FREE PROFINITE SEMIGROUPS OVER SOME CLASSES OF SEMIGROUPS LOCALLY IN $\mathcal{D}$G

JOSÉ CARLOS COSTA[1]

Departamento de Matemática
Universidade do Minho
Campus de Gualtar
4700-320 Braga - Portugal
E-mail: jcosta@math.uminho.pt

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This paper is concerned with the structure of semigroups of implicit operations on various subpseudovarieties $\mathbf{V}$ of $\mathcal{D}\mathbf{ReG} \cap \mathcal{LD}\mathbf{G}$, where $\mathcal{D}\mathbf{ReG}$ and $\mathcal{D}\mathbf{G}$ are the pseudovarieties of all semigroups S in which each regular $\mathcal{D}$-class is, respectively, a rectangular group and a group, and where $\mathcal{LD}\mathbf{G}$ is the pseudovariety of semigroups locally in $\mathcal{D}\mathbf{G}$. As an application, we give a characterization of the variety of languages recognized by semigroups in $\mathbf{V}$ and derive some join decompositions of pseudovarieties.

1 Introduction

The theory of free profinite semigroups, which received its major impetus with the publication of Reiterman's paper [17] in the early eighties, has proven to be an important tool in the study of pseudovarieties of semigroups and on the varieties of recognizable languages associated with them (via Eilenberg's Theorem on varieties [13]). The importance of Reiterman's theorem was immediately understood by Almeida [1, 2, etc] and Azevedo [10] who developed the theory. More recently, this approach has also received the attention of authors like Selmi, Trotter, Volkov, Weil, Zeitoun and others [7, 18, 22, 23, 25].

For a pseudovariety $\mathbf{V}$, denote by $\mathcal{L}\mathbf{V}$ the pseudovariety of all finite semigroups S such that $eSe \in \mathbf{V}$ for each idempotent e of S, and by $\mathcal{D}\mathbf{V}$ the pseudovariety of all finite semigroups S in which each regular $\mathcal{D}$-class is a subsemigroup of S which lies in $\mathbf{V}$. Particularly important in this work are the pseudovarieties $\mathcal{D}\mathbf{ReH}$, $\mathcal{D}\mathbf{RH}$ and $\mathcal{D}\mathbf{LH}$, where, for a pseudovariety $\mathbf{H}$ of groups, $\mathbf{ReH}$, $\mathbf{RH}$ and $\mathbf{LH}$ denote, respectively, the pseudovarieties of rectangular groups, of right groups and of left groups, all of whose subgroups lie in $\mathbf{H}$. We recall that $\mathcal{D}\mathbf{ReG}$ is usually denoted by $\mathcal{D}\mathbf{O}$.

This paper is devoted to the study of implicit operations on some subpseudovarieties of $\mathcal{D}\mathbf{S}$, where $\mathbf{S}$ is the pseudovariety of all finite semigroups, and consists of part of the author's doctoral dissertation [11]. The subpseudovarieties $\mathbf{V}$ of $\mathcal{D}\mathbf{S}$ have a particularly important property (proved by Azevedo [9, 10] extending a similar result of Almeida [2] on $\mathbf{J}$, the pseudovariety of $\mathcal{J}$-trivial semigroups), which is the fact that the implicit operations on $\mathbf{V}$ can be factored as finite products of words and regular elements. For some such pseudovarieties $\mathbf{V}$, a certain form of such a factorization is known to be canonical for $\mathbf{V}$. This is the case, for instance, of $\mathbf{J}$ (Almeida [2]), $\mathbf{J} \cap \mathcal{L}\mathbf{Sl}$ (Selmi [18]) $\mathcal{D}\mathbf{H} \cap \mathcal{E}\mathbf{Com}$, $\mathcal{D}\mathbf{RH}$ (Almeida and Weil [6, 8]) and $\mathbf{R} \cap \mathcal{L}\mathbf{Sl}$ (Costa [12]), where $\mathbf{Sl}$, $\mathcal{E}\mathbf{Com}$ and $\mathbf{R}$ are, respectively, the pseudovarieties of semilattices (i.e. idempotent and

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commutative semigroups), of semigroups in which the idempotents commute and of $\mathcal{R}$-trivial semigroups. However, the general problem of describing canonical factorizations for all subpseudovarieties of $\mathcal{D}\mathbf{S}$ (or even for $\mathcal{D}\mathbf{S}$ itself) is very far from being achieved.

A crucial result in this paper is the characterization of the regular implicit operations on pseudovarieties $\mathbf{V}$ in the interval $[\mathbf{Sl} \vee \mathcal{L}\mathbf{I}, \mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}]$, where $\mathbf{I}$ is the trivial pseudovariety. We prove in Corollary 3.2 that they are characterized by their restrictions to $\mathbf{Sl}, \mathcal{L}\mathbf{I}$ and $\mathbf{V} \cap \mathbf{G}$. We also show that $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}$ is the greatest subpseudovariety of $\mathcal{D}\mathbf{ReG}$ with this property. Note that $\mathbf{V}$ is such that $\mathbf{V} \cap \mathbf{B} = \mathbf{NB}$, where $\mathbf{B}$ and $\mathbf{NB}$ are, respectively, the pseudovarieties of bands and of normal bands. Trotter and Weil [22] proved that the greatest subpseudovariety of $\mathcal{D}\mathbf{A}$, the pseudovariety of semigroups in which all regular elements are idempotents, having intersection $\mathbf{NB}$ with $\mathbf{B}$ is $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J} (= \mathcal{D}\mathbf{A} \cap \mathcal{L}\mathcal{D}\mathbf{G})$. Using their results, one can show that $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}$ is the greatest subpseudovariety of $\mathcal{D}\mathbf{ReG}$ whose intersection with $\mathbf{B}$ is $\mathbf{NB}$. So Corollary 3.2 is somehow related with the result of Trotter and Weil.

This paper is a contribution to the study of the pseudovarieties in the interval $[\mathbf{Sl}, \mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}]$, i.e., the subpseudovarieties of $\mathcal{D}\mathbf{ReG}$ whose intersection with $\mathbf{B}$ is in the interval $[\mathbf{Sl}, \mathbf{NB}]$. More precisely, we study the structure of the semigroups of implicit operations on the pseudovarieties $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$, $\mathbf{R} \cap \mathcal{L}\mathbf{J}$, $\mathbf{V} \cap \mathbf{W}$ and $\mathbf{V} \cap \mathbf{W} \cap \mathcal{E}\mathbf{Com}$, with $\mathbf{V} \in \{\mathcal{D}\mathbf{ReH}, \mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$ and $\mathbf{W} \in \{\mathcal{L}\mathcal{E}\mathbf{Com}, \mathcal{L}\mathbf{ZE}, \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}), \mathbf{Com} * \mathbf{D}\}$, where $*$ denotes the operation of semidirect product of pseudovarieties of semigroups and $\mathbf{Com}$, $\mathbf{D}$ and $\mathbf{ZE}$ are, respectively, the pseudovarieties of commutative semigroups, of semigroups S in which $eS = S$ for each idempotent $e \in S$ and of semigroups in which idempotents are central (i.e., commute with every element). The techniques that we use are in close connection with the ones used by Almeida and Weil [6] in the study of the pseudovarieties of the form $\mathcal{D}\mathbf{H} \cap \mathcal{E}\mathbf{Com}$.

As a consequence of this work, we are able to give combinatorial descriptions of the classes of languages recognized by each of these pseudovarieties $\mathbf{U}$. More precisely, for each finite alphabet A, we describe a set of generators for the Boolean algebra of the languages of A^+ that are recognized by semigroups in $\mathbf{U}$. Excepting the cases $\mathbf{U} = \mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$ and $\mathbf{U} = \mathbf{R} \cap \mathcal{L}\mathbf{J}$, the generators are very simple languages. Depending on the pseudovariety $\mathbf{U}$ considered, they are of the form $u_0 A_1^* \cdots A_{l-1}^* u_{l-1} L_l u_l A_{l+1}^* \cdots A_n^* u_n$ or of the form $u_0 A_1^+ \cdots A_{l-1}^+ u_{l-1} L_l u_l A_{l+1}^+ \cdots A_n^+ u_n$, where $n \geq 0$, the u_i are words over A, L_l is a group language over A_l (if $\mathbf{U}$ is aperiodic, then $L_l = A_l^*$ or $L_l = A_l^+$, respectively), the A_i are non-empty subsets of A, and where the u_i and the A_i satisfy some conditions depending on the pseudovariety involved. Note that several varieties of languages have been described as Boolean combinations of languages of one of the above forms (e.g. piecewise testable languages (Simon [19]), $\mathcal{R}$-trivial languages (Eilenberg [13]), level 2 languages in the Straubing hierarchy (Pin and Straubing [16]), etc).

The previous results also permit us to compute some joins of pseudovarieties. Recall that the join $\mathbf{V} \vee \mathbf{W}$ is the least pseudovariety containing both the pseudovarieties $\mathbf{V}$ and $\mathbf{W}$. Among several equalities we prove that,—in the case $\mathbf{W} = \mathbf{Com} * \mathbf{D}$ for instance,— if $\mathbf{H}$ is a pseudovariety of abelian groups, then

$$\begin{aligned}
\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}) &= (\mathcal{D}\mathbf{A} \cap (\mathbf{Com} * \mathbf{D})) \vee \mathbf{H} \\
&= (\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap (\mathbf{Com} * \mathbf{D}) \\
&\neq (\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})) \vee (\mathcal{D}\mathbf{LH} \cap (\mathbf{Com} * \mathbf{D})).
\end{aligned}$$

This paper is organized as follows. In section 2 we briefly recall some definitions and properties that we shall need in the sequel. Sections 3 to 8 are dedicated to the description of the structure of the semigroups of implicit operations on the various pseudovarieties mentioned above. Finally, section 9 is devoted to the characterization of the corresponding varieties of languages.

2 Preliminaries

We assume the reader is familiar with the basic notions of finite semigroup theory and its relationships with the theory of rational languages and finite automata. For a comprehensive treatment of the theory and for undefined notions and notation, the reader is referred to the books of Almeida [3], Eilenberg [13] and Pin [15], and to the survey [7].

2.1 Generalities

By an *alphabet*, we mean a finite non-empty set A. We denote by $A^{\mathbb{N}}$ (resp. $A^{-\mathbb{N}}$) the set of all words over A that are "infinite to the right" (resp. "infinite to the left"), that is, the set of sequences of letters of A indexed by $\mathbb{N}$ (resp. $-\mathbb{N}$). We denote by $u^{+\infty}$ (resp. $u^{-\infty}$) the infinite word to the right (resp. left) obtained by repeating infinitely often the word $u \in A^+$.

The set of all letters appearing in a word u (finite or infinite) is denoted by $c(u)$ and is called the *content* of u. A word $u \in A^*$ is a *prefix* (resp. *suffix*, *factor*) of a word x (finite or infinite) if there exist words y and z such that $x = uy$ (resp. $x = yu$, $x = yuz$). For each integer k we denote by $p_k(x)$ (resp. $s_k(x)$) the prefix (resp. suffix) of x of length k, if it exists.

It is well known that every finite semigroup S admits an integer k such that s^k is idempotent for every element $s \in S$. Such an integer will be called an *exponent* of S. Notice that if k is an exponent of a finite semigroup S, then every multiple of k is also an exponent of S.

Let $\mathbf{V}$ be a pseudovariety and let A be an alphabet. We denote by $\hat{F}_A(\mathbf{V})$ the free pro-$\mathbf{V}$ semigroup over A. The semigroup $\hat{F}_A(\mathbf{V})$ can be viewed as the completion of a certain uniform structure on the free semigroup A^+ or as the semigroup of A-ary implicit operations on $\mathbf{V}$. For this reason, the elements of $\hat{F}_A(\mathbf{V})$ are usually called (A-*ary*) *implicit operations* (*on* $\mathbf{V}$). It is well known that, for instance, $\hat{F}_A(\mathbf{Sl})$ is the semigroup 2^A of non-empty subsets of A under union. The following important properties of $\hat{F}_A(\mathbf{V})$, will be used freely in this paper.

- There exists a natural injective mapping $\iota : A \to \hat{F}_A(\mathbf{V})$ such that $\iota(A)$ generates a dense subsemigroup of $\hat{F}_A(\mathbf{V})$.
- Any mapping from A into a semigroup S of $\mathbf{V}$ can be uniquely extended to a continuous morphism from $\hat{F}_A(\mathbf{V})$ into S.

In particular, if $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, the identity of A induces a continuous onto homomorphism $\pi : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$, called the *canonical projection of* $\hat{F}_A(\mathbf{V})$ *onto* $\hat{F}_A(\mathbf{W})$. The image $\pi(x)$ of an element $x \in \hat{F}_A(\mathbf{V})$ is called the *restriction of* x *to* $\mathbf{W}$. In particular, when $\mathbf{V}$ is a pseudovariety containing $\mathbf{Sl}$, the canonical projection $c : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{Sl}) = 2^A$ is called the *content homomorphism* on $\mathbf{V}$. As one can easily show, c extends to the elements of $\hat{F}_A(\mathbf{V})$ the notion of content for words of A^+.

For each $x \in \hat{F}_A(\mathbf{V})$, the sequence $(x^{n!})_n$ converges in $\hat{F}_A(\mathbf{V})$. Its limit, denoted by x^ω, is the only idempotent in the topological closure of the subsemigroup generated by x.

Let $\mathbf{V}$ be a pseudovariety and let A be an alphabet. A $\mathbf{V}$-*pseudoidentity* on A is a pair (x, y) of elements of $\hat{F}_A(\mathbf{V})$, and is usually denoted $x = y$. We say that a semigroup $S \in \mathbf{V}$ *satisfies* $x = y$, written $S \models x = y$, if, for any continuous morphism $\mu : \hat{F}_A(\mathbf{V}) \to S$, we have $\mu(x) = \mu(y)$. We say that a subclass $\mathcal{C}$ of $\mathbf{V}$ satisfies a set Σ of $\mathbf{V}$-pseudoidentities, written $\mathcal{C} \models \Sigma$, if each element of $\mathcal{C}$ satisfies each element of Σ. The class of all finite semigroups which satisfy Σ is said to be *defined by* Σ and is denoted $[\![\Sigma]\!]_{\mathbf{V}}$. By a *pseudoidentity* we will mean an $\mathbf{S}$-*pseudoidentity*, and we will also set $[\![\Sigma]\!] = [\![\Sigma]\!]_{\mathbf{S}}$.

For instance, adopting the convention of replacing in a pseudoidentity expressions of the form x^ω, y^ω and z^ω by symbols e, f and g if, respectively, x, y and z do not appear elsewhere in the pseudoidentity, we have the following equalities:

$$
\begin{array}{rclrcl}
\mathbf{A} & = & [\![x^{\omega+1} = x^\omega]\!], & \mathbf{B} & = & [\![x^2 = x]\!] \\
\mathbf{Com} & = & [\![xy = yx]\!], & \mathbf{Com} * \mathbf{D} & = & [\![exfyezf = ezfyexf]\!] \\
\mathbf{D} & = & [\![xe = e]\!], & \mathcal{E}\mathbf{Com} & = & [\![ef = fe]\!] \\
\mathbf{J} & = & [\![(xy)^\omega = (yx)^\omega]\!]_{\mathbf{A}}, & \mathbf{K} & = & [\![ex = e]\!] \\
\mathbf{L} & = & [\![y(xy)^\omega = (xy)^\omega]\!], & \mathbf{LG} & = & [\![ex = x]\!] \\
\mathbf{LNB} & = & [\![xyz = xzy]\!]_{\mathbf{B}}, & \mathbf{NB} & = & [\![xyzx = xzyx]\!]_{\mathbf{B}} \\
\mathbf{R} & = & [\![(xy)^\omega x = (xy)^\omega]\!], & \mathbf{ReG} & = & [\![x = x^{\omega+1}, efe = e]\!] \\
\mathbf{RG} & = & [\![xe = x]\!], & \mathbf{RNB} & = & [\![xyz = yxz]\!]_{\mathbf{B}} \\
\mathbf{Sl} & = & [\![xy = yx]\!]_{\mathbf{B}}, & \mathbf{ZE} & = & [\![ey = ye]\!].
\end{array}
$$

As far as the $\mathcal{D}$ operator is concerned, the following equalities are well known.

$$
\begin{array}{rclrcl}
\mathcal{D}\mathbf{A} & = & [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!]_{\mathbf{A}}, & \mathcal{D}\mathbf{G} & = & [\![(xy)^\omega = (yx)^\omega]\!] \\
\mathcal{D}\mathbf{LG} & = & [\![(xy)^\omega(yx)^\omega = (yx)^\omega]\!], & \mathcal{D}\mathbf{ReG} & = & [\![(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega]\!] \\
\mathcal{D}\mathbf{RG} & = & [\![(xy)^\omega(yx)^\omega = (xy)^\omega]\!]. & & &
\end{array}
$$

Let Σ be a set of pseudoidentities defining a pseudovariety $\mathbf{V}$. Then $\mathcal{L}\mathbf{V}$ is defined by the set of all pseudoidentities which are obtained from Σ by substituting each variable x by $y^\omega x y^\omega$ where y is a variable that does not occur in Σ. For instance, we have that

$$
\begin{array}{rclrcl}
\mathcal{L}\mathcal{D}\mathbf{G} & = & [\![(exeye)^\omega = (eyexe)^\omega]\!], & \mathcal{L}\mathcal{E}\mathbf{Com} & = & [\![(exe)^\omega(eye)^\omega = (eye)^\omega(exe)^\omega]\!] \\
\mathcal{L}\mathbf{I} & = & [\![exe = e]\!], & \mathcal{L}\mathbf{Sl} & = & [\![exexe = exe, exeye = eyexe]\!] \\
\mathcal{L}\mathbf{ZE} & = & [\![(exe)^\omega eye = eye(exe)^\omega]\!]. & & &
\end{array}
$$

The following fundamental theorem is due to Reiterman [17].

Theorem 2.1 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and let* $\mathbf{W}$ *be a subclass of* $\mathbf{V}$. *Then* $\mathbf{W}$ *is a pseudovariety if and only if there exists a set* Σ *of* $\mathbf{V}$-*pseudoidentities such that* $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$. □

2.2 Languages recognized by a pseudovariety $\mathbf{V}$

Let A be an alphabet and let $\mathbf{V}$ be a pseudovariety. A subset L of A^+ is called a *language*. It is said to be *recognizable* (resp. $\mathbf{V}$-*recognizable*) if there exists a finite semigroup S (resp. in $\mathbf{V}$) and a morphism $\mu : A^+ \to S$ such that $L = \mu^{-1}(\mu(L))$. In that case, we say that S *recognizes* L. The *syntactic congruence* of a language L is the congruence $\sim_L$ over A^+ given by

$$u \sim_L v \quad \text{if and only if} \quad xuy \in L \Leftrightarrow xvy \in L \;\; \text{for all } x, y \in A^*.$$

The *syntactic semigroup* of L, denoted by $S(L)$, is the quotient of A^+ by $\sim_L$. We know that L is recognizable (resp. $\mathbf{V}$-recognizable) if and only if $S(L)$ is finite (resp. $S(L) \in \mathbf{V}$). Furthermore, a semigroup S recognizes a language L if and only if $S(L)$ *divides* S (that is, if $S(L)$ is a homomorphic image of a subsemigroup of S). For more details on recognizable languages, the reader is referred to [15, 13].

A *class of (recognizable) languages* is a correspondence $\mathcal{C}$ associating with each alphabet A a set $A^+\mathcal{C}$ of (recognizable) languages of A^+. A *variety of languages* is a class $\mathcal{V}$ of recognizable languages such that

(1) for every alphabet A, $A^+\mathcal{V}$ is closed under finite union, finite intersection and complement;

(2) for every morphism $\varphi : A^+ \to B^+$, $L \in B^+\mathcal{V}$ implies $\varphi^{-1}(L) \in A^+\mathcal{V}$;

(3) if $L \in A^+\mathcal{V}$ and $a \in A$, then $a^{-1}L = \{u \in A^+ \mid au \in L\}$ and $La^{-1} = \{u \in A^+ \mid ua \in L\}$ are in $A^+\mathcal{V}$.

Let $\mathbf{V}$ be a pseudovariety and let $\mathcal{V}$ be the class of recognizable languages which associates with each alphabet A the set $A^+\mathcal{V}$ of $\mathbf{V}$-recognizable languages of A^+. One can show that $\mathcal{V}$ is a variety of languages. Moreover, Eilenberg [13] proved the following fundamental result.

Theorem 2.2 *The correspondence $\mathbf{V} \mapsto \mathcal{V}$ defines a bijective correspondence between pseudovarieties of semigroups and varieties of languages.* □

We say that a family $\mathcal{X}$ of subsets of $\hat{F}_A(\mathbf{V})$ *separates* the points of $\hat{F}_A(\mathbf{V})$ if, for each pair of distinct elements x and y in $\hat{F}_A(\mathbf{V})$, there exists an element X of $\mathcal{X}$ such that either $x \in X$ and $y \notin X$, or $x \notin X$ and $y \in X$. The next result, due to Almeida [3, 7], will be very useful.

Proposition 2.3 *Let A be an alphabet, let $\mathbf{V}$ be a pseudovariety satisfying no non-trivial identity, and let $\mathcal{V}$ be the corresponding variety of languages. Let $\mathcal{L}$ be a subset of $A^+\mathcal{V}$ and let $\overline{\mathcal{L}}$ be the set of the topological closures in $\hat{F}_A(\mathbf{V})$ of the elements of $\mathcal{L}$.*

The Boolean algebra $A^+\mathcal{V}$ is generated by $\mathcal{L}$ if and only if the points of $\hat{F}_A(\mathbf{V})$ are separated by $\overline{\mathcal{L}}$. □

2.3 Subpseudovarieties of $\mathcal{D}\mathbf{S}$

In this paper we will be particularly interested in some subpseudovarieties of $\mathcal{D}\mathbf{S}$. Almeida and Azevedo [5] gave a number of factorization and regularity results for the implicit operations on subpseudovarieties of $\mathcal{D}\mathbf{S}$, which will prove fundamental in this paper. Some of these results are summarized in the following propositions.

Proposition 2.4 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathcal{D}\mathbf{S}$ *containing* $\mathbf{Sl}$ *and let* $x, y \in \hat{F}_A(\mathbf{V})$.

(1) x *can be written as a product of the form* $x = u_0 x_1 u_1 \cdots x_n u_n$ *where the* u_i *are words and the* x_i *are regular implicit operations on* $\mathbf{V}$.

(2) *If* x *and* y *are regular, then* $x \mathcal{J} y$ *if and only if* $c(x) = c(y)$.

(3) *If* $w \in \hat{F}_A(\mathbf{V})$, $c(w) \subseteq c(y)$, $x = wy$ *(resp.* $x = yw$*) and* y *is regular, then* x *is regular and* $x \mathcal{L} y$ *(resp.* $x \mathcal{R} y$*).* □

Proposition 2.5 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathcal{D}\mathbf{ReG}$. *Two regular elements* x *and* y *of* $\hat{F}_A(\mathbf{V})$ *are equal if and only if* $x^\omega = y^\omega$ *and* $\mathbf{V} \cap \mathbf{G}$ *satisfies* $x = y$. □

We will need also the following result (see [3, Corollary 5.6.2]).

