FLORES, P., LANKARANI, H.M., DYNAMIC RESPONSE OF MULTIBODY SYSTEMS WITH MULTIPLE CLEARANCE JOINTS. ASME JOURNAL OF COMPUTATIONAL AND NONLINEAR DYNAMICS, VOL. 7(3), 031003-13, 2012.

ABSTRACT

A general methodology for the dynamic modeling and analysis of planar multibody systems with multiple clearance joints is presented. The inter-connecting bodies that constitute a real physical mechanical joint are modeled as colliding components, whose dynamic behavior is influenced by the geometric, physical and mechanical properties of the contacting surfaces. A continuous contact force model, based on the elastic Hertz theory, together with a dissipative term associated with the internal damping, is utilized to evaluate the intra-joint normal contact forces. The incorporation of the friction phenomenon, based on the classical Coulomb's friction law, is also included in this study. The suitable contact force models are embedded into the dynamic equations of motion for the multibody systems. In the sequel of this process, the fundamental methods to deal with contact-impact events in mechanical systems are presented. Finally, two planar mechanisms with multiple revolute clearance joints are used to demonstrate the accuracy and efficiency of the presented approach and to discuss the main assumptions and procedures adopted. The effects of single versus multiple clearance revolute joints are discussed.

Keywords: Multiple clearance joints, contact-impact forces, multibody dynamics

INTRODUCTION

Over the last few decades, the computer-aided analysis of mechanical systems has emerged as an important scientific part of the applied mechanics field, with important applications in several branches of engineering [1-4]. This has been made possible mainly due to impressive improvements of the computers at both hardware and software levels, due to the creation of robust and accurate computational tools, and due to the demand for analysis of complex multibody mechanical systems. Decades ago, the design of machines and components was based on trial and error and knowledgeable craftsmanship. Later, algebraic methods for analysis eliminated part of the limitations of the trial and error approach, and leaded to documented methods used in the design of mechanical components [5]. In today's industry, there is little room for error and a great need for optimized and cost effective production of components and machines with high reliability and durability.

In the real world, mechanical systems do not have perfect kinematic joints due to the functional tolerances between the adjacent segments [6]. Either due to the loads carried by the mechanical devices or the misalignments that are required for their operation, real joints must be lubricated [7] or include bushings [8], generally made with metals or polymers. By using rubber bushings, a conventional joint is transformed into a joint with clearance allowing for the mobility of the over-constrained system in which it is used for [9]. Therefore, appropriate tribological models must be devised in the framework of their application in general multibody mechanical systems. The characterization of the normal contact forces in the non-lubricated joints is realized by using continuous contact force models [10], while their tangential forces are obtained by using appropriate friction force models [11]. Joints with rubber bushings are usually described by the methodology suggested by Park and Nikravesh [9]. However, this is a linear model which does not include coupling between radial and axial or bending loads. A more advanced model for rubber bushings is realized by obtaining the static nonlinear response of the rubber bushings using finite elements and implementing the force-displacement curves as carpet plots in the computational code that supports the analysis [12].

The undesired consequences of the clearance joints on the behavior of multibody systems have motivated various theoretical and experimental investigations over the last three decades [12-18]. Furthermore, mechanical systems with clearance joints can exhibit a predictable nonlinear dynamic response [19, 20]. This is an important feature for the design and control of these systems. Consequently, proper modeling of clearance joints in mechanical systems is required to achieve better understanding of the dynamic performance of the machines and mechanisms. This aspect has gained much importance due to the demand for the proper design of the real joints in many industrial applications. The general-purpose computational tools used for the design and analysis of multibody systems have a wide number of mechanical systems modeling features which require the description of rigid or flexible bodies for which the geometry, mass, center of mass, moment of inertia, and other relevant properties are defined [21, 22]. The computational codes also provide a large library of kinematic joints that constrain the relative degrees of freedom between connected bodies. The kinematic joints typically available in the commercial programs are represented as ideal joints, for which there are no clearances or deformations associated with them.

Modeling the dynamics of multibody systems with clearances and imperfections is a challenging issue in mechanical design and much work still remains to be done to achieve accuracy of the modeling tools. The primary objective of this work is to explore the mathematical modeling of real joints in the context of multibody dynamics. A study of a planar mechanical system is done in order to quantify the influence of the main parameters on the dynamic response of mechanical systems with multiple clearances joints. For the numerical simulations, the computational time is evaluated in order to analyze the computational efficiency of the presented methodology. It should be highlighted that the clearance joint approach is more accurate and realistic than the common idealized revolute joint model employed by most multibody system frameworks, because it uses more realistic contact representation between the two bodies in the region of the joint, rather than a simple kinematic constraint. Furthermore, when a system has multiple clearance joints, its dynamic response is affected by the clearance, operating conditions and contact properties. In addition, with the present formulation, how a jointed constrained system migrates towards a joint-free unconstrained system is demonstrated in this study. Thus, once the system becomes mechanical-joint force, a reduction in the chaotic behavior is observed in the response of the system. These particular issues have not been addressed before in the literature on planar multibody systems with clearance revolute joints.

