Further results on the inverse along an element in semigroups and rings

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Abstract
In this paper, we introduce a new notion in a semigroup $S$ as an extension of Mary’s inverse. Let $a$, $d \in S$. An element $a$ is called left (resp. right) invertible along $d$ if there exists $b \in S$ such that $bad = d$ (resp. $dab = b$) and $b \leq L_d$ (resp. $b \leq R_d$). An existence criterion of this type inverse is derived. Moreover, several characterizations of left (right) regularity, left (right) $\pi$-regularity and left (right) $\ast$-regularity are given in a semigroup. Further, another existence criterion of this type inverse is given by means of a left (right) invertibility of certain elements in a ring. Finally we study the (left, right) inverse along a product in a ring, and, as an application, Mary’s inverse along a matrix is expressed.

Keywords:
von Neumann regularity, Left (Right) regularity, Left (Right) $\pi$-regularity, Left (Right) $\ast$-regularity, Inverse along an element, Semigroups, Rings

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1. Introduction
Throughout this paper, $S$ is a semigroup. An element $a \in S$ is (von Neumann) regular if there exists $x$ in $S$ such that $axa = a$. Such $x$ is called an inner inverse of $a$. By $a\{1\} = \{x \in S : axa = a\}$ we denote the set of all inner inverses of $a$. An arbitrary element in $a\{1\}$ is denoted by $a^{(1)}$.

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The element $a$ is left (right) regular (see e.g. [2]) if there exists $x$ such that $a = xa^2$ ($a = a^2x$), and strongly regular if it is both left regular and right regular. It is left (right) $\pi$-regular (see e.g. [2]) if there exists $x$ such that $a^n = xa^{n+1}$ ($a^n = a^{n+1}x$) for a positive integer $n$. If $a$ is both left and right $\pi$-regular, then $a$ is strongly $\pi$-regular.

Let $*$ be an involution (anti-isomorphism of degree 2) on $S$, that is, the involution satisfies $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in S$. Let $a \in S$. We call $a$ left (right) $*$-regular if there is $x$ such that $a = aa^*x$ ($a = xaa^*$). A $*$-semigroup $S$ is called left (right) $*$-regular if all elements in $S$ are left (right) $*$-regular. If $x$ satisfies $axa = a$ and $(ax)^* = ax$, then $x$ is a $\{1, 3\}$-inverse of $a$. If $y$ satisfies $aya = a$ and $(ya)^* = ya$, then $y$ is a $\{1, 4\}$-inverse of $a$.

The standard notions of group, Drazin and Moore-Penrose inverse can be referred to the literature [4, 9]. Following [4], an element $a$ is Drazin invertible if and only if it is strongly $\pi$-regular. In particular, $a$ is group invertible if and only if it is strongly regular. It is well known that $a \in S$ is Moore-Penrose invertible if and only if $a \in aa^*S \cap Ssa^*a$ if and only if it is both $\{1, 3\}$ and $\{1, 4\}$-invertible. All these inverses, if they exist, are unique. We denote by $a^\#, a^D$ and $a^\dagger$ the group, Drazin and Moore-Penrose inverses of $a$, respectively.

Mary [6] recently defined a new generalized inverse in a semigroup $S$ called the inverse along an element. Motivated by [6], we introduce in section 2 below a new notion. An existence criterion of this type inverse is derived. Moreover, several characterizations of left (right) regularity, left (right) $\pi$-regularity and left (right) $*$-regularity are given in a semigroup. Also, we prove that $a \in S$ is Moore-Penrose invertible if and only if it is left $*$-regular if and only if it is right $*$-regular. In section 3, another existence criterion of this type inverse is given by means of a left (right) invertibility of certain elements in a ring, and as an application, the formula of the inverse along a matrix is expressed.

2. One-sided inverse along an element in semigroups

Green’s preorders in a semigroup [5] are defined as followed ($S^1$ denotes the monoid generated by $S$)

$a \leq_L b \iff S^1a \subseteq S^1b \iff \text{there exists } x \in S^1 \text{ such that } a = xb$.

$a \leq_R b \iff aS^1 \subseteq bS^1 \iff \text{there exists } x \in S^1 \text{ such that } a = bx$.

$a \leq_H b \iff a \leq_L b \text{ and } a \leq_R b$. 

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We next introduce a notion that is based on Green’s preorders in a semigroup.

