TP Perturbation of TN Matrices and Totally Positive Directions

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July 28, 2015

Abstract

Here, we define and consider (linear) TP-directions and TP-paths for a totally nonnegative matrix, in an effort to more deeply understand perturbation of a TN matrix to a TP matrix. We give circumstances in which a TP-direction exists and an example to show that they do not always exist. A strategy to give (nonlinear) TP-paths is given (and applied to this example). A long term goal is to understand the sparsest TP-perturbation for application to completion problems.

Key Words: Perturbation, Totally nonnegative matrix, Totally positive matrix, TP-direction, TP-path.

AMS Subject Classification: 15B48, 15A83.

1 Introduction

An m-by-n matrix is totally nonnegative (positive), abbreviated TN (TP), if all its minors are nonnegative (positive) real numbers. It is $\mathrm{TN_k}(\mathrm{TP_k})$ if all its minors of order $\leq k$ are nonnegative (positive). A recent general reference about such matrices is [1]. It is known that the TN matrices are the topological closure of the TP matrices. Proofs show that arbitrarily close to each TN matrix there are TP matrices. This is somewhat subtle, and, typically, every entry is changed (which is often unnecessary).

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We are interested in what are the minimal sets of entries of a given TN matrix that must be changed in order to perturb it to TP. This is interesting for at least the following reason: understanding the relation between the TN and TP completion problems. In addition, minimal perturbing sets hold a natural theoretical interest, and will likely be of use in other particular problems [5]. We also ask what may be said about the nature of the perturbations to TP that occur. When can the entries in the perturbing positions be taken to be positive? When the minimal perturbing set is a block, can the perturbation be taken to be TP? When are there homogeneous linear perturbations or linear directions in which all matrices are TP? What about nonlinear parametric perturbations? We have only begun study of these questions here, and focus initially on TP directions.

A simple example that shows that perturbing sets may have few entries is

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which is TN. Perturbation of any single entry can make B TP, either perturbation of the $(1,1)$ or $(2,2)$ entry up or perturbation of the $(1,2)$ or $(2,1)$ entry down. Note that if 2 entries in the same line are perturbed, they could both be perturbed upward (or downward) by small amounts. In the following, after a brief discussion of background information, we define TP-directions and TP-paths for a totally nonnegative matrix, and further study the existence of each of them. For this purpose, the inverse sum property (ISP) and the notion of an even pair are introduced and related to TP-directions.

2 Background

To know that a real matrix is TP, not all minors need be checked. A minor (or submatrix) of a given matrix M is called *contiguous* if both index sets α, β on which it is based are sets of consecutive indices. We use the standard notation $M[\alpha, \beta]$ to denote the submatrix lying in rows α and columns β, and if $|\alpha| = |\beta|$, $\det M[\alpha, \beta]$ denotes the α, β minor. Similarly, $A(\alpha, \beta)$ denotes the submatrix of A resulting from deletion of rows α and columns β. If all the contiguous minors (note that these include the entries) of a real matrix are positive, then the matrix is TP. In fact, not even all the contiguous minors need be checked. A contiguous minor (or submatrix) is called *initial* if at least one of the two consecutive index sets begins with 1. Only the initial minors need be checked for positivity, to verify that a matrix is TP [1, 2, 3, 7]. There are exactly as many initial minors in a matrix as there are entries, and all the initial (or some equivalent set of) minors must be checked to verify TP. For example, the *trailing minors* (contiguous minors in which the last index appears in at least one index set) also suffice because of the forward/backward symmetry of the TP property. However, if something else is already known about a matrix, fewer of the initial minors being positive suffice that it be TP. For example if a matrix is TN, then the *Shapiro minors* being positive imply that it is TP [6, 1]. The Shapiro minors are the initial minors in which the "other" index set includes the last index (both initial and trailing); they are contiguous minors to the northeast or southwest. Generally, checking for TN is more complicated and factorization methods offer a better option [1]. Proofs of many such statements, especially the one about initial minors, best rely on the Sylvester determinantal identity [4, 1]. The very useful special case is the following. Let $A \in M_n(\mathbb{F})$ be the partitioned matrix