METHODS TO DEAL WITH CONTACT-IMPACT EVENTS

The classical problem of the contact mechanics is still an open issue in engineering applications. In particular, the contact-impact modeling and analysis in multibody dynamics has received a great deal of attention over the past few decades and still remains an active field of research and development [23-26]. Contact events happen frequently in multibody systems and in many cases, the performance of mechanical systems is highly dependent on them [11, 27]. In simple terms, the contact mechanics problem must be studied when two bodies that are initially separated come into contact. For this, either the finite element method or the multibody systems approach can be utilized. On one hand, there is no doubt that the finite element analysis is the most powerful and accurate method to solve contact problems [28-30]. On the other hand, the method of multibody systems is the most efficient approach for the dynamic analysis of the gross motion of mechanical systems [31-33].

Regardless of the method used to describe the contact problems of the colliding bodies, it is necessary to model and analyze the contact process. This issue involves two main steps, (*i*) the contact detection and (*ii*) the evaluation of the contact forces, which are the result of collisions between bodies. The contact detection is an important issue in contact modeling of moving bodies, which deals with the determination of when, where and which points are in contact. The most significant task of contact detection is to check whether the potential contact points or surfaces are in contact or not, and allowed the relative interpenetration of the bodies. The efficiency and accuracy of this step depend on the complexity of the contacting surfaces [34], the number of potential colliding bodies [35] and the kinematics of the bodies [36]. In turn, the evaluation of the contact forces can be performed by using different approaches introduced over the last few decades [10, 37, 38]. In the present work, several continuous contact force models are used, for which the local deformations and normal contact forces are treated as continuous events and introduced into the equations of motion of the multibody systems as external generalized forces [1].

It is eminently difficult to find methods and algorithms which can model the highly complex phenomenon of contacting bodies realistically and efficiently enough for multibody systems simulations. From the modeling methodology point of view, several different methods have been introduced to model the contact response in multibody systems. As a rough classification, they can be divided into contact force based models and methods based on geometrical constraints, each of them showing advantages and disadvantages for each particular application [39]. In other words, there are two main approaches to model multibody systems with contact-impact events, namely the regularized models and non-smooth dynamics formulation [10, 11].

Contact force approaches, commonly referred as penalty or compliant methods, own their importance in the context of multibody systems with contacts due to their computational simplicity and efficiency [10]. In these methods, the contact force is expressed as a continuous function of penetration between contacting bodies. This approach is quite simple, very straightforward to implement and presents also a good computational efficiency. In addition, in the regularized approach, there are no impulses at the instant of contact. Therefore, there is no need for impulsive dynamics calculations and the contact loss can easily be determined from position and velocity data. One of the main drawbacks associated with these force models is the difficulty to choose contact parameters such as the equivalent stiffness or the degree of nonlinearity of the penetration, especially for complex contact scenarios and nonmetallic materials. Another disadvantage associated with this formulation is that it can introduce highly-frequency dynamics into the system, due to the presence of stiff springs in compliant surfaces. If these dynamics require the integration algorithm to take smaller steps, then the speed of simulation will be strongly penalized. The penalty formulations can be understood as if each contact region of the contacting bodies is covered with some spring-damper elements scattered over their surfaces. The normal force, including elastic and damping, prevents penetration; i.e., no explicit kinematic constraint is considered but simply force reaction terms are used. The magnitudes of stiffness and deflection of the spring-damper elements are computed based on the penetration, material properties and surface geometries of the colliding bodies. In the work by Khulief and Shabana [40], the required parameters for representing contact force laws are obtained based on the energy balance during contact. This formulation uses a force-displacement law that involves determination of material stiffness and damping coefficients. In the work by Lankarani and Nikravesh [41], two continuous contact force models are presented for which the unknown parameters are evaluated analytically. In their first model, the internal damping of bodies represents the energy dissipation at low impact velocities. However, in their second model, the local plasticity of the surfaces in contact becomes the dominant source of energy dissipation. Dias and Pereira [42] described the contact law using a continuous force model based on the Hertz contact law with hysteresis damping. Furthermore, the effect and importance of structural damping schemes in flexible bodies were also considered. Hunt and Crossley [43] obtained a model for computing the stiffness coefficient from the energy balance relations. The effect of friction in the continuous approach is often taken into consideration by using a regularized Coulomb friction model. An overview of different models of friction together with fundamentals can be found in Oden and Martins [44] and Feeny et al. [45].

An alternative method to treat the contact-impact problems in multibody systems is to use the non-smooth dynamics approach, namely the Linear Complementarity Problem (LCP) [11, 46] and Differential Variational Inequality (DVI) [35, 47]. The complementarity formulations associated with the Moreau's time-stepping algorithm for contact modeling in multibody systems have been used by many researchers [48-51]. Assuming that the contacting bodies are truly rigid, as opposed to locally deformable or penetrable bodies as in the penalty approach, the complementarity formulation resolves the contact dynamics problem by using the unilateral constraints to compute contact impulses or forces to prevent penetration from occurring. Thus, at the core of the complementarity approach is an explicit formulation of the unilateral constraints between the contacting rigid bodies [52]. The basic idea of complementarity in multibody systems can be stated as for a unilateral contact either relative kinematics is zero and the corresponding constraint forces are zero, or vice versa. The product of these two groups of quantities is always zero. This leads to a complementarity problem and constitutes a rule which allows the treatment of multibody systems with unilateral constraints [53-55].