**Definition 2.1.** Let $a, d \in S$. An element $a$ is left invertible along $d$ if there exists $b \in S$ such that $bad = d$ and $b \leq_L d$.

Any $b$ satisfying the conditions in Definition 2.1 is called a left inverse of $a$ along $d$.

**Definition 2.2.** Let $a, d \in S$. An element $a$ is right invertible along $d$ if there exists $b$ such that $dab = d$ and $b \leq_R d$.

In [6], Mary defined a new generalized inverse in a semigroup as follows: An element $b$ is an inverse of $a$ along $d$ if $bad = d = dab$ and $b \leq_H d$. This type inverse is unique, if it exists and denoted by $a^{||d}$. Mary showed in particular that $a^\#, a^D$ and $a^\dagger$ are the inverses of $a$ along $a$, $a^n$ and $a^*$ respectively ([6, Theorem 11]). In [3], Drazin introduced $(b,c)$-inverse in a semigroup. It follows that $(d,d)$-inverse of $a$ is an inverse of $a$ along $d$ (Mary’s inverse). Hence, group inverse, Drazin inverse, Moore-Penrose inverse and Mary’s inverse of $a$ are instances of left or right inverse of $a$ along $d$.

Next, we present an existence criterion of a left inverse along an element.

**Theorem 2.3.** Let $a, d \in S$. Then $a$ is left invertible along $d$ if and only if $d \leq_L dad$.

**Proof.** “⇒” Suppose that $a$ is left invertible along $d$. Then there exists $b$ such that $bad = d$ and $b \leq_L d$. From $b \leq_L d$, it follows that $b = xd$ for some $x \in S^1$. Hence, $d = bad = xdad$, which implies $d \leq_L dad$.

“⇐” $d \leq_L dad$ implies $d = ydad$ for some $y \in S$. Take $b = yd$. Then $b \leq_L d$ and $bad = d$. □

Dually, we can obtain an equivalence for the existence of a right inverse along an element.

**Theorem 2.4.** Let $a, d \in S$. Then $a$ is right invertible along $d$ if and only if $d \leq_R dad$.

Applying Theorems 2.3 and 2.4 and [7, Theorem 2.2], we get the following corollaries.
Corollary 2.5. Let \(a, d \in S\). Then \(a\) is invertible along \(d\) if and only if it is left and right invertible along \(d\).

Corollary 2.6. Let \(d_l, d_r\) and \(d\) be such that \(S^1d_l = S^1d\) and \(d, S^1 = dS^1\). Then \(a\) is invertible along \(d\) if and only if it is left invertible along \(d_l\) and right invertible along \(d_r\).

We consider now the relations between left invertibility along \(d\) and left invertibility, left regularity, left \(\pi\)-regularity and left \(*\)-regularity.

Theorem 2.7. Let \(a \in S\).

(i) If \(S\) is a monoid, then \(a\) is left invertible along 1 if and only it is left invertible.

(ii) \(a\) is left invertible along \(a\) if and only if it is left regular.

(iii) There exists \(n \in \mathbb{N}\) such that \(a\) is left invertible along \(a^n\) if and only if it is left \(\pi\)-regular.

(iv) If \(S\) is a \(*\)-semigroup, then \(a\) is left invertible along \(a^*\) if and only if it is left \(*\)-regular.

Proof. (i) Suppose that \(a\) is left invertible. Then there exists \(b \in S\) such that \(1 = ba\). Also, as \(b = b \ast 1\), then \(b \leq_L 1\) and \(a\) is left invertible along 1.

Conversely, if \(a\) is left invertible along 1, then there exists \(b \in S\) such that \(ba = 1\) and \(a\) is left invertible.

(ii) Assume that \(a\) is left regular. Then exists \(b \in S\), \(a = ba^2\) hence \(a = b^2a^3\) and \(a \leq_L a^3\). By Theorem 2.3, \(a\) is left invertible along \(a\).

Conversely, if \(a\) is left invertible along \(a\), then there is \(b \in S\) such that \(ba = a\) and \(a\) is left regular.

(iii) Let \(a\) be left \(\pi\)-regular. Then there exist \(b \in S\) and an integer \(n\) such that \(a^n = ba^{n+1}\), and by induction \(a^n = b^2a^{n+2} = \ldots = b^{n+1}a^{2n+1}\). Hence \(a \leq_L a^{2n+1}\) and \(a\) is left invertible along \(a^n\) by Theorem 2.3.

