A new very high-order finite volume method to solve the harmonic and the biharmonic operators for one-dimensional geometries is proposed. The main ingredient is the polynomial reconstruction based on local interpolations of mean values providing accurate approximations of the solution up to the sixth-order accuracy. First developed with the harmonic operator, an extension for the biharmonic operator is obtained, which enables to design a very high-order finite volume scheme where the solution is obtained by solving a matrix-free problem. An application in elasticity coupling the two operators is presented. We consider a beam subject to a combination of tensile and bending loads where the main goal is the stress critical point determination for an intramedullary nail.

Keywords: finite volume method, polynomial reconstruction operator, harmonic operator, biharmonic operator, high-order method

1. Introduction

Finite volume method is a popular technique usually employed to carry out numerical approximations for conservation laws (see, for example, (Kroner, 1997; Leveque, 2002; Audusse and Bristeau, 2007; Dumbser and Munz, 2007; Trangenstein, 2009)). In the nineties, the finite volume method received important developments to approximate elliptic and parabolic problems (see (Eymard et al., 2000) and the references herein), where second-order methods were designed for a large range of applications. Recently, a new sixth-order finite volume method has been proposed for convection diffusion problems in (Clain et al., 2013). The technique is based on specific polynomial reconstructions used for the fluxes (Hernández, 2002; Ollivier-Gooch and Altena, 2002; Toro and Hidalgo, 2009; Toro, 2009; Clain et al., 2011; Diot et al., 2011). We here extend the method to the biharmonic (or bi-Laplace) operator which is, up to our knowledge, the first attempt to design a finite volume scheme for a fourth-order differential operator. We restrict to the one-dimensional case to provide a simple but relevant context. We apply the proposed method to an example motivated by a medical apparatus modelling which couples the harmonic and the biharmonic operators.

The organization of the paper is the following. Section 2 is devoted to the harmonic operator, where we introduce the mesh, the generic finite volume formulation, and the polynomial reconstruction operator to design the high-order finite volume scheme. In section 3, we present the biharmonic operator case and the corresponding schemes we designed. Numerical tests are carried out in section 4 to assess the scheme ability to provide high-order accuracy both for the harmonic and the biharmonic operators. A practical application of the method in a simulation of an intramedullary nail is presented in section 5.

2. The harmonic operator

The harmonic operator is an essential building block for modeling phenomena such as elasticity, electromagnetism, and steady-state heat transfer among others examples. In the one dimensional context, the harmonic (or Poisson) operator writes

\[
(−\lambda φ′(x))′ = f(x), \quad x \in \Omega
\]

in domain $\Omega = (0, L)$ with $L \in \mathbb{R}^+$, where $\lambda > 0$ is assume to be constant and $f$ stands for the source term.
term. The harmonic equation is equipped with appropriate conditions on the boundary of the domain. In the present study we consider the following boundary conditions:

\[ \begin{align*}
\phi(0) &= \phi_0 \in \mathbb{R}, \\
\phi(L) &= \phi_r \in \mathbb{R}, \\
-\lambda \phi'(L) &= F \in \mathbb{R},
\end{align*} \]

(2a) \quad \text{(2b)} \quad \text{(2c)}

and shall consider two different situations: we prescribe substituted the exact fluxes \( F \) evaluated.

2.1. Mesh and discretization. To design the finite volume scheme, we introduce meshes \( \mathcal{T}_h \) of \( \Omega \), with \( h \) the mesh parameter, constituted of \( I \) cells \( K_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \), \( i = 1, \ldots, I \), of centroid \( c_i \), where \( x_{\frac{1}{2}} = 0, x_{L+\frac{1}{2}} = L \), \( x_{i+\frac{1}{2}} = x_i + \frac{1}{2} + h_i \) are the interfaces (cf. Fig. 1) and \( h \) is the maximum of the cell lengths. Integration of equation (1) over cell \( K_i \) provides

\[ \frac{1}{h_i} \left( F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} \right) = f_i, \quad i = 1, \ldots, I, \]

where the fluxes are given by \( F_{i+\frac{1}{2}} = -\lambda \phi'(x_{i+\frac{1}{2}}) \) and the mean source term is given by \( f_i = \frac{1}{h_i} \int_{K_i} F(\xi) \, d\xi \).

