The Moore-Penrose inverse of a factorization*

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Abstract

In this paper, we consider the product of matrices PAQ, where A is von Neumann regular and there exist P' and Q' such that $P'PA = A = AQQ'$. We give necessary and sufficient conditions in order to PAQ be Moore-Penrose invertible, extending known characterizations. Finally, an application is given to matrices over separative regular rings.

1 Introduction

Let R be an arbitrary ring with unity 1, $\mathcal{M}_{m\times n}(R)$ be the set of $m \times n$ matrices and $\mathcal{M}_m(R)$ the ring of $m \times m$ matrices over R. Let $*$ be an involution, see [8], on the matrices over R. Given an $m\times n$ matrix A over R, A is *(von Neumann) regular* if there exists an $n \times m$ matrix A^- such that

$$AA^-A = A.$$

The set of von Neumann inverses of A will be denoted by $A\{1\}$. That is,

$$A\{1\} = \{X \in \mathcal{M}_{n\times m}(R) : AXA = A\}.$$

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A is said to be *Moore-Penrose invertible* with respect to * if there exists a (unique) $n \times m$ matrix $A^\dagger$ such that:

$$\begin{aligned} AA^\dagger A &= A, \\ A^\dagger AA^\dagger &= A^\dagger, \\ \left(AA^\dagger\right)^* &= AA^\dagger, \\ \left(A^\dagger A\right)^* &= A^\dagger A. \end{aligned}$$

Also, if $m = n$, then the *group inverse* of A exists if there is a (unique) $A^\#$ such that

$$\begin{aligned} AA^\# A &= A, \\ A^\# AA^\# &= A^\#, \\ AA^\# &= A^\# A. \end{aligned}$$

In this paper, we give an alternative proof of the main result from [6], as well as a more general formula for the computation of the Moore-Penrose inverse of a matrix, extending results from [9], [6] and [3]. As an application we derive the Moore-Penrose inverse of matrices over separative regular rings, using recent results that appear in [1].

2 Results

The following lemma was proved in [7] and will provide a simpler and shorter proof of [6, Theorem 1] in the next theorem.

Lemma 1. *Let $A \in \mathcal{M}_{m\times n}(R)$ be a regular matrix and $B \in \mathcal{M}_m(R)$ such that $AX = B$ is a consistent matrix equation. Then the following conditions are equivalent:*

1. $\Gamma = BAA^- + I_m - AA^-$ *is an invertible matrix for one and hence* all *choices of $A^- \in A\{1\}$.*
2. $\Omega = A^-BA + I_n - A^-A$ *is an invertible matrix for one and hence* all *choices of $A^- \in A\{1\}$.*

Moreover,

$$\Omega^{-1} = A^-AA^-\Gamma^{-1}A + I_n - A^-A$$

and also

$$\Gamma^{-1} = A\Omega^{-1}A^-AA^- + I_m - AA^-.$$

Theorem 2. *Let T be an $m \times n$ matrix over R. The following conditions are equivalent:*

1. *T is von Neumann regular and $TT^*TT^- + I_m - TT^-$ is invertible.*
2. *T is von Neumann regular and $T^-TT^*T + I_n - T^-T$ is invertible.*
3. *The Moore-Penrose inverse $T^\dagger$ exists w.r.t. $*$.*

In that case, besides the expressions for $T^\dagger$ in [6],

$$\begin{aligned} T^\dagger &= T^* \left(TT^*TT^- + I_m - TT^-\right)^{*\,-1} \\ &= \left(T^-TT^*T + I_n - T^-T\right)^{*\,-1} T^*. \end{aligned}$$

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma 1, taking $B = TT^*$.