Proposition 2.6 *Let* $\mathbf{V}$ *be a pseudovariety of semigroups and let* $x \in \hat{F}_A(\mathbf{V}) \setminus A^+$. *Then* $x = yz^\omega w$ *for some* $y, z, w \in \hat{F}_A(\mathbf{V})$. □

We now consider the pseudovariety of *nilpotent* semigroups $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$. It is well known that $\mathbf{N}$ satisfies no non-trivial identity. This means that the natural morphism $\iota : A^+ \to \hat{F}_A(\mathbf{N})$ is injective for each alphabet A. In particular, we may identify the free semigroup A^+ with a subsemigroup of $\hat{F}_A(\mathbf{N})$. Since $\mathbf{N}$ is contained in $\mathbf{K}$, $\mathbf{D}$ and $\mathcal{L}\mathbf{I}$, the same is true for each of these pseudovarieties. Furthermore, it is known (see [3]) that: $\hat{F}_A(\mathbf{N})$ is obtained from A^+ by adding a zero 0; $\hat{F}_A(\mathbf{K}) = A^+ \cup A^{\mathbb{N}}$ and the product in $\hat{F}_A(\mathbf{K})$ is extended from the product in A^+ by letting $ww' = w$ if $w \in A^{\mathbb{N}}$ (dually $\hat{F}_A(\mathbf{D}) = A^+ \cup A^{-\mathbb{N}}$ and the product in $\hat{F}_A(\mathbf{D})$ is extended from the product in A^+ by letting $w'w = w$ if $w \in A^{-\mathbb{N}}$); $\hat{F}_A(\mathcal{L}\mathbf{I}) = A^+ \cup (A^{\mathbb{N}} \times A^{-\mathbb{N}})$ where $A^{\mathbb{N}} \times A^{-\mathbb{N}}$ is a rectangular band and if $u \in A^+$ and $(v, w) \in A^{\mathbb{N}} \times A^{-\mathbb{N}}$, then $u(v, w) = (uv, w)$ and $(v, w)u = (v, wu)$.

Note that if $x = (v, w)$ is an element of $\hat{F}_A(\mathcal{L}\mathbf{I}) \setminus A^+$, then v (resp. w) is the restriction of x to $\mathbf{K}$ (resp. $\mathbf{D}$). In particular, $\mathcal{L}\mathbf{I}$ satisfies a pseudoidentity $x = y$ if and only if $\mathbf{K}$ and $\mathbf{D}$ satisfy $x = y$. This is another way of stating the well known equality $\mathcal{L}\mathbf{I} = \mathbf{K} \vee \mathbf{D}$.

3 Regular elements of $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{LD}\mathbf{G})$

In this section, we give a characterization of the regular elements of the semigroups $\hat{F}_A(\mathbf{V})$ of implicit operations on subpseudovarieties $\mathbf{V}$ of $\mathcal{D}\mathbf{ReG} \cap \mathcal{LD}\mathbf{G}$ and derive some important properties of them.

Proposition 3.1 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathcal{D}\mathbf{ReG} \cap \mathcal{LD}\mathbf{G}$ *containing* $\mathbf{Sl}$ *and* $\mathbf{K}$ *(resp.* $\mathbf{D}$*). Two regular elements* x *and* y *of* $\hat{F}_A(\mathbf{V})$ *are* $\mathcal{R}$*-(resp.* $\mathcal{L}$*-)equivalent if and only if they have the same content and the same restriction to* $\mathbf{K}$ *(resp.* $\mathbf{D}$*).*

Proof. Suppose first that $x \mathcal{R} y$. In particular, $x \mathcal{J} y$ and so by Proposition 2.4, $c(x) = c(y)$. Moreover, $x = yz$ for some $z \in \hat{F}_A(\mathbf{V})$. Since y (and x) is not in A^+, this clearly implies that the restrictions of x and y to $\mathbf{K}$ are equal.

Suppose now that $c(x) = c(y)$ and that $\mathbf{K}$ satisfies $x = y$. We claim that the second condition implies that $x = uz$ and $y = uw$ for some $u, z, w \in \hat{F}_A(\mathbf{V})$ such that $u \notin A^+$. Indeed, if $(x_n)_n$ and $(y_n)_n$ are sequences of A^+ converging, respectively, to x and y in

$\hat{F}_A(\mathbf{V})$, then we can choose subsequences $(x'_n)_n$ and $(y'_n)_n$ of $(x_n)_n$ and $(y_n)_n$, such that $x'_n = u_n z_n$ and $y'_n = u_n w_n$ for some $u_n, z_n, w_n \in A^+$. We may choose u_n such that $|u_n| > n$ and, by compactness of $\hat{F}_A(\mathbf{V})$, we may suppose that the sequences $(u_n)_n$, $(z_n)_n$ and $(w_n)_n$ are convergent in $\hat{F}_A(\mathbf{V})$ proving the claim. Moreover, Proposition 2.6 says that $u = u_1 u_2^\omega u_3$ for some $u_1, u_2, u_3 \in \hat{F}_A(\mathbf{V})$.

Now since $c(x) = c(y)$, we deduce from Proposition 2.4 that x, y, xy and yx are $\mathcal{J}$-equivalent regular elements, and that $xy \,\mathcal{R}\, x$. In particular, xy is a group element because $\mathbf{V} \subseteq \mathcal{D}\mathbf{S}$ and so $xy = (xy)^{\omega+1}$. Furthermore, we deduce successively

$$\begin{array}{rcll} xy & = & (xy)^{\omega+1} & \\ & = & (u_1 u_2^\omega u_3 z u_1 u_2^\omega u_3 w)^{\omega+1} & \\ & = & u_1 (u_2^\omega u_3 \boldsymbol{z} u_1 u_2^\omega u_3 \boldsymbol{w} u_1 u_2^\omega)^\omega u_3 z u_1 u_2^\omega u_3 w & \\ & = & u_1 (u_2^\omega u_3 \boldsymbol{w} u_1 u_2^\omega u_3 \boldsymbol{z} u_1 u_2^\omega)^\omega u_3 z u_1 u_2^\omega u_3 w & \text{since } \mathbf{V} \subseteq \mathcal{L}\mathcal{D}\mathbf{G} \\ & = & (u_1 u_2^\omega u_3 w u_1 u_2^\omega u_3 z)^\omega u_1 u_2^\omega u_3 z u_1 u_2^\omega u_3 w & \\ & = & (yx)^\omega xy. & \end{array}$$

This means that $xy \,\mathcal{R}\, y$ and, consequently, that $x \,\mathcal{R}\, y$. □

Corollary 3.2 *Let* $\mathbf{V}$ *be a subpseudovariety of* $\mathcal{L}\mathcal{D}\mathbf{G}$ *and* $\mathcal{D}\mathbf{ReG}$ *containing* $\mathbf{Sl}$ *and* $\mathcal{L}\mathbf{I}$. *Two regular elements of* $\hat{F}_A(\mathbf{V})$ *are equal if and only if they have the same content and the same restriction to* $\mathcal{L}\mathbf{I}$ *and to* $\mathbf{V} \cap \mathbf{G}$.

Furthermore, $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}$ *is the greatest subpseudovariety of* $\mathcal{D}\mathbf{ReG}$ *with this property.*

Proof. We only need to prove the sufficient condition. Since $c(x) = c(y)$ and $\mathcal{L}\mathbf{I}$ satisfies $x = y$, we have $x \,\mathcal{H}\, y$ from Proposition 3.1. So as the $\mathcal{H}$-class of x is a group (say because x is regular and $\mathbf{V}$ is a subpseudovariety of $\mathcal{D}\mathbf{S}$) we deduce $x^\omega = y^\omega$. Now the equality $x = y$ follows from Proposition 2.5.

Now suppose that $\mathbf{W}$ is a subpseudovariety of $\mathcal{D}\mathbf{ReG}$ not contained in $\mathcal{L}\mathcal{D}\mathbf{G}$. Then there are two distinct idempotents of $\hat{F}_A(\mathbf{W})$ of the form $x^\omega y x^\omega$ and $x^\omega z x^\omega$, respectively, in the same $\mathcal{J}$-class. These elements have clearly the same restriction to $\mathcal{L}\mathbf{I}$ and $\mathbf{W} \cap \mathbf{G}$. Moreover, since they are $\mathcal{J}$-equivalent, they have the same content by Proposition 2.5. □

Let $\mathbf{V}$ be a pseudovariety in the interval $[\mathbf{Sl} \vee \mathcal{L}\mathbf{I}, \mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}]$ and let x be a regular element of $\hat{F}_A(\mathbf{V})$. The previous result shows that x is characterized by its content, say $B \subseteq A$, and by its restrictions to $\mathcal{L}\mathbf{I}$ and to $\mathbf{V} \cap \mathbf{G}$, say $(w, w') \in B^{\mathbb{N}} \times B^{-\mathbb{N}}$ and $g \in \hat{F}_A(\mathbf{V} \cap \mathbf{G})$, respectively. So we will denote x by

$$[w, B, g, w'].$$

In particular, when x is idempotent it will be denoted by $[w, B, 1, w']$. Furthermore, if $\mathbf{V}$ is an aperiodic pseudovariety (i.e., it is such that $\mathbf{V} \cap \mathbf{G} = \mathbf{I}$), then $\mathbf{V}$ is a subpseudovariety of $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathcal{D}\mathbf{G}$. In particular, every regular element of $\hat{F}_A(\mathbf{V})$ is idempotent and it is characterized by its restrictions to $\mathbf{Sl}$ and $\mathcal{L}\mathbf{I}$. In this case we simplify the notation and denote it simply by

$$(w, B, w').$$

Remark. We notice that one can show, as above, that for a pseudovariety $\mathbf{V}$ in the interval $[\mathbf{Sl} \vee \mathbf{K}, \mathcal{D}\mathbf{RG} \cap \mathcal{L}\mathcal{D}\mathbf{G}]$ (resp. $[\mathbf{Sl} \vee \mathbf{D}, \mathcal{D}\mathbf{LG} \cap \mathcal{L}\mathcal{D}\mathbf{G}]$), a regular element $x \in \hat{F}_A(\mathbf{V})$

is characterized by its content, say $B \subseteq A$, and by its restrictions to $\mathbf{K}$ (resp. $\mathbf{D}$) and to $\mathbf{V} \cap \mathbf{G}$, say $w \in B^{\mathbb{N}}$ (resp. $w' \in B^{-\mathbb{N}}$) and $g \in \hat{F}_A(\mathbf{V} \cap \mathbf{G})$, respectively. Thus, x will be denoted by $[w, B, g]$ (resp. $[B, g, w']$). When $\mathbf{V}$ is an aperiodic pseudovariety we denote x simply by (w, B) (*resp.* (B, w')).

Notice also that from the paper of Trotter and Weil [22] one can deduce that $\mathcal{D}\mathbf{ReG} \cap \mathcal{LDG}$ (resp. $\mathcal{D}\mathbf{RG} \cap \mathcal{LDG}$, $\mathcal{D}\mathbf{LG} \cap \mathcal{LDG}$) is the greatest subpseudovariety of $\mathcal{D}\mathbf{ReG}$ (resp. $\mathcal{D}\mathbf{RG}$, $\mathcal{D}\mathbf{LG}$) whose intersection with $\mathbf{B}$ is $\mathbf{NB}$ (resp. $\mathbf{LNB}$, $\mathbf{RNB}$).■

In order to complete our notation for regular elements of semigroups $\hat{F}_A(\mathbf{V})$, we will now consider the case where $\mathbf{V}$ is a subpseudovariety of $\mathcal{D}\mathbf{G}$ containing $\mathbf{Sl}$. It is known (say by Propositions 2.4 and 2.5) that, in this case, a regular element x of $\hat{F}_A(\mathbf{V})$ is characterized by its content B and by its restriction g to $\mathbf{V} \cap \mathbf{G}$. So we denote x by $[B, g]$. If $\mathbf{V}$ is aperiodic (i.e., $\mathbf{V} \subseteq \mathbf{J}$), then every regular element x is idempotent and it is characterized by its content B. So we denote x simply by (B).

Thus, we use the notation (_) for idempotent elements of aperiodic pseudovarieties and [_] for the regular elements of the non-aperiodic pseudovarieties. The regular elements of $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{LDG})$ enjoy the following important properties.

Proposition 3.3 *Let A be an alphabet, let $B, C, D \subseteq A$ be such that $B \cap C \neq \emptyset$ and $D \subseteq B$. Let also $b \in B$. Then, in $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{LDG})$,*

(1) $[w, B, g, w']b = [w, B, gb, w'b]$, $b[w, B, g, w'] = [bw, B, bg, w']$,
$[w, B, g, w'][v, D, f, v'] = [w, B, gf, v']$ *and* $[v, D, f, v'][w, B, g, w'] = [v, B, fg, w']$;

(2) *if one of* $c(w')$ *and* $c(z)$ *is contained in* $B \cap C$, *then*

$$[w, B, g, w'][z, C, h, z'] = [w, B, g, w''][z'', C, h, z']$$

for every $w'' \in B^{-\mathbb{N}}$ *and* $z'' \in C^{\mathbb{N}}$ *such that at least one of* $c(w'')$ *and* $c(z'')$ *is contained in* $B \cap C$.

In particular, $\hat{F}_A(\mathcal{D}\mathbf{RG} \cap \mathcal{LDG})$ *satisfies*

(1′) $[w, B, g]b = [w, B, gb]$, $b[w, B, g] = [bw, B, bg]$, $[w, B, g][v, D, f] = [w, B, gf]$ *and* $[v, D, f][w, B, g] = [v, B, fg]$;

(2′) $[w, B, g][z, C, h] = [w, B, g][z'', C, h]$ *for every* $z, z'' \in C^{\mathbb{N}}$.

Proof. (1) Is an immediate consequence of Proposition 2.4 (3) and of Corollary 3.2.

(2) Suppose, for instance, that $c(w') \subseteq B \cap C$ and let $w'' \in B^{-\mathbb{N}}$ and $z'' \in C^{\mathbb{N}}$ be such that $c(w'') \subseteq B \cap C$ or $c(z'') \subseteq B \cap C$. If $c(z'') \subseteq B \cap C$, we deduce from (1) that $[w, B, g, w'] = [w, B, g, w'][w, B, 1, w''][z'', B \cap C, 1, w']$. So

$$\begin{aligned}
[w, B, g, w'][z, C, h, z'] &= ([w, B, g, w'][w, B, 1, w''])([z'', B \cap C, 1, w'][z, C, h, z']) \\
&= [w, B, g, w''][z'', C, h, z'] \qquad \text{from (1).}
\end{aligned}$$

Suppose now that $c(z'') \not\subseteq B \cap C$ and let $a \in B \cap C$. Then $c(w'') \subseteq B \cap C$ and using what we proved above, we deduce

$$\begin{aligned}
[w, B, g, w'][z, C, h, z'] &= [w, B, g, w'][a^{+\infty}, C, h, z'] \\
&= [w, B, g, w'][a^{+\infty}, B \cap C, 1, w''][z'', C, h, z'] \qquad \text{from (1)} \\
&= [w, B, g, w''][z'', C, h, z'].
\end{aligned}$$

For the proof of ($1'$) and ($2'$), it suffices to consider the canonical projection of $\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG}$ to $\mathcal{D}\mathbf{RG}\cap\mathcal{LDG}$ and note that the restriction to $\mathcal{D}\mathbf{RG}\cap\mathcal{LDG}$ of a regular element $[w,B,g,w']$ of $\hat{F}_A(\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG})$ is the regular element $[w,B,g]$. $\square$

Note that if $\mathbf{V}$ is a subpseudovariety of $\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG}$ containing $\mathbf{Sl}$ and $\mathcal{L}\mathbf{I}$, the restriction to $\mathbf{V}$ of a regular element $[w,B,g,w']$ of $\hat{F}_A(\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG})$ is also denoted by $[w,B,g,w']$. Hence, the previous result is also valid in $\hat{F}_A(\mathbf{V})$.

In the following sections, we will proceed to the description of the semigroups of implicit operations on various subpseudovarieties of $\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG}$, namely the semigroups:

- $\hat{F}_A(\mathcal{D}\mathbf{A}\cap\mathcal{LDG})$ and $\hat{F}_A(\mathbf{R}\cap\mathcal{LDG})$;
- $\hat{F}_A(\mathbf{V}\cap\mathbf{W})$ with $\mathbf{V}\in\{\mathcal{D}\mathbf{ReH},\mathcal{D}\mathbf{RH},\mathcal{D}\mathbf{H}\}$ and $\mathbf{W}\in\{\mathcal{LE}\mathbf{Com},\mathcal{L}\mathbf{ZE},\mathcal{L}(\mathbf{Sl}\vee\mathbf{G}),\mathbf{Com}*\mathbf{D}\}$;
- $\hat{F}_A(\mathcal{D}\mathbf{H}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com})$ with $\mathbf{W}\in\{\mathcal{L}\mathbf{ZE},\mathcal{L}(\mathbf{Sl}\vee\mathbf{G}),\mathbf{Com}*\mathbf{D}\}$.

Note that the non-aperiodic cases $\hat{F}_A(\mathbf{V}\cap\mathcal{LDG})$ with $\mathbf{V}\in\{\mathcal{D}\mathbf{ReH},\mathcal{D}\mathbf{RH},\mathcal{D}\mathbf{H}\}$ (and $\mathbf{H}$ a non-trivial pseudovariety of groups) are not included here, because we were not able to solve them. To give an idea of the inclusion relations between the pseudovarieties involved, we note the following inclusions:

- $\mathcal{L}\mathbf{Sl}\subseteq\mathbf{Com}*\mathbf{D}\subseteq\mathcal{L}\mathbf{Com}\subseteq\mathcal{L}\mathbf{ZE}\subseteq\mathcal{LDG}$;
- $\mathcal{L}\mathbf{Sl}\subseteq\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})\subseteq\mathcal{L}\mathbf{ZE}$;
- $\mathcal{L}\mathbf{ZE}\subseteq\mathcal{LE}\mathbf{Com}$, $\mathcal{LE}\mathbf{Com}\not\subseteq\mathcal{LDG}$ but $\mathcal{D}\mathbf{ReG}\cap\mathcal{LE}\mathbf{Com}\subseteq\mathcal{D}\mathbf{ReG}\cap\mathcal{LDG}$.

4 Implicit operations on $\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J}$

We begin our study with the description of the semigroups $\hat{F}_A(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$ and $\hat{F}_A(\mathbf{R}\cap\mathcal{L}\mathbf{J})$. We prove that every element of each of these semigroups, can be written in a unique form as a product of words and idempotents. We note that, since $\mathbf{J}=\mathcal{D}\mathbf{G}\cap\mathbf{A}$, we have immediately $\mathcal{D}\mathbf{A}\cap\mathcal{LDG}=\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J}$ and $\mathbf{R}\cap\mathcal{LDG}=\mathbf{R}\cap\mathcal{L}\mathbf{J}$. Note also that $\mathbf{J}$ is a subpseudovariety of both $\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J}$ and $\mathbf{R}\cap\mathcal{L}\mathbf{J}$.

Let us begin by considering the case $\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J}$. Let $x\in\hat{F}_A(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$ and let an order be fixed for the letters of the alphabet A. We say that a factorization of x of the form

$$x=u_0(w_1,A_1,w_1')u_1\cdots u_{n-1}(w_n,A_n,w_n')u_n$$

is *normal* if

- $u_i\in A^*$, $u_0\neq 1$ if $x=u_0$;
- for each $1\leq i\leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.
- if u_i ($1\leq i\leq n-1$) is the empty word, then
  - A_i and A_{i+1} are $\subseteq$-incomparable;

– the first letter of w_{i+1} does not lie in A_i;
– if $c(w'_i) \subseteq A_{i+1}$, then $w'_i = u^{-\infty}$ and $w_{i+1} = v^{+\infty}$ where u and v are the least linear (i.e., such that each letter occurs exactly once) words in alphabetical order of content, respectively, $A_i \cap A_{i+1}$ and A_{i+1} such that the first letter of v does not lie in A_i.

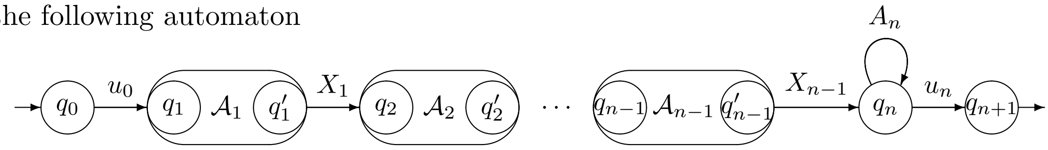
Proposition 4.1 *Every element of $\hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$ admits a normal factorization.*

Proof. Let $x \in \hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$. As a consequence of Proposition 2.4, x admits a factorization of the form $x = u_0(w_1, A_1, w'_1)u_1 \cdots u_{n-1}(w_n, A_n, w'_n)u_n$ as a product of words $u_i \in A^*$ and idempotents (w_i, A_i, w'_i), such that for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i and, if u_i $(1 \leq i \leq n-1)$ is the empty word, then A_i and A_{i+1} are $\subseteq$-incomparable.

Now suppose that $1 \leq i \leq n-1$ is such that $u_i = 1$. Then either one of $c(w'_i)$ and $c(w_{i+1})$ is contained in $A_i \cap A_{i+1}$, or $c(w'_i)$ and $c(w_{i+1})$ are both not contained in $A_i \cap A_{i+1}$. In the first case, letting u and v be the least linear words in alphabetical order of content, respectively, $A_i \cap A_{i+1}$ and A_{i+1} such that the first letter of v does not lie in A_i, we have from Proposition 3.3 that the factor $(w_i, A_i, w'_i)(w_{i+1}, A_{i+1}, w'_{i+1})$ is equal to $(w_i, A_i, u^{-\infty})(v^{+\infty}, A_{i+1}, w'_{i+1})$. In the second case, $w_{i+1} = zz'$ for some words $z \in A^*_{i+1}$ and $z' \in A^{\mathbb{N}}_{i+1}$ such that $c(z) \subseteq A_i$ (if $z \neq 1$) and the first letter of z' does not lie in A_i. Furthermore, $(w_i, A_i, w'_i)(w_{i+1}, A_{i+1}, w'_{i+1}) = (w_i, A_i, w'_i z)(z', A_{i+1}, w'_{i+1})$ by Proposition 3.3. So, for each $1 \leq i \leq n-1$ such that $u_i = 1$, substituting in the factorization of x the factor $(w_i, A_i, w'_i)(w_{i+1}, A_{i+1}, w'_{i+1})$ by $(w_i, A_i, u^{-\infty})(v^{+\infty}, A_{i+1}, w'_{i+1})$ in the first case and by $(w_i, A_i, w'_i z)(z', A_{i+1}, w'_{i+1})$ in the second case, we obtain a normal factorization of x. □

We now describe some automata which we will use to construct test semigroups (the syntactic semigroups of the languages recognized by these automata) to separate distinct factorizations of elements of $\hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$.