Let \( \phi_i \) be an approximation of the mean value of \( \phi \) over \( K_i \) and let gather all the approximations in vector \( \Phi = (\phi_i)_{i=1,\ldots,I} \). To design the numerical scheme, we substitute the exact fluxes \( F_{i+\frac{1}{2}} \) by numerical fluxes up to a 6th-order accuracy \( F_{i+\frac{1}{2}}(\Phi) \) depending on vector \( \Phi \). In the same way, we also approximate the exact mean source term \( f_i \) by \( f_i \) using a 6th-order Gaussian quadrature. The finite volume scheme cast in the residual form writes

\[ E_i(\Phi) = \frac{1}{h_i} \left( F_{i+\frac{1}{2}}(\Phi) - F_{i-\frac{1}{2}}(\Phi) \right) - f_i. \]

2.2. Polynomial reconstruction operator. We now turn to the critical point of the design of very high-order numerical fluxes. The technique proposed in (Clain et al., 2013) for the two-dimensional elliptic problem is here adapted to the one-dimensional case where local polynomial approximations of the underlying solution is evaluated.

At the first stage, we define the stencils associated to the cells and the interfaces. For any cell \( K_i, i = 1, \ldots, I \), we denote by \( S_i \) the associated stencil composed of the \( n \) closest neighbor cells (excluding cell \( K_i \)). In the same way, we denote by \( S_{i-\frac{1}{2}} \) and \( S_{i+\frac{1}{2}} \) the stencils constituted of the \( n \) neighbor cells for inner interfaces \( x_{\frac{1}{2}} = 0 \) and \( x_{L+\frac{1}{2}} = L \), respectively. The second stage consists in defining the polynomial reconstructions based on the entries of vector \( \Phi \) associated to the appropriate stencils. We detail this in the following subsections.

2.2.1. Polynomial reconstruction on cells. Let \( i \in \{1, \ldots, I\} \) and \( \phi_i \) be an approximation of the mean value of \( \phi \) over cell \( K_i \). We define the polynomial conservative reconstruction of degree \( d \) as (see (Clain et al., 2013))

\[ \phi_i(x; d) = \phi_i + \sum_{\alpha=1}^{d} R_{i,\alpha} [(x - c_i)^{\alpha} - M_{i,\alpha}], \]

where we have set \( M_{i,\alpha} = \frac{1}{h_i} \int_{K_i}(x - x_i)^{\alpha} \, dx \) and the vector \( R_i \) gathers the polynomial coefficients \( R_{i,\alpha} \), \( \alpha = 1, \ldots, d \). For a given stencil \( S_i \) and positive weights \( (\omega_{i,j})_{j=1,\ldots,\#S_i} \), we consider the quadratic functional

\[ E_i(R_i) = \sum_{j \in S_i} \omega_{i,j} \left[ \frac{1}{h_j} \int_{K_j} \phi_i(x; d) \, dx - \phi_j \right]^2. \]

We denote by \( \hat{R}_i \) the unique vector that minimizes the quadratic functional and set \( \hat{\phi}_i(x; d) \) as the associated polynomial function that corresponds to the best approximation in the least squares sense of the data of the stencil.