$$A = \begin{bmatrix} a_{11} & a_{12}^T & a_{13} \\ a_{21} & A_{22} & a_{23} \\ a_{31} & a_{32}^T & a_{33} \end{bmatrix}$$

in which $A_{22} \in M_{n-2}(\mathbb{F})$ and a_{11} and a_{33} are scalars. Define the matrices

$$B = \begin{bmatrix} a_{11} & a_{12}^T \\ a_{21} & A_{22} \end{bmatrix}, C = \begin{bmatrix} a_{12}^T & a_{13} \\ A_{22} & a_{23} \end{bmatrix}, D = \begin{bmatrix} a_{21} & A_{22} \\ a_{31} & a_{32}^T \end{bmatrix}, E = \begin{bmatrix} A_{22} & a_{23} \\ a_{32}^T & a_{33} \end{bmatrix}.$$

Then

$$\det A = \frac{\det B \det E - \det C \det D}{\det A_{22}}, \quad \text{when } \det A_{22} \neq 0.$$

Also many of these concepts may be described in terms of compounds (recall that the kth compound matrix of an $m \times n$ matrix A is the $\binom{m}{k} \times \binom{n}{k}$ matrix formed from the determinants of all $k \times k$ submatrices of A, i.e., all $k \times k$ minors, arranged with the submatrix index sets in lexicographic order [4]). A matrix is TN (TP) if all its compounds are entry-wise nonnegative (positive).

A less familiar notion is that a matrix be *totally nonsingular*. This means that all square submatrices are invertible. Most of the nice statements about total positivity do not carry over to total nonsingularity. For example, nonsingularity of the contiguous square submatrices does not suffice for total nonsingularity. Of course a TP matrix is totally nonsingular. Generically (e.g., in the totally nonsingular case), however, a general matrix may be reconstructed from its initial minors using Sylvester.

3 Definitions

We now mention some new ideas specific to our work. We assume throughout that B is a TN matrix to be altered entry-wise.

Definition 1. *If A and B are m-by-n matrices, then A is a* linear B-perturbation *if there is a $\tau > 0$ such that $B + tA$ is TP for all $0 < t < \tau$. If, further, τ may be taken to be $+\infty$, then A is called a* (linear) B-direction.

We note that the existence of a linear B-perturbation for a TN matrix B is a stronger statement than that B may be perturbed to TP.

Definition 2. *If A and B are m-by-n matrices, and $A = A(t)$ is a continuous matrix function such that $A(t) \to 0$ as $t \to 0$, then $A(t)$ is a B-perturbation if there is a $\tau > 0$ such that $B + A(t)$ is TP for $0 < t < \tau$. If, further, τ may be taken to be $+\infty$, then $A(t)$ is called a* B-path.

Thus, a linear B-direction (B-path) is a linear B-perturbation (B-perturbation) for which large as well as "small" changes result in a TP matrix. A B-perturbation may not be TN and may even have negative entries, but a linear B-direction must, at least, be TN and a B-path must eventaully be TN. We will study linear B-directions that are TP and give an algorithm to produce polynomial B-perturbations and B-paths.

Definition 3. *If $A, B \in M_{m,n}(\mathbb{R})$ and A is totally nonsingular, then for index sets α, β such that $|\alpha| = |\beta|$, the α, β count is $C_{\alpha,\beta}(A, B)$, the number of negative real eigenvalues of $A[\alpha, \beta]^{-1}B[\alpha, \beta]$.*

Definition 4. *We say that A and B, as in Definition 3, constitute an even pair if for every pair of index sets α, β, with $|\alpha| = |\beta|$, $C_{\alpha,\beta}(A, B)$ is even. A and B form a 0-pair, if, further, each of these even counts is 0.*

Recall that a real, square, invertible matrix has positive determinant if and only if it has an even number of negative real eigenvalues. This is a motivation for 0 and even pairs.

Next, as usual, we use e to denote the vector of $1's$ of relevant dimension.

