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Numerical Optimization Experiments Using the Hyperbolic Smoothing Strategy to Solve MPCC

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Abstract

In this work we solve Mathematical Programs with Complementarity Constraints using the hyperbolic smoothing strategy. Under this approach, the complementarity condition is relaxed through the use of the hyperbolic smoothing function, involving a positive parameter that can be decreased to zero. An iterative algorithm is implemented in MATLAB language and a set of AMPL problems from MacMPEC database were tested.

Key words: complementarity constraints, hyperbolic smoothing, SQP

1 Introduction

Mathematical Programs with Complementarity Constraints (MPCC) is a subclass of more general Mathematical Programs with Equilibrium Constraints (MPEC). These kind of constraints may come in the form of a game, a variational inequality or as stationary conditions of an optimization problem. The main applications areas are Engineering and Economics [1], [2], [3]. They are so widespread in this areas because the concept of complementarity is synonymous with the notion of system equilibrium. They are very difficult to solve as the usual constraint qualifications necessary to guarantee the algorithms convergence fail in all feasible points [4]. This complexity is caused by the disjunctive nature of the complementarity constraints. They have been proposed some nonlinear approaches to solve MPCC, starting with the smoothing scheme [5], [6], the regularization scheme [7], [8] the interior

point methods [10], the penalty approaches [11], [12], [13] and the "elastic mode" for nonlinear programming in conjunction with a sequential quadratic programming (SQP) algorithm [14]. On this paper we present the hyperbolic smoothing strategy [15] and we apply it for solving MPCC. The proposed method adopts a C^{∞} differential class function, in order to overcome the difficulties on solving the complementarity constraints.
This paper is organized as follows. Next section defines the MPCC problem. Some optimal issues are presented in Section 3. The hyperbolic smoothing technique and the MATLAB algorithm are described in Section 4. Numerical experiments using the hyperbolic smoothing algorithm are reported in Section 5. Some conclusions and future work are exposed in Section 6.

2 Problem definition

We consider Mathematical Program with Complementarity Constraints (MPCC):

$$\begin{array}{ll} \min & f(x) \\ s.t. & c_i(x) = 0,\ i \in E, \\ & c_i(x) \geq 0,\ i \in I, \\ & 0 \leq x_1 \perp x_2 \geq 0, \end{array} \tag{1}$$

where f and c are the nonlinear objective function and the constraint functions, respectively, assumed to be twice continuously differentiable. E and I are two disjoined finite index sets with cardinality p and m, respectively. A decomposition $x = (x_0, x_1, x_2)$ of the variables is used where $x_0 \in \mathbb{R}^n$ (control variables) and $(x_1, x_2) \in \mathbb{R}^{2q}$ (state variables). The expressions $0 \leq x_1 \perp x_2 \geq 0 : \mathbb{R}^{2q} \to \mathbb{R}^q$ are the q complementarity constraints. One attractive way of solving (1) is to consider its equivalent nonlinear programming formulation:

$$\begin{array}{ll} min & f(x) \\ s.t. & c_i(x) = 0,\ i \in E, \\ & c_i(x) \geq 0,\ i \in I, \\ & x_1 \geq 0,\ x_2 \geq 0, \\ & X_1 x_2 \leq 0, \end{array} \tag{2}$$

where X_1 is a diagonal matrix with x_1 as diagonal. On this formulation the complementarity constraints are replaced by a set of nonlinear inequalities, such as $x_{1j}\, x_{2j} \leq 0,\ j = 1, \ldots, q$, enabling the use of standard NLP solvers to solve the complementary constraints.