(3) $\Rightarrow$ (1) Let $T^\dagger$ and T^-, respectively, be the Moore-Penrose inverse and a von Neumann inverse of T. Note that

$$\begin{aligned} T^{\dagger *}T^\dagger \left(TT^*TT^-\right) &= T^{\dagger *}T^*T^{\dagger *}T^*TT^- \\ &= T^{\dagger *}T^*TT^- \\ &= TT^\dagger TT^- \\ &= TT^- \end{aligned}$$

and

$$\begin{aligned} \left(TT^*TT^-\right) TT^\dagger T^{\dagger *}T^- &= TT^*TT^\dagger T^{\dagger *}T^- \\ &= TT^*T^{\dagger *}T^*T^{\dagger *}T^- \\ &= TT^*T^{\dagger *}T^- \\ &= TT^\dagger TT^- \\ &= TT^-. \end{aligned}$$

Therefore,

$$\begin{aligned} I_m &= \left(T^{\dagger *}T^\dagger TT^- + I_m - TT^-\right)\left(TT^*TT^- + I_m - TT^-\right) \\ &= \left(TT^*TT^- + I_m - TT^-\right)\left(TT^\dagger T^{\dagger *}T^- + I_m - TT^-\right) \end{aligned}$$

and $TT^*TT^- + I_m - TT^-$ is invertible.

(1) $\Rightarrow$ (3) Let $U = TT^*TT^- + I_m - TT^-$ and $V = T^-TT^*T + I_n - T^-T$. Assume U is invertible, and consequently V invertible. As

$$UT = TT^*T = TV$$

then

$$TT^*\left(TV^{-1}\right) = T = \left(U^{-1}T\right)T^*T,$$

and therefore T is Moore-Penrose invertible (see [8, Lemma 3]) with

$$\begin{aligned} T^\dagger &= \left(TV^{-1}\right)^* T\left(U^{-1}T\right)^* \\ &= \left(U^{-1}T\right)^* T\left(U^{-1}T\right)^* \\ &= \left(U^{-1}TT^*U^{-1}T\right)^* \\ &= \left(U^{-1}TT^*U^{-1}TT^-T\right)^* \\ &= \left(U^{-1}TT^*TT^-U^{-1}T\right)^* \\ &= \left(TT^-U^{-1}T\right)^* \\ &= \left(U^{-1}T\right)^*. \end{aligned}$$

since $UT = TV$, U commutes with TT^- and $U^{-1}TT^*T = T$. As $U^{-1}T = TV^{-1}$,

$$T^\dagger = \left(TV^{-1}\right)^*.$$

□

Remark. Assume $\mathcal{M}_{m\times n}(R)$ is $*$-*regular*, that is, every matrix A over R is regular (or equivalently, R is a regular ring) and

$$A^*A = 0 \Rightarrow A = 0$$

holds. This implication is equivalent to A is $*$-*cancellable*, i.e.,

$$\begin{aligned} A^*AB &= A^*AC \Rightarrow AB = AC, \\ B'AA^* &= C'AA^* \Rightarrow B'A = C'A, \end{aligned}$$

where B, B', C, C' have appropriate sizes. In this case, and by a result of R. Puystjens and D.W. Robinson (see [8, Lemma 3]), *all* matrices over R are Moore-Penrose invertible. So, for any T belonging to a $*$-regular $\mathcal{M}_{m\times n}(R)$ and for every choice of $T^- \in T\{1\}$,

$$\begin{aligned} U &= TT^*TT^- + I_m - TT^-, \\ V &= T^-TT^*T + I_n - T^-T \end{aligned}$$

are invertible matrices.

Theorem 3. *Let $A \in \mathcal{M}_{m\times n}(R)$ with von Neumann inverse A^-. Let $P \in \mathcal{M}_{p\times m}(R)$ and $Q \in \mathcal{M}_{n\times q}(R)$. The following conditions are equivalent:*

1. $\widetilde{U} = AQQ^*A^*P^*PAA^- + I_m - AA^-$ *is invertible.*

2. $\widetilde{V} = A^-AQQ^*A^*P^*PA + I_n - A^-A$ *is invertible.*

3. $(PAQ)^\dagger$ *exists w.r.t. * and there exist* P', Q' *such that* $P'PA = A = AQQ'$.