One of the first published works on the complementarity problems is due to Signorini [56], who introduced an impenetrability condition in the form of a LCP. Later, Moreau [57] and Panagiotopoulos [58] also applied the concept of complementarity to study non-smooth dynamic systems. Pfeiffer and Glocker [11] extended the developments of Moreau and Panagiotopoulos to multibody dynamics with unilateral contacts, and examined the importance of the complementarity. Indeed, complementarity method has been proven to be a very useful way to formulate problems involving discontinuities [59-61]. In turn, the DVI has been recognized to be a powerful tool to deal with multiple contact problems in multibody dynamics. This approach has the advantage that it does not need the use of small time steps as in the case of penalty approaches, which means that simpler integrator schemes can be used, such as the Euler method [47]. However, the algorithmic procedures that result from DVI approach are of great complexity. This formalism has been used with success by Tasora et al. [35] to model and analyze multibody systems involving hundreds of thousand contacts. The DVI approaches are also interesting in the measure that they can easily deal with friction problems without need to modify Coulomb's friction law.

In short, the different methods to deal with contact-impact events in multibody systems have inherently advantages and disadvantages for each particular application. None of the formulations briefly described above can *a-priori* be said to be superior compared to other for all applications. In fact, a specific multibody problem might be easier to describe by one formulation, but this does not yield a general predominance of this formulation in all situations.

MODELING REVOLUTE CLEARANCE JOINTS

It is known that in standard multibody system models, it is assumed that the connecting points of two bodies, linked by an ideal revolute joint, are coincident. However, the introduction of the clearance separates these two points. Figure 1 shows a typical connection with revolute clearance joints found in a planar multibody system, where the clearance size is exaggerated in order to illustrate the phenomena associated with the revolute joints with clearance, namely, the bouncing effect. In a revolute clearance joint, when a contact occurs between the journal and bearing, a contact force is generated and applied perpendicular to the plane of collision. The force is typically applied as a spring damper element. If this element is linear, the approach is known as the Kelvin Voigt model [62]. If the relation is nonlinear, the model is generally based on the Hertz contact law [63].

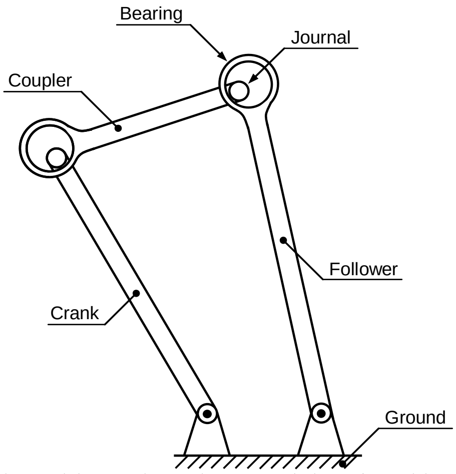

Figure 1: Typical connection with revolute clearance joints found in a planar multibody systems

Figure 2 depicts a revolute joint with clearance, that is, the so-called journal bearing, where the difference in diameters between the bearing and the journal defines the diametric clearance. Several published research works have focused on the different modes of motion of the journal inside the bearing boundaries. Most of them consider a three-mode model for predicting the dynamical response of articulated systems with revolute clearance joints [64]. The three different modes of journal motion inside the bearing are the sustained contact or followed mode, the free flight mode, and the impact mode, which are schematically represented in Figure 2.

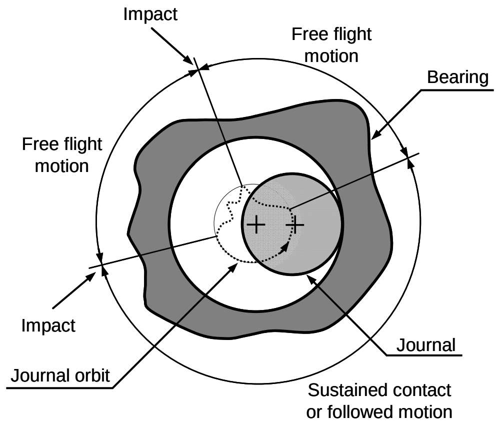

Figure 2: Types of journal motion inside the bearing boundaries

In the sustained contact or followed mode, the journal and the bearing are in permanent contact and a sliding motion relative to one another is assumed to exist. In this mode, the relative indentation varies along the circumference of the bearing. In practice, this mode is ended when the journal and bearing separate from each other and the journal enters in the free flight mode. In the free flight mode, the journal can move freely inside the bearing boundaries, that is, the journal and the bearing are not in contact; consequently, no reaction force is developed at the joint. In the impact mode, which occurs at the termination of the free flight mode, impact forces are applied and removed in the system almost instantaneously. This mode is characterized by a discontinuity in the kinematic and dynamic characteristics of the system, and a significant exchange of momentum occurs between the two impacting bodies. At the termination of the impact mode, the journal can enter either a free flight or a followed mode [65].

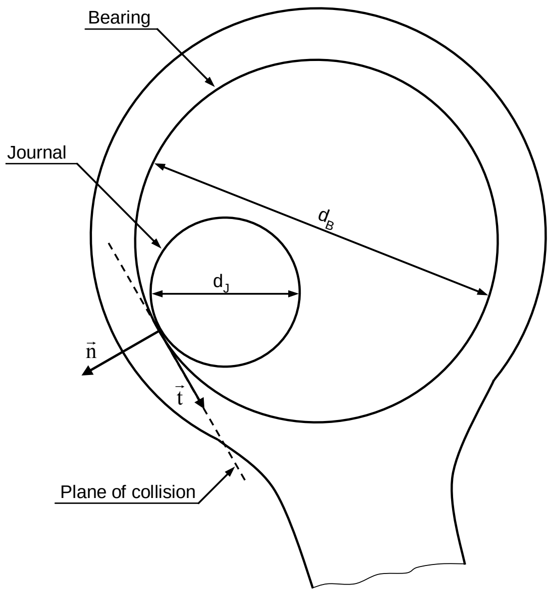

Figure 3: Generic configuration of a revolute joint with clearance in a multibody system

Figure 3 shows a generic configuration of a revolute joint with clearance in a multibody system. In the dynamic simulation, the behavior of the revolute clearance joint is treated as an oblique eccentric collision between the journal and the bearing. The mechanics of this type of collision involves both the relative normal velocity and the relative tangential velocity [66]. When the impact occurs, an appropriate contact law must be applied and being the resulting forces introduced as generalized forces in the system's equations of motion [1].