3 Optimal issues

This section introduces some concepts related to stationarity and first order conditions. The optimality concepts follow the development of [16] and the corresponding proves can

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be consulted in this work. Consider two index sets: $X_1, X_2 \subset \{1, \dots, q\}$ with $X_1 \cup X_2 = \{1, \dots, q\}$, denoting the corresponding complements in $\{1, \dots, q\}$ by $X_1^{\perp}$ e $X_2^{\perp}$. For each pair of index one define the relaxed NLP corresponding to (1):

$$\begin{array}{ll} min & f(x) \\ s.t. & c_i(x) = 0,\ i \in E, \\ & c_i(x) \geq 0,\ i \in I, \\ & x_{1j} = 0,\ \forall j \in X_2^{\perp}, \\ & x_{2j} = 0,\ \forall j \in X_1^{\perp}, \\ & x_{1j} \geq 0,\ \forall j \in X_2, \\ & x_{2j} \geq 0,\ \forall j \in X_1. \end{array} \tag{3}$$

Concepts like constraints qualification, stationarity and second order conditions of the MPCC problem will be defined in terms of (3). The linear independence constraint qualification, LICQ, is extended to MPCC that is MPCC-LICQ:

Definition 1. MPCC-LICQ Consider $x_1, x_2 \geq 0$ and define: $X_1 = \{j : x_{1j} = 0\}$, $X_2 = \{j : x_{2j} = 0\}$. The MPCC problem verifies the MPCC-LICQ at x if the corresponding (3) verifies the LICQ.

If x^* is a local solution of (3) and satisfies $x_1^{*T} x_2^* = 0$, then x^* is also a local solution of original MPCC.

There are several kinds of stationarity defined for MPCC problem. Among them, the strong stationarity is the following one:

Definition 2. Strong stationarity x^* is a strong stationary point if exist Lagrange multipliers λ, $\widehat{\nu}_1$ and $\widehat{\nu}_2$ so that:

$$\begin{array}{l} \nabla f^* - [\nabla(c_i^*), i \in E \ : \ \nabla(c_i^*), i \in I]\, \lambda - \begin{pmatrix} 0 \\ \widehat{\nu}_1 \\ \widehat{\nu}_2 \end{pmatrix} = 0, \\ c_i^* = 0, i \in E, \\ c_i^* \geq 0, i \in I, \\ x_1^* \geq 0, \\ x_2^* \geq 0, \\ x_{1j}^* = 0 \text{ or } x_{2j}^* = 0, \\ \lambda_i \geq 0, i \in I, \\ c_i \lambda_i = 0, \\ x_{1j}^* \widehat{\nu}_{1j} = 0, \\ x_{2j}^* \widehat{\nu}_{2j} = 0, \\ \text{if } x_{1j}^* = x_{2j}^* = 0 \text{ then } \widehat{\nu}_{1j} \geq 0 \text{ and } \widehat{\nu}_{2j} \geq 0. \end{array} \tag{4}$$

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Note that (4) are the first order optimality conditions of the problem (3) at x^*. As theoretical support, we summarized some known results concerning constraint qualifications and first order optimality conditions of MPCC. Based on these ideas, a computational implementation of a hyperbolic smoothing strategy was developed. Details of the corresponding hyperbolic smoothing algorithm are in next section.

4 Hyperbolic smoothing

They have been proposed several smoothing approaches, the most obvious smoothing analysed by [9] is to replace $X_1x_2 \leq 0$ by $X_1x_2 \leq \epsilon_k$, and solve a sequence of NLPs, decreasing ϵ_k to zero. Another similar approach studied by [7] is to gather the complementarity constraints into a single constraint by $x_1^T x_2 \leq \epsilon_k$. Other alternative is to penalize the complementarity constraints [12], solving a sequence of NLPs where the objective is modified as

$$\min f(x) + \rho_k x_1^T x_2$$

for a sequence of increasing penalty parameters $\rho_k > 0$.
Another smoothing idea [6] is to replace the complementarity constraints by the smoothed function,

$$\psi_\mu(x_{1j}, x_{2j}) = \sqrt{(x_{1j} - x_{2j})^2 + 4\mu} - x_{1j} - x_{2j} = 0,$$