2.2.2. Polynomial reconstruction at the outer interfaces. For the left boundary interface \( x_{\frac{1}{2}} \), we adapt the previous polynomial reconstruction in order to obtain the polynomial \( \hat{\phi}_{L+\frac{1}{2}}(x; d) \) as

\[ \hat{\phi}_{\frac{1}{2}}(x; d) = \phi_L + \sum_{\alpha=1}^{d} \hat{R}_{\frac{1}{2},\alpha} (x - x_{\frac{1}{2}})^{\alpha}, \]

where vector \( \hat{R}_{\frac{1}{2}} \) gathers the polynomial coefficients \( \hat{R}_{\frac{1}{2},\alpha}, \alpha = 1, \ldots, d \). For a given stencil \( S_{\frac{1}{2}} \) and positive weights \( (\omega_{\frac{1}{2},j})_{j=1,\ldots,\#S_{\frac{1}{2}}} \), we consider the quadratic functional

\[ E_{\frac{1}{2}}(\hat{R}_{\frac{1}{2}}) = \sum_{j \in S_{\frac{1}{2}}} \omega_{\frac{1}{2},j} \left[ \frac{1}{h_j} \int_{K_j} \hat{\phi}_{\frac{1}{2}}(x; d) \, dx - \phi_j \right]^2. \]

We denote by \( \hat{R}_{\frac{1}{2}} \) the unique vector which minimizes the quadratic functional and set \( \hat{\phi}_{\frac{1}{2}}(x; d) \) as the associated polynomial function that corresponds to the best approximation in the least squares sense of the data of the stencil.

We proceed in the same way for polynomial \( \hat{\phi}_{I+\frac{1}{2}}(x; d) \) associated to the interface \( x_{I+\frac{1}{2}} \).

2.3. High-order fluxes. Having all the polynomial reconstructions in hand, we detail the numerical fluxes for the harmonic operator with respect to the interfaces:
6th-order finite volume method for the 1D biharmonic operator: application to the intramedullary nail simulation

The biharmonic operator (also called bilaplacian operator) writes

\[ (-\mu \psi^{(4)}(x))'' = g(x), \quad x \in \Omega \]  

(3)

in domain \( \Omega = [0, L] \) with \( L \in \mathbb{R}^+ \), where \( \mu > 0 \) is assumed to be constant and \( g \) is the source term. The biharmonic equation is equipped with appropriate conditions on the boundary of the domain. In the present study we consider the following boundary conditions:

\[ \psi(0) = \psi_T \in \mathbb{R}, \quad (4a) \]
\[ \psi(L) = \psi_B \in \mathbb{R}, \quad (4b) \]
\[ \psi'(0) = \psi_{\ell\ell} \in \mathbb{R}, \quad (4c) \]
\[ \psi'(L) = \psi_{rr} \in \mathbb{R}, \quad (4d) \]
\[ -\mu \psi''(0) = M_{i} \in \mathbb{R}, \quad (4e) \]
\[ -\mu \psi''(L) = M_{i} \in \mathbb{R}, \quad (4f) \]
\[ -\mu \psi'''(L) = G \in \mathbb{R}. \quad (4g) \]

and shall consider three different situations: we prescribe (i) (4a), (4b), (4c), and (4d), or (ii) (4a), (4b), (4e), and (4f), or (iii) (4a), (4c), (4f), and (4g).

3.1. Discretization. The mesh and the finite volume discretization follows as in section 2.1 and integration of equation (3) over cell \( K_i \) writes

\[ \frac{1}{h_i} \left( G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}} \right) = \tilde{g}_i, \quad i = 1, \ldots, I, \]

where the fluxes are given by \( G_{i+\frac{1}{2}} = -\mu \psi'''(x_{i+\frac{1}{2}}) \) and the mean source term is given by \( \tilde{g}_i = \frac{1}{h_i} \int_{K_i} g(\xi) \, d\xi \). We substitute the exact fluxes on interfaces \( G_{i+\frac{1}{2}} \) by an up to 6th-order accuracy numerical flux \( \tilde{G}_{i+\frac{1}{2}}(\Psi) \) based on vector \( \Psi = (\psi_i)_{i=1, \ldots, I} \) gathering approximations of the mean values of \( \psi \). As in the previous section, the exact mean source term \( \tilde{g}_i \) is approximated by an 6th-order Gaussian quadrature denoted \( g_i \) and the finite volume scheme cast in the residual form writes

\[ G_{i+\frac{1}{2}}(\Psi) - G_{i-\frac{1}{2}}(\Psi) = \tilde{g}_i. \]