Definition 5. *A totally nonsingular matrix $A \in M_{m,n}(\mathbb{R})$ has the (strict) inverse sum property (ISP), if, for all index sets $\alpha \subseteq \{1, ..., m\}, \beta \subseteq \{1, ...n\}$, with $|\alpha| = |\beta|$,*

$$e^T A[\alpha, \beta]^{-1} e \geq 0 \ (> 0),$$

i.e., the sum of the entries of the inverse of every square submatrix is nonnegative (positive).

If the above inequalities hold for the initial submatrces of A, A is said to have the initial (strict) ISP. Note that the ISP implies that the entries of a matrix are positive.

4 B-directions and linear B-perturbations

We begin with some elementary properties of B-directions and B-perturbations. First, note that, since TP is a property inherited by submatrices, so are B-directions and perturbations.

Proposition 6. *If A is a linear B-direction (perturbation), then for any index sets α, β, $A[\alpha, \beta]$ is a linear $B[\alpha, \beta]$-direction (perturbation), in which α and β are sets of indices of the same cardinality.*

Since increasing the northwestern-most or southeastern-most entry of a TP matrix preserves total positivity [1, p.198], we have

Proposition 7. *If A and B are m-by-n matrices and A is a linear B-direction (perturbation), then for any $s \geq 0$, $A + sE_{11}$ and $A + sE_{m,n}$ are linear B-directions (perturbations).*

By simple algebraic manipulation, we also have

Proposition 8. *If A is a linear B-direction (perturbation), then any positive linear combination of A and B is a linear B-direction (perturbation).*

Note that if A and B are m-by-n matrices and A is TP, then $B + tA$ is TP for sufficiently large t. Despite this, and the above results, it is possible for a TP matrix to be a linear B-perturbation for a TN matrix B, without being a B-direction.

Example 9. *Let $B = \begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Then B is TN and A is TP. A calculation shows that A is a B-perturbation, for $\tau = 1$; $B + tA$ is not TP for $1 \leq t \leq 2$, and $B + tA$ is TP for $t > 2$.*

Of course, multiplication by a positive scalar preserves the ISP, but positive diagonal scaling does not.

Example 10. *The matrix*

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

is TP and has the ISP. But

$$\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 30 & 20 \end{bmatrix}$$

is, of course, TP, but does not have the ISP.

The above shows that a matrix may be TP but not have the ISP. Conversely, a matrix may have the ISP without being TP.

Example 11. *The matrix $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ is not TP, as its determinant is -1, but, as $e^T A^{-1} e = 1$ and the entries of A are positive, A does have the ISP.*

Given a TN matrix B, the natural existence question of a TP matrix A such that $A + B$ is TP is answered by a process similar to the bordering by which TP matrices may be constructed (see the cover of [1]). It is clear that for A to be a B-direction, B must be TN (small t) and A must be (at least) TN (large t). We now turn to characterizations of linear B-perturbations and B-directions.

Proposition 12. *Let A, B be n-by-n real matrices and $t \in \mathbb{R}$ be such that* $\det A > 0$ *and $B + tA$ is invertible. Then* $\det(B + tA) = \det A \det(A^{-1}B + tI)$*, so that* $\det(B + tA) > 0$ *if and only if $A^{-1}B + tI$ has an even number of negative real eigenvalues, i.e., $A^{-1}B$ has an even number of real eigenvalues $< -t$.*

Theorem 13. *Suppose $B \in M_{m,n}(\mathbb{R})$ is TN and $A \in M_{m,n}(\mathbb{R})$ is TP. Then A is a linear B-perturbation if and only if A, B is an even pair.*

Proof. If A, B is not an even pair, then there are index sets α, β with $|\alpha| = |\beta|$ and with $A[\alpha, \beta]^{-1}B[\alpha, \beta]$ having odd negative eigenvalues $-\lambda_{2k+1} \leq ... \leq -\lambda_1$, and for $0 < t < \lambda_1$,

$$A[\alpha, \beta]^{-1}B[\alpha, \beta] + tI$$

would have negative determinant, as would $B[\alpha, \beta] + tA[\alpha, \beta]$. Thus, $B + tA$ would not be TP for small t. On the other hand, if A, B is an even pair (and not a 0-pair), and λ is the algebraically largest negative real eigenvalue among the matrices $A[\alpha, \beta]^{-1}B[\alpha, \beta]$ with $|\alpha| = |\beta|$, then $\tau = -\lambda$ verifies that A is a linear B-perturbation (with tolerance τ). If A, B is actually a 0-pair, we may take any $\tau > 0$ to verify that A is a linear B-perturbation. □