for $j = 1, \ldots, q$, where $\mu > 0$ is a parameter that decreases to zero.
On this work we consider the following NLP:

$$\begin{array}{ll} min & f(x) \\ s.t. & c_i(x) = 0,\ i \in E, \\ & c_i(x) \geq 0,\ i \in I, \\ & x_1 \geq 0,\ x_2 \geq 0, \\ & \phi(x_1, x_2) \leq 0, \end{array} \tag{5}$$

where $\phi(x_1, x_2) = (\varphi_\tau(x_{11}, x_{21}), \ldots, \varphi_\tau(x_{1q}, x_{2q}))$ is a vector and φ_τ is the hyperbolic smoothing function defined as follows:

$$\varphi_\tau(x_{1j}, x_{2j}) = \frac{1}{2}\left(x_{1j}x_{2j} + \sqrt{(x_{1j}x_{2j})^2 + \tau^2}\right),$$

for $j = 1, \ldots, q$ and $\tau \to 0$. An algorithm was implemented (Algorithm 1) to iteratively solve problem (5) with $\tau \to 0$. This algorithm has two iterative procedures, the inner one is performed by `fmincon` routine from MATLAB Optimization toolbox, that uses the SQP method.

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Algorithm 1 Hyperbolic smoothing

Take initial values x_0, $\tau_0 > 0$ and tolerances ϵ_1, ϵ_2.
for $k = 0, 1, 2, \ldots$ **do**
  Solve the minimization problem (5) with x_k, τ_k obtaining x_{k+1}.
  if $\|\nabla L(x_{k+1}, \ldots)\| \leq \epsilon_1$ and $\|x_1^T x_2\| \leq \epsilon_2$ **then**
    STOP.
  else
    $\tau_{k+1} = r\tau_k$, $0 < r < 1$.
  end if
end for

To evaluate the stop criterium in the algorithm, we consider the following equality in the solution x^*:

$$\nabla L(x^*, \delta, \gamma, \xi) = \nabla f(x^*) - \sum_{i=1}^{m} \delta_i \nabla c_i(x^*) - \sum_{i=1}^{p} \gamma_i \nabla c_i(x^*) + \sum_{j=1}^{q} \xi_i \nabla \varphi_{\tau,j}(x^*)$$

where for $j = 1, \ldots, q$ and $x \in \mathbb{R}^n$ we have

$$\nabla \varphi_{\tau,j}(x^*) = \frac{1}{2}\left(\nabla x_{1j}x_{2j} + \frac{x_{1j}x_{2j}\nabla x_{1j}x_{2j}}{\sqrt{(x_{1j}x_{2j})^2 + \tau^2}}\right).$$

The Lagrange multipliers δ, γ and ξ are an output of the `fmincon` routine from MATLAB. The tolerances used in the stop criterium are $\epsilon_1 = \epsilon_2 = 10^{-4}$. The initial choices, $\tau_0 = 0.25$ and $r = 0.25$ were considered. Next section reports the numerical results using 45 test problems.

5 Numerical results

This section describes the numerical experiments with an implementation of the hyperbolic smoothing scheme for problem (1). The computational experiments were made on a *2.26 GHz Intel Core 2 Duo* with 8GB of RAM, MAC OS 10.6.8 operating system. The MATLAB version used was 7.11.0 (R2010b). The `fmincon` routine is connected to the modeling language AMPL [17] by a MATLAB mex interface and the test problems are from MacMPEC database [18].
Table 1 reports the numerical results achieved by Algorithm 1, the first column indicates the name of the test problem, from column 2 to 5, the problem dimensions are presented. Column f^* shows the final objective function value and column $\|\nabla L\|$ presents the norm of the Lagrangian function of problem (5). The last three columns give information about the performance of the algorithm. Column `int` presents the number of internal iterations performed by the `fmincon` routine from MATLAB, column `ext` shows the number of external