3.2. Polynomial reconstruction operator. The delicate point is the introduction of the boundary conditions in the reconstructions. Indeed, one has to handle two conditions at each outer interfaces but it is not possible to support the two conditions on the same function, namely on functions \( \tilde{\psi}_{i+\frac{1}{2}}(x; d) \) and \( \tilde{\psi}_{i+\frac{1}{2}}(x; d) \).

As a consequence functions \( \tilde{\psi}_{i+\frac{1}{2}}(x; d) \) or \( \tilde{\psi}_{i+\frac{1}{2}}(x; d) \) will also support some boundary conditions to provide an invertible linear system.

Fig. 1. Mesh and notations.

- for a inner interface the flux is given by

\[ F_{i+\frac{1}{2}} = -\lambda \tilde{\Phi}_{i+\frac{1}{2}}(0), \]

- the flux on the left boundary interface is given by

\[ F_{\frac{1}{2}} = -\lambda \tilde{\Phi}_{\frac{1}{2}}(L) \]

for condition (2c) or by

\[ F_{\frac{1}{2}} = F \]

for the case (2a).

Since \( F_{i+\frac{1}{2}} \) linearly depends on vector \( \Phi \), the residual operator \( \Phi \rightarrow F(\Phi) \) is an affine operator. Gathering all the components of the residual in vector \( F(\Phi) \), we obtain an affine operator from \( \Phi \) linearly depends on vector \( \Phi^* \) gives polynomial approximations up to the sixth-order. Notice that the method is matrix-free and the linear problem is solved by applying a GMRES procedure as explained in (Clain et al., 2013).

3. The biharmonic operator

\[ \begin{align*}
  K_1 & \quad c_1 \\
  x_\frac{1}{2} \equiv 0 & \quad x_\frac{3}{2} \quad x_{i-\frac{1}{2}} \quad x_{i+\frac{1}{2}} \quad x_{I-\frac{1}{2}} \\
  K_i & \quad c_i \\
  x_{I+\frac{1}{2}} \equiv L \\
  h_i
\end{align*} \]
3.2.1. Polynomial reconstruction on inner cells. For cells $K_i$, $i = 2, \ldots, I - 1$, the reconstructions are performed exactly as in section 2.2.1 and provide polynomial functions $\hat{\psi}_j(x; d)$.

3.2.2. Polynomial reconstruction at the outer interfaces. To take into account boundary conditions (4a) and (4b), we reconstruct the polynomials $\hat{\psi}_1(x; d)$ and $\hat{\psi}_{I+\frac{1}{2}}(x; d)$, respectively, as described in section 2.2.2.

3.2.3. Polynomial reconstruction on the first cell and last cell. We reach the important point where we shall construct the polynomial functions $\hat{\psi}_1(x; d)$ and $\hat{\psi}_{I+\frac{1}{2}}(x; d)$ in order to take into account boundary conditions (4d)-(4e) and boundary conditions (4f), respectively.

Let $\psi_1$ be an approximation of the mean value of $\hat{\psi}$ over cell $K_1$. The conservative polynomial reconstruction associated with the first cell $K_1$ writes

$$\psi_1(x; d) = \psi_1 + \sum_{\alpha=1}^{d} \hat{R}_{1,\alpha} [(x - c_1)^{\alpha} - M_{1,\alpha}],$$

where vector $\hat{R}_1$ gathers the polynomial coefficients $R_{1,\alpha}$, $\alpha = 1, \ldots, d$. For a given stencil $\hat{S}_1$ and positive weights $(\omega_{i,j})_{j=1,\ldots,\#S_1}$, we shall consider two kinds of functionals depending of the boundary condition we shall apply. To introduce condition (4e), the quadratic functional writes