Theorem 14. *Suppose that $B \in M_{m,n}(\mathbb{R})$ is TN and $A \in M_{m,n}(\mathbb{R})$ is TP. Then A is a (linear) B-direction if and only if A, B is a* 0*-pair.*

Proof. That A is a B-direction means that A is a linear B-perturbation with $\tau = +\infty$ (i.e., any $\tau > 0$ verifies the definition of B-perturbation). The proof of the previous theorem shows that this happens precisely for a 0-pair. □

Remark. Since positivity of the initial minors implies the positivity of all minors, then the definitions of B-perturbation and B-direction need only consider the initial submatrices of A and B. The two theorems then indicate that the notions of even pair and 0 pair need only take these submatrices into account.

This has the curious implication that if A is TP and B is TN, then if for all initial submatrices $A[\alpha, \beta]^{-1}B[\alpha, \beta]$ have an even number of (0) negative real eigenvalues, then the same will hold for all square submarices. We do not know the extent to which this holds beyond our hypotheses that A be TP and B be TN.

As usual, we let J denote the matrix all of whose entries are 1. If the number of rows and columns is unclear from context, we denote them with subscripts as in $J_{m,n}$. Of course, the matrix J is TN.

Corollary 15. *A TP matrix A is a (linear) J-directon if and only if A has the ISP.*

Proof. For each pair of index sets α, β, with $|\alpha| = |\beta|$, $\text{rank}A[\alpha, \beta]^{-1}J[\alpha, \beta] = 1$, because $\text{rank}J[\alpha, \beta] = 1$ and $A[\alpha, \beta]^{-1}$ is nonsingular. Since $\text{Tr}(A[\alpha, \beta]^{-1}J[\alpha, \beta]) = e^T A[\alpha, \beta]^{-1}e$, the only possible nonzero eigenvalue of $A[\alpha, \beta]^{-1}J[\alpha, \beta]$ is $e^T A[\alpha, \beta]^{-1}e$. Thus, A and J form a 0-pair if and only if A has the ISP, and the claim of the corollary follows from the prior theorem □

If J is replaced by a more general TN matrix, the content of the corollary need no longer be valid.

Example 16. *The matrix* $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ *is TP and has the ISP, and* $B = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$ *is TN, but A is not a B-direction.*

We may say a bit more in the case of J.

Theorem 17. *The following statements are equivalent for a TP matrix A:*

(1) A is a J-direction;

(2) A is a J-perturbation;

(3) A, J is an even pair;

(4) A, J is a 0-pair;

(5) A has the ISP; and

(6) A has the initial ISP.

Proof. (1) obviously implies (2). (2) implies (3) by Theorem 13. (3) implies (4) because J has rank 1, so that $A[\alpha, \beta]^{-1}J[\alpha, \beta]$ cannot have more than 1 negative real eigenvalue. (4) implies (5) follows by Theorem 14 and Corollary 15. (5) implies (6) trivially. (6) implies (1) because the initial minors of $J + tA$ are positive for all $t > 0$. □

Not surprisingly, we may similarly characterize TP directions for arbitrary rank 1 matrices.

Corollary 18. *Suppose that K is a positive rank 1 TN matrix. Then there are positive diagonal matrices D and E such that $DKE = J$. Now, A is a (linear) TP K-direction if and only if $D^{-1}AE^{-1}$ has the ISP.*

Now, for J, and thus positive rank 1 matrices generally, there do always exist TP directions. To see this, we first give some lemmas.