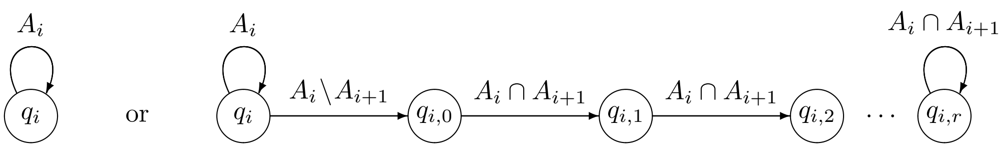
Let $r, n \geq 0$ be two integers and let $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ be such that, for all $1 \leq i \leq n-1$: if $u_i \neq 1$ then $c(u_i)$ is not contained in either A_i or A_{i+1}; if $u_i = 1$ then A_i and A_{i+1} are $\subseteq$-incomparable. Let $\mathcal{A} = \mathcal{A}(r; u_0, A_1, u_1, \ldots, A_n, u_n)$ be the following automaton



where, for each $1 \leq i \leq n-1$,

$$X_i = \begin{cases} u_i & \text{if } u_i \neq 1 \\ A_{i+1} \setminus A_i & \text{if } u_i = 1 \end{cases}$$

and the automaton $\mathcal{A}_i$ is either



when, respectively, $u_i \neq 1$ or $u_i = 1$. Note that the state q'_i is q_i in the first case and is

$q_{i,r}$ in the second one. Note also that $\mathcal{A}_i$ is an automaton on the alphabet A_i. In the figure of automaton $\mathcal{A}$, the initial, q_0, and final, q_{n+1}, states are pointed out by arrows. We will follow this convention throughout the paper.

Lemma 4.2 *Let L be the language recognized by the automaton $\mathcal{A}$ above. Then $S(L)$ lies in $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$. Moreover, if $w \in A^+, k > |u_0 \cdots u_n| + 3n - 2 + lr$ (where l is the number of indices $1 \leq i \leq n-1$ such that $u_i = 1$) and w^k is the label of a path $\mathcal{T}$ in $\mathcal{A}$, then there exists $1 \leq i \leq n$ such that $w \in A_i^+$ and $\mathcal{T}$ visits state q_i (or state $q_{i,r}$ when it exists) and does not visit either state q_{i-1}, if $i > 1$, or state q_{i+1}, if $i < n$.*

In particular, if $r = 0$ and the first letter of u_j ($1 \leq j \leq n$) does not lie in A_j, then $S(L) \in \mathbf{R}$. In this case, if w, k and $\mathcal{T}$ are as above, then $\mathcal{T}$ ends in state q_i (or state $q_{i,0}$), with i as above.

Proof. Because of the choice of k, it is clear that the path $\mathcal{T}$ visits a state p having a loop and stays in p for at least $|w|$ steps. If $p = q_{i,r}$ for some $1 \leq i \leq n-1$, then $w \in (A_i \cap A_{i+1})^+$ and so, in particular, $w \in A_i^+$. Otherwise, $p = q_i$ for some $1 \leq i \leq n$ and so $w \in A_i^+$. In both cases $\mathcal{T}$ does not visit either state q_{i-1} (when $i > 1$) or state q_{i+1} (when $i < n$) because in that case $\mathcal{T}$ would contain a transition labeled with a word not in A_i^+.

Let us now show that $S(L)$ lies in $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$. For that, it suffices to show that, for all $x, y, z \in A^+$ and m large enough, $(xy)^m(yx)^m(xy)^m \sim_L (xy)^m$, $x^{m+1} \sim_L x^m$ and $(x^m y x^m z x^m)^m \sim_L (x^m z x^m y x^m)^m$. Without loss of generality, we may suppose that m is an exponent of $S(L)$ (so that, for all $w \in A^+$, $w^m \sim_L w^m w^m$, that is, the syntactic image of w^m is an idempotent of $S(L)$).

Let $x, y, z \in A^+$. To prove that $(xy)^m(yx)^m(xy)^m \sim_L (xy)^m$ it suffices to show the following condition:

> $(xy)^m(yx)^m(xy)^m$ is the label of a path $\mathcal{P}$ in $\mathcal{A}$ (say from a state p to a state q) if and only if there is a path $\mathcal{Q}$ in $\mathcal{A}$ labeled $(xy)^m$ and co-terminal with $\mathcal{P}$ (that is, from p to q).

So let us first suppose that $\mathcal{P}$ exists. Then from the above, $xy \in A_i^+$ for some $1 \leq i \leq n$ and $\mathcal{P}$ visits state q_i or state $q_{i,r}$ (when it exists) and does not visit either state q_{i-1} (if $i > 1$) or state q_{i+1} (if $i < n$). Suppose that $q_{i,r}$ exists (i.e., that $1 \leq i < n$ and $u_i = 1$) and that $\mathcal{P}$ visits the states of the form $q_{i,j}$ ($0 \leq j \leq r$) for at least $|xy|$ steps. Then $xy \in (A_i \cap A_{i+1})^+$ and so $\mathcal{P}$ is entirely between the states $q_{i,0}$ and $q_{i,r}$. Therefore, the subpath of $\mathcal{P}$ labeled $(yx)^m(xy)^m$ is entirely in $q_{i,r}$ and so the existence of $\mathcal{Q}$ is clear. Now suppose that $\mathcal{P}$ visits the states of the form $q_{i,j}$ ($0 \leq j \leq r$) for at most $|xy| - 1$ steps so that $\mathcal{P}$ visits state q_i. Therefore, since $\mathcal{P}$ does not visit state q_{i-1} (when $i > 1$) and, as above, it can not visit the states of the form $q_{i-1,j}$ ($0 \leq j \leq r$), if they exist, for more than $|xy| - 1$ steps, we deduce that at most $max\{|xy|, |u_{i-1}|\} - 1$ of the steps of $\mathcal{P}$ take place strictly between the states q_{i-1} and q_i. Hence, the subpath of $\mathcal{P}$ labeled $(yx)^m$ is entirely in q_i. So the existence of $\mathcal{Q}$ is also clear in this case. (In fact, what is clear is the existence of a path labeled $(xy)^m(xy)^m$ co-terminal with $\mathcal{P}$. But, since we are considering m such that $(xy)^m \sim_L (xy)^m(xy)^m$, the existence of $\mathcal{Q}$ is guaranteed.) The case when $q_{i,r}$ does not exist can be treated analogously. Similarly, one can show that the existence of $\mathcal{Q}$ implies the existence of $\mathcal{P}$, proving that $(xy)^m(yx)^m(xy)^m \sim_L (xy)^m$. That $x^{m+1} \sim_L x^m$ can be proved analogously.

Now suppose that $\mathcal{P}$ is a path in $\mathcal{A}$ labeled $(x^m yx^m zx^m)^m$ so that $c(x)\cup c(y)\cup c(z) \subseteq A_i$ for some $1 \leq i \leq n$. Then either $i < n$, $u_i = 1$ and $\mathcal{P}$ takes place entirely between the states $q_{i,0}$ and $q_{i,r}$, and the existence of a path $\mathcal{Q}$ in $\mathcal{A}$ co-terminal with $\mathcal{P}$ and labeled $(x^m zx^m yx^m)^m$ is immediate, or at least $|x^m yx^m zx^m|$ steps of $\mathcal{P}$ take place in state q_i and at most $|yx^m zx^m|(= |x^m yx^m z|)$ steps of $\mathcal{P}$ take place strictly between the states q_i and q_{i+1} (resp. between the states q_{i-1} and q_i). In this case, let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ be the subpaths of $\mathcal{P}$ labeled, respectively, $x^m yx^m z$, $x^m(x^m yx^m zx^m)^{m-2}x^m$ and $yx^m zx^m$. Hence, $\mathcal{P}_2$ is entirely in q_i so that $\mathcal{P}_1$ ends in q_i and $\mathcal{P}_3$ begins in q_i. Moreover, the subpath of $\mathcal{P}_1$ labeled $yx^m z$ begins in q_i or in $q_{i-1,r}$ (when it exists). In both cases it is clear that there is a path $\mathcal{P}'_1$ co-terminal with $\mathcal{P}_1$ and labeled $x^m zx^m y$. Analogously, there is a path $\mathcal{P}'_3$ co-terminal with $\mathcal{P}_3$ and labeled $zx^m yx^m$. Since, trivially, there is a path labeled $x^m(x^m zx^m yx^m)^{m-2}x^m$, entirely in q_i, we deduce the existence of a path $\mathcal{Q}$ (co-terminal with $\mathcal{P}$ and labeled $(x^m zx^m yx^m)^m$). By symmetry, we deduce that $(x^m yx^m zx^m)^m \sim_L (x^m zx^m yx^m)^m$.

Finally, suppose that $r = 0$ and that the first letter of u_j $(1 \leq j \leq n)$ does not lie in A_j. As above, these conditions clearly imply that, for some $1 \leq i \leq n$, $w \in A_i^+$ and $\mathcal{T}$ ends in state q_i or state $q_{i,0}$ (in this case $i < n$). To prove that $S(L) \in \mathbf{R}$, let us show that $(xy)^m x \sim_L (xy)^m$. For that, let $\mathcal{P}$ be a path in $\mathcal{A}$ labeled $(xy)^m x$. This path ends in some state q_i or state $q_{i,0}$. In the first case, the assertion that there is some path $\mathcal{Q}$ in $\mathcal{A}$ labeled $(xy)^m$ and co-terminal with $\mathcal{P}$ is immediate. In the second case, either $xy \in (A_i \cap A_{i+1})^+$, and so $\mathcal{P}$ is entirely in $q_{i,0}$ (and the existence of such a path $\mathcal{Q}$ is trivial), or there is at least a letter of xy in $A_i \setminus A_{i+1}$ and $\mathcal{P}$ stays in $q_{i,0}$ for at most $|yx|-1$ steps. In this case, the existence of the desired path $\mathcal{Q}$ is also ensured (this path can pass from state q_i to state $q_{i,0}$ using, for instance, the last occurrence not in A_{i+1} of a letter of the word $(xy)^m$). The proof of the converse is similar and so we conclude that $(xy)^m x \sim_L (xy)^m$, proving that $S(L) \in \mathbf{R}$. □

Now we are able to prove the following characterization of the semigroups of implicit operations on $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$.

Theorem 4.3 *Let $x, y \in \hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$ and let $x = u_0(w_1, A_1, w'_1)u_1 \cdots (w_n, A_n, w'_n)u_n$ and $y = v_0(z_1, B_1, z'_1)v_1 \cdots (z_m, B_m, z'_m)v_m$ be factorizations in normal form. Then $x = y$ if and only if $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ and $w'_i = z'_i$ for all i.*

Proof. Let $r \geq 1$ be an integer such that $r > |v_i|$ for every $1 \leq i \leq n$ and $c(s_r(w'_i)) \not\subseteq A_{i+1}$ for every $1 \leq i \leq n-1$ such that $u_i = 1$ and $c(w'_i) \not\subseteq A_{i+1}$. Consider the automaton $\mathcal{A} = \mathcal{A}(r; u_0 p_r(w_1), A_1, u'_1, \ldots, u'_{n-1}, A_n, s_r(w'_n)u_n)$ where, for each $1 \leq i \leq n-1$, u'_i is equal to:

- $s_r(w'_i)u_i p_r(w_{i+1})$ if $u_i \neq 1$, or $u_i = 1$ and $c(w'_i) \not\subseteq A_{i+1}$;
- 1 if $u_i = 1$ and $c(w'_i) \subseteq A_{i+1}$.

Note that by definition of normal factorization of x, for each $1 \leq i \leq n-1$, if $u'_i = 1$ then A_i and A_{i+1} are $\subseteq$-incomparable and if $u'_i \neq 1$ then $c(u'_i) \not\subseteq A_i, A_{i+1}$.

Let L be the language recognized by $\mathcal{A}$ and let $\mu : A^+ \to S$ be its syntactic homomorphism. By Lemma 4.2, $S \in \mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$. So let $\hat{\mu} : \hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}) \to S$ be the unique continuous homomorphic extension of μ, and let $k > |u_0 \cdots u_n| + 3n - 2 + lr$ (where l is the number of indices $1 \leq i \leq n-1$ such that $u'_i = 1$) be an exponent of S (so that for all $w \in A^+$ the syntactic image of w^k is an idempotent of S).

For each $1 \leq i \leq n$, let $\bar{w}_i \in A_i^{\mathbb{N}}$ and $\bar{w}'_i \in A_i^{-\mathbb{N}}$ be such that $w_i = p_r(w_i)\bar{w}_i$ and $w'_i = \bar{w}'_i s_r(w'_i)$ so that $(w_i, A_i, w'_i) = p_r(w_i)(\bar{w}_i, A_i, \bar{w}'_i)s_r(w'_i)$. Since $(\bar{w}_i, A_i, \bar{w}'_i)$ is idempotent, its image in S, $\hat{\mu}(\bar{w}_i, A_i, \bar{w}'_i)$ is also idempotent. By density of A^+ in $\hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$, there is a word x_i such that $c(x_i) = A_i$ and $\hat{\mu}(\bar{w}_i, A_i, \bar{w}'_i) = \mu(x_i^k)$. Now it is not very difficult to verify that

$$w = u_0 p_r(w_1) x_1^k s_r(w'_1) u_1 p_r(w_2) x_2^k \cdots s_r(w'_{n-1}) u_{n-1} p_r(w_n) x_n^k s_r(w'_n) u_n$$

is a word recognized by $\mathcal{A}$, whence $w \in L$. On the other hand, we have $\hat{\mu}(x) = \mu(w)$.

Consider now words $\bar{z}_i \in B_i^{\mathbb{N}}$ $(1 \leq i \leq m)$ and $\bar{z}'_i \in B_i^{-\mathbb{N}}$ such that $z_i = p_r(z_i)\bar{z}_i$ and $z'_i = \bar{z}'_i s_r(z'_i)$ so that $(z_i, B_i, z'_i) = p_r(z_i)(\bar{z}_i, B_i, \bar{z}'_i)s_r(z'_i)$. Consider also words y_i such that $c(y_i) = B_i$ and $\hat{\mu}(\bar{z}_i, B_i, \bar{z}'_i) = \mu(y_i^k)$. Let

$$w' = v_0 p_r(z_1) y_1^k s_r(z'_1) v_1 p_r(z_2) y_2^k \cdots y_m^k s_r(z'_m) v_m.$$

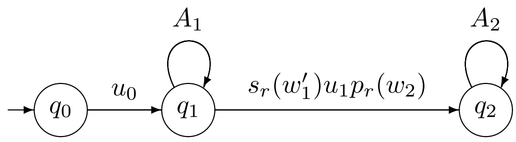
We have $\hat{\mu}(y) = \mu(w')$ and, as $x = y$, $\hat{\mu}(x) = \hat{\mu}(y)$. Therefore, $\mu(w) = \mu(w')$ whence $w' \in L$ and so w' is recognized by $\mathcal{A}$.

Let $\mathcal{P}$ be a successful path in $\mathcal{A}$ (i.e., which goes from q_0 to q_{n+1}) labeled w' and, for each $1 \leq i \leq m$, let $\mathcal{P}_i$ be the subpath of $\mathcal{P}$ labeled $v_0 p_r(z_1) y_1^k s_r(z'_1) v_1 p_r(z_2) y_2^k \cdots y_i^k$. From Lemma 4.2, we deduce that the path $\mathcal{P}_i$ visits state q_{j_i} — for some $1 \leq j_i \leq n$ such that $B_i \subseteq A_{j_i}$ — and does not visit state q_{j_i+1} (if $j_i < n$). Furthermore, the subpath $\mathcal{P}'_i$ of $\mathcal{P}_i$ labeled $p_r(z_i) y_i^k$ does not visit state q_{j_i-1} (if $j_i > 1$).

In particular, the path $\mathcal{P}_1$ visits state q_1 and so the word $u_0 p_r(w_1)$ is a prefix of $v_0 p_r(z_1) y_1^k$. Now since $r > |v_0|$, also the path $\mathcal{P}'_1$ visits state q_1. Hence, $j_1 = 1$ and $B_1 \subseteq A_1$. By symmetry it follows that $A_1 = B_1$. Now since the last letter of u_0 (if it exists) does not lie in $A_1 = B_1$, we deduce that u_0 is a prefix of v_0. Again by symmetry it follows that $u_0 = v_0$ and consequently that $p_r(w_1) = p_r(z_1)$. Since this holds for r arbitrarily large, we conclude that $w_1 = z_1$.

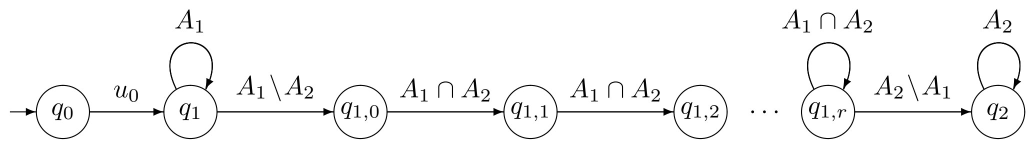
Now as the first letter of the word $v_1 p_r(z_2)$ does not lie in $B_1 = A_1$ (note that, as the factorization of y is normal, if $v_1 = 1$ then the first letter of z_2 does not belong to B_1) we have $j_2 > 1$. Let us consider the two possible cases for u'_1.

First case Suppose, first, that $u'_1 \neq 1$, i.e., that $u_1 \neq 1$, or $u_1 = 1$ and $c(w'_1) \not\subseteq A_2$. Then automaton $\mathcal{A}$ begins like this



Therefore, the word $s_r(w'_1) u_1 p_r(w_2)$ is a factor of $y_1^k s_r(z'_1) v_1 p_r(z_2) y_2^k$. Since the first letters of $u_1 p_r(w_2)$ and $v_1 p_r(z_2)$, respectively, do not lie in $A_1 = B_1$, we deduce that $s_r(w'_1) = s_r(z'_1)$ and that $u_1 p_r(w_2)$ is a prefix of $v_1 p_r(z_2) y_2^k$. Now as above, this implies that $w'_1 = z'_1$, $j_2 = 2$ and $B_2 \subseteq A_2$. Moreover, since the last letter of u_1 (if it exists) does not lie in A_2, and so does not lie also in B_2, we deduce that u_1 is a prefix of v_1. If $v_1 \neq 1$ we can apply symmetry to deduce that $u_1 = v_1$. If $v_1 = 1$, we have trivially $u_1 = v_1$. Now this equality implies that $p_r(w_2)$ is a prefix of $p_r(z_2) y_2^k$ so that $p_r(w_2) = p_r(z_2)$. Therefore, as above $w_2 = z_2$. Note that, in this case, it remains to prove the inclusion $A_2 \subseteq B_2$.

Second case Suppose now that $u'_1 = 1$, i.e., that $u_1 = 1$ and $c(w'_1) \subseteq A_2$. In particular, $w'_1 = u^{-\infty}$ and $w_2 = v^{+\infty}$ where u and v are the least linear words in alphabetical order of content, respectively, $A_1 \cap A_2$ and A_2 such that the first letter of v does not lie in A_1. We may also suppose that $v_1 = 1$ since otherwise, we could apply an argument as above to deduce that v_1 would be a prefix of u_1 and so u_1 would not be equal to the empty word. In this case, the beginning of the automaton $\mathcal{A}$ is the following.



Therefore, in path $\mathcal{P}_2$, the first letter of $p_r(z_2)$ is read in the transition from state $q_{1,r}$ to state q_2, and $s_r(z'_1)$ is read in the transitions between state $q_{1,0}$ and state $q_{1,r}$. This means, in particular, that $j_2 = 2$ so that $B_2 \subseteq A_2$. So in both cases ($u'_1 = 1$ and $u'_1 \neq 1$) we have $B_2 \subseteq A_2$. Hence, $B_2 \subseteq A_2$ and applying symmetry we deduce that $A_2 = B_2$. In the case $u'_1 = 1$ we are considering, we also deduce that $c(s_r(z'_1)) \subseteq A_1 \cap A_2$. Since r is arbitrarily large, this implies that $c(z'_1) \subseteq A_2$. So since we are dealing with normal factorizations and $v_1 = 1$, we have $z'_1 = u^{-\infty} = w'_1$ and $z_2 = v^{+\infty} = w_2$.

Therefore, we have proved that $w'_1 = z'_1$, $u_1 = v_1$, $w_2 = z_2$ and $A_2 = B_2$. Iterating the above argument, we deduce that $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ and $w'_i = z'_i$ for all i. $\square$

This last proof shows, in particular, that the syntactic semigroups of the languages recognized by the automata $\mathcal{A}(r; u_0, A_1, \ldots, A_n, u_n)$, as above, suffice to separate distinct implicit operations on $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$.

Corollary 4.4 *The pseudovariety $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$ is generated by the syntactic semigroups of the languages recognized by the automata $\mathcal{A}(r; u_0, A_1, \ldots, A_n, u_n)$ where $r, n \geq 0$ and, for some alphabet A, $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ are such that, for each $1 \leq i \leq n-1$: if $u_i \neq 1$ then $c(u_i)$ is not contained in either A_i or A_{i+1}; if $u_i = 1$ then A_i and A_{i+1} are $\subseteq$-incomparable.* $\square$

Almeida and Azevedo [4] showed that $\mathbf{R} \vee \mathbf{L} = [\![(xy)^\omega x(zx)^\omega = (xy)^\omega (zx)^\omega]\!]$. If a, b and c are distinct letters of an alphabet A, in $\hat{F}_A(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$ we have

$$\begin{aligned} (ab)^\omega a(ca)^\omega &= ((ab)^{+\infty}, \{a,b\}, (ab)^{-\infty}) a ((ca)^{+\infty}, \{a,c\}, (ca)^{-\infty}) \\ &= ((ab)^{+\infty}, \{a,b\}, (ba)^{-\infty})((ca)^{+\infty}, \{a,c\}, (ca)^{-\infty}) \end{aligned}$$

and

$$(ab)^\omega (ca)^\omega = ((ab)^{+\infty}, \{a,b\}, (ab)^{-\infty})((ca)^{+\infty}, \{a,c\}, (ca)^{-\infty}).$$

Hence, by Theorem 4.3, $(ab)^\omega a(ca)^\omega \neq (ab)^\omega (ca)^\omega$ and so $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$ does not satisfy the pseudoidentity $(xy)^\omega x\,(zx)^\omega = (xy)^\omega (zx)^\omega$. This proves that $(\mathbf{R} \vee \mathbf{L}) \cap \mathcal{L}\mathbf{J} \neq \mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$.

Let us now consider the case $\mathbf{R} \cap \mathcal{L}\mathbf{J}$. Let $x \in \hat{F}_A(\mathbf{R} \cap \mathcal{L}\mathbf{J})$. We say that a factorization of x of the form $x = u_0(w_1, A_1)u_1 \cdots u_{n-1}(w_n, A_n)u_n$ is *normal* if

- $u_i \in A^*$, $u_0 \neq 1$ if $x = u_0$;
- for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in $c(x_i)$.
- if u_i $(1 \leq i \leq n-1)$ is the empty word, then
  - A_i and A_{i+1} are $\subseteq$-incomparable;
  - if $A_i \cap A_{i+1} \neq \emptyset$, then $w_{i+1} = v^{+\infty}$ where v is the least linear word in alphabetical order of content A_{i+1} such that the first letter of v does not lie in A_i.

Using the $(\mathbf{R} \cap \mathcal{L}\mathbf{J})$-recognizable languages described on Lemma 4.2 and applying similar arguments as those of the proof of Theorem 4.3, one can show that the implicit operations on $\mathbf{R} \cap \mathcal{L}\mathbf{J}$ are characterized by the following result.

Theorem 4.5 *Every element of $\hat{F}_A(\mathbf{R} \cap \mathcal{L}\mathbf{J})$ admits a normal factorization. Let $x, y \in \hat{F}_A(\mathbf{R} \cap \mathcal{L}\mathbf{J})$ and let $x = u_0(w_1, A_1)u_1 \cdots (w_n, A_n)u_n$ and $y = v_0(z_1, B_1)v_1 \cdots (z_m, B_m)v_m$ be factorizations in normal form. Then $x = y$ if and only if $n = m$, $u_i = v_i$, $w_i = z_i$ and $A_i = B_i$ for all i.* $\square$

Naturally, a left-right dual of this last theorem could be stated for the pseudovariety $\mathbf{L} \cap \mathcal{L}\mathbf{J}$.

