R_{\oplus} \Leftrightarrow a \in R^{\#} \cap R^{(1,3)} \cap R^{(1,4)} \Leftrightarrow a \in R^{\#} \cap R^{\dagger}$.

We next give a result regarding commutativity. Firstly, we show the following lemma.

Lemma 2.2. *Let $a, x \in R$ with $xa = ax$ and $xa^* = a^*x$. If $a^{(1,3)}$ exists, then*

$$aa^{(1,3)}x = xaa^{(1,3)}.$$

Proof. From $xa = ax$, it follows that

$$\begin{aligned} xaa^{(1,3)} &= axa^{(1,3)} = aa^{(1,3)}axa^{(1,3)} \\ &= aa^{(1,3)}xaa^{(1,3)}. \end{aligned}$$

The condition $xa^* = a^*x$ implies that

$$\begin{aligned} aa^{(1,3)}x &= (a^{(1,3)})^*a^*x = (a^{(1,3)})^*xa^* \\ &= (a^{(1,3)})^*x(aa^{(1,3)}a)^* = (a^{(1,3)})^*xa^*aa^{(1,3)} \\ &= (a^{(1,3)})^*a^*xaa^{(1,3)} \\ &= aa^{(1,3)}xaa^{(1,3)}. \end{aligned}$$

Hence, $aa^{(1,3)}x = xaa^{(1,3)}$. $\square$

Applying Lemma 2.2, we obtain the following result.

Theorem 2.3. *Let $a, x \in R$ with $xa = ax$ and $xa^* = a^*x$. If $a^{\oplus}$ exists, then $a^{\oplus}x = xa^{\oplus}$.*

Proof. Since $a^{\oplus} = a^{\#}aa^{(1,3)}$ and $a^{\#}x = xa^{\#}$, it follows that

$$a^{\oplus}x = a^{\#}aa^{(1,3)}x = a^{\#}xaa^{(1,3)} = xa^{\#}aa^{(1,3)} = xa^{\oplus}.$$

Hence, $a^{\oplus}x = xa^{\oplus}$. $\square$

Baksalary and Trenkler [1] asked the following question: Given complex matrices A and B, if $A^{\oplus}$, $B^{\oplus}$ and $(AB)^{\oplus}$ exist, does it follow that $(AB)^{\oplus} = B^{\oplus}A^{\oplus}$. Later, Cohen, Herman and Jayaraman [3] presented several counterexamples for this problem.

Next, we show that the reverse order law for core inverse holds under certain conditions in a general ring case.

Theorem 2.4. *Let $a, b \in R$ with $ab = ba$ and $ab^* = b^*a$. If $a^{\oplus}$ and $b^{\oplus}$ exist, then $(ab)^{\oplus}$ exists and $(ab)^{\oplus} = b^{\oplus}a^{\oplus} = a^{\oplus}b^{\oplus}$.*

Proof. It follows from Theorem 2.3 that $b^{\oplus}a = ab^{\oplus}$ and $a^{\oplus}b = ba^{\oplus}$.

Also, the conditions $b^*a = ab^*$ and $a^*b^* = b^*a^*$ guarantee that $b^*a^{\oplus} = a^{\oplus}b^*$, which together with $a^{\oplus}b = ba^{\oplus}$ imply $a^{\oplus}b^{\oplus} = b^{\oplus}a^{\oplus}$ according to Theorem 2.3.

Once given the above conditions, it is straightforward to check that

(1) By Lemma 2.2, we have $abb^{(1,3)} = bb^{(1,3)}a$. Hence, $abb^{\oplus}a^{\oplus}ab = abb^{(1,3)}aa^{\#}b = bb^{(1,3)}aaa^{\#}b = bb^{(1,3)}ba = ab$.