Moreover,

$$\begin{aligned}(PAQ)^\dagger &= \left(P\widetilde{U}^{-1}AQ\right)^* \\ &= \left(PA\widetilde{V}^{-1}Q\right)^*.\end{aligned}$$

Proof. (1) $\Leftrightarrow$ (2).
If $\widetilde{U}$ is invertible then $AQQ^*A^*P^*AA^-$ is invertible in the ring $AA^-\mathcal{M}_mAA^-$. That is, there exists $X \in AA^-\mathcal{M}_mAA^-$ for which

$$AQQ^*A^*P^*PAA^-X = AA^- = XAQQ^*A^*P^*PAA^-.$$

Then

$$A^-AQQ^*A^*P^*PA\left(A^-XA\right) = A^-A = A^-XAQQ^*A^*P^*PA$$

which implies $A^-XA \in A^-A\mathcal{M}_nA^-A$ is an inverse of $A^-AQQ^*A^*P^*PA$ in $A^-A\mathcal{M}_nA^-A$. Therefore, $A^-AQQ^*A^*P^*PA + I_n + A^-A$ is an invertible matrix.

(3) $\Rightarrow$ (1).
In the first place, we remark that

$$PAQ\,(PAQ)^*+I-PAQ\,(PAQ)^\dagger = PAQ\,(PAQ)^*\,PAQ\,(PAQ)^\dagger+I-PAQ\,(PAQ)^\dagger$$

has inverse

$$((PAQ)^*)^\dagger\,(PAQ)^\dagger + I - PAQ\,(PAQ)^\dagger.$$

As $(PAQ)^\dagger$ is in particular a von Neumann inverse of PAQ, then

$$PAQ\,(PAQ)^*\,PAQ\,(PAQ)^- + I - PAQ\,(PAQ)^-$$

is invertible for any choice of $(PAQ)^- \in PAQ\{1\}$.
It is clear that $Q'A^-P'$ is a von Neumann inverse of PAQ. As $(PAQ)^\dagger$ exists, then

$$PAQ\,(PAQ)^*\,PAQ\left(Q'A^-P'\right) + I_p - PAQ\left(Q'A^-P'\right)$$

is invertible, i.e.,

$$K = PAQQ^*A^*P^*PAA^-P' + I_p - PAA^-P'$$

is invertible. Setting $E = PAA^-P'$, and since $E^2 = E$ and K is invertible, then

$$\begin{aligned} W &= PAQQ^*A^*P^*PAA^-P' \\ &= EKE \end{aligned}$$

is invertible in the ring $E\mathcal{M}_p(R)E$. So, there exists a $X \in E\mathcal{M}_p(R)E$ such that

$$E = WX, \quad (1)$$
$$E = XW. \quad (2)$$

By (1), and as $EX = X$,

$$\begin{aligned} PAA^-P' &= E \\ &= WX \\ &= WEX \\ &= \left(WPAA^-\right)P'X \\ &= \left(PAQQ^*A^*P^*PAA^-\right)P'X. \end{aligned}$$

Multiplying on the left by P' and on the right by PAA^-, we have

$$\left(AQQ^*A^*P^*PAA^-\right)P'XPAA^- = AA^-$$

and therefore

$$\left[\left(AA^-\right)AQQ^*A^*P^*P\left(AA^-\right)\right]\left[\left(AA^-\right)P'XP\left(AA^-\right)\right] = AA^- \quad (3)$$

By (2), and as $XE = X$,

$$\begin{aligned} PAA^-P' &= E \\ &= XW \\ &= XEW \\ &= XPAA^-P'W \\ &= XP\left(AQQ^*A^*P^*PAA^-P'\right). \end{aligned}$$

Multiplying on the left by AA^-P' and on the right by PAA^-,

$$\left[\left(AA^-\right)P'XP\left(AA^-\right)\right]\left[\left(AA^-\right)AQQ^*A^*P^*P\left(AA^-\right)\right] = AA^-. \quad (4)$$

Combining (3) and (4), it follows that $AQQ^*A^*P^*PAA^-$ is invertible in the ring $AA^-\mathcal{M}_m(R)\,AA^-$ and therefore $AQQ^*A^*P^*PAA^- + I_m - AA^-$ is an invertible matrix.