In regard to Figure 3, the relative penetration vector between the journal and bearing walls can be defined as:

$$\boldsymbol{\delta} = \mathbf{e} - \left(\frac{d_B - d_J}{2} \right) \mathbf{n} \tag{1}$$

where **e** is the eccentricity vector, i.e., the vector that defines the distance between the journal and bearing centers; d_B and d_J are the bearing and journal diameters; and **n** is the vector that defines the normal direction to the plane of collision.

The magnitude of the indentation can be evaluated as

$$\delta = e - c \tag{2}$$

in which c represents the radial clearance defined as

$$c = \frac{d_B - d_J}{2} \tag{3}$$

Typical contact-impact force models with dissipated effects are dependent on the contact velocities, and therefore, it is important to evaluate these velocities in order to account for the dissipative effects during the contact-impact process. In particular, in the continuous force contact model, it is necessary to calculate the relative velocity between colliding surfaces. The relative velocity vector between bearing and journal can be expressed by:

$$\mathbf{v} = \frac{d\left(\left\|\mathbf{r}_B^C - \mathbf{r}_J^C\right\|\right)}{dt}\mathbf{n} + \left(\left\|\mathbf{r}_B^C - \mathbf{r}_J^C\right\|\right)\left(\omega_B - \omega_J\right)\mathbf{t} \tag{4}$$

where $\mathbf{r}_B^C$ and $\mathbf{r}_J^C$ are the position vectors of the contact points in the bearing and journal, **n** and **t** are the normal and tangential vectors, and ω_B and ω_J are the bearing and journal angular velocities. It should be stated that vector **t** is obtained by rotating the vector **n** in the counter clockwise direction by 90º.

The relative velocity between the contact points is projected onto the plane of contact and onto the normal plane of contact, yielding a relative tangential velocity, v_T, and a relative normal velocity, v_N. The normal relative velocity determines whether the contact bodies are approaching or separating. Similarly, the tangential relative velocity determines whether the contact bodies are sliding or sticking. The relative scalar velocities, normal and tangential to the plane of collision, are found by projecting the relative impact velocity onto each one of these directions

$$v_N = \mathbf{v}^{\mathrm{T}}\mathbf{n} \tag{5}$$

$$v_T = \mathbf{v}^{\mathrm{T}}\mathbf{t} \tag{6}$$

In the presented approach, although a revolute joint with clearance does not constrain any degree-of-freedom from the mechanical system, as an ideal joint does, it imposes force-induced restrictions, limiting the journal to move within the bearing. However, this is not the case when the constraint-based contact formulation is considered, in the measure that the contact is modeled at the kinematic level when the contact between the bodies is detected. Thus, when the clearance is present in a revolute joint, two kinematic constraints associated with an ideal joint are removed and two degrees of freedom are introduced instead. The dynamics of the joint is then controlled by the contact-impact forces developed between the journal and bearing. Thus, whilst a perfect revolute joint in a mechanical system imposes kinematic constraints, a revolute clearance joint leads to force constraints. Therefore, joints with clearance can be defined as force-joints instead of kinematic joints. When a contact between the journal and bearing takes place, contact-impact forces act at the contact points. The contributions of these contact-impact forces to the external forces applied to the mechanical system are found by projecting them onto the *X* and *Y* directions. Since these forces do not act through the center of mass of the bearing and journal bodies, the moment components for each body also needs to be evaluated.

In short, in the dynamic analysis, the deformation is known at every time step from the configuration of the system and the forces are evaluated based on these state variables. With the variation of the contact force during the contact period, the system's dynamic response is obtained by simply including the updated forces into the system's equations of motion. Since the equations of motion are integrated over the period of contact, this approach results in a rather accurate response. Furthermore, this methodology accounts for the changes in the system's configuration during the contact periods. The methodology is quite straightforward and generic to apply in the dynamic modeling and analysis of any mechanical system. To the authors' knowledge, none of the formulations briefly described in this section can a-priori be said to be superior compared to other for all applications. It is a fact that a specific multibody problem might be easier to describe by one formulation, but this does not yield a general predominance of this formulation in all situations. However, in the present work, due to its simplicity, easiness to implement and computational efficiency, the penalty approach has been selected as the method to be considered to model the contact-impact events. Additionally, it must be stated that the multibody systems formulation used in this study is generic and more appropriate to be associated with penalty approach.