(2) Since $abb^{(1,3)} = bb^{(1,3)}a$, it follows that $b^{\oplus}a^{\oplus} = b^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^{\#}bb^{(1,3)}aa^{\#}a^{(1,3)} = b^{\#}abb^{(1,3)}a^{\#}a^{(1,3)} = ab^{\#}bb^{(1,3)}a^{\#}a^{(1,3)} = abb^{\#}b^{(1,3)}a^{\#}a^{(1,3)}$ and $ab = b^{\#}b^2a = b^{\#}bb^{(1,3)}b^2a = b^{\oplus}ab^2 = b^{\oplus}a^{\#}aa^{(1,3)}a^2b^2 = b^{\oplus}a^{\oplus}a^2b^2$.

Hence, $abR = b^{\oplus}a^{\oplus}R$.

(3) If x in Lemma 2.2 is group invertible, then $aa^{(1,3)}x^{\#} = x^{\#}aa^{(1,3)}$. We have

$b^{\oplus}a^{\oplus} = b^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^{\#}a^{\#}bb^{(1,3)}aa^{(1,3)} = b^{\#}a^{\#}(aa^{(1,3)}bb^{(1,3)})^* = b^{\#}a^{\#}(baa^{(1,3)}b^{(1,3)})^* = b^{\#}a^{\#}(a^{(1,3)}b^{(1,3)})^*(ab)^*$ and

$(ab)^* = b^*a^*aa^{(1,3)} = a^*b^*aa^{(1,3)} = a^*b^*bb^{(1,3)}aa^{(1,3)} = b^*a^*aa^{\#}abb^{(1,3)}a^{(1,3)} = b^*a^*abb^{(1,3)}a^{\#}aa^{(1,3)} = b^*a^*abb^{\#}bb^{(1,3)}a^{\#}aa^{(1,3)} = b^*a^*abb^{\oplus}a^{\oplus}$.

So,

$$Rb^{\oplus}a^{\oplus} = R(ab)^*.$$

Thus, $(ab)^{\oplus} = b^{\oplus}a^{\oplus} = a^{\oplus}b^{\oplus}$. $\square$

Herein, we first state several lemmas which paly an important role in the sequel.

Lemma 2.5. *Let $a, b \in R$. Then*

(i) *If $(1 + ab)x = 1$, then $(1 + ba)(1 - bxa) = 1$.*

(ii) *If $y(1 + ab) = 1$, then $(1 - bya)(1 + ba) = 1$.*

Lemma 2.6. [12, Theorems 2.16, 2.19 and 2.20] *Let S be a $*$-semigroup and $a \in S$. Then the following conditions are equivalent*:

(i) $a \in S^{\dagger}$.

(ii) $a = aa^*ax$ *for some* $x \in S$.

(iii) $a = yaa^*a$ *for some* $y \in S$.

In this case, $a^{\dagger} = a^*ax^2a^* = a^*y^2aa^*$.

Lemma 2.7. (see e.g. [5, Lemma 5.1]) *Let $a \in R$. Then $a \in R^{\dagger}$ if and only if there exist $x, y \in R$ such that $axa = a = aya$, $(ax)^* = ax$ and $(ya)^* = ya$. In this case, $a^{\dagger} = yax$.*

In the following theorem, new characterizations of the Moore-Penrose inverse are given by one-sided invertibilities.

Theorem 2.8. *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a \in R^{\dagger}$.

(ii) $aa^* + 1 - aa^-$ *is right invertible.*

(iii) $a^*a + 1 - a^-a$ *is right invertible.*

(iv) $aa^*aa^- + 1 - aa^-$ *is right invertible.*

(v) $a^-aa^*a + 1 - a^-a$ *is right invertible.*

(vi) $aa^* + 1 - aa^-$ *is left invertible.*

(vii) $a^*a + 1 - a^-a$ *is left invertible.*

(viii) $aa^*aa^- + 1 - aa^-$ *is left invertible.*

(ix) $a^-aa^*a + 1 - a^-a$ *is left invertible.*

Proof. (ii) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iv), (iii) $\Leftrightarrow$ (v), (vi) $\Leftrightarrow$ (vii), (vi) $\Leftrightarrow$ (viii) and (vii) $\Leftrightarrow$ (ix) are followed from Lemma 2.5.