5 Implicit operations on $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$

In this section, we concentrate our attention on subpseudovarieties of $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$, namely the pseudovarieties of the form $\mathbf{W} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$ where $\mathbf{H}$ is a pseudovariety of groups and $\mathbf{W}$ is one of $\mathcal{D}\mathbf{ReH}$, $\mathcal{D}\mathbf{RH}$ and $\mathcal{D}\mathbf{H}$. Note that $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$ is a subpseudovariety of $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{D}\mathbf{G}$. Indeed, we have

$$\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com} \subseteq \mathcal{L}(\mathcal{D}\mathbf{ReG} \cap \mathcal{E}\mathbf{Com}) = \mathcal{L}(\mathcal{D}\mathbf{G} \cap \mathcal{E}\mathbf{Com}) \subseteq \mathcal{L}\mathcal{D}\mathbf{G},$$

since $\mathcal{D}\mathbf{ReG} \cap \mathcal{E}\mathbf{Com} = \mathcal{D}\mathbf{G} \cap \mathcal{E}\mathbf{Com}$. Also note that $\mathbf{J}$ is not a subpseudovariety of $\mathcal{L}\mathcal{E}\mathbf{Com}$ because it does not satisfy the pseudoidentity $(exe)^\omega(eye)^\omega = (eye)^\omega(exe)^\omega$ which defines $\mathcal{L}\mathcal{E}\mathbf{Com}$.

Besides the properties given by Proposition 3.3, the regular elements of the semigroup $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$ enjoy also the following important one.

Proposition 5.1 *Let A be an alphabet and let B and C be subalphabets of A such that $B \cap C \neq \emptyset$. In $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$, if one of $c(w')$ and $c(z)$ is contained in $B \cap C$, then $[w, B, g, w'][z, C, h, z'] = [w, B \cup C, gh, z']$.*

In particular, $\hat{F}_A(\mathcal{D}\mathbf{RG} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$ satisfies $[w, B, g][z, C, h] = [w, B \cup C, gh]$ for every $z \in C^{\mathbb{N}}$.

Proof. Put $p = [w, B, g, w']$ and $q = [z, C, h, z']$. Suppose first that both $c(w')$ and $c(z)$ are contained in $B \cap C$. Also let r be the idempotent $[z, B \cap C, 1, w']$. Then $(rpr)^\omega(rqr)^\omega$ is idempotent since $\mathbf{V} \subseteq \mathcal{L}\mathcal{E}\mathbf{Com}$ and so $(rpr)^\omega(rqr)^\omega = [z, B \cup C, 1, w']$ by Corollary 3.2. Moreover, $p = p(rpr)^\omega$ and $q = (rqr)^\omega q$ by Proposition 3.3. Thus, $pq = p(rpr)^\omega(rqr)^\omega q = p[z, B \cup C, 1, w']q = [w, B \cup C, gh, z']$ again by Proposition 3.3.

Suppose now that, for instance, $c(z) \subseteq B \cap C$ (and not necessarily $c(w') \subseteq B \cap C$) and let $a \in B \cap C$. Then by Proposition 3.3, $q = [z, B \cap C, 1, a^{-\infty}][a^{+\infty}, C, h, z']$. So $pq = p[z, B \cap C, 1, a^{-\infty}][a^{+\infty}, C, h, z'] = [w, B, g, a^{-\infty}][a^{+\infty}, C, h, z'] = [w, B \cup C, gh, z']$ since $c(a^{-\infty}) = c(a^{+\infty}) = \{a\} \subseteq B \cap C$.

The second part of the result is a natural consequence of the first one. □

The second part of this result says that the product of any two regular elements of $\hat{F}_A(\mathcal{D}\mathbf{RG} \cap \mathcal{LE}\mathbf{Com})$ with non-disjoint contents, is a regular element. In the case of the product xy of two regular elements x and y of $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{LE}\mathbf{Com})$ with non-disjoint contents, we only are sure to obtain a regular element if one of $c(x')$ and $c(y')$ is contained in $c(x) \cap c(y)$, where x' and y' are, respectively, the restrictions of x and y to $\mathbf{D}$ and $\mathbf{K}$. As we shall see, only under these conditions will the product xy be a regular element.

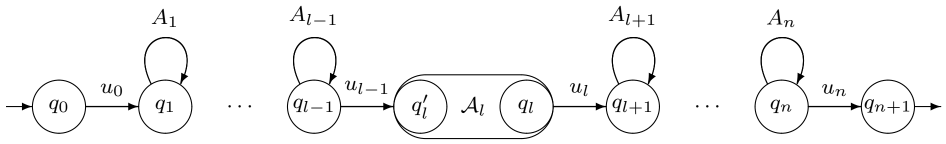
We begin by considering the cases $\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com}$ where $\mathbf{H}$ is a pseudovariety of groups. We say that a factorization of an element $x \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com})$ of the form

$$x = u_0[w_1, A_1, g_1, w_1']u_1 \cdots u_{n-1}[w_n, A_n, g_n, w_n']u_n$$

is *normal* if: $u_i \in A^*$, $u_0 \neq 1$ if $x = u_0$; if u_i $(1 \leq i \leq n-1)$ is the empty word, then the first letter of w_{i+1} does not lie in A_i and $c(w_i') \not\subseteq A_{i+1}$; for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in $c(x_i)$.

Propositions 2.4 and 5.1 guarantee that every element of $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com})$ admits a normal factorization. In order to separate distinct factorizations, we will need some adequate automata which we now describe.

For $n \geq 0$, let $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ be such that $u_i \neq 1$ $(1 \leq i \leq n-1)$. Let $l \in \{1, \ldots, n\}$, let $\mathcal{A}_l$ be a permutation automaton on the alphabet A_l with set of states Q_l and let $q_l, q_l' \in Q_l$. Finally, let $\mathcal{C} = \mathcal{C}(u_0, A_1, u_1, \ldots, \mathcal{A}_l; q_l'; q_l, u_l, \ldots, A_n, u_n)$ be the following automaton.



In order to simplify notations, we denote $Q_i = \{q_i\}$ for all $1 \leq i \leq n$ with $i \neq l$.

Before the proof of a lemma, note that $\mathcal{LE}\mathbf{Com} = [\![e(exe)^\omega(eye)^\omega e = e(eye)^\omega(exe)^\omega e]\!]$.

Lemma 5.2 *Let L be the language recognized by the automaton $\mathcal{C}$ above, and suppose that it satisfies the following extra condition: for each $1 \leq i \leq n-1$, $c(u_i)$ is not contained in either A_i or A_{i+1}. Then $S(L)$ lies in $\mathcal{D}\mathbf{ReG} \cap \mathcal{LE}\mathbf{Com}$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$.*

Moreover, if $w \in A^+$, k is an exponent of $S(L)$ such that $k > |u_0 \cdots u_n| + n$ and w^k is the label of a path in $\mathcal{C}$, then there exists $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$ and the path visits Q_i but does not visit either Q_{i-1} (if $i > 1$) or Q_{i+1} (if $i < n$).

Proof. The second part of the lemma and the fact that $S(L)$ verifies the pseudoidentity $(xy)^\omega(yx)^\omega(xy)^\omega = (xy)^\omega$ defining $\mathcal{D}\mathbf{ReG}$ can be proved as in Lemma 4.2. Now from the remark immediately before the lemma, to show that $S(L)$ lies in $\mathcal{LE}\mathbf{Com}$ it suffices to show that $x^k(x^kyx^k)^k(x^kzx^k)^kx^k \sim_L x^k(x^kzx^k)^k(x^kyx^k)^kx^k$ for all $x, y, z \in A^+$. For this, it suffices to prove that

$x^k(x^kyx^k)^k(x^kzx^k)^kx^k$ is the label of a path $\mathcal{P}$ in $\mathcal{C}$ if and only if there is a path $\mathcal{Q}$ in $\mathcal{C}$ labeled $x^k(x^kzx^k)^k(x^kyx^k)^kx^k$ co-terminal with $\mathcal{P}$.

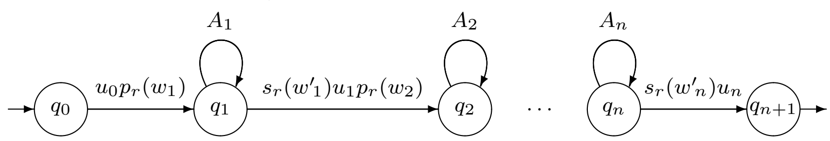
Let $x, y, z \in A^+$, suppose that $\mathcal{P}$ exists and consider the two subpaths $\mathcal{P}_1$ and $\mathcal{P}_2$ of $\mathcal{P}$ labeled, respectively, $x^k(x^kyx^k)^k$ and $(x^kzx^k)^kx^k$. By the second part of the lemma, since $\mathcal{P}$ is a path in $\mathcal{C}$, there are $1 \le i \le j \le n$ such that $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) visits Q_i (resp. Q_j) and does not visit Q_{i-1} nor Q_{i+1} (resp. Q_{j-1} nor Q_{j+1}). Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are consecutive paths, it follows that either $i = j$ or $i+1 = j$. We claim that $i = j$. Indeed, let us suppose that $i+1 = j$. Then $c(x) \cup c(y) \subseteq A_i$, $c(x) \cup c(z) \subseteq A_{i+1}$ and, because of the choice of k, the subpath of $\mathcal{P}$ labeled $v = yx^kx^kz$ is a path from Q_i to Q_{i+1}. Hence, u_i is a factor of v whence it is a factor of one of yx^kx^k and x^kx^kz. But this contradicts the hypothesis on the content of u_i since in that case, $c(u_i) \subseteq A_i$ or $c(u_i) \subseteq A_{i+1}$. Hence $i = j$ and so the existence of path $\mathcal{Q}$ is clear. Indeed, the subpath $\mathcal{P}'$ of $\mathcal{P}$ labeled $(x^kyx^k)^k(x^kzx^k)^k$ is entirely in Q_i. Therefore, if $Q_i = \{q_i\}$ this is immediate. If Q_i is not singular (so that $i = l$ and Q_l is the state set of automaton $\mathcal{A}_l$), we deduce, since k is an exponent of $S(L)$, that $\mathcal{P}'$ is a path in $\mathcal{A}_l$ from a state $q \in Q_l$ to the same state q. We also deduce that there is a path labeled $(x^kzx^k)^k(x^kyx^k)^k$ from q to q. By symmetry, it follows that $x^k(x^kyx^k)^k(x^kzx^k)^kx^k \sim_L x^k(x^kzx^k)^k(x^kyx^k)^kx^k$ proving that $S(L) \in \mathcal{LE}\mathbf{Com}$.

To conclude the proof, consider the syntactic morphism $\mu : A^+ \to S(L)$. Let $w \in A^+$ and suppose that $\mu(w)$ is a regular element of $S(L)$ so that $\mu(w) = \mu(w^{k+1})$. Now let $w' \in A^+$ be such that $\mu(ww'w) = \mu(w)$ and $\mu(w'ww') = \mu(w')$. Then as above, one can show that, for every $1 \le i \le n$, $w \in A_i^+$ if and only if $w' \in A_i^+$. Hence, the subsemigroup of $S(L)$ consisting of its regular elements divides the direct product of the subsemilattice of 2^A generated by the A_i with the semigroups of the form $u'_{i-1}G_iu'_i$, where G_i is the trivial group if $i \neq l$ and is the group $S(\mathcal{A}_l)$ otherwise, and u'_{i-1} (resp. u'_i) is a sufix of u_{i-1} (resp. prefix of u_i) with content contained in A_i. The subgroups of these semigroups are subgroups of $S(\mathcal{A}_l)$ and so the proof is concluded. □

Before we present the characterization of the implicit operations on $\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com}$, we recall the notion of the Cayley graph of a group. Let G be an A-generated group. The *Cayley graph of* G is the labeled graph whose set of vertices is G, and, for every $g \in G$ and $a \in A$, there exists an edge, labeled a, from vertex g to vertex ga.

Theorem 5.3 *Let* $\mathbf{H}$ *be a pseudovariety of groups, let* $x, y \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com})$ *and let* $x = u_0[w_1, A_1, g_1, w'_1]u_1 \cdots u_{n-1}[w_n, A_n, g_n, w'_n]u_n$ *and* $y = v_0[z_1, B_1, h_1, z'_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m, z'_m]v_m$ *be factorizations in normal form. Then* $x = y$ *if and only if* $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$, $g_i = h_i$ *and* $w'_i = z'_i$ *for all* i.

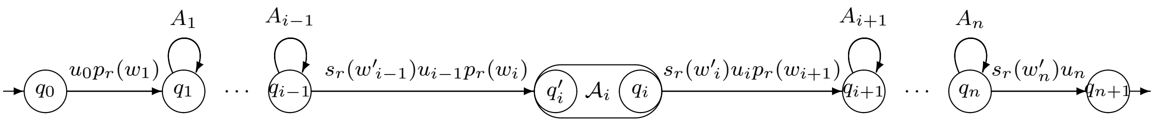
Proof. Consider the following automaton $\mathcal{C}$



where $r \ge 1$ is an integer such that $r > |v_j|$ for all $1 \le j \le m$ and such that, for all $1 \le i \le n-1$ with $u_i = 1$, the content of the word $s_r(w'_i)$ is not contained in A_{i+1}. This guarantees that the content of the word $s_r(w'_i)u_ip_r(w_{i+1})$ is not contained in either A_i or A_{i+1} and that the automaton $\mathcal{C}$ is as in the conditions of Lemma 5.2. Hence, the syntactic semigroup S of the language L recognized by $\mathcal{C}$ is in $\mathcal{D}\mathbf{A} \cap \mathcal{LE}\mathbf{Com}$ and so $S \in \mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com}$. Now using similar (and somewhat simpler) arguments to those in the proof of Theorem 4.3, one can show that $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ and $w'_i = z'_i$ for all i.

Now note that $\mathbf{H} = (\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com}) \cap \mathbf{G}$. For every $1 \leq i \leq n$, let $\bar{w}_i \in A_i^{\mathbb{N}}$, $\bar{w}'_i \in A_i^{-\mathbb{N}}$ and $\bar{g}_i, \bar{h}_i \in \hat{F}_{A_i}(\mathbf{H})$ be such that $[w_i, A_i, g_i, w'_i] = p_r(w_i)[\bar{w}_i, A_i, \bar{g}_i, \bar{w}'_i]s_r(w'_i)$ and $[w_i, A_i, h_i, w'_i] = p_r(w_i)[\bar{w}_i, A_i, \bar{h}_i, \bar{w}'_i]s_r(w'_i)$. Set $\bar{x}_i = [\bar{w}_i, A_i, \bar{g}_i, \bar{w}'_i]$ and $\bar{y}_i = [\bar{w}_i, A_i, \bar{h}_i, \bar{w}'_i]$.

Let us now fix an $i \in \{1, \ldots, n\}$ and consider an A_i-generated group G of $\mathbf{H}$. Let $\mathcal{A}_i$ be the Cayley graph of G over A_i. Note that the transition semigroup of $\mathcal{A}_i$ is G. Let $A_i = \{a_{i,1}, \ldots, a_{i,n_i}\}$ and let $\mathcal{C}'$ be the following automaton



where the states q'_i and q_i are, respectively, the elements 1 and $(\bar{x}_i)_G(a_{i,1}, \ldots, a_{i,n_i})$ of G. Denote by $\mu : A^+ \to S$ the syntactic homomorphism of the language recognized by $\mathcal{C}'$ and by $\hat{\mu}$ its continuous homomorphic extension to $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com})$ (which exists by Lemma 5.2).

Moreover, consider an exponent $k > |u_0 \cdots u_n| + n$ of S and, for all $j \in \{1, \ldots, n\}$, words x_j and y_j such that $c(x_j) = c(y_j) = A_j$, $\hat{\mu}(\bar{x}_j) = \mu(x_j^k)$ and $\hat{\mu}(\bar{y}_j) = \mu(y_j^k)$. We then have

- $\hat{\mu}(u_0p_r(w_1)\bar{y}_1s_r(w'_1)u_1p_r(w_2)\cdots\bar{y}_{i-1}s_r(w'_{i-1})u_{i-1}p_r(w_i)) =$
  $\mu(u_0p_r(w_1)y_1^ks_r(w'_1)u_1p_r(w_2)\cdots y_{i-1}^ks_r(w'_{i-1})u_{i-1}p_r(w_i))$,
- $\hat{\mu}(s_r(w'_i)u_ip_r(w_{i+1})\bar{y}_{i+1}\cdots\bar{y}_ns_r(w'_n)u_n) = \mu(s_r(w'_i)u_ip_r(w_{i+1})y_{i+1}^k\cdots y_n^ks_r(w'_n)u_n)$,
- $\hat{\mu}(x) = \mu(u_0p_r(w_1)x_1^k\cdots x_n^ks_r(w'_n)u_n)$.

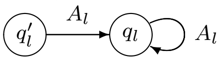
Using the equalities proved so far, one can verify that

- $u_0p_r(w_1)y_1^k\cdots y_{i-1}^ks_r(w'_{i-1})u_{i-1}p_r(w_i)$,
- $s_r(w'_i)u_ip_r(w_{i+1})y_{i+1}^k\cdots y_n^ks_r(w'_n)u_n$,
- $u_0p_r(w_1)x_1^k\cdots x_n^ks_r(w'_n)u_n$,

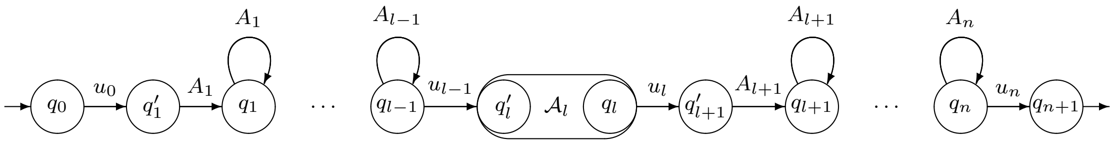
are the labels of paths in $\mathcal{C}'$ from, respectively, q_0 to q'_i, q_i to q_{n+1} and q_0 to q_{n+1}. Since $\hat{\mu}(x) = \hat{\mu}(y) = \hat{\mu}(u_0p_r(w_1)\bar{y}_1\cdots\bar{y}_ns_r(w'_n)u_n)$ and $\mathcal{A}_i$ is a permutation automaton, it follows that $(\bar{y}_i)_G(a_{i,1}, \ldots, a_{i,n_i}) = (\bar{x}_i)_G(a_{i,1}, \ldots, a_{i,n_i})$, which shows that $\bar{g}_i = \bar{h}_i$. Hence, $g_i = h_i$ and the proof is concluded. □

One can verify, similarly to the case $\mathcal{L}\mathbf{J}$ above, that the equality $(\mathbf{R} \vee \mathbf{L}) \cap \mathcal{LE}\mathbf{Com} = \mathcal{D}\mathbf{A} \cap \mathcal{LE}\mathbf{Com}$ does not hold.

Let now $\mathbf{V}$ be one of the pseudovarieties $\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com}$ and $\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com}$. We say that a factorization of an element $x \in \hat{F}_A(\mathbf{V})$ of the form $x = u_0x_1u_1\cdots u_{n-1}x_nu_n$ is *normal* if: $u_i \in A^*$, $u_0 \neq 1$ if $x = u_0$; $x_i \in \hat{F}_A(\mathbf{V})$ $(1 \leq i \leq n)$ is regular; if u_i $(1 \leq i \leq n-1)$ is the empty word, then $c(x_i) \cap c(x_{i+1}) = \emptyset$; for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in $c(x_i)$. Propositions 2.4 and 5.1 guarantee that every element of $\hat{F}_A(\mathbf{V})$ admits a normal factorization.

We begin by describing the semigroups $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com})$. For that we are going to consider the following automata. For $n \geq 0$, let $u_0, \ldots, u_n \in A^*$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$ be such that, if $u_i = 1$ $(1 \leq i \leq n-1)$ then $A_i \cap A_{i+1} = \emptyset$ and, for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i. Let $l \in \{1, \ldots, n\}$ and let $\mathcal{A}_l$ be either the automaton 

, or a non-trivial permutation automaton on the alphabet A_l with set of states Q_l and we let $q'_l, q_l \in Q_l$ be two distinct states. Finally, let $\mathcal{D} = \mathcal{D}(u_0, A_1, u_1, \ldots, u_{l-1}, \mathcal{A}_l; q'_l; q_l, u_l, \ldots, A_n, u_n)$ be the following automaton



An analysis of the structure of $\mathcal{D}$ can be made like in Lemmas 4.2 and 5.2, proving the following lemma.

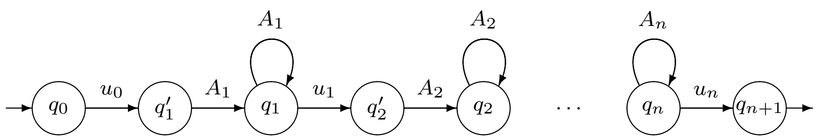
Lemma 5.4 *Consider the automaton $\mathcal{D}$ above and let L be the language recognized by $\mathcal{D}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{G} \cap \mathcal{LE}\mathbf{Com}$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$.*

Moreover, if $w \in A^+$, k is an exponent of $S(L)$ such that $k > |u_0 \cdots u_n| + n$ and w^k is the label of a path in $\mathcal{D}$ not beginning nor ending by a transition labeled by the empty word, then there exists $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$ and that path begins in q'_i or q_i (in Q_l if $i = l$) and ends in q_i (in Q_l if $i = l$). □

Now we are able to prove the following characterization.

Theorem 5.5 *Let $\mathbf{H}$ be a pseudovariety of groups, let $x, y \in \hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com})$ and let $x = u_0[A_1, g_1]u_1 \cdots u_{n-1}[A_n, g_n]u_n$ and $y = v_0[B_1, h_1]v_1 \cdots v_{m-1}[B_m, h_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if $n = m$, $u_i = v_i$, $A_i = B_i$ and $g_i = h_i$ for all i.*

Proof. Consider the following automaton $\mathcal{D}$.



Let L be the language recognized by $\mathcal{D}$ and let $\mu : A^+ \to S$ be its syntactic homomorphism. By Lemma 5.4, $S \in \mathbf{J} \cap \mathcal{LE}\mathbf{Com}$ and so $S \in \mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com}$. So let $\hat{\mu} : \hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com}) \to S$ be the unique continuous homomorphic extension of μ.

Let $k > |u_0 \cdots u_n| + n$ be an exponent of S and let $1 \leq i \leq n$. Since $[A_i, g_i]$ is regular, its image in S, $\hat{\mu}([A_i, g_i])$ is idempotent. By density of A^+ in $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com})$, there is a word x_i such that $c(x_i) = A_i$ and $\hat{\mu}([A_i, g_i]) = \mu(x_i^k)$. Now it is immediate that $w = u_0 x_1^k u_1 \cdots x_n^k u_n$ is a word recognized by $\mathcal{D}$, whence $w \in L$. On the other hand, we have $\hat{\mu}(x) = \mu(w)$.