Lemma 19. *Suppose that C is a k-by-$(k+1)$ TP matrix. Then there is a $k+1$ row vector g such that $\begin{bmatrix} g \\ C \end{bmatrix}$ is a $(k+1)$-by-$(k+1)$ TP matrix. Moreover, the entries of g may be chosen so as to increase from right to left, and a greater rate of increase, producing g', will also make $\begin{bmatrix} g' \\ C \end{bmatrix}$ TP.*

Recall that if $A = (a_{ij})$ is n-by-n and invertible, then $adj(A) = (\det A)A^{-1}$. We have that

Lemma 20. *$e^T adj(A)e$ is a linear function of each row of A, and in particular of the first row of A.*

Lemma 21. *The coefficient of a_{11} in $e^T adj(A)e$ as a function of the first row of A is $e^T adj(A(1,1))e$. And this coefficient is positive if $A(1,1)$ is TP and satisfies the ISP.*

Lemma 22. *There exist J-directions.*

Proof. It suffices to construct a TP matrix A for which the trailing inverse sums are positive. This may be done sequentially using the above lemmas starting with a positive, right-to-left increasing, last row, and then bordering right to left, making each successive entry large enough to ensure both TP and positive inverse sum (for the maximum trailing minor). Note that making $e^T adj(A)e > 0$ is the same as $e^T A^{-1} e > 0$ as $\det A > 0$. $\square$

Example 23. *A TP matrix A, such as Lemma 22 guarantees, may be constructed as follows: The last row and column are all 1s; the entries of $(n-1)^{th}$ row are $a_{n-1,j} = n+1-j$, $j = 1, ..., n-1$; the entries of $(n-1)^{th}$ column are $a_{i,n-1} = n+1-i$, $i = 1, ..., n-1$ and the other entries are $a_{ij} = a_{i,j+1} + a_{i+1,j}$.*

For example a 5×5 J-direction is
$$\begin{bmatrix} 70 & 35 & 15 & 5 & 1 \\ 35 & 20 & 10 & 4 & 1 \\ 15 & 10 & 6 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Corollary 24. *If B is positive and rank 1, then there exist B-directions.*

It follows that there exist TP matrices of any size, with the ISP. It would be interesting to more explicitly characterize these.

Of course, if the TN matrix B is actually TP, then B itself is a (linear) B-direction.

Theorem 25. *If $B \in M_{m,n}(\mathbb{R}), m \geq n$, is TN and any one of, TP_1 and rank 1; TP; or TP_{n-1}, then B has a TP direction.*

Proof. Here we only consider $m = n$, when $m > n$ see Lemma 19. The first case is concluded by Corollary 24, and in the second case, just let the TP matrix be B itself. In the third case, since B is $\mathrm{TP_{n-1}}$, we may only perturb great enough the $(1, 1)$ entry of B to make it TP, then the perturbed matrix is a B-direction. □

If B is a positive 3-by-3 TN matrix, the only situations, in which Theorem 25 does not guarantee a TP direction for B, are those in which B is TP_1 and not TP_2 and rank either 2 or 3. (So, some initial 2-by-2 minor is 0, and the determinant may be 0.) In either of these cases, straightforward calculation verifies that there is a TP direction for B. So, every positive 3-by-3 TN matrix B does have (linear) B-directions. However, this need not happen for every n.

The first case not covered is a 4-by-4, TP_1, TN, rank 2 matrix. Consider the following example.

Example 26. *Let* $B = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. *Then B is a TN matrix of rank* 2*.*

Let $A = (a_{ij})$ be a 4-by-4 TP matrix. Then

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} + t \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} ta_{11}+4 & ta_{12}+3 & ta_{13}+2 & ta_{14}+1 \\ ta_{21}+1 & ta_{22}+1 & ta_{23}+1 & ta_{24}+1 \\ ta_{31}+1 & ta_{32}+1 & ta_{33}+1 & ta_{34}+1 \\ ta_{41}+1 & ta_{42}+2 & ta_{43}+3 & ta_{44}+4 \end{bmatrix}.$$

Notice that the coefficient of t of the minor $123; 123$ *is* $-[(a_{21} - 2a_{22} + a_{23}) - (a_{31} - 2a_{32} + a_{33})]$*, but at the same time the coefficient of t of the minor* $234; 123$ *is* $[(a_{21} - 2a_{22} + a_{23}) - (a_{31} - 2a_{32} + a_{33})]$*. Since neither of the minors has a constant term, the minors* $123; 123$ *and* $234; 123$ *cann't be positive simultaneously for all positive t. Thus, there is no (linear) B-direction.*

It is a natural question, then which rank 2 TN matrices B have (linear) B-directions, and what is the role of 0 entries.