(i) $\Rightarrow$ (ii) If $a \in R^{\dagger}$, then there exists $x \in R$ such that $a = aa^*ax$ from Lemma 2.6. As $(aa^*aa^- + 1 - aa^-)(axa^- + 1 - aa^-) = 1$, then $aa^*aa^- + 1 - aa^-$ is right invertible. Hence, $aa^* + 1 - aa^-$ is right invertible by Lemma 2.5.

(ii)$\Rightarrow$ (i) As $aa^* + 1 - aa^-$ is right invertible, then $a^*a + 1 - a^-a$ is also right invertible by Lemma 2.5. Hence, there is $s \in R$ such that $(a^*a + 1 - a^-a)s = 1$. We have $a = a(a^*a + 1 - a^-a)s = aa^*as \in aa^*aR$. So $a \in R^{\dagger}$ by Lemma 2.6.

(i) $\Rightarrow$ (vi) It is similar to the proof of (i) $\Rightarrow$ (ii).

(vi) $\Rightarrow$ (i) As $aa^* + 1 - aa^-$ is left invertible, then $t(aa^* + 1 - aa^-) = 1$ for some $t \in R$. Also, $a = 1 \cdot a = t(aa^* + 1 - aa^-)a = taa^*a \in Raa^*a$, which ensures $a \in R^{\dagger}$ according to Lemma 2.6. $\square$

We get the following result from Theorem 2.8.

Corollary 2.9. [7, Theorem 1.2] *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a \in R^\dagger$.

(ii) $aa^* + 1 - aa^-$ *is invertible.*

(iii) $a^*a + 1 - a^-a$ *is invertible.*

(iv) $aa^*aa^- + 1 - aa^-$ *is invertible.*

(v) $a^-aa^*a + 1 - a^-a$ *is invertible.*

Theorem 2.10. *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a \in R^\dagger$ *and* $aR = a^2R$.

(ii) $u = aa^*a + 1 - aa^-$ *is right invertible.*

(iii) $v = a^*a^2 + 1 - a^-a$ *is right invertible.*

Proof. (i) $\Rightarrow$ (ii) As $aR = a^2R$, then $a + 1 - aa^-$ is right invertible by [9, Theorem 1]. Also, $a \in R^\dagger$ can conclude $aa^*aa^- + 1 - aa^-$ is invertible by Corollary 2.9. Hence, $u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is right invertible.

(ii) $\Leftrightarrow$ (iii) Follows from Lemma 2.5.

(iii) $\Rightarrow$ (i) Since v is right invertible, there exists $v_1 \in R$ such that $vv_1 = 1$. Then $a = avv_1 = a(a^*a^2 + 1 - a^-a)v_1 = aa^*a^2v_1 \in aa^*aR$ and hence $a \in R^\dagger$ by Lemma 2.6. It follows from Corollary 2.9 that $a \in R^\dagger$ implies that $w = a^*a + 1 - a^-a \in R^{-1}$. As $v = (a^*a + 1 - a^-a)(a^-a^2 + 1 - a^-a)$ is right invertible, then $a^-a^2 + 1 - a^-a = w^{-1}v$ is right invertible, and hence $a + 1 - a^-a$ is also right invertible. So, $aR = a^2R$ by [9, Theorem 1]. $\square$

Remark 2.11. In general, $a \in R^\dagger$ and $aR = a^2R$ can not imply $a \in R^\#$. Such as, let R be the ring of all bi-finite infinite complex matrices with transpose as involution, where an infinite matrix is said to be bi-finite if it is both row-finite and column-finite. Let $a = \Sigma_{i=1}^{\infty} e_{i,i+1} \in R$, where $e_{i,j}$ denotes the infinite matrix whose (i,j)-entry is 1 and other entries are zero. Then $aa^* = 1$ and $a^*a = \Sigma_{i=2}^{\infty} e_{i,i}$. So, $a^\dagger = a^*$ and $aR = a^2R$. But $a \notin R^\#$. In fact, if $a \in R^\#$, then $a^\#a = aa^\# = aa^\#aa^* = aa^* = 1$, which implies a is invertible. Contradiction.

Dually, we have the following result.

Theorem 2.12. *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a \in R^\dagger$ *and* $Ra = Ra^2$.