The converse part is straightforward.

(iv) Assume that \(a\) is left \(*\)-regular. Then there exists \(x \in S\) such that \(a = aa^*ax\) and hence \(a^* = x^*a^*aa^*\), which implies that \(a\) is left invertible along \(a^*\) by Theorem 2.3.

Conversely, if \(a\) is left invertible along \(a^*\), it follows from Theorem 2.3 that \(a^* \leq_L a^*aa^*\). Hence, \(a = aa^*ay\) for some \(y \in S\) and \(a\) is left \(*\)-regular. □
Applying Theorems 2.3 and 2.7, we give some characterizations of left invertibility and left generalized invertibilities in the following corollary.

**Corollary 2.8.** Let \( a \in S \). Then

(i) If \( S \) is a monoid, \( a \) is left invertible if and only if \( 1 \leq_L a \).

(ii) \( a \) is left regular if and only if \( a \leq_L a^3 \).

(iii) \( a \) is left \( \pi \)-regular if and only if \( a^m \leq_L a^{2m+1} \), for a positive integer \( m \).

(iv) If \( S \) is a \( * \)-semigroup, then \( a \) is left \( * \)-regular if and only if \( a^* \leq_L a^*aa^* \).

Dually, we have the following result.

**Theorem 2.9.** Let \( a \in S \). Then

(i) If \( S \) is a monoid, \( a \) is right invertible along 1 if and only if it is right invertible.

(ii) \( a \) is right invertible along \( a \) if and only if it is right regular.

(iii) \( a \) is right invertible along \( a^m \) if and only if it is right \( \pi \)-regular.

(iv) If \( S \) is a \( * \)-semigroup, then \( a \) is right invertible along \( a^* \) if and only if it is right \( * \)-regular.

By Theorems 2.4 and 2.9, we have

**Corollary 2.10.** Let \( a \in S \). Then

(i) If \( S \) is a monoid, \( a \) is right invertible if and only if \( 1 \leq_R a \).

(ii) \( a \) is right regular if and only if \( a \leq_R a^3 \).

(iii) \( a \) is right \( \pi \)-regular if and only if \( a^m \leq_R a^{2m+1} \), for a positive integer \( m \).

(iv) If \( S \) is a \( * \)-semigroup, then \( a \) is right \( * \)-regular if and only if \( a^* \leq_R a^*aa^* \).

**Remark 2.11.** Let \( S \) be a non Dedekind finite ring with \( ab = 1 \neq ba \). Then \( a \) is right invertible along \( a \ (a^n) \) by Theorem 2.4, but one can show that it is not left invertible along \( a \ (a^n) \). However, in a \( * \)-semigroup, we prove that every right \( * \)-regular element is left \( * \)-regular (see Theorem 2.16 below).

We present characterizations of \( \{1,3\} \)-inverse, \( \{1,4\} \)-inverse, left \( * \)-regularity and right \( * \)-regularity of an element in a \( * \)-semigroup with an identity element.

The conditions (i) and (ii) in Proposition 2.12 below were essentially proved in [11, Lemma 2.2] in a ring with involution case.
Proposition 2.12. Let $S$ be a $*$-semigroup and let $a \in S^1$. Then

(i) $a$ has a $\{1, 3\}$-inverse if and only if $S^1a = S^1a^*a$.
(ii) $a$ has a $\{1, 4\}$-inverse if and only if $aS^1 = aa^*S^1$.
(iii) $a$ is left $*$-regular if and only if $aS^1 = aa^*S^1$.
(iv) $a$ is right $*$-regular if and only if $S^1a = S^1aa^*$.

Remark 2.13. Proposition 2.12 does not hold in the case that there is no identity element. Indeed, let $S$ be a null semigroup ($xy = 0, \forall x, y \in S$) distinct from $\{0\}$. Then 0 is the only von Neumann regular element but $(\forall a \in S) Sa = 0 = Saa^*a = Sa^*a$ for instance.

Remark 2.14. If $a$ is left $*$-regular, then $a$ has a $\{1, 4\}$-inverse by Proposition 2.12. But the converse does not necessarily hold. Let $S = M_2(\mathbb{C})$ and the involution is the transpose. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $AA^*S = AS$, which implies that $A$ is $\{1, 4\}$-invertible. However $AA^*AS = 0$. So, $A$ is not left $*$-regular.