(1) $\Rightarrow$ (3) If $\widetilde{U} = AQQ^*A^*P^*PAA^- + I_m - AA^-$ is invertible, then as $AA^-\widetilde{U} = AQQ^*A^*P^*PAA^-$,

$$\begin{aligned} A &= AA^-A \\ &= AA^-\widetilde{U}\widetilde{U}^{-1}A \\ &= AQ\left(Q^*A^*P^*PAA^-\widetilde{U}^{-1}A\right) \end{aligned}$$

and we take $Q' = Q^*A^*P^*PAA^-\widetilde{U}^{-1}A$. Moreover, since $\widetilde{U}A = AQQ^*A^*P^*PA$ and $\widetilde{U}$ is invertible,

$$A = \left(\widetilde{U}^{-1}AQQ^*A^*P^*\right)PA$$

and we can take $P' = \widetilde{U}^{-1}AQQ^*A^*P^*$. To show that $(PAQ)^\dagger$ exists it is sufficient to show that

$$PAQ\,(PAQ)^*\,PAQ\,(PAQ)^- + I_p - PAQ\,(PAQ)^-$$

is invertible for one choice of $(PAQ)^-$, in this case for $(PAQ)^- = Q'A^-P'$. As $\widetilde{U}$ is invertible in the ring $\mathcal{M}_m(R)$ then $AA^-\widetilde{U}AA^-$ is invertible in the ring $AA^-\mathcal{M}_m(R)\,AA^-$. So, there exists a X in $AA^-\mathcal{M}_m(R)\,AA^-$ such that

$$X\left(AA^-\right)\widetilde{U}\left(AA^-\right) = \left(AA^-\right)\widetilde{U}\left(AA^-\right)X = AA^-.$$

So,

$$\left[\left(AA^-\right)X\left(AA^-\right)\right]\left[\left(AA^-\right)AQQ^*A^*P^*P\left(AA^-\right)\right] = AA^-,$$

and since $AA^- = (AA^-)^2 = (AA^-)\,P'P\,(AA^-) = P'PAA^-$ and $A = P'PA$, it follows that

$$\left[\left(AA^-P'\right)PAA^-XP'\left(PAA^-P'\right)\right]\left[\left(PAA^-P'\right)PAQQ^*A^*P^*\left(PAA^-\right)\right] = AA^-.$$

Multiplying on the left by P and on the right by P',

$$\left[\left(PAA^-P'\right)PAA^-XP'\left(PAA^-P'\right)\right]\left[\left(PAA^-P'\right)PAQQ^*A^*P^*\left(PAA^-P'\right)\right] = PAA^-P'.$$

Analogously, as

$$\left[\left(AA^-\right)AQQ^*A^*P^*P\left(AA^-\right)\right]\left[\left(AA^-\right)X\left(AA^-\right)\right] = AA^-$$

then

$$\left[\left(AA^-P'\right)PAQQ^*A^*P^*\left(PAA^-P'\right)\right]\left[\left(PAA^-P'\right)PAA^-XP'\left(PAA^-\right)\right] = AA^-,$$

and multiplying on the left by P and on the right by P',

$$\left[\left(PAA^{-}P'\right)PAQQ^{*}A^{*}P^{*}\left(PAA^{-}P'\right)\right]\left[\left(PAA^{-}P'\right)PAA^{-}XP'\left(PAA^{-}P'\right)\right]=PAA^{-}P'.$$