$$\tilde{E}_1(\hat{R}_1) = \sum_{j \in \hat{S}_1} \omega_{i,j} \left[ \frac{1}{h_{i,j}} \int_{K_j} \psi_1(x; d) \, dx - \tilde{\psi}_j \right]^2 + \left[ \hat{\psi}_1'(0) - \tilde{\psi}_j \right]^2,$$

whereas to consider condition (4e) we set

$$\tilde{E}_1(\hat{R}_1) = \sum_{j \in \hat{S}_1} \omega_{i,j} \left[ \frac{1}{h_{i,j}} \int_{K_j} \psi_1(x; d) \, dx - \psi_j \right]^2 + \left[ \hat{\psi}_1''(0) + \frac{M_{1,\mu}}{\mu} \right]^2.$$

We denote by $\hat{R}_1$ the unique vector which minimizes the quadratic functional (5) for the boundary condition (4d) and the quadratic functional (6) for the boundary condition (4e) and set $\hat{\psi}_1(x; d)$ the associated polynomial function.

We proceed in the same way for polynomial $\hat{\psi}_J(x; d)$ associated to the last cell $K_J$.

3.3. High-order finite volumes scheme. Having in hand all the polynomial reconstructions, we compute the fluxes for the biharmonic operator:

- for an inner interface the flux writes
  $$\hat{G}_{i+\frac{1}{2}} = -\mu \hat{\psi}_i''' \left( x_i + \frac{1}{2} \right),$$
  $$i = 1, \ldots, I - 1;$$

- on the left boundary, one has
  $$\hat{G}_{\frac{1}{2}} = -\mu \hat{\psi}_{\frac{1}{2}}'''(0);$$

- for the right boundary, we prescribe
  $$\hat{G}_{I+\frac{1}{2}} = -\mu \hat{\psi}_{I+\frac{1}{2}}'''(L)$$
  for condition (4f) or
  $$\hat{G}_{I+\frac{1}{2}} = G$$
  for condition (4g).

4. Numerical results

Quantitative and qualitative assessments of the scheme robustness and accuracy both for the harmonic and the biharmonic operator are addressed in this section. To evaluate the error between the exact solution and the numerical solution we introduce the $L^\infty$ norms

$$E_{\infty}(I) = \max_{i=1}^{I} |\phi_i - \bar{\phi}_i| \text{ and } E_{\infty}(I) = \max_{i=1}^{I} |\psi_i - \bar{\psi}_i|,$$

where $\bar{\phi}_i$ and $\bar{\psi}_i$ are the exact mean values of $\phi$ and $\psi$, respectively, over cell $K_i$. The orders of convergence based on the $L^\infty$-norms, given by

$$O_{\infty}(I_1, I_2) = \frac{\log(E_{\infty}(I_1) / E_{\infty}(I_2))}{\log(I_1/I_2)},$$

are also provided. The notation $P_d(n)$ means that we employ a $d$-degree polynomial reconstructions involving $n$ cells stencils. The weights we will consider are summarized with the notation $\omega_{i,j} = q/r$, $q, r \in R^+$, with the following meaning: if $i$ and $j$ are contiguous cells, then $\omega_{i,j} = q$; otherwise, $\omega_{i,j} = r$ (this notation extends in the natural way for the cases $\omega_{2,3}$ and $\omega_{4,5}$). In the present study all the computations have been carried out with weights $3|1$ for stability reasons. Moreover, to reduce the computational effort, a preconditioning matrix is used as proposed in (Clain et al., 2013) for the steady-state problems as well as when dealing with time-dependent implicit schemes.