Now, we construct a $*$-semigroup to illustrate various relations in Proposition 2.12.

Example 2.15. Let $A = \{1, 2, 3\}$. Then every map from $A$ to $A$ can be written as $
abla = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$, where $i, j, k \in A$. If $S$ is a semigroup generated by $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, then $S = \{x, x^2, y, xy, yx\}$. Set $x^* = x$, $(x^2)^* = x^2$, $y^* = y$, $(xy)^* = yx$ and $(yx)^* = xy$, then $*$ is an involution on $S$. Moreover, we get

(i) $x$ is regular but neither $\{1, 3\}$ nor $\{1, 4\}$-invertible.
(ii) $y$ and $x^2$ are projectors and hence Moore-Penrose invertible.
(iii) $xy$ is $\{1, 4\}$-invertible but neither $\{1, 3\}$-invertible nor left $*$-regular.
(iv) $yx$ is $\{1, 3\}$-invertible but neither $\{1, 4\}$-invertible nor right $*$-regular.

Theorem 2.16. Let $S$ be a $*$-semigroup and let $a \in S$. Then the following conditions are equivalent:

(i) $a$ is Moore-Penrose invertible.
(ii) $a$ is left $*$-regular.
(iii) $a$ is right $*$-regular.
Proof. (i)⇒(ii) Let $a^\dagger$ be the Moore-Penrose inverse of $a$. Then $a = a(a^\dagger a)^* = aa^*(a^\dagger a^\dagger)^* = aa^*a^\dagger(a^\dagger)^*$ and hence $a$ is left $*$-regular.

(ii)⇔(iii) Assume that $a$ is left $*$-regular. There exists $x \in S$ such that $a = aa^*ax$ and hence $a^* = x^*a^*aa^*$. Since $(ax)^*a = (ax)aa^*(ax)$, it follows that $(ax)^*a = [(ax)^*a]^* = a^*ax$. Hence, we have $a = aa^*ax = a(ax)^*a = ax^*a^*a = (ax^*a^*)aa^*a$. So, $a$ is right $*$-regular.

The converse part follows by a similar way.

(iii)⇒(i) Let $a$ be right $*$-regular and hence left $*$-regular. We have $a \in aa^*S \cap Sa^*a$. Thus, $a$ is Moore-Penrose invertible. □

Recall that a semigroup $S$ is called $*$-regular if all elements in $S$ are Moore-Penrose invertible. Hence, we get

**Corollary 2.17.** Let $S$ be a $*$-semigroup. Then $S$ is $*$-regular if and only every element in $S$ is left (right) $*$-regular.

The following lemma was given by Penrose in complex matrices (see [9, p. 407]), it indeed holds in a $*$-semigroup.

**Lemma 2.18.** Let $S$ be a $*$-semigroup and let $a \in S$. If $axa = a = aya$, $(ax)^* = ax$ and $(ya)^* = ya$ for some $x, y \in S$. Then $a$ is Moore-Penrose invertible and $a^\dagger = yax$.

We now present the formula of the Moore-Penrose inverse of a left (right) $*$-regular element.

**Theorem 2.19.** Let $S$ be a $*$-semigroup and let $a \in S$. If $a = aa^*ax$ for some $x \in S$, then $a$ is Moore-Penrose invertible and $a^\dagger = a^*ax^2a^*$.

Proof. If $a = aa^*ax$, then $(ax)^*$ is a $\{1,4\}$-inverse of $a$ according to [11, Lemma 2.2]. By the proof (ii)⇔(iii) in Theorem 2.16, it is known that $a = (ax^*a^*a)aa^*$ and $(ax^*a^*a)^*$ is a $\{1,3\}$-inverse of $a$. By virtue of Lemma 2.18, it follows that $a$ is Moore-Penrose invertible and $a^\dagger = (ax)^*a(aa^*x^*a^*)^* = a^*ax^2a^*$.

Dually, we have the following result.

**Theorem 2.20.** Let $S$ be a $*$-semigroup and let $a \in S$. If $a = yaa^*$ for some $y \in S$, then $a$ is Moore-Penrose invertible and $a^\dagger = a^*y^2aa^*$. 

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We then recover and improve some known characterizations of generalized invertibility in a semigroup.

**Corollary 2.21.** [6, Theorem 11] Let $a \in S$. Then

(i) $a$ is invertible if and only if it is invertible along $1$. In this case, $a^{-1} = a\|1$.