NUMERICAL MODELS FOR CONTACT FORCES

Internal impact, as it occurs in a revolute clearance joint, is one of the most common types of dynamic loading conditions giving rise to impulsive forces, which in turn excites higher vibration modes affecting the dynamic characteristics of the mechanical systems. For a revolute joint with clearance, the contact between the journal and bearing can be modeled by the well known Hertz contact law [63]

$$F_N = K\delta^n \tag{7}$$

where K is a generalized stiffness coefficient, and δ is the relative indentation. The exponent n is typically 1.5 or higher. The parameter K depends on the material and geometric properties of the contacting surfaces. For two spheres in contact the generalized stiffness coefficient is a function of the radii of the spheres i and j and the material properties as [67]

$$K = \frac{4}{3(\sigma_i + \sigma_j)}\left[\frac{R_i R_j}{R_i + R_j}\right]^{\frac{1}{2}} \tag{8}$$

where the material parameters σ_i and σ_j are given by

$$\sigma_k = \frac{1-\nu_k^2}{E_k}, \quad (k=i, j) \tag{9}$$

the quantities ν_k and E_k are the Poisson's ratio and the Young's modulus associated with each sphere, respectively.

Hertz contact law given by Eq. (7) is a pure elastic model, and it does not include any energy dissipation. Lankarani and Nikravesh [10] extended the Hertz contact law to include energy loss due to internal damping expressed as

$$F_N = K\delta^n\left[1 + \frac{3(1-c_r^2)}{4}\frac{\dot{\delta}}{\dot{\delta}^{(-)}}\right] \tag{10}$$

where the stiffness coefficient K is given by Eq. (8), c_r is the restitution coefficient, $\dot{\delta}$ is the instantaneous relative penetration velocity. and $\dot{\delta}^{(-)}$ is the initial impact velocity.

The Coulomb's friction law of sliding friction represents the most fundamental and simplest model of friction between dry contacting surfaces. When sliding takes place, the Coulomb's law states that the tangential friction force F_T is proportional to the magnitude of the normal contact force, F_N, at the contact point by introducing a coefficient of friction c_f [68]. The Coulomb's friction law is independent of relative tangential velocity. In practice, this is not true, because friction forces can depend on many parameters such as material properties, temperature, surfaces cleanliness and sliding velocity. Thus, a friction force-velocity relation is desirable. Furthermore, the application of the original Coulomb's friction law in a general-purpose computational program may lead to numerical difficulties, because it is a highly nonlinear phenomenon which may involve switching between sliding and stiction conditions. In order to overcome such difficulties, a modified Coulomb law is utilized [69]

$$F_T = -c_f c_d F_N \frac{\mathbf{v}_T}{\|\mathbf{v}_T\|} \tag{11}$$

where c_f is the friction coefficient, F_N is the normal force, $\mathbf{v}_T$ is the relative tangential velocity, and c_d is a dynamic correction coefficient, which is expressed as

$$c_d = \begin{cases} 0 & \text{if} \quad v_T \le v_0 \\ \dfrac{v_T - v_0}{v_1 - v_0} & \text{if} \quad v_0 \le v_T \le v_1 \\ 1 & \text{if} \quad v_T \ge v_1 \end{cases} \tag{12}$$

where v_0 and v_1 are given tolerances for the tangential velocity [69]. The dynamic correction factor c_d prevents that the friction force to change direction in the presence of almost null values of the tangential velocity. This would be perceived by the integration algorithm as a dynamic response with high frequency contents, forcing it to reduce the time step size. In terms of computation and use of numerical integrators, the increase of system's stiffness due to the presence of contact needs to be highlighted against the reduction in the system's stiffness due to the elimination of joint constraints. The direct integration algorithm with Baumgarte stabilization and Adams-Bashforth numerical integrator has been utilized in this study due to the simplicity of implementation. Further research should also pursued to examine the solutions when other computational/numerical schemes, such as the coordinate-partitioning, use of integrators for stiff systems due to the presence of contact/impact such as Gear, etc are implemented.

MULTIBODY SYSTEMS FORMULATION

The equations of motion for a dynamic multibody system subjected to holonomic constraints associated with kinematic joints can be stated in the form [1]

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{g} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{13}$$

With the reference frame placed at the center of mass for each body, $\mathbf{M}$ is the system mass matrix, $\mathbf{\Phi}_{\mathbf{q}}$ is the constraint Jacobian matrix of the kinematic constraint equations, the vector $\ddot{\mathbf{q}}$ contains the generalized state accelerations, $\boldsymbol{\lambda}$ is the vector that contains the Lagrange multipliers associated with the kinematic constraints, $\mathbf{g}$ is the vector of generalized forces and $\boldsymbol{\gamma}$ is the vector of quadratic velocity terms [1]. Equation (13) is formed as a combination of the equations of motion and kinemtaic constraint equations, often referred to as a mixed set of differential and algebraic equations of motion.

A set of initial conditions on positions and velocities is required to start the dynamic simulation. The selection of the appropriate initial conditions plays a key role in the prediction of the dynamic performance of mechanical system. In the present work, the initial conditions are based on the results of kinematic simulation of mechanical system in which all the joints are assumed to be ideal, that is, without clearance. The subsequent initial conditions for each time step in the simulation are obtained in the usual manner from the conditions of the previous time step [1]. In order to stabilize or to keep the constraints violation under control, Eq. (13) is solved using the Baumgarte stabilization technique [70, 71]. In turn, the integration process is performed in this study using a predictor-corrector algorithm with both variable step size and order [72].