(ii) $u = aa^*a + 1 - a^-a$ *is left invertible.*

(iii) $v = a^2a^* + 1 - aa^-$ *is left invertible.*

Lemma 2.13. ([6, Proposition 2.1] and [9, Corollary 2]) *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a^\#$ *exists.*

(ii) $a + 1 - aa^-$ *is invertible.*

(iii) $a + 1 - a^-a$ *is invertible.*

(iv) $a^2 + 1 - aa^-$ *is invertible.*

We next give existence criteria and representations of core inverse and dual core inverse by units in a ring.

Theorem 2.14. *Let $a \in R$ be regular with inner inverse a^-. Then the following conditions are equivalent*:

(i) $a \in R^\# \cap R^\dagger$.

(ii) $a \in R^{\oplus} \cap R_{\oplus}$.

(iii) $u = aa^*a + 1 - aa^-$ *is invertible.*

(iv) $v = aa^*a + 1 - a^-a$ *is invertible.*

(v) $s = a^*a^2 + 1 - a^-a$ *is invertible.*

(vi) $t = a^2a^* + 1 - aa^-$ *is invertible.*

In this case,

$$\begin{aligned} &a^{\oplus} = u^{-1}aa^*, \ a_{\oplus} = a^*av^{-1}, \\ &a^\dagger = (t^{-1}a^2)^* = (a^2s^{-1})^* \text{ and} \\ &a^\# = (aa^*t^{-1})^2a = a(s^{-1}a^*a)^2. \end{aligned}$$

Proof. (i) $\Leftrightarrow$ (ii) By Proposition 2.1.

(iii) $\Leftrightarrow$ (v) and (iv) $\Leftrightarrow$ (vi) are obtained by Lemma 2.5.

(i) $\Rightarrow$ (iii) In virtue of Lemma 2.13 and Corollary 2.9, $a \in R^\# \cap R^\dagger$ implies that $a + 1 - aa^-$ and $aa^*aa^- + 1 - aa^-$ are both invertible. Hence, $u = aa^*a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible.

(iii) $\Rightarrow$ (i) Suppose that $u = aa^*a + 1 - aa^-$ is invertible. Then $a \in R^\dagger$ from Theorem 2.10 and hence $aa^*aa^- + 1 - aa^-$ is invertible by Corollary 2.9. As $u = (aa^*aa^- + 1 - aa^-)(a + 1 - aa^-)$ is invertible, then $a + 1 - aa^- = (aa^*aa^- + 1 - aa^-)^{-1}u$ is invertible, i.e., $a \in R^\#$ by Lemma 2.13.

(i) $\Leftrightarrow$ (iv) can be obtained by a similar proof of (i) $\Leftrightarrow$ (iii).

Next, we give representations of $a^{\oplus}$, $a_{\oplus}$, $a^{\dagger}$ and $a^{\#}$, respectively. Herein, we recall in [4, Proposition 7] and [11, Corollary 5] that $a \in R^{\#}$ if and only if $a = a^2x$ and $a = ya^2$ for some $x, y \in R$. In this case, $a^{\#} = yax = y^2a = ax^2$.

Since $ua = aa^*a^2$, $a = (u^{-1}aa^*)a^2$. As $a^{\#}$ exists, then $a^{\#} = (u^{-1}aa^*)^2a$. By Proposition 2.1, we have

$$\begin{aligned} a^{\oplus} &= a^{\#}aa^{(1,3)} = u^{-1}aa^*u^{-1}aa^*a^2a^{(1,3)} \\ &= u^{-1}aa^*aa^{(1,3)} = u^{-1}aa^*(aa^{(1,3)})^* \\ &= u^{-1}aa^*. \end{aligned}$$

Similarly, it follows that $a^{\#} = a(a^*av^{-1})^2$ and $a_{\oplus} = a^*av^{-1}$.