Consider now words y_i ($1 \leq i \leq m$) such that $c(y_i) = B_i$ and $\hat{\mu}([B_i, h_i]) = \mu({y_i}^k)$. Let $w' = v_0 y_1^k v_1 \cdots y_m^k v_m$. We then have that $\hat{\mu}(y) = \mu(w')$ and, as $x = y$, $\hat{\mu}(x) = \hat{\mu}(y)$. So $w' \in L$. Let $\mathcal{P}$ be a successful path in $\mathcal{D}$ labeled w' and, for each $1 \leq i \leq m$, let $\mathcal{P}_i$ be the subpath of $\mathcal{P}$ labeled $v_0 p_r(z_1) y_1^k v_1 y_2^k \cdots y_i^k$. By Lemma 5.4, the path $\mathcal{P}_i$ ends in state q_{j_i}, for some $1 \leq j_i \leq n$ such that $B_i \subseteq A_{j_i}$. Furthermore, if $j_i > 1$, the subpath $\mathcal{P}'_i$ of $\mathcal{P}_i$ labeled y_i^k, whose first transition is not labeled by the empty word, does not visit state $q_{j_i - 1}$.

In particular, the path $\mathcal{P}_1$ visits state q_1. Therefore u_0 is a prefix of $v_0 y_1^k$. Furthermore, if $j_1 > 1$, the path $\mathcal{P}'_1$ does not visit state $q_{j_1 - 1}$. Hence, if $j_1 > 1$, u_0 is clearly a prefix of v_0. In the case that j_1 equals 1, we have $B_1 \subseteq A_1$ and, as the last letter of u_0 does not lie in A_1 (and so does not belong to the content of y_1^k), we also deduce that u_0 is a prefix of v_0. By symmetry it follows that $u_0 = v_0$ so that $j_1 = 1$ and $B_1 \subseteq A_1$. Again by symmetry we deduce $A_1 = B_1$.

Now the path $\mathcal{P}_2$ ends in state q_{j_2}, for some $1 \leq j_2 \leq n$ such that $B_2 \subseteq A_{j_2}$. Since the first letter of the word $v_1 y_2^k$ does not belong to $B_1 = A_1$ (note that as the factorization of y is normal, if $v_1 = 1$ then $B_1 \cap B_2 = \emptyset$), it follows that $j_2 > 1$. Now as above one can show that $j_2 = 2$, $u_1 = v_1$ and $A_2 = B_2$.

Iterating the above argument, we deduce that $n = m$, $u_i = v_i$ and $A_i = B_i$ for all i.

The proof of the equalities $g_i = h_i$ is similar to the proof of the same equalities in Theorem 5.3. We only point out the fact that after choosing an A_i-generated group G (non-trivial) of $\mathbf{H}$ and considering its Cayley graph $\mathcal{A}_i$ over A_i, we have to choose in $\mathcal{A}_i$ two distinct states. One of them is 1. Now letting $x_i = [A_i, g_i]$ and $y_i = [A_i, h_i]$, if $r = (y_i)_G(a_{i,1}, \ldots, a_{i,n_i})$ and $s = (x_i)_G(a_{i,1}, \ldots, a_{i,n_i})$ are both equal to 1, then there is nothing to prove. So without loss of generality, we may suppose that $s \neq 1$ and so we choose s to be the other state. The proof continues like in Theorem 5.3, proving that r and s must be equal and so g_i and h_i too. $\square$

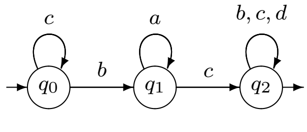
Similar results hold for the pseudovarieties $\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com}$.

Lemma 5.6 *Consider an automaton $\mathcal{C} = \mathcal{C}(u_0, A_1, \ldots, \mathcal{A}_l; q'_l; q_l, \ldots, A_n, u_n)$ as defined after Proposition 5.1 above, satisfying the following extra conditions: for each $1 \leq i \leq n$, the first letter of u_i does not lie in A_i; for each $1 \leq i \leq n-1$ the last letter of u_i lies in A_{i+1} and if $c(u_i) \subseteq A_{i+1}$, then $A_i \cap A_{i+1} = \emptyset$. Let L be the language recognized by $\mathcal{C}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{RG} \cap \mathcal{LE}\mathbf{Com}$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$.*

Moreover, if $w \in A^+, k > |u_0 \cdots u_n| + n$ is an exponent of $S(L)$ and w^k is the label of a path $\mathcal{T}$ in $\mathcal{C}$, then there exists $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$ and $\mathcal{T}$ ends in Q_i. Furthermore, either $\mathcal{T}$ does not visit Q_{i-1} (if $i > 1$), or $\mathcal{T}$ begins in Q_{i-1} and it leaves Q_{i-1} at the first step.

Proof. The lemma can be proved like other similar results. We only note that a path labeled $v = (x^k y x^k)^k (x^k z x^k)^k$ stays for at least one step in at most one Q_i. So $c(xyz) \subseteq A_i^+$ and the path stays out of q_i for at most $|u_{i-1}|$ steps. $\square$

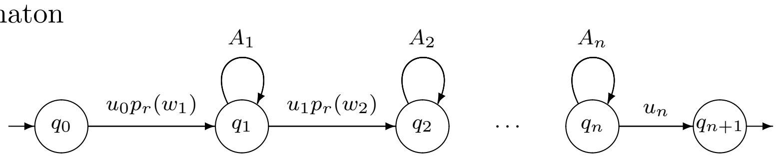
Example 5.7 *Notice that the condition "for each $1 \leq i \leq n-1$ the last letter of u_i lies in A_{i+1}" on the automaton $\mathcal{C}$ of last lemma avoids, for instance, a situation like this*



where a, b, c and d are distinct letters of an alphabet A. The syntactical semigroup of the language recognized by this automaton does not verify the equality $(c^\omega bc^\omega)^\omega(c^\omega dc^\omega)^\omega = (c^\omega dc^\omega)^\omega(c^\omega bc^\omega)^\omega$ *and so does not lie in* $\mathcal{LE}\mathbf{Com}$*.* ■

Theorem 5.8 *Let* $\mathbf{H}$ *be a pseudovariety of groups, let* $x, y \in \hat{F}_A(\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com})$ *and let* $x = u_0[w_1, A_1, g_1]u_1 \cdots [w_n, A_n, g_n]u_n$ *and* $y = v_0[z_1, B_1, h_1]v_1 \cdots [z_m, B_m, h_m]v_m$ *be factorizations in normal form. Then* $x = y$ *if and only if* $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ *and* $g_i = h_i$ *for all* i*.*

Proof. Using the canonical projection of $\hat{F}_A(\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com})$ onto $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com})$, we deduce immediately that $n = m$, $u_i = v_i$, $A_i = B_i$ and $g_i = h_i$ for all i. To prove the equality $p_r(w_i) = p_r(z_i)$ for each $1 \le i \le n$ and each $r \ge 1$, it suffices to consider the following automaton



— which recognizes a language whose syntactical semigroup belongs to $\mathbf{R} \cap \mathcal{LE}\mathbf{Com}$, by Lemma 5.6 —, and to proceed like in the proof of Theorem 5.3. This shows that $w_i = z_i$ for all i. □

A dual result is valid for the pseudovarieties $\mathcal{D}\mathbf{LH} \cap \mathcal{LE}\mathbf{Com}$.

6 Implicit operations on $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$

This section is devoted to the study of the semigroups $\hat{F}_A(\mathbf{V})$ where $\mathbf{V}$ is a pseudovariety of the form $\mathbf{V} = \mathbf{W} \cap \mathcal{L}\mathbf{ZE}$ or $\mathbf{V} = \mathbf{W} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$ with $\mathbf{W} \in \{\mathcal{D}\mathbf{ReH}, \mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$. As in the previous sections, we describe "normal" forms for the elements of $\hat{F}_A(\mathbf{V})$ and prove that they are unique. We apply these results to the computation of certain joins.

Let $\mathbf{CR} = [\![x^{\omega+1} = x]\!]$ be the pseudovariety of completely regular semigroups, i.e., semigroups whose $\mathcal{H}$-classes are all groups. It is well-known that the pseudovariety $\mathbf{ZE} \cap \mathbf{CR} = \mathbf{Sl} \vee \mathbf{G}$ and that $\mathbf{ZE} \subseteq \mathcal{E}\mathbf{Com}$. Therefore, it follows immediately that $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) \subseteq \mathcal{L}\mathbf{ZE} \subseteq \mathcal{LE}\mathbf{Com}$.

In the last section, where we studied the semigroups of the form $\hat{F}_A(\mathbf{W} \cap \mathcal{LE}\mathbf{Com})$, we had to separate the description of the normal factorizations into two cases. The case $\mathcal{D}\mathbf{ReH} \cap \mathcal{LE}\mathbf{Com}$ on the one hand and the cases $\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com}$ and $\mathcal{D}\mathbf{H} \cap \mathcal{LE}\mathbf{Com}$ on the other. As we shall see, this is not necessary for the semigroups $\hat{F}_A(\mathbf{W} \cap \mathcal{L}\mathbf{ZE})$ (nor for $\hat{F}_A(\mathbf{W} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))$). The normal factorizations $x = u_0x_1u_1 \cdots x_nu_n$ described for elements $x \in \hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE})$ will be such that, if $\pi : \hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE}) \to \hat{F}_A(\mathbf{W} \cap \mathcal{L}\mathbf{ZE})$ is the canonical projection, then $\pi(x) = u_0\pi(x_1)u_1 \cdots \pi(x_n)u_n$ is a normal factorization of the element $\pi(x) \in \hat{F}_A(\mathbf{W} \cap \mathcal{L}\mathbf{ZE})$. The definition of normal factorization will be "inspired" by the following result.

Proposition 6.1 *Let* A *be an alphabet, let* $B, C \subseteq A$ *be such that* $B \cap C \neq \emptyset$ *and let* $x \in \hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE})^1$*. Then, in* $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE})$*,*

$$[w, B, g, w']x[z, C, h, z'] = [w, B \cup C, g, w']x[z, B \cup C, h, z'].$$

In particular, $[w, B, g, w'][z, C, h, z'] = [w, B \cup C, gh, z']$*.*

Furthermore, if $x \in \hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))^1$*, then in* $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))$ *we have that* $[w, B, g, w']x[z, C, h, z'] = [w, B \cup c(x) \cup C, gx'h, z']$ *where* x' *is the restriction of* x *to* $\mathbf{G}$.

Proof. Let us first suppose that $C \subseteq B$ and put $p = [w, B, g, w']$ and $q = [z, C, h, z']$. We have $p = pp^\omega(q^\omega p^\omega q^\omega)^\omega q^\omega p^\omega$ and $q = q^\omega q$ from Proposition 3.3. So

$$pxq = pp^\omega(q^\omega p^\omega q^\omega)^\omega q^\omega p^\omega x q^\omega q = pp^\omega q^\omega p^\omega x q^\omega (q^\omega p^\omega q^\omega)^\omega q$$

since $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE} \subseteq \mathcal{L}\mathbf{ZE}$. Hence, again from Proposition 3.3, it follows that $pxq = [w, B, g, w']x[z, B, h, z']$. It can be proved analogously that the result also holds when $B \subseteq C$.

Let us now prove the particular case $x = 1$. Let $a \in B \cap C$. Then $p = pa^\omega a^\omega p^\omega$ from Proposition 3.3. Hence, $pq = pa^\omega a^\omega p^\omega q = p[a^{+\infty}, C, 1, a^{-\infty}][a^{+\infty}, B, 1, a^{-\infty}]p^\omega q$ from the above since $c(a^\omega) = \{a\} \subseteq B \cap C$. Similarly, one can prove that the product $[a^{+\infty}, C, 1, a^{-\infty}][a^{+\infty}, B, 1, a^{-\infty}]$ is an idempotent element. So by Corollary 3.2, it is equal to $[a^{+\infty}, B \cup C, 1, a^{-\infty}]$. The equality $pq = [w, B \cup C, gh, z']$ is now a simple consequence of Proposition 3.3.

Finally, we prove the general case. We have, $p = pa^\omega p^\omega$ and $q = q^\omega a^\omega q$. Therefore, from the particular cases proved above, we deduce

$$\begin{aligned} pxq &= pa^\omega p^\omega x q^\omega a^\omega q \\ &= p[a^{+\infty}, C, 1, a^{-\infty}]p^\omega x q^\omega [a^{+\infty}, B, 1, a^{-\infty}]q \\ &= p[a^{+\infty}, B \cup C, 1, w']x[z, B \cup C, 1, a^{-\infty}]q \\ &= [w, B \cup C, g, w']x[z, B \cup C, h, z']. \end{aligned}$$

To conclude the proof it remains to show the result for $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. With the same notations as above, we have $pxq = pa^\omega p^\omega x q^\omega a^\omega q$. Now $a^\omega p^\omega x q^\omega a^\omega$ is a group element since $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) \subseteq \mathcal{L}\mathbf{CR}$. So it is a regular element of content $B \cup c(x) \cup C$ and the result follows from Proposition 3.3. □

Almeida and Weil [8] proved that, for each pseudovariety $\mathbf{H}$ of groups, the join $\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}$ is strictly contained in $\mathcal{D}\mathbf{ReH}$. Indeed, they proved that

$$\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH} = [\![(xy)^\omega x^\omega (zx)^\omega = (xy)^\omega (zx)^\omega]\!] \cap \mathcal{D}\mathbf{ReH}.$$

However, when intersected with $\mathcal{L}\mathbf{ZE}$, we obtain an equality, which is an easy consequence of Proposition 6.1.

Corollary 6.2 *For each pseudovariety of groups* $\mathbf{H}$*, the equality* $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap \mathcal{L}\mathbf{ZE} = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ *holds.*

Proof. The inclusion $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap \mathcal{L}\mathbf{ZE} \subseteq \mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ is clear. Now to prove the inverse inclusion, it suffices to prove that $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ satisfies the pseudoidentity $(xy)^\omega x^\omega (zx)^\omega = (xy)^\omega (zx)^\omega$. For that, it suffices to show that this equality holds in $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE})$ when x, y and z are letters of A. In $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE})$, $(xy)^\omega = [(xy)^{+\infty}, \{x, y\}, 1, (xy)^{-\infty}]$ and $(zx)^\omega = [(zx)^{+\infty}, \{x, z\}, 1, (zx)^{-\infty}]$. Hence, it follows from Proposition 6.1, that $(xy)^\omega x^\omega (zx)^\omega = [(xy)^{+\infty}, \{x, y, z\}, 1, (zx)^{-\infty}] = (xy)^\omega (zx)^\omega$. □

Another consequence of Proposition 6.1 is that if $x = u_0x_1u_1\cdots u_{n-1}x_nu_n$ is a factorization of an element x of $\hat{F}_A(\mathcal{D}\mathbf{ReG}\cap\mathcal{L}\mathbf{ZE})$ (resp. $\hat{F}_A(\mathcal{D}\mathbf{ReG}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G}))$) in terms of words u_i and regular elements x_i, then we may suppose that the contents of the regular elements are pairwise equal or disjoint (resp. pairwise disjoint). So we will consider the following notion of normal factorization for the implicit operations on pseudovarieties of the form $\mathbf{V} = \mathbf{W}\cap\mathcal{L}\mathbf{ZE}$ (resp. $\mathbf{V} = \mathbf{W}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$), with $\mathbf{W} \in \{\mathcal{D}\mathbf{ReH}, \mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$.

In the cases where $\mathbf{V} = \mathbf{W}\cap\mathcal{L}\mathbf{ZE}$, we say that a factorization of an element $x \in \hat{F}_A(\mathbf{V})$ of the form $x = u_0x_1u_1\cdots u_{n-1}x_nu_n$ is *normal* if:

- $u_i \in A^*$, $u_0 \neq 1$ if $x = u_0$;
- $x_i \in \hat{F}_A(\mathbf{V})$ $(1 \leq i \leq n)$ is regular;
- for each pair $1 \leq i,j \leq n$, $c(x_i) = c(x_j)$ or $c(x_i)\cap c(x_j) = \emptyset$;
- if u_i $(1 \leq i \leq n-1)$ is the empty word, then $c(x_i)\cap c(x_{i+1}) = \emptyset$;
- for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in $c(x_i)$.

In the cases where $\mathbf{V} = \mathbf{W}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$, the definition of *normal* factorization of an element x of $\hat{F}_A(\mathbf{V})$ is the same as that above except that the condition "for each pair $1 \leq i,j \leq n$, $c(x_i) = c(x_j)$ or $c(x_i)\cap c(x_j) = \emptyset$" is replaced by the condition "for each pair $1 \leq i,j \leq n$ with $i \neq j$, $c(x_i)\cap c(x_j) = \emptyset$".

Note that these definitions of normal factorization differ from that of elements of $\hat{F}_A(\mathbf{W}\cap\mathcal{L}\mathcal{E}\mathbf{Com})$, with $\mathbf{W} \in \{\mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$, only in the imposition of the following condition (putting $c(x_i) = A_i$)

$$A_i = A_j \text{ or } A_i\cap A_j = \emptyset \text{ for each pair } 1 \leq i,j \leq n \tag{1}$$

for the cases $\hat{F}_A(\mathbf{W}\cap\mathcal{L}\mathbf{ZE})$ and of the condition

$$A_i\cap A_j = \emptyset \text{ for each pair } i \neq j \tag{2}$$

for the cases $\hat{F}_A(\mathbf{W}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G}))$.

Proposition 6.3 *Let* $\mathbf{V} = \mathbf{W}\cap\mathcal{L}\mathbf{ZE}$ *or* $\mathbf{V} = \mathbf{W}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$*, where* $\mathbf{W}$ *is one of* $\mathcal{D}\mathbf{ReH}$*,* $\mathcal{D}\mathbf{RH}$ *and* $\mathcal{D}\mathbf{H}$*. Every element of* $\hat{F}_A(\mathbf{V})$ *admits a normal factorization.* □

With this notion of normal factorization, a study entirely similar to that conducted for the subpseudovarieties of $\mathcal{D}\mathbf{ReG}\cap\mathcal{L}\mathcal{E}\mathbf{Com}$ can be made for the subpseudovarieties of $\mathcal{D}\mathbf{ReG}\cap\mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{ReG}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$, leading to the following results. They will be presented usually without proofs because they are analogous to other, similar results.

The automata used to separate distinct factorizations of elements of a semigroup of the form $\hat{F}_A(\mathcal{D}\mathbf{ReH}\cap\mathcal{L}\mathbf{ZE})$ (resp. $\hat{F}_A(\mathcal{D}\mathbf{ReH}\cap\mathcal{L}(\mathbf{Sl}\vee\mathbf{G}))$) are the automata $\mathcal{C} = \mathcal{C}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$ as in Lemma 5.2 where, of course, the A_i's must satisfy condition (1) (resp. condition (2)).

Lemma 6.4 *Let* $\mathcal{C} = \mathcal{C}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l', \ldots, A_n, u_n)$ *be an automaton as defined after Proposition 5.1 above, satisfying the extra conditions:* $A_i = A_j$ *or* $A_i\cap A_j = \emptyset$ *for all* $1 \leq i,j \leq n$ *and, for each* $1 \leq i \leq n-1$*,* $c(u_i)$ *is not contained in either* A_i *or*

A_{i+1}. Let L be the language recognized by $\mathcal{C}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{ReG} \cap \mathcal{L}\mathbf{ZE}$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$. If $A_i \cap A_j = \emptyset$ for each pair $i \neq j$, then $S(L) \in \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$.

Moreover, if $w \in A^+, k > |u_0 \cdots u_n| + n$ is an exponent of $S(L)$ and w^k is the label of a path in $\mathcal{C}$, then there exists $i \in \{1, \ldots, n\}$ such that $w \in A_i^+$ and this path visits Q_i but does not visit either Q_{i-1} (if $i > 1$) or Q_{i+1} (if $i < n$).

Proof. We only recall the proof that $S(L)$ lies in $\mathcal{L}\mathbf{ZE}$. Admitting that $S(L) \in \mathcal{D}\mathbf{ReG}$ is already proved, to show that $S(L) \in \mathcal{L}\mathbf{ZE}$ it suffices, as in Lemma 5.2, to show that $x^k(x^kyx^k)^kx^kzx^k \sim_L x^kzx^k(x^kyx^k)^kx^k$ for all $x, y, z \in A^+$, or more generally that

> $x^k(x^kyx^k)^kx^kzx^k$ is the label of a path $\mathcal{P}$ in $\mathcal{C}$ if and only if there is a path $\mathcal{P}'$ in $\mathcal{C}$ labeled $x^kzx^k(x^kyx^k)^kx^k$ and co-terminal with $\mathcal{P}$.

Let $x, y, z \in A^+$ and suppose that $\mathcal{P}$ exists. Consider the three subpaths $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ of $\mathcal{P}$ labeled, respectively $x^k(x^kyx^k)^kx^k$, z and x^k. To prove the existence of $\mathcal{P}'$ it suffices to show that there are paths $\mathcal{P}_1'$ and $\mathcal{P}_3'$ labeled x^k and $x^k(x^kyx^k)^kx^k$, respectively, co-terminal with $\mathcal{P}_1$ and $\mathcal{P}_3$. Since $\mathcal{P}$ is a path, there are $1 \leq i \leq j \leq n$ such that $c(x) \cup c(y) \subseteq A_i$, $c(x) \subseteq A_j$ and i and j are, respectively, the least and the greatest indices such that $\mathcal{P}$ visits Q_i and Q_j. In particular, $A_i \cap A_j \neq \emptyset$ and consequently $A_i = A_j$. Furthermore, the path $\mathcal{P}_1$ does not visit either Q_{i-1} or Q_{i+1}. So it is clear that $\mathcal{P}_1'$ exists. Analogously, since the path $\mathcal{P}_3$ does not visit either Q_{j-1} or Q_{j+1} and $c(x) \cup c(y) \subseteq A_j$, $\mathcal{P}_3'$ exists, proving the existence of $\mathcal{P}'$. By symmetry, we deduce that $x^k(x^kyx^k)^kx^kzx^k \sim_L x^kzx^k(x^kyx^k)^kx^k$ whence $S(L) \in \mathcal{L}\mathbf{ZE}$. □

Theorem 6.5 *Let $\mathbf{H}$ be a pseudovariety of groups and let $\mathbf{V}$ be one of $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. Let $x, y \in \hat{F}_A(\mathbf{V})$ and let $x = u_0[w_1, A_1, g_1, w'_1]u_1 \cdots u_{n-1} [w_n, A_n, g_n, w'_n]u_n$ and $y = v_0[z_1, B_1, h_1, z'_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m, z'_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$, $g_i = h_i$ and $w'_i = z'_i$ for all i.* □

Similar results hold for the pseudovarieties $\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}$ (and $\mathcal{D}\mathbf{RH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$). To separate two distinct normal factorizations with respect to $\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}$ it suffices to consider again the automata $\mathcal{C} = \mathcal{C}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$, defined after Proposition 5.1 and impose the adequate conditions.