(ii) $a$ is group invertible if and only if it is invertible along $a$. In this case, $a^# = a\|a$.

(iii) $a$ is Drazin invertible if and only if there exists an integer $n$, $a$ is invertible along $a^n$. In this case, $a^D = a\|a^n$.

(iv) $a$ is Moore-Penrose invertible if and only if it is left (right) invertible along $a^*$. In this case, $a^\dagger = a\|a^*$.

3. One-sided inverse along a product in rings

In this section, we present equivalent conditions for the existence of one-sided inverse along a product in a ring $R$. In what follows, $R$ is always an associative ring with unity $1$.

First, we begin with a well-known lemma.

**Lemma 3.1.** Let $a, b, c \in R$.

(i) If $(1 + ab)c = 1$, then $(1 + ba)(1 - bca) = 1$.

(ii) If $c(1 + ab) = 1$, then $(1 - bca)(1 + ba) = 1$.

It follows from Lemma 3.1 that $1 + ab$ is (left, right) invertible if and only if $1 + ba$ is (left, right) invertible and $(1 + ba)^{-1} = 1 - b(1 + ab)^{-1}a$. This result is known as Jacobson’s Lemma.

Let $a \in R$. By $a^{-1}_L$ and $a^{-1}_R$ we denote a left inverse and a right inverse of $a$, respectively. Next, we present an existence criterion of a left inverse along a product by means of one-sided invertibility of certain elements.

**Theorem 3.2.** Let $p, a, q, m \in R$ with $m$ regular. If $m \leq_L pm$ and $m \leq_R mq$, then the following conditions are equivalent:

(i) $a$ is left invertible along $pmq$.

(ii) $u = mqap + 1 - mm^{(1)}$ is left invertible.

(iii) $v = qapm + 1 - m^{(1)}m$ is left invertible.

In this case, $pu^{-1}_L mq$ is a left inverse of $a$ along $pmq$. 

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Proof. It follows from Lemma 3.1 that (ii)$\Leftrightarrow$ (iii).

(i)$\Rightarrow$(ii) Suppose that $a$ is left invertible along $pmq$. From Theorem 2.3, we get $pmq \leq_L pmqapmq$. Hence, there exists $x \in R$ such that

$$pmq = xpmqapmq.$$  

(∗)

By $m \leq_R mq$, there exists $q' \in R$ such that $m = mqq'$. Similarly, $m \leq_L pm$ guarantees that $m = p'pm$ for some $p' \in R$. Multiplying the equation (∗) by $q'$ on the right yields $pm = xpmqapm$. Set $y = mm(1)p'xpmm(1) + 1 - mm(1)$, we obtain $y(mm(1)p'mq+1 - mm(1)) = 1$. Indeed, we have

$$y(mm(1)p'mq+1 - mm(1))$$

$$= (mm(1)p'xpmm(1) + 1 - mm(1))(mm(1)p'mq+1 - mm(1))$$

$$= mm(1)p'xpmqapm + 1 - mm(1)$$

$$= mm(1) + 1 - mm(1)$$

$$= 1.$$

Consequently, $mm(1)p'mq+1 - mm(1)$ is left invertible. Again, Lemma 3.1 ensures that $mm(1)p'mq+1 - mm(1)$ is left invertible.

(ii)$\Rightarrow$(i) Suppose now that $u$ is left invertible. Then there is $u'$ such that $u'u = 1$. Since $um = mqapm$, it follows that $m = u'mqapm$. Also, by $m \leq_L pm$, there exists $p' \in R$ such that $p'pm = m$ and hence $pmq = pu'mqapmq = pu'p'mqapmq$. Take $b = pu'p'mq$, then $b \leq_L pmq$, that is, $a$ is left invertible along $pmq$. Hence, $b = pu^{-1}mq$ is a left inverse of $a$ along $pmq$. □

As a special corollary of Theorem 3.2, we get

Corollary 3.3. Let $a, m \in R$ with $m$ regular. Then the following conditions are equivalent:

(i) $a$ is left invertible along $m$.
(ii) $u = ma + 1 - mm(1)$ is left invertible.
(iii) $v = am + 1 - m(1)m$ is left invertible.

In this case, $u^{-1}_t m$ is a left inverse of $a$ along $m$.