As $as = aa^*a^2$ and $ta = a^2a^*a$, then we have $a = aa^*(a^2s^{-1}) = (t^{-1}a^2)a^*a$. It follows from Lemma 2.7 that $a \in R^{\dagger}$ and

$$\begin{aligned} a^{\dagger} &= (a^2s^{-1})^*a(t^{-1}a^2)^* = (s^{-1})^*(a^2)^*a(a^2)^*(t^{-1})^* \\ &= (s^{-1})^*(aa^*a^2)^*a^*(t^{-1})^* = (s^{-1})^*(as)^*a^*(t^{-1})^* \\ &= (a^*)^2(t^{-1})^* \\ &= (t^{-1}a^2)^*. \end{aligned}$$

Similarly, $a^{\dagger} = (a^2s^{-1})^*$.

Noting $sa^-a = a^*a^2$, we have $a^-a = s^{-1}a^*a^2$ and $a = aa^-a = (as^{-1}a^*)a^2$. Hence, it follows that $a^{\#} = (as^{-1}a^*)^2a = a(s^{-1}a^*a)^2$ since $a \in R^{\#}$.

We can also get $a^{\#} = (aa^*t^{-1})^2a$ by a similar way. $\square$

Theorem 2.15. *Let $a \in R$ be regular. Then the following conditions are equivalent*:

(i) $a^{\oplus}$ *exists.*

(ii) $a + 1 - aa^-$ *and* $a^* + 1 - aa^=$ *are invertible for some* $a^-, a^= \in a\{1\}$.

(iii) $a + 1 - aa^-$ *is invertible and* $a^* + 1 - aa^=$ *is left invertible for some* $a^-, a^= \in a\{1\}$.

(iv) $a^*a + 1 - aa^-$ *and* $(a^*)^2 + 1 - aa^=$ *are invertible for some* $a^-, a^= \in a\{1\}$.

(v) $a^*a + 1 - aa^-$ *and* $(a^*)^2 + 1 - aa^=$ *are left invertible for some* $a^-, a^= \in a\{1\}$.

In this case, $a^{\oplus} = (a^*a + 1 - aa^-)^{-1}a^* = a[((a^*)^2 + 1 - aa^=)^{-1}]^*$.

Proof. (i) $\Rightarrow$ (ii) Since $a \in R^{\oplus}$, $a \in R^{(1,3)}$ by Proposition 2.1. Let a^-, $a^= \in a\{1, 3\}$. Then $a + 1 - aa^-$ and $a + 1 - aa^=$ are invertible by Lemma 2.13 and hence $a^* + 1 - aa^= = (a + 1 - aa^=)^*$ is invertible.

(ii) $\Rightarrow$ (iii) It is clear.

(iii) $\Rightarrow$ (i) As $a^* + 1 - aa^=$ is left invertible, then there exists $s \in R$ such that $s(a^* + 1 - aa^=) = 1$. Hence, $a = s(a^* + 1 - aa^=)a = sa^*a \in Ra^*a$, i.e., $a^{(1,3)}$ exists by [13, Lemma 2.2]. Also, $a + 1 - aa^- \in R^{-1}$ concludes that $a \in R^{\#}$ exists by Lemma 2.13. So, $a \in R^{\oplus}$ by Proposition 2.1.

(i) $\Rightarrow$ (iv) Let a^-, $a^= \in a\{1,3\}$. Then $a + 1 - aa^-$ and $a^* + 1 - aa^=$ are invertible. Hence, $a^*a + 1 - aa^- = (a^* + 1 - aa^=)(a + 1 - aa^-)$ is invertible.

Also, it follows from Lemma 2.13 that $a^2 + 1 - aa^= \in R^{-1}$ since $a \in R^{\#}$. So, $(a^*)^2 + 1 - aa^= = (a^2 + 1 - aa^=)^* \in R^{-1}$.

(iv) $\Rightarrow$ (v) Clearly.

(v) $\Rightarrow$ (i) Since $a^*a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^=$ are both left invertible, there exist $m, n \in R$ such that $m(a^*a + 1 - aa^-) = 1 = n((a^*)^2 + 1 - aa^=)$. As $a = m(a^*a + 1 - aa^-)a = ma^*a^2$ and $a = n((a^*)^2 + 1 - aa^=)a = n(a^*)^2a$, then $ma^* = m(n(a^*)^2a)^* = (ma^*a^2)n^* = an^*$.