Lemma 6.6 *Let $\mathcal{C} = \mathcal{C}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$ be an automaton, as defined after Proposition 5.1, satisfying the extra conditions: for all $1 \leq i, j \leq n$, $A_i = A_j$ or $A_i \cap A_j = \emptyset$, the first letter of u_i does not lie in A_i and, for each $1 \leq i \leq n-1$, if $c(u_i) \subseteq A_{i+1}$, then $A_i \cap A_{i+1} = \emptyset$. Let L be the language recognized by $\mathcal{C}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{RG} \cap \mathcal{L}\mathbf{ZE}$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$. If $A_i \cap A_j = \emptyset$ for each pair $i \neq j$, then $S(L) \in \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$.* □

Observe that the automaton $\mathcal{C}$ in this last lemma is not obtained from the automaton of Lemma 5.6 by the imposition of the extra condition (1). Indeed, the automaton of Lemma 5.6 satisfies also the condition "for each $1 \leq i \leq n-1$ the last letter of u_i lies in A_{i+1}". Note that the syntactical semigroup of the language recognized by the automaton of Example 5.7 does not lie in $\mathcal{L}\mathbf{ZE}$. This happens because the automaton does not verify the condition "$A_i = A_j$ or $A_i \cap A_j = \emptyset$ for all $1 \leq i, j \leq n$" of last lemma.

Theorem 6.7 *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $\mathbf{V}$ *be one of* $\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}$ *and* $\mathcal{D}\mathbf{RH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. *Let* $x, y \in \hat{F}_A(\mathbf{V})$ *and let* $x = u_0[w_1, A_1, g_1]u_1 \cdots u_{n-1}[w_n, A_n, g_n]u_n$ *and* $y = v_0[z_1, B_1, h_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m]v_m$ *be factorizations in normal form. Then* $x = y$ *if and only if* $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ *and* $g_i = h_i$ *for all* i. □

In Corollary 6.2, we proved that $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap \mathcal{L}\mathbf{ZE} = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ (and so also $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$). Now note that from Theorem 6.5 and from the last theorem and its analogue for $\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE}$, if x and y are two elements of $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE})$, then x and y are equal if and only if their restrictions to both $\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE}$ are equal. A similar argument is valid for $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. So, from Reiterman's Theorem, we have also the following equalities.

Corollary 6.8 *Let* $\mathbf{H}$ *be a pseudovariety of groups. Then*

- $(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}) \vee (\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE}) = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$;
- $(\mathcal{D}\mathbf{RH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})) \vee (\mathcal{D}\mathbf{LH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})) = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. □

In order to conclude the study of this section, we consider now the cases $\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathbf{ZE}$ and $\mathcal{D}\mathbf{H} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. The automata used to separate distinct factorizations of elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathbf{ZE})$ (resp. $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))$) are the automata $\mathcal{D} = \mathcal{D}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$ as in Lemma 5.4, where the A_i's satisfy also the condition (1) (resp. condition (2)).

Lemma 6.9 *Consider the automaton* $\mathcal{D} = \mathcal{D}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$ *as in Lemma 5.4 with* $A_i = A_j$ *or* $A_i \cap A_j = \emptyset$ *for each pair* $1 \leq i, j \leq n$. *Let* L *be the language recognized by the automaton* $\mathcal{D}$. *Then* $S(L)$ *lies in* $\mathcal{D}\mathbf{G} \cap \mathcal{L}\mathbf{ZE}$ *and its subgroups lie in the pseudovariety generated by the transition group* $S(\mathcal{A}_l)$. *If* $A_i \cap A_j = \emptyset$ *for each pair* $i \neq j$, *then* $S(L) \in \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. □

Theorem 6.10 *Let* $\mathbf{H}$ *be a pseudovariety of groups and let* $\mathbf{V}$ *be one of* $\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathbf{ZE}$ *and* $\mathcal{D}\mathbf{H} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$. *Let* $x, y \in \hat{F}_A(\mathbf{V})$ *and let* $x = u_0[A_1, g_1]u_1 \cdots u_{n-1}[A_n, g_n]u_n$ *and* $y = v_0[B_1, h_1]v_1 \cdots v_{m-1}[B_m, h_m]v_m$ *be factorizations in normal form. Then* $x = y$ *if and only if* $n = m$, $u_i = v_i$, $A_i = B_i$ *and* $g_i = h_i$ *for all* i. □

In the case of the pseudovarieties involving $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$, we can also deduce the following join decompositions.

Corollary 6.11 *Let* $\mathbf{H}$ *be a pseudovariety of abelian groups. Then*

- $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) = (\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}) \vee \mathbf{H}$;
- $\mathcal{D}\mathbf{RH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) = (\mathbf{R} \cap \mathcal{L}\mathbf{Sl}) \vee \mathbf{H}$;
- $\mathcal{D}\mathbf{H} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) = (\mathbf{J} \cap \mathcal{L}\mathbf{Sl}) \vee \mathbf{H}$.

To prove this result we will use the following known result.

Proposition 6.12 *Let* $A = \{a_1, \ldots, a_n\}$ *be an alphabet and let* $\mathbf{V}$ *be a pseudovariety of commutative semigroups. Then* $\hat{F}_A(\mathbf{V})^1$ *is isomorphic to the direct product* $\hat{F}_{\{a_1\}}(\mathbf{V})^1 \times \ldots \times \hat{F}_{\{a_n\}}(\mathbf{V})^1$. □

Proof of Corollary 6.11. We give, for instance, the proof of the first equality. Let $x, y \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))$. From Reiterman's Theorem, it suffices to prove that if $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$ and $\mathbf{H}$ satisfy $x = y$, then $x = y$. So let $x = u_0[w_1, A_1, g_1, w'_1]u_1 \cdots u_{n-1} [w_n, A_n, g_n, w'_n]u_n$ and $y = v_0[z_1, B_1, h_1, z'_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m, z'_m]v_m$ be factorizations in normal form and suppose that $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$ and $\mathbf{H}$ satisfy $x = y$. The restrictions of x and y to $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$ are, respectively, $u_0[w_1, A_1, w'_1]u_1 \cdots [w_n, A_n, w'_n]u_n$ and $v_0[z_1, B_1, z'_1]v_1 \cdots [z_m, B_m, z'_m]v_m$ and these factorizations are in normal form. Then by Theorem 6.5, we deduce immediately $n = m$, $u_i = v_i$, $w_i = z_i$, $A_i = B_i$ and $w'_i = z'_i$ for all i.

Now since $\mathbf{H} = (\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})) \cap \mathbf{G}$ and $\mathbf{H}$ satisfies $x = y$, we have in $\hat{F}_A(\mathbf{H})$, $u_0 g_1 u_1 g_2 \cdots u_{n-1} g_n u_n = v_0 h_1 v_1 h_2 \cdots v_{n-1} h_n v_n$. From the above we know that $u_i = v_i$ for all i. Hence, by commutativity and cancellativity of $\hat{F}_A(\mathbf{H})$ we deduce, $g_1 g_2 \cdots g_n = h_1 h_2 \cdots h_n$. Therefore, since the contents of the regular elements are pairwise disjoint, we deduce from Proposition 6.12 that $g_i = h_i$ for all i. This shows that $x = y$ and concludes the proof. □

Notice that a similar result does not hold for $\mathcal{L}\mathbf{ZE}$. For instance, the equality $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE} = (\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{ZE}) \vee \mathbf{H}$ does not hold for any non-trivial pseudovariety of groups $\mathbf{H}$. Indeed, let $x, y \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE})$ be such that $c(x) \cap c(y) = \emptyset$ and x be a regular element that is not idempotent. Then both $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{ZE}$ and $\mathbf{H}$ satisfy $xy^\omega x^\omega = x^\omega y^\omega x$ but, by Theorem 6.5, $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ does not.

Note that $\mathbf{V} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}) = \mathbf{V} \cap \mathcal{L}\mathbf{Sl}$ for any aperiodic pseudovariety $\mathbf{V}$. The aperiodic cases $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$, $\mathbf{R} \cap \mathcal{L}\mathbf{Sl}$ and $\mathbf{J} \cap \mathcal{L}\mathbf{Sl}$, considered in this section, are the object of the author's article [12].

7 Implicit operations on $\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D})$

This section is concerned with the structure of the semigroups of implicit operations on subpseudovarieties $\mathbf{V}$ of $\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D})$ of the form $\mathbf{V} = \mathbf{W} \cap (\mathbf{Com} * \mathbf{D})$ with $\mathbf{W}$ as usual. Remark that $\mathbf{Com} * \mathbf{D}$ is a subpseudovariety of $\mathcal{L}\mathbf{ZE}$. Indeed, it is clear that $\mathbf{Com} * \mathbf{D} \subseteq \mathcal{L}\mathbf{Com} \subseteq \mathcal{L}\mathbf{ZE}$. Once again we describe "normal" factorizations, in terms of words and regular elements, for the elements of $\hat{F}_A(\mathbf{V})$. Contrary to the cases considered so far, these factorizations are not necessarily unique. However, we prove that given two elements of $\hat{F}_A(\mathbf{V})$, written in such a "normal" form, we can decide if they are equal or not.

Naturally, the definition of normal form for an element of $\hat{F}_A(\mathbf{W} \cap (\mathbf{Com} * \mathbf{D}))$ is obtained from the same notion for an element of $\hat{F}_A(\mathbf{W} \cap \mathcal{L}\mathbf{ZE})$ making a small adjustment, "dictated" by the following result.

Proposition 7.1 *Let A be an alphabet, let $x, y, z \in \hat{F}_A(\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D}))$*[1] *and let $B, C \subseteq A$. Then, in $\hat{F}_A(\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D}))$,*

(1) $[w_1, B, g_1, w'_1]x[w_2, B, g_2, w'_2] = [w_1, B, g_1 g_2, w'_1]x[w_2, B, 1, w'_2] =$
$[w_1, B, 1, w'_1]x[w_2, B, g_1 g_2, w'_2]$;

(2) $[w_1, B, g_1, w'_1]x[z_1, C, h_1, z'_1]y[w_2, B, g_2, w'_2]z[z_2, C, h_2, z'_2] =$
$[w_1, B, g_1, w'_2]z[z_2, C, h_1, z'_1]y[w_2, B, g_2, w'_1]x[z_1, C, h_2, z'_2]$.

In particular, letting $[z_1, C, h_1, z'_1] = [w_2, B, 1, w'_2]$, $[z_2, C, h_2, z'_2] = [w_3, B, g_3, w'_3]$ *and* $y = 1$, *we see that*

$$[w_1, B, g_1, w'_1]x[w_2, B, g_2, w'_2]z[w_3, B, g_3, w'_3] =$$
$$[w_1, B, g_1, w'_2]z[w_3, B, g_2, w'_1]x[w_2, B, g_3, w'_3].$$

Proof. Put $\mathbf{V} = \mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D})$, $p_i = [w_i, B, g_i, w'_i]$ and $q_i = [z_i, C, h_i, z'_i]$ for $i = 1, 2$. We have $p_1 = p_1 p_1^\omega p_2^\omega p_1^\omega$ and $p_2 = p_2^\omega p_2 p_2^\omega$ from Proposition 3.3, since $c(p_1) = c(p_2) = B$. Therefore, $p_1 x p_2 = p_1 p_1^\omega \boldsymbol{p_2^\omega} \underline{p_1^\omega x} \boldsymbol{p_2^\omega} \underline{p_2} \boldsymbol{p_2^\omega} = p_1 p_1^\omega \boldsymbol{p_2^\omega} \underline{p_2} \boldsymbol{p_2^\omega} \underline{p_1^\omega x} \boldsymbol{p_2^\omega}$ since $\mathbf{V} \subseteq \mathcal{L}\mathbf{Com}$. It follows that $p_1 x p_2 = [w_1, B, g_1 g_2, w'_1]x[w_2, B, 1, w'_2]$. The second equality of (1) can be proved symmetrically. Now we have

$$\begin{aligned}
p_1 x q_1 y p_2 z q_2 &= p_1 \boldsymbol{p_1^\omega} \underline{x} \boldsymbol{q_1^\omega} q_1 y p_2 \boldsymbol{p_1^\omega} \underline{p_2^\omega z q_2^\omega} \boldsymbol{q_1^\omega} q_2^\omega q_2 \\
&\qquad\qquad \text{since } c(p_1) = c(p_2) = B \text{ and } c(q_1) = c(q_2) = C \\
&= p_1 \boldsymbol{p_1^\omega} \underline{p_2^\omega z q_2^\omega} \boldsymbol{q_1^\omega} q_1 y p_2 \boldsymbol{p_1^\omega} \underline{x} \boldsymbol{q_1^\omega} q_2^\omega q_2 \qquad \text{since } \mathbf{V} \subseteq \mathbf{Com} * \mathbf{D} \\
&= [w_1, B, g_1, w'_2]z[z_2, C, h_1, z'_1]y[w_2, B, g_2, w'_1]x[z_1, C, h_2, z'_2]. \square
\end{aligned}$$

If $\mathbf{V}$ is one of the pseudovarieties $\mathbf{W} \cap (\mathbf{Com} * \mathbf{D})$, with $\mathbf{W} \in \{\mathcal{D}\mathbf{ReH}, \mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$, we say that a factorization $x = u_0 x_1 u_1 \cdots u_{n-1} x_n u_n$ of an element $x \in \hat{F}_A(\mathbf{V})$ is *normal* if:

- $u_i \in A^*$, $u_0 \neq 1$ if $x = u_0$;
- $x_i \in \hat{F}_A(\mathbf{V})$ $(1 \leq i \leq n)$ is regular;
- for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in $c(x_i)$;
- for each pair $1 \leq i, j \leq n$, $c(x_i) = c(x_j)$ or $c(x_i) \cap c(x_j) = \emptyset$;
- if u_i $(1 \leq i \leq n-1)$ is the empty word, then $c(x_i) \cap c(x_{i+1}) = \emptyset$;
- if $c(x_i) = c(x_j)$ for some $i \neq j$, then one of x_i and x_j is idempotent (i.e., with the same content, there is at most one regular element that is not idempotent).

Note that this definition of normal factorization differs from that of elements of $\hat{F}_A(\mathbf{W} \cap \mathcal{L}\mathbf{ZE})$ only in the imposition of the following condition:

$$\text{if } c(x_i) = c(x_j) \text{ for some } i \neq j, \text{ then one of } x_i \text{ and } x_j \text{ is idempotent.} \qquad (3)$$

The imposition of this condition is a natural consequence of point (1) of Proposition 7.1.

Proposition 7.2 *Let* $\mathbf{V} = \mathbf{W} \cap (\mathbf{Com} * \mathbf{D})$ *with* $\mathbf{W} \in \{\mathcal{D}\mathbf{ReH}, \mathcal{D}\mathbf{RH}, \mathcal{D}\mathbf{H}\}$. *Every element of* $\hat{F}_A(\mathbf{V})$ *admits a normal factorization.* $\square$

Example 7.3 *Let* $\mathbf{V}$ *be one of the pseudovarieties* $\mathbf{W} \cap (\mathbf{Com} * \mathbf{D})$. *We note that a normal factorization of an element* $x \in \hat{F}_A(\mathbf{V})$ *is not necessarily unique. For instance, suppose that* x *is of the form* $x = y^\omega a z^\omega a y^\omega a^2 z^\omega$ *for some* $y, z \in \hat{F}_A(\mathbf{V})$ *and* $a \in A$ *such that* $c(y) \cap c(z) = \emptyset$ *and* $a \notin c(y), c(z)$. *Then this factorization of* x *is in normal form and, since* $\mathbf{V} \subseteq \mathbf{Com} * \mathbf{D}$, $x = y^\omega a^2 z^\omega a y^\omega a z^\omega$ *is another normal factorization of* x. ■

The first semigroups of implicit operations to be described in this section will be the semigroups $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$. Note that for any pseudovariety $\mathbf{H}$ of groups, $(\mathbf{Com} * \mathbf{D}) \cap \mathbf{H} = \mathbf{H} \cap \mathbf{Ab}$, where $\mathbf{Ab}$ is the pseudovariety of all abelian groups. So in this section, it suffices to consider $\mathbf{H}$ a pseudovariety of abelian groups. We will see, given two elements $x = u_0 x_1 u_1 \cdots x_n u_n$ and $y = v_0 y_1 v_1 \cdots y_m v_m$ of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$, written in normal form, that to decide whether x and y are the same element, it does not suffice to look at each factor u_i and x_i (and v_j and y_j) by itself. We also have to compare the factors of x and y of the form $x_i u_i x_{i+1}$ and $y_j v_j y_{j+1}$. The result we want to prove is the following.

Theorem 7.4 *Let $\mathbf{H}$ be a pseudovariety of abelian groups, let x and y be two elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$ and let $x = u_0[A_1, g_1]u_1 \cdots [A_n, g_n]u_n$ and $y = v_0[B_1, h_1]v_1 \cdots [B_m, h_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if*

(1) *$n = m$, $u_0 = v_0$, $A_1 = B_1$, $A_n = B_m$, $u_n = v_m$;*

(2) *if $n \geq 2$, then there is a permutation α of the set $\{1, \dots, n-1\}$ such that, for every $1 \leq i \leq n-1$, the triple (B_i, v_i, B_{i+1}) is equal to $(A_{\alpha(i)}, u_{\alpha(i)}, A_{\alpha(i)+1})$;*

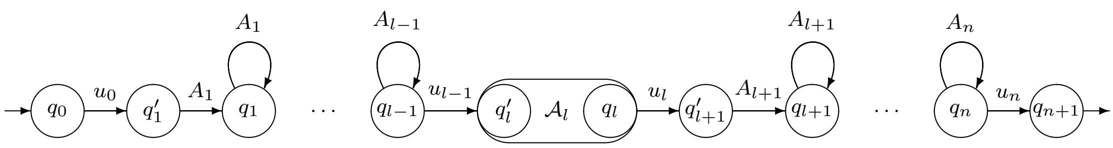
(3) *for every $1 \leq i \leq n$, $h_i = g_{\beta(i)}$ for some permutation β of the set $\{1, \dots, n\}$.*

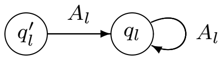
To prove this result we need, as usual, to define some suitable automata to separate factorizations of distinct elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$. Evidently, these automata are not supposed to separate, for instance, the normal factorizations

$$x = y^\omega a z^\omega a y^\omega a^2 z^\omega \qquad \text{and} \qquad x = y^\omega a^2 z^\omega a y^\omega a z^\omega,$$

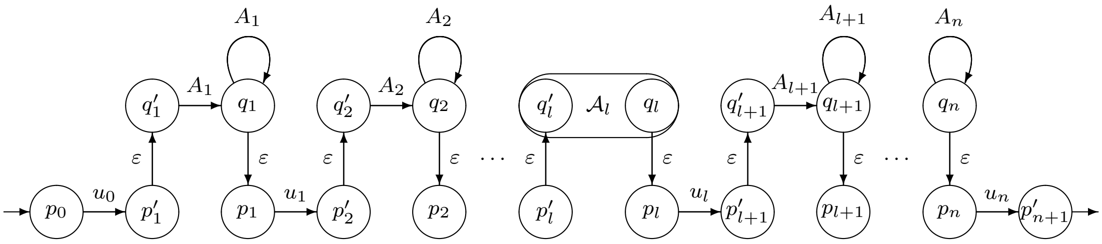
as in Example 7.3, of an element $x \in \hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$. Each such automaton, say $\mathcal{G}$, is constructed as a "union" of a finite number of certain forms of automata $\mathcal{D}$ as in Lemma 6.9 (in the sense that the language recognized by $\mathcal{G}$ is the union of the languages recognized by these automata $\mathcal{D}$). Roughly speaking, this automaton $\mathcal{G}$ will be obtained from a unique automaton $\mathcal{D}$ but we have to permit the transitions $q_i \xrightarrow{u_i} q'_{i+1}$ to be traversed by a path in an order different from their "natural" order (naturally, not all orders will be allowed). Let us be more precise.

Let $\mathcal{D} = \mathcal{D}(u_0, A_1, u_1, \dots, u_{l-1}, \mathcal{A}_l; q'_l; q_l, u_l, \dots, A_n, u_n)$ be the automaton

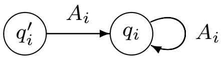


as in Lemma 6.9, where $u_0, \dots, u_n \in A^*$ and $\emptyset \neq A_1, \dots, A_n \subseteq A$ are such that: for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i; if $u_i = 1$ $(1 \leq i \leq n-1)$ then $A_i \cap A_{i+1} = \emptyset$; $A_i = A_j$ or $A_i \cap A_j = \emptyset$ for each pair $1 \leq i, j \leq n$; $\mathcal{A}_l$ $(1 \leq l \leq n)$ is either the automaton 

, or a non-trivial permutation automaton on the alphabet A_l with state set Q_l and $q'_l, q_l \in Q_l$ two distinct states. Now consider the following automaton $\mathcal{D}'$.

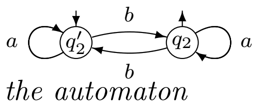


Notice that the language recognized by $\mathcal{D}'$ is the same as the language recognized by $\mathcal{D}$. Finally, let $\mathcal{G} = \mathcal{G}(u_0, A_1, u_1, \ldots, u_{l-1}, \mathcal{A}_l; q'_l; q_l, u_l, \ldots, A_n, u_n)$ be the automaton obtained from $\mathcal{D}'$ by the addition of the following transitions: if $A_i = A_j$ for some $i \neq j$, then there are in $\mathcal{G}$ transitions $p'_i \xrightarrow{\varepsilon} q'_j$, $p'_j \xrightarrow{\varepsilon} q'_i$, $q_i \xrightarrow{\varepsilon} p_j$ and $q_j \xrightarrow{\varepsilon} p_i$.

For each $1 \le i \le n$ with $i \neq l$, denote the automaton  by $\mathcal{A}_i$. We say that a word $w \in A^+$ is *recognized* by the automaton $\mathcal{G}$ if w is the label of a path $\mathcal{P}$ in the automaton which goes from p_0 to p'_{n+1} and which passes in each state p_i $(0 \le i \le n)$ and q'_j $(1 \le j \le n)$ exactly once (i.e., it goes through each transition $p_i \xrightarrow{u_i} p'_{i+1}$ and each automaton $\mathcal{A}_j$ exactly once). Such a path is said to be *successful*. Notice that the path $\mathcal{P}$ is of the form

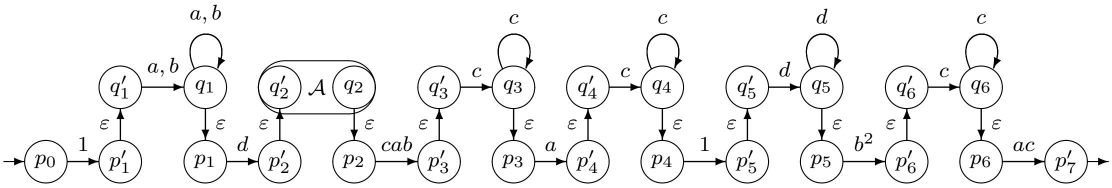
$$p_0 \xrightarrow{u_0} p'_1 \xrightarrow{\varepsilon} q'_{j_1} \xrightarrow{w_{j_1}} q_{j_1} \xrightarrow{\varepsilon} p_{r_1} \xrightarrow{u_{r_1}} p'_{r_1+1} \xrightarrow{\varepsilon} q'_{j_2} \xrightarrow{w_{j_2}} q_{j_2} \cdots q'_{j_n} \xrightarrow{w_{j_n}} q_{j_n} \xrightarrow{\varepsilon} p_n \xrightarrow{u_n} p'_{n+1}$$

where $q'_{j_k} \xrightarrow{w_{j_k}} q_{j_k}$ represents a path in automaton $\mathcal{A}_{j_k}$ labeled w_{j_k}. Hence w is of the form $w = u_0 w_{j_1} u_{r_1} w_{j_2} u_{r_2} \cdots w_{j_n} u_n$ and $w_{j_k} \in A^+_{j_k}$ for all $1 \le k \le n$. Notice also that $A_1 = A_{j_1} = A_{r_1}$, $A_{r_{k-1}+1} = A_{j_k} = A_{r_k}$ for all $2 \le k \le n-1$ and $A_n = A_{j_n} = A_{r_{n-1}+1}$.