Example 1. We first assess the numerical scheme accuracy for the harmonic operator equipped with boundary conditions (2a) and (2b). Taking $f(x) = e^x$ and $\lambda = 1$, the exact solution writes $\phi(x) = -e^x + (e-1)x + 1$. We report in Table 1 the errors and the convergence
rates between the numerical solution and the exact solution. Effective second-, fourth-, and sixth-order of convergence are achieved when dealing with the \( P_1(2) \), \( P_3(4) \), and \( P_5(6) \) reconstructions, respectively. Notice that the sixth-order scheme exceed the IEEE-754 standard capacity of a double for the 160 cells mesh due to the very high accuracy of the reconstruction technique.

Example 2. A second example for the harmonic operator concerns the Dirichlet-Neumann boundary conditions (2a) and (2c). Taking \( f(x) = e^x \) and \( \lambda = 1 \), we get \( F = 1 \) and the exact solution writes \( \phi(x) = -e^x + (e - 1)x + 1 \).

As in the previous case, Table 2 shows that the scheme achieved an effective second-, fourth-, and sixth-order of convergence for the \( P_1(2) \), \( P_3(4) \), and \( P_5(6) \) polynomial reconstructions, respectively. Notice that Tables 1 and 2 provide similar errors and that the IEEE-754 standard limitation is reached for too finer meshes.

Example 3. We now turn to the biharmonic operator. Taking \( \mu = 1 \) and \( g(x) = e^x \), the exact solution writes \( \psi(x) = -e^x - (e - 3)x^3 - (5 - 2e)x^2 + x + 1 \). Thanks to the exact solution, we prescribe the boundary conditions (4a), (4b), (4c), and (4d).

Table 3 reports the convergence orders and the scheme accuracy for the \( P_3(4) \), \( P_5(6) \), and \( P_7(8) \) polynomial reconstructions. We observe some differences with respect to the harmonic operator. For example, the scheme hardly achieved an effective sixth-order convergence rate with the fifth-degree polynomial reconstruction (slightly larger than 5) in contrast with the harmonic case which delivers the optimal order. Using a seventh-degree polynomial reconstruction, the IEEE-754 standard limitation is, one more time, patent.

Example 4. We deal with a similar situation taking \( \mu = 1 \) and \( g(x) = e^x \) but Neumann conditions are now prescribed. The exact solution writes \( \psi(x) = -e^x + \left( \frac{\mu - 1}{6} \right)x^3 + x^2 + \left( \frac{5\mu - 11}{6} \right)x + 1 \) and we consider the approximation of the biharmonic operator equipped with the boundary conditions (4a), (4b), (4c), and (4d).

We print in Table 4 the errors and convergence orders. Clear differences appears with respect to the former case. The scheme only reached the sixth-order accuracy with the \( P_7(8) \) reconstruction. Unlike the harmonic operator, the convergence orders for the biharmonic operator differ with the choice of the boundary conditions. Comparisons between Tables 3 and 4 emphasize such differences. As an example, the \( P_3(4) \) reconstruction achieves an effective third-order accuracy for Example 3 whereas the same reconstruction only provides a second-order for Example 4.

Example 5. The last test we address concerns the biharmonic operator equipped with boundary conditions (4a), (4c), (4f), and (4g). The exact solution writes \( \psi(x) = -e^x + \left( \frac{\mu - 1}{6} \right)x^3 + x^2 + x + 1 \) with \( g(x) = e^x \), \( \mu = 1 \), and \( G = 1 \). Errors and convergence rates are shown in Table 5 where we observe an effective second-, fourth-, and sixth-order of convergence for the \( P_3(4) \), \( P_5(6) \), and \( P_7(8) \) reconstructions, respectively. The high preconditioning number (no preconditioning matrix has been employed in the numerical simulations) associated to the IEEE-754 limitation is responsible of the low threshold of the error saturation (around \( 10^{-10} \)).

5. Study case

Numerical simulation of the intramedullary nail stress (a metal rod used to treat fractures of long bones of the body such as the femur or the tibia) is an interesting example where one has to couple the harmonic and biharmonic operators. The device attempts to stabilize and align the fracture until the full consolidation of the bone submitted to internal stresses deriving from the traction or pressure applied on the bone.