Let $x = ma^* = an^*$. Then x is the core inverse of a. Indeed, we have

(1) $(ax)^* = ax$ since $ax = n(a^*)^2a(an^*) = (a^2n^*)^*a^2n^*$.

(2) $axa = (ax)^*a = (a^*ax)^* = (a^*a^2n^*)^* = n(a^*)^2a = a$.

(3) $xax = (ma^*)a(an^*) = (ma^*a^2)n^* = an^* = x$.

(4) $xa^2 = ma^*a^2 = a$.

(5) $ax^2 = ax(an^*) = (axa)n^* = an^* = x$.

It follows from [10, Theorem 2.14] that $x = a^{\oplus}$.

We next give the formulae of $a^{\oplus}$. In process of (v) $\Rightarrow$ (i), $a^*a + 1 - aa^-$ and $(a^*)^2 + 1 - aa^=$ are both invertible from (iv) $\Leftrightarrow$ (v). Hence, $m = (a^*a + 1 - aa^-)^{-1}$ and $n = ((a^*)^2 + 1 - aa^=)^{-1}$.

We obtain

$$\begin{aligned} a^{\oplus} &= ma^* = (a^*a + 1 - aa^-)^{-1}a^* \\ &= an^* = a[((a^*)^2 + 1 - aa^=)^{-1}]^*. \end{aligned}$$

The proof is completed. □

Proposition 2.16. *Let $a \in R$ be regular. If $a^* + 1 - aa^-$ is invertible for any $a^- \in a\{1\}$, then $a^{\oplus}$ exists.*

Proof. If $u = a^* + 1 - aa^-$ is invertible, then $a = u^{-1}a^*a \in Ra^*a$, hence a is $\{1,3\}$-invertible by [13, Lemma 2.2].

As $a + 1 - aa^{(1,3)} = (a^* + 1 - aa^{(1,3)})^*$ is invertible for $a^{(1,3)} \in a\{1\}$, then $a \in R^{\#}$ by Lemma 2.13. So, $a^{\oplus}$ exists from Proposition 2.1. □

Proposition 2.17. *Let $a \in R$ be regular. If $(a^*)^2 + 1 - aa^-$ is invertible for any $a^- \in a\{1\}$, then $a^{\oplus}$ exists.*

Proof. Let $u = (a^*)^2 + 1 - aa^-$. Then $ua = (a^*)^2 a$, it follows $a = u^{-1}(a^*)^2 a \in Ra^*a$. So, a is $\{1,3\}$-invertible by [13, Lemma 2.2].

Also, $a^2 + 1 - aa^{(1,3)} = ((a^*)^2 + 1 - aa^{(1,3)})^* \in R^{-1}$ guarantees that $a \in R^{\#}$ from Lemma 2.13. Hence, it follows from Proposition 2.1 that $a^{\oplus}$ exists. $\square$

The converse statements of Propositions 2.16 and 2.17 may not be true. In following Example 2.18, we find that a is core invertible, but there exist some $a^- \in a\{1\}$ such that $a^* + 1 - aa^-$, $(a^*)^2 + 1 - aa^-$ and $a^*a + 1 - aa^-$ are all not invertible.

Example 2.18. Let $M_2(\mathbb{C})$ be the ring of 2 by 2 complex matrices and let involution $*$ be the conjugate transpose. Given $A = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \in M_2(\mathbb{C})$, then $A^2 = -A$ and hence $A^{\#}$ exists. So, $A^{\oplus}$ exists. Taking $A^- = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ 0 & 0 \end{bmatrix}$, then $A^* + I - AA^- = \frac{1}{3}\begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$, $(A^*)^2 + I - AA^- = \frac{1}{3}\begin{bmatrix} -2 & -4 \\ 4 & 8 \end{bmatrix}$ and $A^*A + I - AA^- = \frac{1}{3}\begin{bmatrix} 7 & -13 \\ -14 & 26 \end{bmatrix}$ are not invertible.