Example 7.5 *Let $A = \{a, b, c, d\}$, let $\mathcal{A} =$*  *be a permutation automaton on the alphabet $\{a, b\}$, and consider the automaton*

$$\mathcal{G} = \mathcal{G}(1, \{a, b\}, d, \mathcal{A}; q'_2; q_2, cab, \{c\}, a, \{c\}, 1, \{d\}, b^2, \{c\}, ac).$$

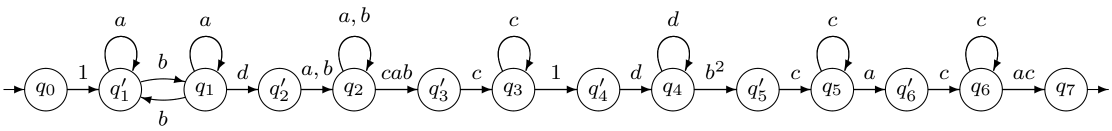
The automaton $\mathcal{G}$ can be represented as follows



where we omit the ε-transitions $p'_1 \to q'_2$, $p'_2 \to q'_1$, $q_1 \to p_2$, $q_2 \to p_1$, $p'_3 \to q'_4$, etc. The language recognized by $\mathcal{G}$ is

$$L = (\{a, b\}^+ dL' \cup L'd\{a, b\}^+) cabc^+ (ac^+d^+b^2 \cup d^+b^2c^+a) c^+ ac,$$

where $L' = a^*ba^*(ba^*ba^*)^*$ is the language recognized by $\mathcal{A}$. Note that L is the union of four $(\mathcal{D}\mathbf{G} \cap \mathcal{L}\mathbf{ZE})$-recognizable languages. For instance, $L'd\{a, b\}^+cabc^+d^+b^2c^+ac^+ac$ is the language recognized by the automaton

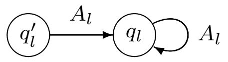


which is as in the conditions of Lemma 6.9. ■

Lemma 7.6 *Consider the automaton $\mathcal{G}$ above and let L be the language recognized by $\mathcal{G}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{G} \cap (\mathbf{Com} * \mathbf{D})$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$.*

Moreover, if $w \in A^+$, $k > |u_0 \cdots u_n| + n$ is an exponent of $S(L)$ and w^k is the label of a path in $\mathcal{G}$, then there exists a path $\mathcal{P}$ and $i \in \{1, \ldots, n\}$ such that: $\mathcal{P}$ is labeled w^k and contains no ε-transitions; $w \in A_i^+$ and the path $\mathcal{P}$ begins in q_i' or q_i (in Q_l if $i = l$) and ends in q_i (in Q_l if $i = l$). □

Now we are able to prove the announced result.

Proof of Theorem 7.4. Suppose first that $x = y$. Let $l \in \{1, \ldots, n\}$ and let $\mathcal{G}$ be the automaton $\mathcal{G}(u_0, A_1, u_1, \ldots, u_{l-1}, \mathcal{A}_l; q_l'; q_l, u_l, \ldots, A_n, u_n)$ where $\mathcal{A}_l$ is the automaton 

, let L be the language recognized by $\mathcal{G}$ and let S be the syntactic semigroup of L. By Lemma 7.6, S lies in $\mathbf{J} \cap (\mathbf{Com} * \mathbf{D})$ and so also is in $\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D})$.

Let $k > |u_0 \cdots u_n| + n$ be an exponent of S. Like in the proof of Theorem 5.5, one can show that $\mathcal{G}$ recognizes a word w' of the form $w' = v_0 y_1^k v_1 y_2^k \cdots y_m^k v_m$ where, for each $1 \le i \le m$, y_i is a word of content B_i. Let $\mathcal{P}$ be a successful path in $\mathcal{A}$ labeled w'. In particular, for each $1 \le i \le m$, ${y_i}^k$ is the label of a subpath $\mathcal{Q}$ of $\mathcal{P}$ and, by Lemma 7.6, there exists $1 \le j_i \le n$ such that $y_i \in A_{j_i}^+$ whence $B_i \subseteq A_{j_i}$. By symmetry, we also have $A_{j_i} \subseteq B_r$ for some $1 \le r \le m$. This implies $B_i \subseteq B_r$ and so by definition of normal factorization, we deduce $B_i = B_r$. Then $B_i = A_{j_i}$. Now as the last (resp. first) letter of the word $y_{i-1}^k v_{i-1}$ (resp. $v_i y_{i+1}^k$) does not lie in B_i, we may suppose without loss of generality that the path $\mathcal{Q}$ is of the form $t_1 \xrightarrow{\varepsilon} q_{j_i}' \overset{y_i^k}{\dashrightarrow} q_{j_i} \xrightarrow{\varepsilon} t_2$ for some states t_1 and t_2. This implies that $m \le n$ and, by symmetry, it follows that $n = m$. Hence, the path $\mathcal{P}$ is of the form

$$p_0 \xrightarrow{u_0} p_1' \xrightarrow{\varepsilon} q_{j_1}' \overset{y_1^k}{\dashrightarrow} q_{j_1} \xrightarrow{\varepsilon} p_{r_1} \xrightarrow{u_{r_1}} p_{r_1+1}' \xrightarrow{\varepsilon} q_{j_2}' \overset{y_2^k}{\dashrightarrow} q_{j_2} \cdots q_{j_n}' \overset{y_n^k}{\dashrightarrow} q_{j_n} \xrightarrow{\varepsilon} p_n \xrightarrow{u_n} p_{n+1}'$$

and the correspondence $i \mapsto r_i$ is a permutation of the set $\{1, \ldots, n-1\}$. Now as in the other proofs, we deduce that $u_0 = v_0$, $u_n = v_n$, $A_1 = A_{j_1} = B_1$, $A_n = A_{j_n} = B_n$ and $v_i = u_{r_i}$ for every $1 \le i \le n-1$. Furthermore, since u_{r_i} is the label of a transition from p_{r_i} to p_{r_i+1}', meaning that y_i^k (resp. y_{i+1}^k) is a word of content A_{r_i} (resp. A_{r_i+1}), we deduce that $B_i = A_{r_i}$ and that $B_{i+1} = A_{r_i+1}$ which shows that points (1) and (2) hold.

Now put $x_i = [A_i, g_i]$ and $y_i = [A_i, h_i]$ for all $1 \le i \le n$. To show that point (3) holds, let us suppose that x_i is not idempotent for some $1 \le i \le n$, i.e., that $g_i \ne 1$. Recall that with content A_i and not idempotent, there is no other x_j and there is at most one y_j. Using similar arguments as those of the proof of Theorem 5.5, one can show that, for each A_i-generated group G of $\mathbf{H}$, $(x_i)_G$ coincides with $(y_j)_G$ for some $1 \le j \le n$ such that $A_i = A_j$. Now since x_i is not idempotent, we deduce that there is some $1 \le k \le n$ such that y_k is not idempotent and that $(x_i)_G$ coincides with $(y_k)_G$ for every G. Hence $g_i = h_k$ and point (3) holds. (Note that, alternatively, we could prove that (3) holds using analogous arguments as those used in the proof of Corollary 6.11, since $(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D})) \cap \mathbf{G}$ is a pseudovariety of abelian groups and there is at most one x_i and one y_j with a fixed content C and not idempotent.)

Let us now prove the converse. If $n \leq 1$ or $n = 2$ and $A_1 \neq A_2$ it is clear that $x = y$. If $n = 2$ and $A_1 = A_2$, then the equality $x = y$ is a simple consequence of point (1) of Proposition 7.1. So suppose that $n \geq 3$. In the sequel we will say that a normal factorization of y of the form $y = w_0[C_1, f_1]w_1 \cdots w_{n-1}[C_n, f_n]w_n$ satisfies condition $(*)$ if it satisfies conditions (1) to (3) of the statement with w_i, C_i and f_i in the place of v_i, B_i and h_i respectively. In order to simplify notation we will also denote by $\bar{C}$ (with $C \subseteq A$) all regular elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$ with a fixed content C. This notation is somewhat ambiguous since it "hides" the restriction of the element to $\mathbf{H}$. Nevertheless, since we are dealing with normal factorizations, there is at most one regular element with content C (for $C \subseteq A$) that is not idempotent. Moreover, by point (1) of Proposition 7.1, two regular elements with the same content in a factorization can exchange their positions.

We will begin by proving that y admits a normal factorization of the form

$$y = u_0\bar{A}_1u_1\bar{A}_2w_2\bar{C}_3 \cdots \bar{C}_{n-1}w_{n-1}\bar{A}_nu_n \tag{4}$$

satisfying condition $(*)$.

Suppose that $u_1 \neq v_1$ or $A_2 \neq B_2$. By hypothesis, there exists some $2 \leq k \leq n-1$ such that $A_1 = B_k$, $u_1 = v_k$ and $A_2 = B_{k+1}$. So

$$y = u_0\bar{A}_1v_1\bar{B}_2 \cdots \bar{B}_{k-1}v_{k-1}\underline{\bar{A}_1u_1\bar{A}_2}v_{k+1}\bar{B}_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n. \tag{5}$$

If $A_2 = A_1$, we deduce from point (2) of Proposition 7.1, that

$$\begin{aligned} y &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}\underline{v_1\bar{B}_2 \cdots v_{k-1}}\bar{\boldsymbol{A}}_{\boldsymbol{1}}\underline{u_1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n \\ &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}\underline{u_1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}\underline{v_1\bar{B}_2 \cdots v_{k-1}}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n \\ &= u_0\bar{A}_1u_1\bar{A}_2v_1\bar{B}_2 \cdots v_{k-1}\bar{A}_1v_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n \end{aligned}$$

and this factorization is of form (4) and clearly satisfies condition $(*)$.

Suppose now that $A_2 \neq A_1$. Since the case $B_2 = B_1 (= A_1)$ can be treated the same way as the case $A_2 = A_1$, we may also suppose $B_2 \neq A_1$. In this case, $k > 2$ necessarily holds. If $A_2 = B_2$, using point (2) of Proposition 7.1, we deduce from (5) that

$$\begin{aligned} y &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_2 \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{k+1} \cdots v_{n-1}\bar{A}_nu_n \\ &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_2 \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{k+1} \cdots v_{n-1}\bar{A}_nu_n. \end{aligned}$$

Suppose now that $A_2 \neq B_2$. If there exists $k+1 < i \leq n$ such that $A_1 = B_i$, we have that

$$\begin{aligned} y &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{B}_2 \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{A}_2v_{k+1} \cdots v_{i-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_i \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n \\ &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{A}_2v_{k+1} \cdots v_{i-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{B}_2 \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_i \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n. \end{aligned}$$

Let us now assume that, for all $k+1 < i \leq n$, $A_1 \neq B_i$. If there exists $2 \leq i < k$ such that $A_2 = B_i$, we have that

$$\begin{aligned} y &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{B}_2 \cdots v_{i-1}\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{i+1} \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n \\ &= u_0\bar{\boldsymbol{A}}_{\boldsymbol{1}}u_1\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{i+1} \cdots v_{k-1}\bar{\boldsymbol{A}}_{\boldsymbol{1}}v_1\bar{B}_2 \cdots v_{i-1}\bar{\boldsymbol{A}}_{\boldsymbol{2}}v_{k+1} \cdots \bar{B}_{n-1}v_{n-1}\bar{A}_nu_n. \end{aligned}$$

Assume now that for all $2 \le i < k$, $A_2 \ne B_i$. Since $B_1 = A_1$ and $A_1 \ne A_2$, the triple (A_1, v_1, B_2) is equal to the triple (A_l, u_l, A_{l+1}) for some $3 \le l \le n-1$. So

$$x = u_0\bar{A}_1u_1\bar{A}_2\cdots\bar{A}_{l-1}u_{l-1}\underline{\bar{A}_1v_1\bar{B}_2}u_{l+1}\bar{A}_{l+1}\cdots u_{n-1}\bar{A}_nu_n.$$

More precisely, l cannot equal 3. Indeed, suppose by way of contradiction that $l = 3$. Then

$$x = u_0\bar{A}_1u_1\bar{A}_2u_2\bar{A}_1v_1\bar{B}_2u_4\bar{A}_5\cdots u_{n-1}\bar{A}_nu_n,$$

and there exists some $3 \le i \le n-1$ such that $A_2 = B_i$ and $A_1 = B_{i+1}$ (and $u_2 = v_i$). As a consequence, there is either some $k+1 < i \le n$ such that $A_1 = B_i$, or some $2 \le i < k$ such that $A_2 = B_i$. But this contradicts our assumptions that we have been considering. So $l > 3$.

Now we claim that there exists $3 \le h \le l-1$ such that $A_h = B_r = B_s \ne A_1$ for some $2 \le r < k$ and $k+1 < s \le n$. Indeed, we have

$$\forall i \in \{2,\ldots,l-1\}\exists j \in \{1,\ldots,n-1\},\ A_i = B_j \text{ and } A_{i+1} = B_{j+1}. \tag{6}$$

In particular, for $i = 2$, we have $A_2 = B_j$ and $A_3 = B_{j+1}$ for some $1 \le j \le n-1$. Since, by hypothesis, $A_2 \ne B_j$ for all $2 \le j < k$ we have $j \ge k$. Furthermore, $B_k = A_1 \ne A_2$. So $j \ne k$ and we have $A_3 = B_j$ for some $k+1 < j \le n$. Note that this implies $A_3 \ne A_1$ by the assumptions we have made. Consider now $i = l-1$ in (6). We have $A_{l-1} = B_j$ and $(A_1 =)A_l = B_{j+1}$ for some $1 \le j \le n-1$. But $B_{k+1} = A_2 \ne A_1$. So $j \ne k$ and since $A_1 \ne B_j$ for all $j > k+1$, we deduce that $A_{l-1} = B_j$ for some $1 \le j < k$. Now it is clear by (6) that the claim is valid. Hence, y is equal to

$$\begin{aligned} & u_0\boldsymbol{\bar{A}_1}v_1\bar{B}_2\cdots v_{r-1}\boldsymbol{\bar{A}_h}v_{r+1}\cdots v_{k-1}\boldsymbol{\bar{A}_1}u_1\bar{A}_2v_{k+1}\cdots v_{s-1}\boldsymbol{\bar{A}_h}v_{s+1}\cdots v_{n-1}\bar{A}_nu_n \\ = \; & u_0\boldsymbol{\bar{A}_1}u_1\bar{A}_2v_{k+1}\cdots v_{s-1}\boldsymbol{\bar{A}_h}v_{r+1}\cdots v_{k-1}\boldsymbol{\bar{A}_1}v_1\bar{B}_2\cdots v_{r-1}\boldsymbol{\bar{A}_h}v_{s+1}\cdots v_{n-1}\bar{A}_nu_n. \end{aligned}$$

So in all cases y admits a normal factorization of form (4) satisfying condition $(*)$.

Iterating the above argument, one proves that y admits a factorization of the form $u_0\bar{A}_1u_1\bar{A}_2u_2\bar{A}_3\cdots u_{n-1}\bar{A}_nu_n$ satisfying condition $(*)$. Now using point (3) of the statement, we deduce from point (1) of Proposition 7.1 that $x = y$. □

This last proof shows, in particular, that given two elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$ written in normal form, these elements are the same if and only if we can pass from one normal form to the other using a finite number of times the following "rewriting rules"

$$x_1[B,g_1]x_2[C,h_1]x_3[B,g_2]x_4[C,h_2]x_5 \mapsto x_1[B,g_1]x_4[C,h_1]x_3[B,g_2]x_2[C,h_2]x_5$$

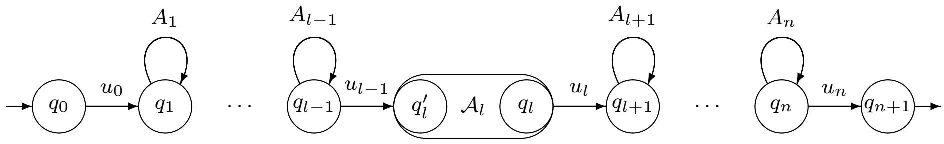
$$x_1[B,g_1]x_2[B,g_2]x_3[B,g_3]x_4 \mapsto x_1[B,g_1]x_3[B,g_2]x_2[B,g_3]x_4$$

$$x_1[B,1]x_2[B,g]x_3 \mapsto x_1[B,g]x_2[B,1]x_3$$

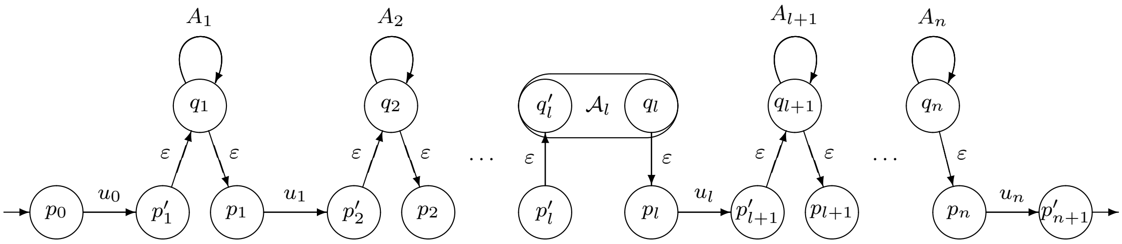
$$x_1[B,g]x_2[B,1]x_3 \mapsto x_1[B,1]x_2[B,g]x_3$$

given by Proposition 7.1.

The previous results can be easily adapted to the cases $\mathbf{V} = \mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$ and $\mathbf{V} = \mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})$. The automata used to separate factorizations of distinct elements of $\hat{F}_A(\mathbf{V})$ are the following. Let $\mathcal{C} = \mathcal{C}(u_0, A_1, u_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$ be an automaton



as defined after Proposition 5.1, where $u_0, \ldots, u_n \in A^*$, $\emptyset \neq A_1, \ldots, A_n \subseteq A$ are such that $u_i \neq 1$ $(1 \leq i \leq n-1)$ and $\mathcal{A}_l$ is a permutation automaton on the alphabet A_l with state set Q_l and $q_l, q'_l \in Q_l$ (contrary to the case $\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D})$, in this case the states q_l and q'_l may coincide). Let also $A_i = A_j$ or $A_i \cap A_j = \emptyset$ for each pair $1 \leq i, j \leq n$. Now consider the following automaton $\mathcal{C}'$



Finally, let $\mathcal{H} = \mathcal{H}(u_0, A_1, u_1, \ldots, \mathcal{A}_l; q'_l; q_l, \ldots, A_n, u_n)$ be the automaton obtained from $\mathcal{C}'$ by the addition of the following transitions: if $A_i = A_j$ for some $i \neq j$, then there are in $\mathcal{H}$ the following transitions $p'_i \xrightarrow{\varepsilon} q_j$ (resp. $p'_i \xrightarrow{\varepsilon} q'_l$ if $j = l$), $q_i \xrightarrow{\varepsilon} p_j$ and vice-versa.

Lemma 7.7 *Consider an automaton $\mathcal{H} = \mathcal{H}(u_0, A_1, \ldots, \mathcal{A}_l; q'_l; q_l, \ldots, A_n, u_n)$ as above satisfying the following extra condition $(*)$: for each $1 \leq i \leq n-1$, $c(u_i)$ is not contained in either A_i or A_{i+1}. Let L be the language recognized by $\mathcal{H}$. Then $S(L)$ lies in $\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D})$ and its subgroups lie in the pseudovariety generated by the transition group $S(\mathcal{A}_l)$. If instead of the condition $(*)$, $\mathcal{H}$ satisfies the condition "for each $1 \leq i \leq n$, the first letter of u_i does not lie in A_i and, if $c(u_i) \subseteq A_{i+1}$ $(1 \leq i \leq n-1)$, then $A_i \cap A_{i+1} = \emptyset$", then $S(L) \in \mathcal{D}\mathbf{RG} \cap (\mathbf{Com} * \mathbf{D})$.* □

Naturally, the pseudovarieties $\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$ and $\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})$ admit the following analogues of Theorem 7.4. Their proofs can be easily adapted from the proof of Theorem 7.4 and others and so we omit them.

Theorem 7.8 *Let $\mathbf{H}$ be a pseudovariety of abelian groups, let x and y be two elements of $\hat{F}_A(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$ and let $x = u_0[w_1, A_1, g_1, w'_1]u_1 \cdots u_{n-1}[w_n, A_n, g_n, w'_n]u_n$ and $y = v_0[z_1, B_1, h_1, z'_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m, z'_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if*

(1) *$n = m$, $u_0 = v_0$, $w_1 = z_1$, $A_1 = B_1$, $A_n = B_m$, $w'_n = z'_m$, $u_n = v_m$;*

(2) *if $n \geq 2$, then $(B_i, z'_i, v_i, z_{i+1}, B_{i+1}) = (A_{\alpha(i)}, w'_{\alpha(i)}, u_{\alpha(i)}, w_{\alpha(i)+1}, A_{\alpha(i)+1})$ for some permutation α of the set $\{1, \ldots, n-1\}$ and for all $1 \leq i \leq n-1$;*

(3) *for every $1 \leq i \leq n$, $h_i = g_{\beta(i)}$ for some permutation β of the set $\{1, \ldots, n\}$.* □

Theorem 7.9 *Let $\mathbf{H}$ be a pseudovariety of abelian groups, let x and y be two elements of $\hat{F}_A(\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D}))$ and let $x = u_0[w_1, A_1, g_1]u_1 \cdots u_{n-1}[w_n, A_n, g_n]u_n$ and $y = v_0[z_1, B_1, h_1]v_1 \cdots v_{m-1}[z_m, B_m, h_m]v_m$ be factorizations in normal form. Then $x = y$ if and only if*

(1) *$n = m$, $u_0 = v_0$, $w_1 = z_1$, $A_1 = B_1$, $A_n = B_m$, $u_n = v_m$;*

(2) *if $n \geq 2$, then $(B_i, v_i, z_{i+1}, B_{i+1}) = (A_{\alpha(i)}, u_{\alpha(i)}, w_{\alpha(i)+1}, A_{\alpha(i)+1})$ for some permutation α of the set $\{1, \ldots, n-1\}$ and for all $1 \leq i \leq n-1$;*

(3) *for every $1 \leq i \leq n$, $h_i = g_{\beta(i)}$ for some permutation β of the set $\{1, \ldots, n\}$.* □

One can verify from Theorem 7.8 that if $x, y \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$, then $x = y$ if and only if $\mathcal{D}\mathbf{A} \cap (\mathbf{Com} * \mathbf{D})$ and $\mathbf{H}$ satisfy $x = y$. Analogous remarks are valid for $\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})$ and $\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D})$, proving the following join decompositions.

Corollary 7.10 *Let $\mathbf{H}$ be a pseudovariety of abelian groups. Then*

- $\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}) = (\mathcal{D}\mathbf{A} \cap (\mathbf{Com} * \mathbf{D})) \vee \mathbf{H}$;
- $\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D}) = (\mathbf{R} \cap (\mathbf{Com} * \mathbf{D})) \vee \mathbf{H}$;
- $\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}) = (\mathbf{J} \cap (\mathbf{Com} * \mathbf{D})) \vee \mathbf{H}$. □

A left-right dual of Theorem 7.9 is valid for the pseudovarieties $\mathcal{D}\mathbf{LH} \cap (\mathbf{Com} * \mathbf{D})$. From Corollary 6.2, we deduce that $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap (\mathbf{Com} * \mathbf{D}) = \mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$. Now we show that $(\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})) \vee (\mathcal{D}\mathbf{LH} \cap (\mathbf{Com} * \mathbf{D}))$ is strictly contained in $(\mathcal{D}\mathbf{RH} \vee \mathcal{D}\mathbf{LH}) \cap (\mathbf{Com} * \mathbf{D})$.