5.1. The model. In Fig. 2 (Ramos and Simoes, 2009) it is showed four different types of femoral fractures and the application of an intramedullary nail. As a test case, we shall consider the two first configurations displayed in Fig. 2.

![Fig. 2. Femoral fractures and the use of intramedullary nails.](image-url)
Table 1. Results for Example 1

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Table 2. Results for Example 2

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Table 3. Results for Example 3

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Table 4. Results for Example 4

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<th>( P_5(6) )</th>
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<th>( P_7(8) )</th>
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<td>( O_\infty )</td>
<td>( E_\infty )</td>
<td>( O_\infty )</td>
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<td>( O_\infty )</td>
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Table 5. Results for Example 5

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<th>( P_5(6) )</th>
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<th>( P_7(8) )</th>
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<td>( O_\infty )</td>
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<td>( O_\infty )</td>
<td>( E_\infty )</td>
<td>( O_\infty )</td>
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Sixth-order finite volume method for the 1D biharmonic operator: application to the intramedullary nail simulation

Usually, intramedullary nails have a ring-shaped cross-section and are made of stainless steel or titanium. The body has a small curvature we shall assume null in the present study, leading to a straight beam as domain. We also state that there is a single resultant force \( F_0 \) in the upper screw and there is no rotational loads (cf. Fig. 3). We consider that the left edge of the intramedullary nail is fixed with two screws while two forces, \( F_1 \) and \( F_2 \), resulting from the decomposition of the force \( F_0 \) following the two axis, are applied on the opposite edge. A bending moment \( M \) is also prescribed on the right edge (cf. Fig. 3). In Fig. 4 it is sketched out the simplified unidimensional geometry we shall deal with where the applied forces and moment are represented.

![Figure 3](image3.png)

**Fig. 3.** The body of the intramedullary nail (bounded by a rectangle) and forces.

We briefly present the model where the two operators are coupled. Since there is no loads in the \( z \) direction and, therefore, the \( z \)-component of the stress vector vanishes, we only compute the stresses in the \( xOy \) plane and the two-dimensional Cauchy’s stress tensor \([T]\) is given by

\[
[T] = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix},
\]

where \( \sigma \) stands for the normal stress while \( \tau \) stands for the shear stress. We neglect \( \tau_{xy} \) and \( \tau_{yx} \) due to their small influence (Branco, 2011) whereas \( \sigma_x \) and \( \sigma_y \) derive from the Hooke’s Law for an isotropic material and are given by

\[
\sigma_x = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] \quad (7)
\]

and

\[
\sigma_y = \frac{E}{(1 + \nu)(1 - 2\nu)} [(1 - \nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)], \quad (8)
\]

with \( E \) the Young’s Modulus, \( \nu \) the Poisson’s ratio while \( \varepsilon_x, \varepsilon_y, \) and \( \varepsilon_z \) stand for the strain components in the \( x, y, \) and \( z \)-direction, respectively. Provided that \( \varepsilon_y = \varepsilon_z = -\nu \varepsilon_x \), we deduce that \( \sigma_y = 0 \). Let \( u = u(x) \) and \( v = v(x) \) denote the horizontal and the vertical displacements of the neutral axis of the beam. We assume that each cross-section of the beam remains orthogonal to the neutral axis (Euler-Bernoulli beam theory for small strains) such that the angular displacement of the cross-section is given by \( \frac{dv}{dx} \). Therefore, the strain \( \varepsilon_x \) in the \( xOy \) plane writes

\[
\varepsilon_x(x, y) = \frac{\partial u(x)}{\partial x} - \frac{\partial^2 v(x)}{\partial x^2} y. \quad (9)
\]

Let consider a ring-shaped section beam of length \( L \) where \( D_i \) and \( D_o \) are the inner diameter and the outer diameter respectively, \( i.e. \ x \in [0, L] \) and \( y \in \left[ -\frac{D_o}{2}, -\frac{D_o}{2} \right] \cup \left[ \frac{D_i}{2}, \frac{D_i}{2} \right] \). One has to determine \( u \) and \( v \) to compute the strain component \( \varepsilon_x \) and then to compute the stress component \( \sigma_x \) with equation (7).