Therefore,

$$\left(PAA^{-}P'\right)PAQQ^{*}A^{*}P^{*}\left(PAA^{-}P'\right)$$

is invertible in the ring $\left(PAA^{-}P'\right)\mathcal{M}_{p}\left(R\right)\left(PAA^{-}P'\right)$ and consequently

$$\left(PAA^{-}P'\right)PAQQ^{*}A^{*}P^{*}\left(PAA^{-}P'\right)+I_{p}-PAA^{-}P'$$

is an invertible matrix. That is,

$$PAQ\left(PAQ\right)^{*}PAQ\left(Q'A^{-}P'\right)+I_{p}-PAQ\left(Q'A^{-}P'\right)$$

is an invertible matrix.

Let $U=PAQ\left(PAQ\right)^{*}PAQ\left(Q'A^{-}P'\right)+I_{p}-PAQ\left(Q'A^{-}P'\right)$. As $UPAA^{-}=PAA^{-}\widetilde{U}$ and the invertibility of $\widetilde{U}$ implies the invertibility of U, then

$$U^{-1}PAA^{-}=PAA^{-}\widetilde{U}^{-1}.$$

Furthermore, and since AA^{-} commutes with $\widetilde{U}$, then $AA^{-}\widetilde{U}^{-1}=\widetilde{U}^{-1}AA^{-}$. So,

$$\begin{aligned}(PAQ)^{\dagger} &= \left(U^{-1}PAA^{-}AQ\right)^{*}\\ &= \left(PAA^{-}\widetilde{U}^{-1}AQ\right)^{*}\\ &= \left(P\widetilde{U}^{-1}AQ\right)^{*}.\end{aligned}$$

In addition, $\widetilde{U}A=A\widetilde{V}$ and thus $A\widetilde{V}^{-1}=\widetilde{U}^{-1}A$. So,

$$(PAQ)^{\dagger}=\left(PA\widetilde{V}^{-1}Q\right)^{*}. \ \square$$

Remark. Using the same notation of the previous proof, it is known (see [6]) that if U (and therefore V) is invertible then PAQ is Moore-Penrose invertible with

$$(PAQ)^{\dagger}=\left(PAQ\right)^{*}\left(UU^{*}\right)^{-1}\left(PAQ\left(PAQ\right)^{*}\right).$$

As $UPAA^{-}=PAA^{-}\widetilde{U}$ and the invertibility of $\widetilde{U}$ implies the invertibility of U, then

$$U^{-1}PAA^{-}=PAA^{-}\widetilde{U}^{-1}.$$

Furthermore, and since AA^- commutes with $\widetilde{U}$, then $AA^-\widetilde{U}^{-1} = \widetilde{U}^{-1}AA^-$. So,

$$\begin{aligned}
(PAQ)^\dagger &= Q^*A^*P^*U^{*-1}U^{-1}PAQ\,(PAQ)^* \\
&= Q^*A^*\widetilde{U}^{*-1}\left(A^-\right)^* A^*P^*PAA^-\widetilde{U}^{-1}AQ\,(PAQ)^* \\
&= Q^*A^*\widetilde{U}^{*-1}P^*P\widetilde{U}^{-1}AQ\,(PAQ)^* \\
&= (AQ)^*\left(P\widetilde{U}^{-1}\right)^* P\widetilde{U}^{-1}AQ\,(PAQ)^* \\
&= \left(P\widetilde{U}^{-1}AQ\right)^* P\widetilde{U}^{-1}AQ\,(PAQ)^*.
\end{aligned}$$

In addition, $\widetilde{U}A = A\widetilde{V}$ and thus $A\widetilde{V}^{-1} = \widetilde{U}^{-1}A$. So,

$$(PAQ)^\dagger = \left(PA\widetilde{V}^{-1}Q\right)^* PA\widetilde{V}^{-1}Q\,(PAQ)^*.$$

Theorem 4. *If PAQ is a matrix product for which there exist matrices P' and Q' such that $P'PA = A = AQQ'$, then the Moore-Penrose inverse of PAQ exists if and only if $(PA)^{1,3}$ and $(AQ)^{1,4}$ exist, in which case*

$$(PAQ)^\dagger = (AQ)^{1,4}\,A\,(PA)^{1,3}.$$

Proof.