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Problem	n	m	p	q	f^*	$\|\nabla L\|$	int	ext	nfe
bard1	5	3	1	3	17.000	1.781910e-15	63	10	661
bard3	6	2	3	1	-12.679	8.149116e-09	19	18	161
bard1m	6	3	1	3	17.000	9.949933e-16	40	10	480
bard2m	12	4	5	3	-6598.000	7.831440e-06	104	10	2076
bilevel2	16	9	4	8	-6600.000	5.560937e-05	229	11	5275
bilevel2m	16	9	4	8	-6600.000	5.560937e-05	229	11	5275
dempe	3	1	1	1	28.258	5.577955e-06	175	26	3952
desilva	6	2	2	2	-1.000	7.450581e-09	35	18	677
df1	2	3	0	1	0.000	9.425165e-09	13	10	54
ex9.1.1	13	5	7	5	-13.000	1.159107e-15	27	20	419
ex9.1.2	8	2	5	2	-3.000	1.364484e-07	35	22	521
ex9.1.4	8	2	5	2	-37.000	4.322747e-15	22	17	229
ex9.1.5	13	5	7	5	-1.000	6.182457e-15	27	21	421
ex9.1.8	11	4	5	3	-3.250	4.965068e-16	23	19	311
ex9.1.10	11	4	5	3	-3.250	4.965068e-16	23	19	311
ex9.2.9	9	3	5	3	2.000	5.787607e-16	15	10	281
flp2	4	2	0	2	0.000	1.568440e-07	58	12	706
flp4-1	80	60	0	30	0.000	5.132135e-12	11	10	902
flp4-2	110	110	0	60	0.000	9.102046e-12	11	10	1232
flp4-3	140	170	0	70	0.000	7.670026e-12	11	10	1562
flp4-4	200	250	0	100	0.000	9.583691e-12	11	10	2222
incid-set1-8	118	54	50	49	0.000	1.147902e-14	15	11	1949
incid-set1c-8	117	61	49	49	0.000	1.288618e-07	14	10	1844
jr1	2	1	0	1	0.500	2.349058e-08	63	3	457
kth1	2	1	0	1	0.000	0	3	2	12
kth2	2	1	0	1	0.000	2.014697e-08	13	10	53
kth3	2	1	0	1	0.500	6.852287e-06	23	10	236
nash1a	6	2	2	2	0.000	1.701382e-07	13	10	104
nash1b	6	2	2	2	0.000	2.359488e-07	16	10	187
nash1c	6	2	2	2	0.000	2.359719e-07	14	10	130
nash1d	6	2	2	2	0.000	2.359995e-07	16	10	153
outrata31	5	4	0	4	3.208	2.704044e-08	85	10	903
outrata32	5	4	0	4	3.449	2.823092e-08	139	15	3419
outrata33	5	4	0	4	4.604	6.415710e-07	144	12	2674
outrata34	5	4	0	4	6.593	4.768317e-07	172	17	2662
qpec1	30	20	0	20	80.000	1.055822e-14	13	10	416
scholtes1	3	1	0	1	2.000	5.176421e-08	107	10	2137
scholtes2	3	1	0	1	15.000	1.336380e-07	107	10	2137
scholtes3	2	1	0	1	0.500	2.279238e-07	36	11	380
scholtes5	3	2	0	2	1.000	7.377520e-06	223	10	1987
scale1	2	1	0	1	1.000	9.643328e-05	107	21	824
scale2	2	1	0	1	1.000	1.491230e-06	74	17	729
scale3	2	1	0	1	1.000	2.210586e-05	28	12	177
scale5	2	1	0	1	100.000	7.805362e-05	43	11	422
sl1	8	3	2	3	0.0001	3.266939e-08	13	10	157
stackelberg1	3	1	1	1	-3266.667	1.101177e-06	65	9	946

Table 1: Numerical results.

iterations and the last column reports the number of function evaluations. The solutions obtained by Algorithm 1 are similar to the ones reported in MacMPEC database with good accuracy.