5.2. Elastic beam theory. We now introduce the mathematical models to compute the unknowns \( u \) and \( v \). The bending phenomenon occurs when an external load is applied perpendicularly to the longitudinal axis whereas the tensile occurs when an external load is applied in the same direction as the longitudinal axis. It results that the bending leads to a vertical displacement of the beam while the tensile leads to a horizontal displacement. Since the total displacement is the sum of the displacements caused by each body force apart (mechanical superposition principle), we split the intramedullary nail problem into two subproblems: the tensile problem and the bending problem. The first one considers the load \( F_1 \) associated to the horizontal displacement \( u \) and the second one considers the loads \( F_2 \) and \( M \) associated to the vertical displacement \( v \). According to the Elasticity theory, the tensile problem writes

\[
( -EAu')' = 0,
\]

where \( A \) is the cross-sectional area of the beam. The equation is equipped with the boundary conditions

\[
u(0) = 0, \quad -EAu'(L) = F_1.
\]

This problems is a specific case of the harmonic operator where \( \lambda = EA, \phi = u, f = 0, F = F_1, \) and \( \phi \theta = 0 \).

On the other hand, according to the Euler-Bernoulli theory, the bending problem writes

\[
( -EIv''(x))'' = g,
\]

where \( I \) stands for the second moment of area relatively to its longitudinal axis and perpendicular to the bending.
plane (in the case of a ring-shaped cross-section, the second moment of area is given by \( I = \frac{\pi}{4}(D_o^4 - D_i^4) \)). Function \( g \) is the vertical gravity applied along the x-axis we shall neglect in this practical application. The equation is equipped with the boundary conditions

\[
\begin{align*}
  v(0) &= 0, \\
  v'(0) &= 0, \\
  -EIv''(L) &= M, \\
  -EIv'''(L) &= F_2.
\end{align*}
\]

This problem is a specific case of the biharmonic operator where \( \mu = EI, \psi = v, g = 0, G = F_2, \psi_t = 0, \) and \( \psi_{tt} = 0. \)

To perform the numerical simulation, we consider a stainless steel intramedullary nail with 20 cm of length, 5 mm of inner diameter, and 11 mm of outer diameter and the upper screw length is 5 cm. The Young’s modulus and the Poisson’s ratio for stainless steel are 200 GPa and 0.3, respectively. According to (Barreira et al., 2002), for a daily physical activity such as walking down stairs and for a person weighing 70 kg, the maximum force applied in the femur head is about 1784 N with a direction shifted 13° from the cortical plane. Assuming that 50% of this force is supported by the intramedullary nail, we evaluate \( F_1 = 869.15 \) N, \( F_2 = 200.65 \) N and \( M = 26.85 \) N m. We considered a 80 cells mesh and a fifth-degree polynomial reconstruction both for the tensile and the bending. Fig. 5 shows the stress field of the longitudinal section of the intramedullary nail body. The stress critical point is located in the upper-right corner of the longitudinal section with a stress of −436.8 MPa. The negative value means a compressive stress.

### 6. Conclusions

We presented an adaptation of the very high-order finite volume scheme introduced by (Clain et al., 2013) for the one-dimensional harmonic and biharmonic operators, where a specific discretization for each of these operators should be considered in order to obtain a sixth-order of convergence. Numerical simulations have been carried out to assess the method efficiency to provide a sixth-order of convergence scheme. An application in the elasticity context has also been proposed to show that the presented method may be a future alternative to the classical finite element method.

### Acknowledgment

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### References


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Fig. 5. Stress field of the longitudinal section of the intramedullary nail.


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