The use of numerical algorithms with automated adjust step size is particularly important in contact problems whose dynamic response is quite complex due to the suddenly change in the system's kinematic configuration. In such events, the use of a constant time step is computationally inefficient and the system could be overlooked due to insufficient time resolution. The automated time step size adaptability is therefore a crucial part of the dynamic solution procedure. Furthermore, the abrupt configuration changes caused by the rapid variation of contact forces are source of stiffness, since the natural frequency of the system is widely spread. Thus, the time step size must be adjusted in order to capture the fast and low frequency components of the system response [1, 39].

RESULTS AND DISCUSSION

The main purpose of this section is to assess the influence of the presented approach on the dynamic response of a planar multibody system with multiple clearance revolute joints. The effects of misalignment, lubrication, flexibility of bodies, contact approach, and contact measurement for comparison, are out of scope of the present study, and can be subject of other future research developments. This section contains the results obtained from computational simulations of two planar mechanical systems with multiple clearance revolute joints, namely a slider-crank mechanism and a quick-return mechanism. Figure 4 shows the generic configuration of the slider-crank mechanism. The system consists of four rigid bodies that represent the crank, connecting rod, slider and ground, one ideal revolute joint and one ideal translational joint. The length and inertia characteristics of this mechanism are listed in Table 1. The acceleration due to gravity is taken as acting in the negative Y direction and the mechanism is defined as moving in a vertical plane. The existence of one, two or three clearance revolute joints are considered in the present analysis. When the system is modeled with one, two or three clearance revolute joints makes a system with a total of three, five and seven degrees-of-freedom, respectively.

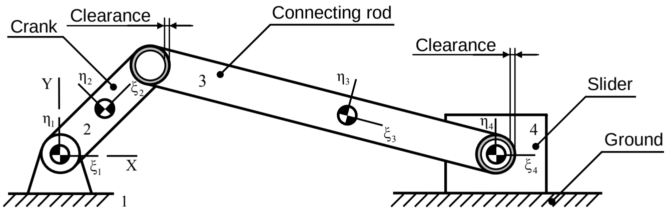

Figure 4: Slider-crank mechanism with two clearance joints

The crank, which is the driving link, rotates with a constant angular speed of 5000 rpm. The initial configuration of the mechanism is defined with the crank and the connecting rod collinear, and the journal and bearing centers coincident. Furthermore, the initial positions and velocities necessary to start the dynamic analysis are obtained from kinematic simulation of the slider-crank mechanism in which all the joints are considered to be ideal. Table 2 presents the values of the main parameters used for the different models, required to characterize the problem and for the numerical methods, required to solve the system's equations of motion.

Table 1: Geometric and inertial properties of the slider-crank mechanism

Body Nr.	Length [m]	Mass [kg]	Moment of inertia [kgm^2]
2	0.05	0.30	0.00010
3	0.12	0.21	0.00025
4	-	0.14	0.00010

Table 2: Parameters used in the dynamic simulations

Nominal bearing diameter	20.0 mm	Poisson's ratio	0.3
Standard diametric clearance	0.05 mm	Integrator algorithm	Gear
Coefficient of restitution	0.9	Baumgarte – α, β	5
Coefficient of friction	0.01	Integration step size	1×10^{-6} s
Young's modulus	207 GPa	Total time of simulation	10.0 s

The diametric clearance size of the non-ideal revolute joints is taken to be equal to 0.05 mm, which corresponds to the clearance size in a typical journal-bearing with nominal dimensions used in the present work. For example, for a journal-bearing in which the nominal diameter falls into the interval between 18 and 30 mm, the minimum and maximum recommended diametric clearance sizes are 0.02 and 0.06, respectively [73].

In what follows, several numerical results are presented and analyzed to demonstrate the computational implementation and efficiency of the described methodologies. The dynamic response of the slider-crank mechanism is quantified by plotting the slider velocity and acceleration, joint reaction forces and the reaction moment that acts on the crank. In addition, the journal center orbit inside the bearing boundaries and the maps that relates the eccentricity and eccentricity velocity are plotted. The global results are relative to two complete crank revolutions after the steady configurations state has been reached. Furthermore, the results for slider-crank mechanism are compared with the dynamic behavior when the system is modeled with ideal joints only. The influence of the number of clearance joints on the dynamic behavior of the slider-crank mechanism is investigated. In the first simulation of this mechanism, only one joint is modeled as clearance revolute joint, namely, the joint between the connecting rod and the slider.