Dually, we have
Theorem 3.4. Let \( p, a, q, m \in R \) with \( m \) regular. If \( m \leq \_ \_ pm \) and \( m \leq \_ \_ R \) \( mq \), then the following conditions are equivalent:

(i) \( a \) is right invertible along \( pmq \).
(ii) \( u = mqap + 1 - mm(1) \) is right invertible.
(iii) \( v = qapm + 1 - m(1)m \) is right invertible.

In this case, \( pmv_r^{-1}q \) is a right inverse of \( a \) along \( pmq \).

Corollary 3.5. Let \( a, m \in R \) with \( m \) regular. Then the following conditions are equivalent:

(i) \( a \) is right invertible along \( m \).
(ii) \( u = ma + 1 - mm(1) \) is right invertible.
(iii) \( v = am + 1 - m(1)m \) is right invertible.

In this case, \( mv_r^{-1} \) is a right inverse of \( a \) along \( m \).

An involution \( * \) in a ring \( R \) is an anti-isomorphism of degree 2 which satisfies \((a^*)^* = a\), \((ab)^* = b^*a^* \) and \((a + b)^* = a^* + b^* \), for all \( a, b \in R \).

Let \( S \) be a ring with involution in Theorem 2.16. We have

Corollary 3.6. Let \( R \) be a ring with involution and let \( a \in R \). Then

(i) \( a \) is left \( * \)-regular if and only if it is right \( * \)-regular.
(ii) \( R \) is \( * \)-regular if and only if every element in \( R \) is left (right) \( * \)-regular.

Recall that a ring \( R \) is called strongly \( \pi \)-regular if each element \( a \in R \) is left (right) \( \pi \)-regular (see e.g. [1]). In particular, \( R \) is called strongly regular if each element \( a \in R \) is left (right) regular. We next give new characterizations of strongly (\( \pi \)) regular rings, \( * \) -regular rings, by one-sided invertibility along an element.

Corollary 3.7. Let \( a \in R \). Then

(i) \( R \) is a strongly regular ring if and only if every element \( a \) is left (right) invertible along \( a \).
(ii) \( R \) is a strongly \( \pi \)-regular ring if and only if every element \( a \) is left (right) invertible along \( a^n \) for some positive \( n \).
(iii) \( R \) is a \( * \)-regular ring if and only if every element \( a \) is left (right) invertible along \( a^* \).

We have seen that \( a \) is both left and right invertible along \( pmq \) if and only if it is invertible along \( pmq \). Moreover, the inverse of \( a \) along \( pmq \) is unique (Corollary 2.5). Hence we have
Corollary 3.8. ([10, Theorem 2.2] Let \( p, a, q, m \in R \) with \( m \) regular. If \( m \leq L pm \) and \( m \leq R mq \), then the following conditions are equivalent:

(i) \( a ^{||pmq} \) exists.
(ii) \( u = mqap + 1 - mm(1) \) is invertible.
(iii) \( v = qapm + 1 - m(1)m \) is invertible.

In this case,
\[
a ^{||pmq} = pu^{-1}mq = pmv^{-1}q.
\]

By taking \( p = q = 1 \) we get

Corollary 3.9. ([7, Theorem 3.2] and [8, Theorem 1.3]) Let \( m \in R \) be regular. Then the following are equivalent:

(i) \( a \) is invertible along \( m \).
(ii) \( u = ma + 1 - mm(1) \) is invertible.
(iii) \( v = am + 1 - m(1)m \) is invertible.

In this case,
\[
a ^{||m} = u^{-1}m = mv^{-1}.
\]

We finally give some applications of the inverse along a product by its existence criterion. More results on the inverse along a matrix can be referred to references [8, 10]. By the symbol \( R_{2\times 2} \) we denote the ring of \( 2 \times 2 \) matrices over a ring \( R \).

Let \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} \in R_{2\times 2} \) with \( D \) regular and assume that \( d_4 \) in matrix \( D \) is invertible. Then we have the following decomposition
\[
D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & d_3d_4^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & d_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ d_4^{-1}d_2 & 1 \end{bmatrix} =: PMQ,
\]
where \( s = d_1 - d_3d_4^{-1}d_2 \) is the Schur complement of \( d_4 \) in the matrix \( D \). It is well known that \( D \) is regular if and only if \( M \) is regular. Similarly, if \( d_1 \) is invertible, \( d_4 - d_2d_1^{-1}d_3 \) is called the Schur complement of \( d_1 \) in the matrix \( D \).