6 Conclusions and future work

An iterative algorithm in MATLAB language to solve MPCC was implemented. The algorithm aims to compute a local optimal solution joining the hyperbolic smoothing the SQP strategy. The algorithm is still in an improvement phase but some conclusions can already be taken: the promising numerical results present good accuracy of the solutions when compared with the ones provided from the MacMPEC test problem database. As future work, it is intended to test the method on large scale test problems and compare the hyperbolic

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smoothing strategy with others smoothing methods suggested in literature.

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References

[1] Z. Luo, J. Pang and D. Ralph, *Mathematical programs with equilibrium constraints*, Cambridge University Press, 1996.

[2] M. Ferris and J. Pang, *Engineering and economic applications of complementarity constraints*, SIAM Review **39** (1997) 669–713.

[3] J. Outrata, M. Kocvara and J. Zowe, *Nonsmooth approach to optimization problems with equilibrium constraints*, Kluwer Academic Publishers, Dordrecht, 1998.

[4] X. Chen, M. Florian, *The nonlinear bilevel programming problem: formulations, regularity and optimality conditions*, Optimization **32** (1995) 193–209.

[5] M. Fukushima, and J. Pang, *Convergence of a smoothing continuation method for mathematical programs with complementarity constraints*, Lectures Notes in Economics and Mathematical Systems, Springer-Verlag **447** (1999) 99–110.

[6] F. Facchinei, H. Jiang and L. Qi, *A smoothing method for mathematical programs with equilibrium constraints*, Mathematical Programming **85** (1999) 107–134.

[7] S. Scholtes, *Convergence proprieties of a regularization schemes for mathematical programs with complementarity constraints*, SIAM Journal on Optimization **11** (2001) 918–936.

[8] M. Monteiro and H. Rodrigues, *Combining the regularization strategy and the SQP to solve MPCC – A MATLAB implementation*, Journal of Computational and Applied Mathematics **235** (2011) 5348–5356.

[9] H. Scheel and S. Scholtes, *Mathematical programs with complementarity constraints: Stationarity, optimality and sensitivity*, Mathematics of Operations Research **25** (2001) 1–22.

[10] RAGHUNATHAN, A. AND BIEGLER, L., *Interior point methods for mathematical programs with complementarity constraints*, CAPD technical report, Department of Chemical Engineering, Carnegie Mellon University, June 2003

[11] X. HU AND D. RALPH, *Convergence of a penalty method for mathematical programs with complementarity constraints*, Journal of Optimization Theory and Application **123** (2004) 365–390.

[12] D. RALPH AND S. WRIGHT, *Some proprieties of regularization and penalization schemes for MPECs*, Optimization Methods and Software **19** (2004) 527–556.

[13] M. MONTEIRO AND J. MEIRA, *A penalty method and a regularization strategy to solve MPCC*, International Journal of Computers Mathematics **88** (2011) 145–149.

[14] M. ANITESCU, *On using the elastic mode in nonlinear programming approaches to mathematical programs with complementarity constraints*, SIAM Journal on Optimization **15** (2005) 1203–1236.

[15] A. XAVIER AND A. OLIVEIRA, *Optimal Covering of Plane Domains by Circles Via Hyperbolic Smoothing*, Journal of Global Optimization **31** (2005) 493–504

[16] R. FLETCHER, S. LEYFFER, D. RALPH AND S. SCHOLTES, *Local convergence of SQP methods for mathematical programs with equilibrium constraints*, SIAM Journal on Optimization **17** (2006) 259–286.

[17] R. FOURER AND B. KERNIGHAN, *AMPL: A modeling language for mathematical programming*, Duxburg Press, Massachusetts, 1993.

[18] S. LEYFFER, *MacMPEC AMPL collection of mathematical programs with equilibrium constraints*, http://wiki.mcs.anl.gov/leyffer/index.php/MacMPEC

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