In the next section we show that for the B of Example 26, there is a (polynomial) B-path and suggest a general method for constructing (sparse) B-paths.

5 Polynomial B-perturbations and B-Paths for TN matrices

We suppose now that B is an entry-wise positive TN matrix that we wish to perturb to TP via a polynomial matrix. We allow the polynomial matrix to be sparse, but it may also be arranged that it be full, as well. We proceed by analogy to the classical method of constructing TP matrices, as exhibited on the cover of [1]. We proceed right to left from the southeastern-most entry to the northwestern-most entry one row at a time. Of course, the upper left entry of each trailing minor that is 0 must be perturbed (upward). It suffices to make each of these minors positive. In particular, as in the case of TP construction, it suffices to make positive the determinant of the maximal southeastern minor that is altered (at each step). This one will be a trailing minor. Our perturbations will be polynomials, in a single variable, that take on the value 0 at the argument 0, i.e., the constant term is 0. But as we have seen, these polynomials may have to be nonlinear.

We consider Example 26 from the last section to illustrate the process. We actually find a B-path, where there was no (linear) B-direction.

Example 27. *Let*

$$B = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Then B is TN of rank 2 (but not TP). Since the last two rows form a TP matrix, the "first" entry that requires perturbation is in the $(2,3)$ position. Call the perturbation x. The entry in the $(2,2)$ position also requires perturbation as the (trailing) 234; 234 minor of B is 0. A perturbation of $2x$ would leave this minor 0, so make the perturbation $2x + x^2$. The $(2,1)$ position also requires perturbation. Since a perturbation of $3x + 2x^2$ would make the (trailing) $2,3,4;1,2,3$ minor 0, choose a perturbation of $3x + 2x^2 + x^3$. So far, our perturbed matrix is

$$\begin{bmatrix} 4 & 3 & 2 & 1 \\ 1+3x+2x^2+x^3 & 1+2x+x^2 & 1+x & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

To achieve a B-perturbation, the $(1,2)$ and $(1,1)$ entries require perturbation in that order. However, to achieve a B-path, the $(1,3)$ entry need be perturbed by at least x to keep the 12;34 minor positive for all x. Without this, continuing in a similar manner gives the B-perturbation

$$\begin{bmatrix} 4+2x+3x^2+x^3+x^4 & 3+x+x^2 & 2 & 1 \\ 1+3x+2x^2+x^3 & 1+2x+x^2 & 1+x & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

The initial minors of size at least 2 are

12; 12:	$1 + x^2 + x^3 + 3x^4 + 2x^5 + x^6$	123; 123:	$x^4 + x^5 + x^6$
12; 23:	$1 + x^3$	123; 234:	x^3
12; 34:	$1 - x$	234; 123:	x^3
23; 12:	$x + x^2 + x^3$	1234; 1234:	x^6
34; 12:	1		

Because the 12;34 minor is $1 - x$, this is not a B-path. If we had continued by first perturbing the $(1, 3)$ entry and continuing, we would have produced

$$\begin{bmatrix} 4 + 8x + 5x^2 + 4x^3 + x^4 + x^5 & 3 + 5x + 2x^2 + x^3 & 2 + x & 1 \\ 1 + 3x + 2x^2 + x^3 & 1 + 2x + x^2 & 1 + x & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

This is a B-path as the trailing minors of size at least 2-by-2 are

34; 34:	1	23; 34:	x	234; 123:	x^3
34; 23:	1	23; 12:	x	123; 234:	x^4
34; 12:	1	234; 234:	x^2	1234; 1234:	x^7.

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