Corollary 7.11 *Let $\mathbf{H}$ be a pseudovariety of abelian groups. Then*

$$(\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})) \vee (\mathcal{D}\mathbf{LH} \cap (\mathbf{Com} * \mathbf{D})) \neq \mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}).$$

Proof. Let, for instance, $A = \{a, b, c\}$, $B = \{a, b\}$, $w \in B^{\mathbb{N}}$, $w' \in B^{-\mathbb{N}}$ and $x, y \in \hat{F}_A(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$ with $x = [w, B, 1, a^{-\infty}b]c[a^{+\infty}, B, 1, a^{-\infty}]c[ba^{+\infty}, B, 1, w']$ and $y = [w, B, 1, a^{-\infty}b]c[ba^{+\infty}, B, 1, a^{-\infty}]c[a^{+\infty}, B, 1, w']$.

Then x and y are different by Theorem 7.8 but their restrictions to both $\mathcal{D}\mathbf{RH}$ $\cap(\mathbf{Com} * \mathbf{D})$ and $\mathcal{D}\mathbf{LH} \cap (\mathbf{Com} * \mathbf{D})$ are equal. Indeed, the restrictions of x and y to $\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D})$, for instance, are respectively $[w, B, 1]c[a^{+\infty}, B, 1]c[ba^{+\infty}, B, 1]$ and $[w, B, 1]c[ba^{+\infty}, B, 1]c[a^{+\infty}, B, 1]$, which are clearly the same by Theorem 7.9. So the result follows from Reiterman's Theorem. □

8 Implicit operations on $\mathcal{D}\mathbf{G} \cap \mathcal{L}\mathbf{ZE} \cap \mathcal{E}\mathbf{Com}$

Consider the pseudovariety $\mathcal{E}\mathbf{Com}$ of all finite semigroups in which the idempotents commute. Observe that $\mathcal{D}\mathbf{S} \cap \mathcal{E}\mathbf{Com} = \mathcal{D}\mathbf{G} \cap \mathcal{E}\mathbf{Com}$. Recall that in a semigroup having commuting idempotents, the product of regular elements is again a regular element, since the product of two idempotents in such a semigroup is a regular element. This allows us to consider a notion of *normal* factorization for elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathbf{W} \cap \mathcal{E}\mathbf{Com})$ (where $\mathbf{W}$ is one of $\mathcal{L}\mathbf{ZE}$, $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$ and $\mathbf{Com} * \mathbf{D}$) by imposing the extra condition

$$u_i \neq 1 \text{ for every } 1 \leq i \leq n-1$$

in the definition of normal factorization for elements of $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathbf{W})$. Note that $\mathcal{E}\mathbf{Com} \subseteq \mathcal{L}\mathcal{E}\mathbf{Com}$ and that the semigroups $\hat{F}_A(\mathcal{D}\mathbf{H} \cap \mathcal{E}\mathbf{Com})$ were studied by Almeida and Weil [6, 7].

With this definition of normal factorization for the implicit operations on $\mathcal{D}\mathbf{H}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com}$, and doing a study absolutely similar to that conducted for $\mathcal{D}\mathbf{H}\cap\mathbf{W}$, one can show that the analogous statements, obtained from Theorems 6.10 and 7.4 by a simple substitution of $\mathcal{D}\mathbf{H}\cap\mathbf{W}$ by $\mathcal{D}\mathbf{H}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com}$, are valid.

Note that $\mathcal{L}\mathbf{ZE}\cap\mathcal{E}\mathbf{Com}$ is the pseudovariety intermediate between $\mathbf{ZE}$ and $\mathcal{E}\mathbf{Com}$, defined by the pseudoidentity $(exe)f = f(exe)$.

Recall (see [6]) that a non-trivial pseudovariety of groups $\mathbf{H}$ is *arborescent* if $(\mathbf{H}\cap\mathbf{Ab}) *\mathbf{H} = \mathbf{H}$. As examples of arborescent pseudovarieties of groups, we can mention the pseudovarieties closed under semidirect product (see Gildenhuys and Ribes [14]). On the other hand, the pseudovariety $\mathbf{Ab}$, for instance, is not arborescent. The condition "$u_i \neq 1$ for every $1 \leq i \leq n-1$" in the definition of normal factorization of the elements of $\hat{F}_A(\mathcal{D}\mathbf{H}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com})$ permits the following join decompositions.

Corollary 8.1 *Let $\mathbf{W}$ be one of the pseudovarieties $\mathcal{L}\mathbf{ZE}$, $\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$ and $\mathbf{Com}*\mathbf{D}$ and let $\mathbf{H}$ be an arborescent pseudovariety of groups. In the cases $\mathbf{W}=\mathcal{L}(\mathbf{Sl}\vee\mathbf{G})$ and $\mathbf{W}=\mathbf{Com}*\mathbf{D}$, $\mathbf{H}$ can also be abelian. Then*

$$\mathcal{D}\mathbf{H}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com} = (\mathbf{J}\cap\mathbf{W}\cap\mathcal{E}\mathbf{Com})\vee\mathbf{H}.\Box$$

The proof of the "arborescent" part of this result can be done exactly as in the proof of Theorem 4.1 in [6]. The "abelian" part is similar to Corollaries 6.11 and 7.10. Note that the "arborescent" condition in the case $\mathbf{W}=\mathbf{Com}*\mathbf{D}$ is superfluous since $(\mathbf{Com}*\mathbf{D})\cap\mathbf{G}=\mathbf{Ab}$.

9 The corresponding varieties of languages

In this section we give combinatorial descriptions of the varieties of languages associated, via Eilenberg's Theorem, with the pseudovarieties studied in the previous sections. We present, for each of these varieties, a set of generators. In this section we fix an alphabet A.

9.1 $(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$-recognizable languages

Denote by $\mathcal{K}(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$ (resp. $\mathcal{K}(\mathbf{R}\cap\mathcal{L}\mathbf{J})$) the class of all languages of the form

$$u_0A_1^*X_1A_2^*\ldots X_{n-1}A_n^*u_n$$

such that $n,r\geq 0$, $u_0,\ldots,u_n\in A^*$, $\emptyset\neq A_1,\ldots,A_n\subseteq A$ and for all $1\leq i\leq n-1$:

$$X_i = \begin{cases} u_i & \text{if } u_i\neq 1\\ (A_i\setminus A_{i+1})(A_i\cap A_{i+1})^{\geq r} & \text{if } u_i=1;\end{cases}$$

if $u_i\neq 1$ then $c(u_i)$ is not contained in either A_i or A_{i+1}; if $u_i=1$ then A_i and A_{i+1} are $\subseteq$-incomparable (resp. $r=0$ and for all $1\leq j\leq n$ such that $u_j\neq 1$, the first letter of u_j does not lie in A_j).

Note that the languages of the class $\mathcal{K}(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$ are precisely the languages recognized by the automata $\mathcal{A}(r;u_0,A_1,u_1,\ldots,A_n,u_n)$ as in Lemma 4.2, and thus they are $(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$-recognizable. The description of the class of all $(\mathcal{D}\mathbf{A}\cap\mathcal{L}\mathbf{J})$-recognizable languages of A^+ is now an easy consequence of Eilenberg's correspondence and of Corollary 4.4. Similar results hold for $\mathbf{R}\cap\mathcal{L}\mathbf{J}$.

Theorem 9.1 *Let A be an alphabet. The class of languages in A^+ which are recognized by semigroups in $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$ (resp. $\mathbf{R} \cap \mathcal{L}\mathbf{J}$), is the Boolean algebra generated by $\mathcal{K}(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J})$ (resp. $\mathcal{K}(\mathbf{R} \cap \mathcal{L}\mathbf{J})$).* □

9.2 $(\mathcal{D}\mathbf{ReG} \cap \mathbf{W})$-recognizable languages, $\mathbf{W} = \mathcal{L}\mathcal{E}\mathbf{Com}, \mathcal{L}\mathbf{ZE}, \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$

Denote by $\mathcal{K}(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$ (resp. $\mathcal{K}(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$) the class of all languages of the form

$$u_0 A_1^* u_1 \dots A_{l-1}^* u_{l-1} L_l u_l A_{l+1}^* \dots A_n^* u_n$$

such that:

- $n \geq 0$, $1 \leq l \leq n$, $u_0, \dots, u_n \in A^*$, $u_i \neq 1$ for all $1 \leq i \leq n-1$;
- L_l is a group language over A_l whose syntactic semigroup lies in $\mathbf{H}$ and whose minimal automaton has only one terminal state;
- $\emptyset \neq A_1, \dots, A_n \subseteq A$ and, for each $1 \leq i \leq n-1$, $c(u_i)$ is not contained in either A_i or A_{i+1} (resp. for each $1 \leq i \leq n$, the first letter of u_i does not lie in A_i and, for each $1 \leq i \leq n-1$, the last letter of u_i lies in A_{i+1} and, if $c(u_i) \subseteq A_{i+1}$ then $A_i \cap A_{i+1} = \emptyset$).

Then just as above we have the following result.

Theorem 9.2 *Let $\mathbf{H}$ be a pseudovariety of groups. The class of languages in A^+, recognized by semigroups in $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$ (resp. $\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$), is the Boolean algebra generated by $\mathcal{K}(\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$ (resp. $\mathcal{K}(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$).* □

Now denote by $\mathcal{K}(\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$ the class of all languages of the form

$$u_0 A_1^+ u_1 \dots A_{l-1}^+ u_{l-1} L_l u_l A_{l+1}^+ \dots A_n^+ u_n$$

with:

- $n \geq 0$, $1 \leq l \leq n$, $u_0, \dots, u_n \in A^*$;
- L_l is either A_l^+ or is a group language over A_l whose syntactic semigroup lies in $\mathbf{H}$ and whose minimal automaton is not trivial and has only one terminal state, distinct from the initial one;
- $\emptyset \neq A_1, \dots, A_n \subseteq A$;
- if $u_i = 1$ then $A_i \cap A_{i+1} = \emptyset$ and for each $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.

Theorem 9.3 *Let $\mathbf{H}$ be a pseudovariety of groups. The class of languages in A^+ which are recognized by semigroups in $\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$, is the Boolean algebra generated by $\mathcal{K}(\mathcal{D}\mathbf{H} \cap \mathcal{L}\mathcal{E}\mathbf{Com})$.* □

A similar situation arises when we consider the pseudovariety $\mathcal{L}\mathbf{ZE}$ (resp. $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$) in the place of $\mathcal{L}\mathcal{E}\mathbf{Com}$. The corresponding languages are obtained by the addition of the condition "$A_i = A_j$ or $A_i \cap A_j = \emptyset$ for all $1 \leq i, j \leq n$" (resp. "$A_i \cap A_j = \emptyset$ for each pair $i \neq j$"). In the case of the $(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE})$- and $(\mathcal{D}\mathbf{RH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G}))$- recognizable languages, we also have to drop the condition "for each $1 \leq i \leq n-1$, the last letter of u_i lies in A_{i+1}" in the definition of the $(\mathcal{D}\mathbf{RH} \cap \mathcal{LE}\mathbf{Com})$-recognizable languages.

One can also use the join decompositions of Corollaries 6.8 and 6.11 to give alternative descriptions of those languages. For instance, denote by $\mathcal{K}(\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE})$ the left-right dual of $\mathcal{K}(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE})$. Then the join decomposition $(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}) \vee (\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE}) = \mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$ given by Corollary 6.8 permits us to give the alternative description of the following result.

Theorem 9.4 *Let* $\mathbf{H}$ *be a pseudovariety of groups. The class of languages in* A^+ *which are recognized by semigroups in* $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}\mathbf{ZE}$, *is the Boolean algebra generated by* $\mathcal{K}(\mathcal{D}\mathbf{RH} \cap \mathcal{L}\mathbf{ZE}) \cup \mathcal{K}(\mathcal{D}\mathbf{LH} \cap \mathcal{L}\mathbf{ZE})$. □

Also, in the case of $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$, for instance, we deduce from Corollary 6.11 the following alternative description.

Theorem 9.5 *Let* $\mathbf{H}$ *be a pseudovariety of groups. The class of languages in* A^+, *recognized by semigroups in* $\mathcal{D}\mathbf{ReH} \cap \mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$, *is the Boolean algebra generated by* $\mathcal{K}(\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl})$ *and by the languages recognized by semigroups in* $\mathbf{H}$. □

9.3 $(\mathcal{D}\mathbf{ReG} \cap (\mathbf{Com} * \mathbf{D}))$-recognizable languages

Let $\mathbf{H}$ be a pseudovariety of abelian groups and consider a language of the form

$$u_0 A_1^* u_1 \ldots A_{l-1}^* u_{l-1} L_l u_l A_{l+1}^* \ldots A_n^* u_n$$

such that:

- $n \geq 0$, $1 \leq l \leq n$, $u_0, \ldots, u_n \in A^*$, $u_i \neq 1$ for all $1 \leq i \leq n-1$;
- L_l is a group language over A_l whose syntactic semigroup lies in $\mathbf{H}$ and whose minimal automaton (say $\mathcal{A}_l$) has only one terminal state;
- $\emptyset \neq A_1, \ldots, A_n \subseteq A$;
- $A_i = A_j$ or $A_i \cap A_j = \emptyset$ for all $1 \leq i, j \leq n$;
- for each $1 \leq i \leq n-1$, $c(u_i)$ is not contained in either A_i or A_{i+1}.

Now denote by $L(u_0, A_1, \ldots, A_{l-1}, u_{l-1}, L_l, u_l, A_{l+1}, \ldots, A_n, u_n)$ the (finite) union of all languages of the form

$$u_0 K_1 v_1 K_2 \cdots v_{n-1} K_n u_n$$

such that:

- for each $1 \leq i \leq n$, K_i is a language over a subalphabet B_i of A;
- $B_1 = A_1$, $B_n = A_n$ and, if $n \geq 2$, there is a permutation α of the set $\{1, \ldots, n-1\}$ such that $(B_i, v_i, B_{i+1}) = (A_{\alpha(i)}, u_{\alpha(i)}, A_{\alpha(i)+1})$ for all $1 \leq i \leq n-1$;
- $K_j = L_l$ for some $1 \leq j \leq n$ and $K_i = B_i^*$ for every $i \neq j$.

Note that the language $L(u_0, A_1, \ldots, L_l, u_l, \ldots, A_n, u_n)$ is precisely the language recognized by the automaton $\mathcal{H}(u_0, A_1, \ldots, \mathcal{A}_l; q_l'; q_l, \ldots, A_n, u_n)$, where q_l' and q_l are, respectively, the initial and the final states of $\mathcal{A}_l$. Thus it is $(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$-recognizable, by Lemma 7.7. Now denote by $\mathcal{K}(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$ the class of all the languages $L(u_0, A_1, \ldots, L_l, u_l, \ldots, A_n, u_n)$.

Theorem 9.6 *Let* $\mathbf{H}$ *be a pseudovariety of abelian groups. The class of languages in* A^+*, recognized by semigroups in* $\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$*, is the Boolean algebra generated by* $\mathcal{K}(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$*.* □

The $(\mathcal{D}\mathbf{RH} \cap (\mathbf{Com} * \mathbf{D}))$-recognizable languages are described analogously. It suffices to substitute the condition "for each $1 \le i \le n-1$, $c(u_i)$ is not contained in either A_i or A_{i+1}" in the definition of $\mathcal{K}(\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D}))$ above, by the condition "for each $1 \le i \le n$, the first letter of u_i does not lie in A_i and if $c(u_i) \subseteq A_{i+1}$ $(1 \le i \le n-1)$, then $A_i \cap A_{i+1} = \emptyset$".

Consider now a language of the form

$$u_0 A_1^+ u_1 \dots A_{l-1}^+ u_{l-1} L_l u_l A_{l+1}^+ \dots A_n^+ u_n$$

with:

- $n \ge 0$, $1 \le l \le n$, $u_0, \dots, u_n \in A^*$;
- L_l is either A_l^+ or is a group language over A_l whose syntactic semigroup lies in $\mathbf{H}$ and whose minimal automaton (say $\mathcal{A}_l$) is not trivial and has only one terminal state, distinct from the initial one;
- $\emptyset \ne A_1, \dots, A_n \subseteq A$;
- $A_i = A_j$ or $A_i \cap A_j = \emptyset$ for all $1 \le i, j \le n$;
- if $u_i = 1$ then $A_i \cap A_{i+1} = \emptyset$;
- for each $1 \le i \le n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.

Denote by $L'(u_0, A_1, \dots, A_{l-1}, u_{l-1}, L_l, u_l, A_{l+1}, \dots, A_n, u_n)$ the (finite) union of all languages of the form

$$u_0 K_1 v_1 K_2 \cdots v_{n-1} K_n u_n$$

such that:

- for each $1 \le i \le n$, K_i is a language over a subalphabet B_i of A;
- $B_1 = A_1$, $B_n = A_n$ and, if $n \ge 2$, there is a permutation α of the set $\{1, \dots, n-1\}$ such that $(B_i, v_i, B_{i+1}) = (A_{\alpha(i)}, u_{\alpha(i)}, A_{\alpha(i)+1})$ for all $1 \le i \le n-1$;
- $K_j = L_l$ for some $1 \le j \le n$ and $K_i = B_i^+$ for every $i \ne j$.

The language $L'(u_0, A_1, \dots, L_l, u_l, \dots, A_n, u_n)$ is the language recognized by the automaton $\mathcal{G}(u_0, A_1, \dots, \mathcal{A}_l; q_l'; q_l, \dots, A_n, u_n)$, where q_l' and q_l are, respectively, the initial and the final states of $\mathcal{A}_l$. Thus by Lemma 7.6, it is $(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$-recognizable. Denote by $\mathcal{K}(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$ the class of all the languages $L'(u_0, A_1, \dots, A_n, u_n)$.

Theorem 9.7 *Let* $\mathbf{H}$ *be a pseudovariety of abelian groups. The class of languages in* A^+*, recognized by semigroups in* $\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D})$*, is the Boolean algebra generated by* $\mathcal{K}(\mathcal{D}\mathbf{H} \cap (\mathbf{Com} * \mathbf{D}))$*.* □

In the case, for instance, of the pseudovarieties $\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$, Corollary 7.10 permits the following alternative description.

Theorem 9.8 *Let* $\mathbf{H}$ *be a pseudovariety of abelian groups. The class of languages in* A^+*, recognized by semigroups in* $\mathcal{D}\mathbf{ReH} \cap (\mathbf{Com} * \mathbf{D})$*, is the Boolean algebra generated by* $\mathcal{K}(\mathcal{D}\mathbf{A} \cap (\mathbf{Com} * \mathbf{D}))$ *and by the languages recognized by semigroups in* $\mathbf{H}$*.* □

The $(\mathcal{D}\mathbf{H} \cap \mathbf{W} \cap \mathcal{E}\mathbf{Com})$-recognizable languages (where $\mathbf{W}$ is one of $\mathcal{L}\mathbf{ZE}$, $\mathcal{L}(\mathbf{Sl} \vee \mathbf{G})$ or $\mathbf{Com} * \mathbf{D}$) are described likewise to $(\mathcal{D}\mathbf{H} \cap \mathbf{W})$-recognizable languages: it suffices to add on the condition that "$u_i \neq 1$ for all $1 \leq i \leq n-1$".

9.4 Comparative tables

We summarize in two tables some of the results of this section. We restrict ourselves to aperiodic pseudovarieties. We present first varieties of languages which are Boolean combinations of languages of the form

$$u_0 A_1^* u_1 A_2^* u_2 \cdots A_n^* u_n$$

where $n \geq 0$, $u_0, \ldots, u_n \in A^*$ with $u_i \neq 1$ $(1 \leq i \leq n-1)$ and $\emptyset \neq A_1, \ldots, A_n \subseteq A$. These varieties are described, furthermore, by the imposition of certain conditions on the u_i's and the A_i's.

Pseudovariety	Conditions
$\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$	$c(u_i) \not\subseteq A_i, A_{i+1}$ $(1 \leq i \leq n-1)$
$\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{ZE}$	$c(u_i) \not\subseteq A_i, A_{i+1}$ $(1 \leq i \leq n-1)$ $A_i = A_j$ or $A_i \cap A_j = \emptyset$
$\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$	$c(u_i) \not\subseteq A_i, A_{i+1}$ $(1 \leq i \leq n-1)$ $A_i \cap A_j = \emptyset$ $(i \neq j)$
$\mathbf{R} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$	$p_1(u_i) \notin A_i$ $(1 \leq i \leq n)$ $s_1(u_i) \in A_{i+1}$ $(1 \leq i \leq n-1)$ if $c(u_i) \subseteq A_{i+1}$ $(1 \leq i \leq n-1)$, then $A_i \cap A_{i+1} = \emptyset$
$\mathbf{R} \cap \mathcal{L}\mathbf{ZE}$	$p_1(u_i) \notin A_i$ $(1 \leq i \leq n)$ if $c(u_i) \subseteq A_{i+1}$ $(1 \leq i \leq n-1)$, then $A_i \cap A_{i+1} = \emptyset$ $A_i = A_j$ or $A_i \cap A_j = \emptyset$
$\mathbf{R} \cap \mathcal{L}\mathbf{Sl}$	$p_1(u_i) \notin A_i$ $(1 \leq i \leq n)$ $A_i \cap A_j = \emptyset$ $(i \neq j)$

Now we present varieties of languages which are Boolean combinations of languages of the form

$$u_0 A_1^+ u_1 A_2^+ u_2 \cdots A_n^+ u_n$$

where $n \geq 0$, $u_0, \ldots, u_n \in A^*$, $\emptyset \neq A_1, \ldots, A_n \subseteq A$, and for every $1 \leq i \leq n$ such that u_i (resp. u_{i-1}) is not the empty word, the first (resp. last) letter of u_i (resp. u_{i-1}) does not lie in A_i.

Pseudovariety	Conditions
$\mathbf{J} \cap \mathcal{L}\mathcal{E}\mathbf{Com}$	if $u_i = 1$ $(1 \leq i \leq n-1)$, then $A_i \cap A_{i+1} = \emptyset$
$\mathbf{J} \cap \mathcal{L}\mathbf{ZE}$	if $u_i = 1$ $(1 \leq i \leq n-1)$, then $A_i \cap A_{i+1} = \emptyset$ $A_i = A_j$ or $A_i \cap A_j = \emptyset$
$\mathbf{J} \cap \mathcal{L}\mathbf{Sl}$	$A_i \cap A_j = \emptyset$ $(i \neq j)$
$\mathbf{J} \cap \mathcal{L}\mathbf{ZE} \cap \mathcal{E}\mathbf{Com}$	$u_i \neq 1$ $(1 \leq i \leq n-1)$ $A_i = A_j$ or $A_i \cap A_j = \emptyset$
$\mathbf{J} \cap \mathcal{L}\mathbf{Sl} \cap \mathcal{E}\mathbf{Com}$	$u_i \neq 1$ $(1 \leq i \leq n-1)$ $A_i \cap A_j = \emptyset$ $(i \neq j)$

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