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A fractional Malthusian growth model with variable order using an optimization approach

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Abstract

The goal of this work is to show, based on concrete data, that fractional differential equations with variable fractional order are more efficient to model the world population growth than the classical differential equation, or even a fractional differential equation with constant order. With these new models, we can predict more efficiently the population growth based on the present data.

Key words: Fractional calculus, fractional differential equation, least squares, unconstrained optimization

1 Introduction

The Malthusian growth model was proposed in 1798 by the English economist Thomas Malthus in his book *An Essay on the Principle of Population*. The theory states that the population number has exponential growth based on a constant rate, applied to ideal circumstances or to a short period of time, when an individual lives in region with no constraints on food and with no natural enemies. In this case, if $N(t)$ represents the size of the population at an instant t, the dynamic differential equation

$$N'(t) = P \cdot N(t)$$

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models the growth of the population. The constant P, called the Malthusian parameter, is given by the difference between the fertility and the mortality rates, assuming that these rates are constant in time. If N_0 is the initial level of the population, the function

$$N(t) = N_0 \exp(Pt) \tag{1}$$

gives the exact number of individuals at a given time t. Although there are several models to describe the dynamics of the population growth, the Malthusian model has the advantage that it is given by a linear differential equation. Later, when we model the same problem but using a fractional differential equation, we know the analytic expression of solution to the problem.

To test the different models that we purpose here, we will see how close they are to real data, by fitting the solution with dependence on some parameters with the observations. One of the most used methods is the least squares technique. Suppose that the data consists in m points, say $(t_1, x_1), \dots, (t_m, x_m)$, and we intend to fit these values in a theoretical model $t \mapsto x(t)$, where the form x is known but it depends on some unknown parameters $\beta_1, \dots, \beta_k$. If we consider in each step the error $d_i := x_i - x(t_i)$, for $i = 1, \dots, m$, then the total error is given by

$$E := \sum_{i=1}^{m} (d_i)^2.$$

The goal is to find the values of the parameters $\beta_1, \dots, \beta_k$ for which E attains a minimum value.

2 World population

In [1], a fractional approach was considered to model the World Population Growth. Starting with the classical model

$$N'(t) = P \cdot N(t), \tag{2}$$

the ordinary derivative was replaced by the Caputo fractional derivative, and the dynamic was described by the fractional differential equation (for fractional calculus theory, see [2, 3])

$${}^C D^{\alpha}_{0+} N(t) = P \cdot N(t), \quad t \geq 0,\ \alpha \in (0,1). \tag{3}$$

The solution to the fractional problem is given by the function

$$N(t) = N_0 E_\alpha(P t^\alpha), \tag{4}$$

where $E_\alpha(\cdot)$ denotes the Mittag–Leffler function:

$$E_\alpha(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{R}.$$

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Then, using the data available from the United Nations [4] from year 1910 until 2010, the best values for the parameters were found. The objective is to minimize the sum of the squares of the offsets, and to get a better accuracy for the model, the fractional order α was considered free, without any constraints. For the classical model (2), the values obtained were

$$P \approx 1.3501 \times 10^{-2} \quad \text{with the error} \quad E_{classical} \approx 7.0795 \times 10^5.$$

When we considered the problem modeled by the fractional differential equation (3), the values were

$$\alpha \approx 1.3933, \quad P \approx 3.4399 \times 10^{-3} \quad \text{with the error} \quad E_{fractional} \approx 2.0506 \times 10^5.$$

So, from these results, we see that the fractional approach is more efficient in modelling the problem than the ordinary one. The next step is to consider even a more general approach to this problem, by considering the fractional order to be a function depending on time $t \mapsto \alpha(t)$. Motivated by Eq. (4), and considering the Mittag–Leffler function with variable order

$$E_{\alpha(t)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha(t)+1)}, \quad z \in \mathbb{R},$$

we purpose the following theoretical model to study the world population problem:

$$N(t) = N_0 E_{\alpha(t)}(Pt^{\alpha(t)}). \tag{5}$$

Observe that, when $\alpha(\cdot)$ is constant, $\alpha(t) \equiv \alpha$, then Eq. (5) reduces to Eq. (4), which in turn when $\alpha \to 1$, we obtain the classical model (1). We test model (5) by the closeness to the observed data, from which we infer the values of the parameters. We compare the fractional model with constant order with a new one, with variable fractional orders. For example, when considering the order

$$\alpha_1(t) := at^2 + bt + c$$

the best values for this fractional order are

$$a \approx -4.4865 \times 10^{-5}, \quad b \approx 7.5332 \times 10^{-3}, \quad c \approx 8.5596 \times 10^{-1} \quad \text{and} \quad P \approx 7.5849 \times 10^{-3}$$

with error $E \approx 1.4813 \times 10^4$. The results are shown in Figure 1.

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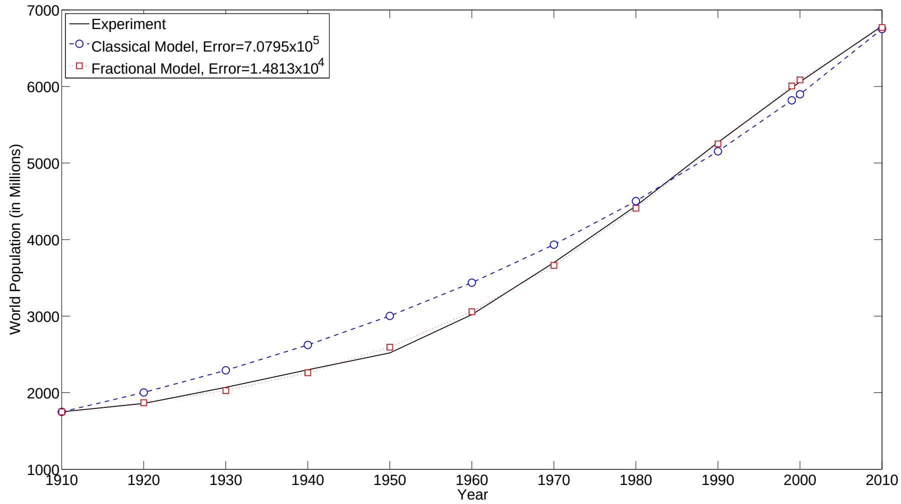

Figure 1: World Population: data, classical and fractional models.

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