Remark 2.19. Even $a^*a + 1 - aa^- \in R^{-1}$ for any $a^- \in a\{1\}$, a may not be core invertible. Let R be a ring which is the same as the infinite matrix ring in Remark 2.11 and let $a = \Sigma_{i=1}^{\infty} e_{i+1,i}$. Then $a^*a = 1$, $aa^* = \Sigma_{i=2}^{\infty} e_{i,i}$ and $a^{\dagger} = a^*$. It is easy to know that $a^- = \Sigma_{i=1}^{\infty} e_{i,i+1} + \Sigma_{i=1}^{n} a_i e_{i,1}$ for some n and $a_i \in \mathbb{C}$. So, $a^*a + 1 - aa^- = 2 - aa^- = 2e_{1,1} - \Sigma_{i=1}^{n} a_i e_{i+1,1} + \Sigma_{i=2}^{\infty} e_{i,i}$ and $(a^*a + 1 - aa^-)^{-1} = \frac{1}{2} e_{1,1} + \Sigma_{i=1}^{n} a_i e_{i+1,1} + \Sigma_{i=2}^{\infty} e_{i,i}$. But $a \notin R^{\#}$, hence $a \notin R^{\oplus}$.

Proposition 2.20. *Let $a \in R^{\#}$. Then $a \in R^{\dagger}$ if and only if $a^* + 1 - aa^{\#} \in R^{-1}$.*

Proof. " $\Rightarrow$ " Note that $a \in R^{\dagger}$ implies $a^*a + 1 - a^{\#}a \in R^{-1}$ by Corollary 2.9. As $a \in R^{\#}$, then $a + 1 - aa^{\dagger} \in R^{-1}$ from Lemma 2.13. Since $a^*a + 1 - a^{\#}a = (a^* + 1 - aa^{\#})(a + 1 - aa^{\dagger}) \in R^{-1}$, it follows that $a^* + 1 - aa^{\#} \in R^{-1}$.

" $\Leftarrow$ " Let $u = a^* + 1 - aa^{\#}$ be invertible. Then $ua = a^*a$ and $au = aa^*$. Hence, $a = u^{-1}a^*a = aa^*u^{-1} = a(u^{-1}a^*a)^*u^{-1} = aa^*a(u^{-1})^*u^{-1} \in aa^*aR$. So, $a \in R^{\dagger}$ by Lemma 2.6. $\square$

Recall that a ring R is called Dedekind-finite ring if $ab = 1$ implies $ba = 1$, for all $a, b \in R$. We next give characterizations of core inverse in such a ring.

Proposition 2.21. *Let R be a Dedekind-finite ring. Then the following conditions are equivalent:*

(i) *$a^{\oplus}$ exists.*

(ii) *$a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for any $a^{(1,3)}$.*

(iii) *$a \in R^{(1,3)}$ and $a^*a + 1 - aa^{(1,3)}$ is invertible for some $a^{(1,3)}$.*

In this case, $a^{\oplus} = (a^*a + 1 - aa^{(1,3)})^{-1}a^*$.

Proof. (i) $\Rightarrow$ (ii) By Theorem 2.15 (i) $\Rightarrow$ (iv).

(ii) $\Rightarrow$ (iii) Clearly.

(iii) $\Rightarrow$ (i) Let $u = a + 1 - aa^{(1,3)}$. Then $u^*u = a^*a + 1 - aa^{(1,3)} \in R^{-1}$. As R is a Dedekind-finite ring, then $u \in R^{-1}$, which guarantees $a \in R^{\#}$ by Lemma 2.13. Hence, $a \in R^{\#} \cap R^{(1,3)}$ is core invertible from Proposition 2.1. Now, $a^{\oplus} = (a^*a + 1 - aa^{(1,3)})^{-1}a^*$ by Theorem 2.15. $\square$

Corollary 2.22. *Let R be a Dedekind-finite ring. If $a \in R^{\dagger}$, then $a \in R^{\oplus}$ if and only if $a^*a + 1 - aa^{\dagger} \in R^{-1}$. In this case, $a^{\oplus} = (a^*a + 1 - aa^{\dagger})^{-1}a^*$.*

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