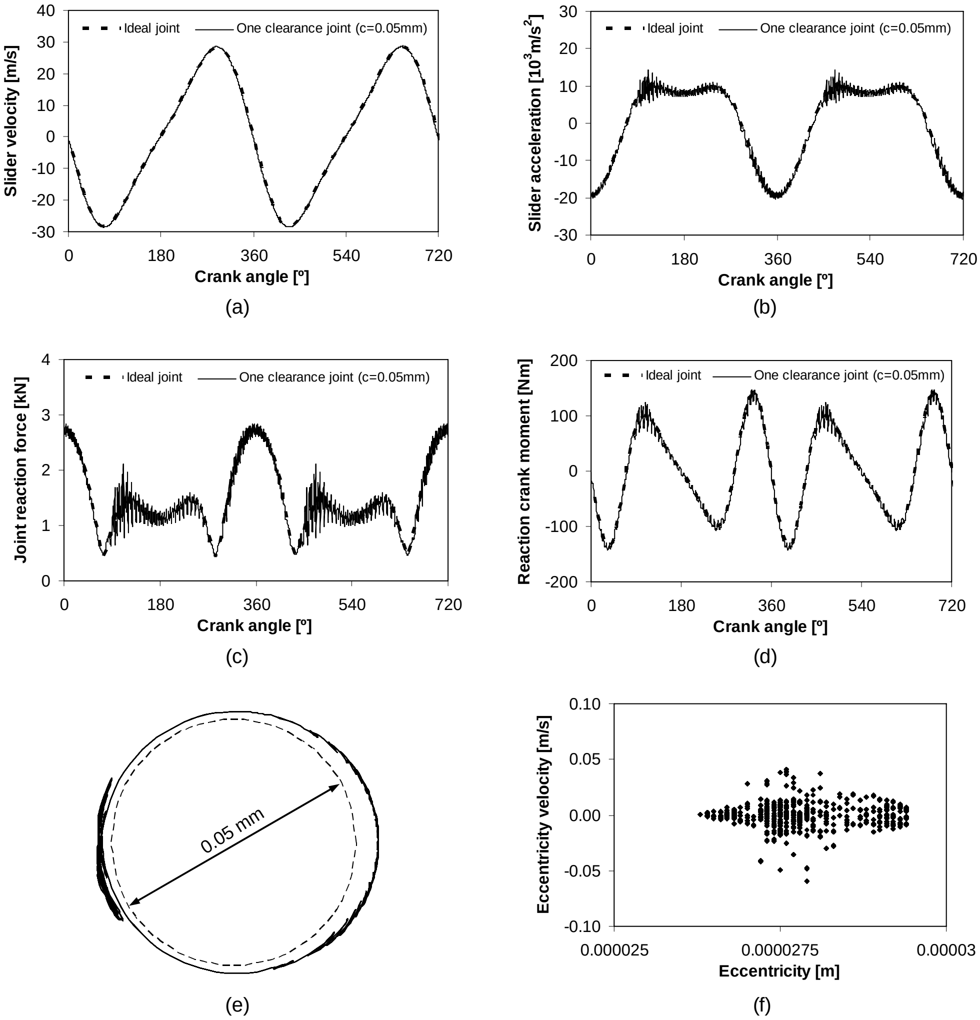

Figure 5: (a) Slider velocity; (b) Slider acceleration; (c) Joint reaction force at the clearance joint; (d) Crank moment required to maintain its angular velocity constant; (e) Journal center trajectory relative to the bearing center; (f) Poincaré map.
All for one clearance joint of 0.05 mm between the connecting rod and the slider

From Figure 5(a), it can be observed that the existence of a clearance joint does not influence the slider velocity in a significant manner. In sharp contrast, the slider acceleration is strongly influenced by the contact-impact forces produced between the journal and bearing contacting surfaces in the clearance joint. The slider acceleration is subjected to peaks caused by contact forces that are propagated through the rigid bodies of the mechanism, as observed in Fig. 5(b) on the acceleration of the slider. The same phenomena can be observed in the joint reaction forces and crank moments, represented by Figs. 5(c) and 5(d), respectively. As far as the trajectory of the journal center relative to the bearing center is concerned, only one type of motion between the two bodies is observed, namely the sustained or followed contact mode. The relative penetration depth between the journal and bearing is visible by the lines of the journal path that are plotted outside the clearance circle represented by the dashed line, as illustrated in Fig. 5(e). From the

Poincaré map of Fig. 5(f) it can be deduced that the behavior of the slider-crank mechanism modeled with a clearance joint tends to be aperiodic, or even chaotic, due to the nonlinear nature of the contact-impact phenomena associated with the joint components. This can be explained based on the spread out nature of the Poincaré map. In fact, non periodic responses are extremely sensitive to initial conditions and are densely filled by orbits or points in the Poincaré map. A complex looking phase in a Poincaré map is one indicator of the chaotic motion. Quasi-periodic orbits fill up the Poincaré maps as the chaotic orbits, but they do so in a fully predictable manner since there is no sensitive dependence on the initial conditions [74].

The influence of the number of clearance joints on the dynamic response of the slider-crank mechanism is analyzed in the following paragraphs. The simulation of the mechanism is performed with one, two and three revolute clearance joints. Figure 6 shows the global results for the case in which the joint connecting the crank and connecting rod is modeled as clearance joint with 0.05 mm for diametric clearance size. The system is again operating at 5000 rpm. In general, the results, as expected, are dependent on the contact-impact phenomena that take place between the journal and bearing. From Figure 6(e) it can be observed that the journal and bearing are in permanent contact mode. The system's response is also chaotic, as illustrated in Figure 6(f). Figure 6 shows the different responses of the system for the different location of the single clearance joint. It shows that the clearance joint between the crank and the connecting rod produces higher impact forces and then more oscillatory response.