Assume, in the first place, $(PA)^{1,3}$ and $(AQ)^{1,4}$ exist. Then

$$AQ = AQ\,(AQ)^{1,4}\,AQ = AQ\,(AQ)^*\left((AQ)^{1,4}\right)^*,$$

and hence

$$PAQ = PAQ\,(PAQ)^*\left(P'\right)^*\left((AQ)^{1,4}\right)^*.$$

Analogously,

$$PA = PA\,(PA)^{1,3}\,PA = \left((PA)^{1,3}\right)^*(PA)^*\,PA$$

and hence

$$PAQ = \left((PA)^{1,3}\right)^*\left(Q'\right)^*(PAQ)^*\,PAQ.$$

We therefore have,

$$(PAQ)^\dagger = (AQ)^{1,4}\,P'PAQQ'\,(PA)^{1,3} = (AQ)^{1,4}\,A\,(PA)^{1,3}.$$

Conversely, assume $(PAQ)^\dagger$ exists. By one hand,

$$PAQ = PAQ\,(PAQ)^*\left((PAQ)^\dagger\right)^*$$

which implies $AQ = AQ\,(AQ)^*\,X$ is a consistent matrix equation on X. We will show that $X^* \in AQ\,\{1,4\}$. Indeed it is a von Neumann inverse of AQ as

$$AQ = AQX^*AQ\,(AQ)^*\,X = AQX^*AQ,$$

and the idempotent X^*AQ is symmetric since

$$\begin{aligned} X^*AQ &= X^*AQX^*AQ \\ &= X^*AQX^*AQ\,(AQ)^*\,X \\ &= X^*AQ\,(AQ)^*\,X. \end{aligned}$$

Similar arguments show that $(PA)^{1,3}$ exists if $(PAQ)^\dagger$ exists. □

3 Matrices over separative regular rings

Throughout this section, R is a *separative* regular ring, i.e., for any finitely generated projective R-modules A and B, the following cancellation property holds:

$$A \oplus A \cong A \oplus B \cong B \oplus B \Rightarrow A \cong B.$$

A recent result states that every *square* matrix over a separative regular ring admits a diagonal reduction, i.e., is equivalent to a diagonal matrix (see [1, Theorem 2.5]). This means that for square matrices over separative regular rings the Moore-Penrose inverse can be characterized by [6, Theorem 2].

For nonsquare matrices over separative regular rings the characterization of the Moore-Penrose inverse can now be done in the following way:

Let $A_{m\times n} \in \mathcal{M}_{m\times n}(R)$, with $m < n$. Then we can complete it to a square matrix by adding zeros, and it follows from [1] that there exist invertible matrices P, Q and a diagonal matrix D such that

$$\begin{bmatrix} A_{m\times n} \\ 0_{(n-m)\times n} \end{bmatrix} = PDQ. \tag{5}$$

Therefore

$$A_{m\times n} = \left(\begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix} P\right) DQ. \tag{6}$$

We are now in the conditions of Theorem 3 since $P' = P^{-1}\begin{bmatrix} I_m \\ 0_{(n-m)\times m} \end{bmatrix}$ is a matrix such that

$$P'\left(\begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix} P\right) D = D.$$

We therefore can apply Theorem 3 to the factorization (6). That is, $A^{\dagger}_{m\times n}$ exists if and only if

$$DQ \left[\begin{array}{cc} A^*_{n\times m} & 0_{n\times(n-m)} \end{array} \right] PDD^- + I_n - DD^-$$

is invertible for one and hence all choices of von Neumann inverses D^- of D.

For the case $n < m$, the outline of the application is analogous.

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