According to Corollary 3.8, it is known that \( A ^{||D} \) exists if and only if \( U = MQAP + I - MM(1) \) is invertible. One can get \( I - MM(1) = \begin{bmatrix} 1 - ss(1) & 0 \\ 0 & 0 \end{bmatrix} \) by a direct calculation.
We also get that \( MQAP = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \), where

\[
\begin{align*}
\alpha &= sa(ad_3d_4^{-1} + c), \\
\beta &= (d_2a + d_4b)d_3d_4^{-1} + d_2c + d_4d.
\end{align*}
\]

Hence, it follows that \( U = \begin{bmatrix} u \\ d_2a + d_4b \end{bmatrix} \), where \( u = sa + 1 - ss^{(1)} \).

If \( a^s \) exists, applying Corollary 3.9, it follows that \( u \) is invertible. Using the Schur complement, we have

\[
U = \begin{bmatrix} u \\ d_2a + d_4b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ (d_2a + d_4b)u^{-1} & \xi \end{bmatrix} \begin{bmatrix} 1 & u^{-1} \alpha \\ 0 & 1 \end{bmatrix},
\]

where \( \xi = \beta - (d_2a + d_4b)a^s(ad_3d_4^{-1} + c) \). Moreover, \( U \) is invertible if and only if \( \xi \) is invertible.

In this case,

\[
U^{-1} = \begin{bmatrix} 1 & -u^{-1}\alpha \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u^{-1} & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -(d_2a + d_4b)u^{-1} & 1 \end{bmatrix}.
\]

Thus, \( A^D \) exists if and only if \( \xi = \beta - (d_2a + d_4b)a^s(ad_3d_4^{-1} + c) \) is invertible. Moreover, we get

\[
A^D = PU^{-1}MQ = \begin{bmatrix} x_1s + x_3d_2 \\ x_2s + \xi^{-1}d_2 \end{bmatrix} \begin{bmatrix} x_3d_4 \\ \xi^{-1}d_4 \end{bmatrix},
\]

where

\[
\begin{align*}
x_1 &= u^{-1} + (u^{-1}\alpha - d_3d_4^{-1})\xi^{-1}(d_2a + d_4b)u^{-1}, \\
x_2 &= -\xi^{-1}(d_2a + d_4b)u^{-1}, \\
x_3 &= d_3d_4\xi^{-1} - u^{-1}\alpha\xi^{-1}.
\end{align*}
\]

**Remark 3.10.** Even if \( a^s \) does not exist, \( A^D \) may exist. For instance, take \( A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \), \( D = \begin{bmatrix} d_1 & d_3 \\ d_2 & d_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2} \). Since \( s = d_1 - d_3d_4^{-1}d_2 = 1 \), it follows that \( sa + 1 - ss^{(1)} = 0 \). Hence, \( a^s \) does not exist by Corollary 3.9. However, \( A \) is invertible along \( D \) since they are both invertible.
We close this section with some further remarks:

(i) In Theorem 3.2, since \( v_i^{-1}(1 + (qap - m^{(1)})m) = 1 \), it follows that \( 1 - mv_i^{-1}(qap - m^{(1)}) \) is a left inverse of \( u \) by Lemma 3.1. Hence, we can give the representation of a left inverse of \( a \) along \( pmq \) by \( v_i^{-1} \).

(ii) We give another proof for Corollary 3.6(i). Assume that \( a \) is left \( * \)-regular (we have \( a = aa^*ax \) for some \( x \in R \)). Then it is left invertible along \( a^* \) according to Theorem 2.7. Moreover, \( a \) is regular, and \( (ax)^* \) is an inner inverse (indeed a \( \{1, 4\} \)-inverse) of \( a \). Indeed, it follows that \( [(ax)^*a]^* = a^*ax = (ax)^*a \) and \( a(ax)^*a = aa^*ax = a \) since \( a^*ax = (aa^*ax)^*ax = (ax)^*aa^*ax = (ax)^*a \). By Corollary 3.3, \( u = a^*a + 1 - a^*(a^*) = a^*a + 1 - (a^{(1)}a)^* \) is left invertible. Hence, we can pick an inner inverse \( (ax)^* \) of \( a \) such that \( a^{(1)}a \) is symmetric. Then \( u = u^* \) is right invertible, and by Corollary 3.5, it follows that \( a \) is right invertible along \( a^* \).

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