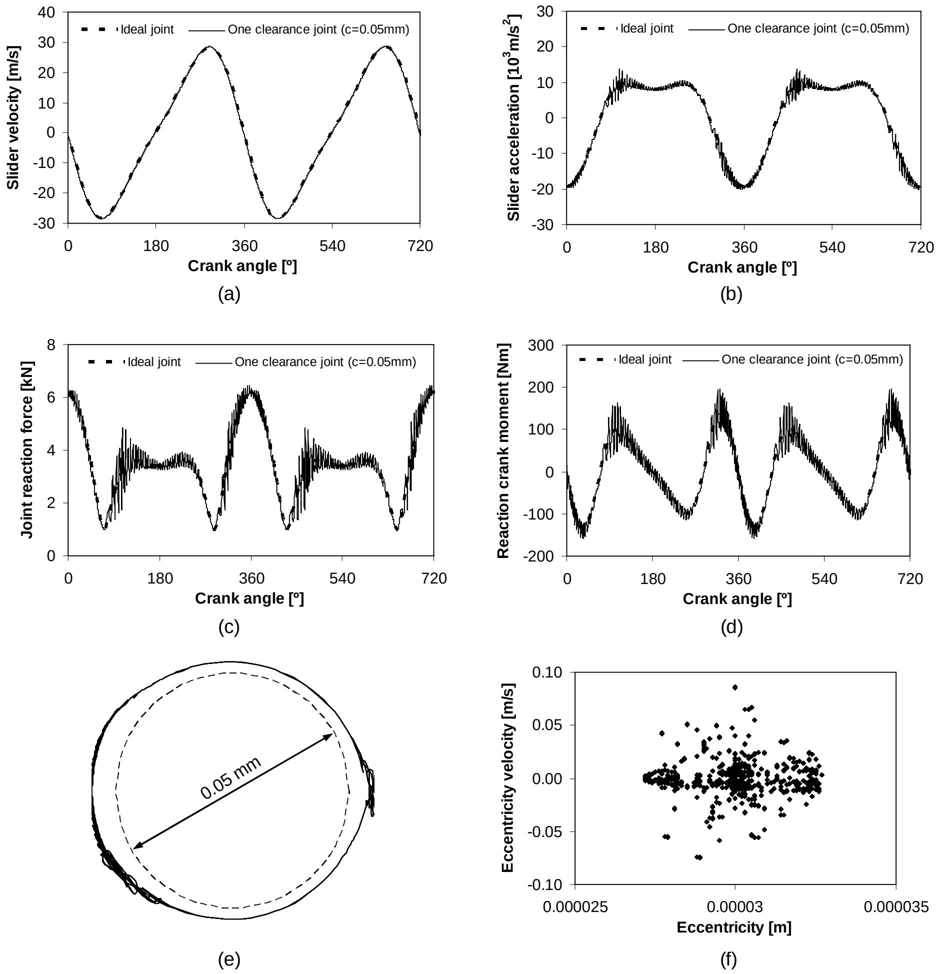

Figure 6: (a) Slider velocity; (b) Slider acceleration; (c) Joint reaction force at the clearance joint; (d) Crank moment required to maintain its angular velocity constant; (e) Journal center trajectory relative to the bearing center; (f) Poincaré map.
All for one clearance joint of 0.05 mm between the crank rod and the connecting rod

Figures 7 to 10 illustrate the results obtained when the slider-crank mechanism is modeled with one, two and three clearance joints. It should be noted that the journal trajectories and Poincaré maps have similar look for all the cases simulated, as illustrated in Figs. 9 and 10. In general, it can be stated that as the number of clearance joints increase, the responses seem to have more pronounced peaks. However, as observed as the number of clearance joints increases to maximum, the system is gradually converted a constrained multibody system governed by a differential-algebraic system, to an unconstrained or joint-force system governed by a system of ordinary differential equations. As observed in case (d) of Figures 7-10, the chaotic behavior diminishes as the system becomes joint-force (when all the three joints have clearance).

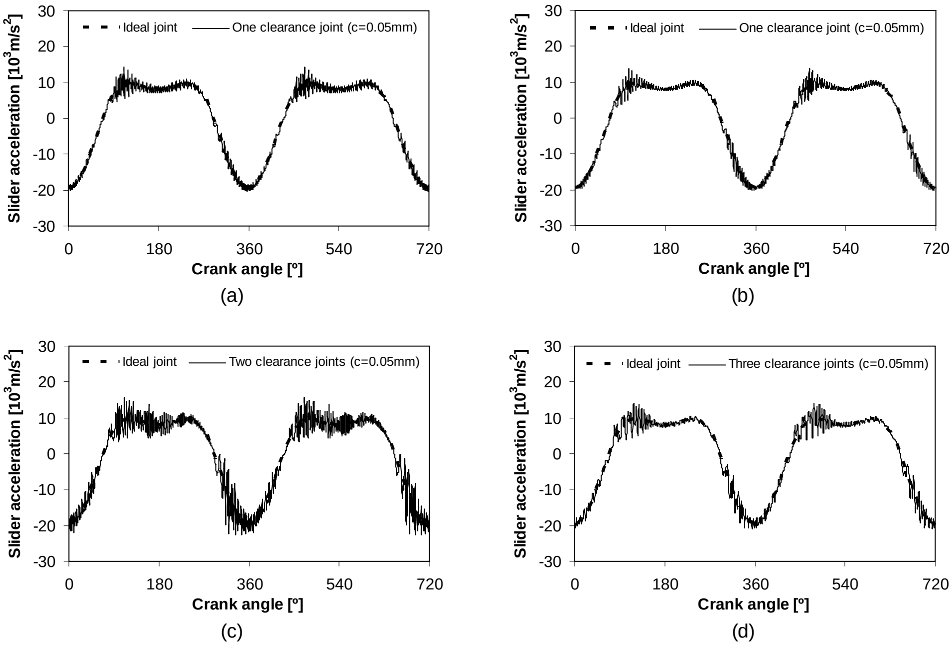

Figure 7: Slider acceleration for different number of clearance joints: (a) One clearance joint between connecting rod and slider; (b) One clearance joint between crank and connecting rod; (c) One clearance joint between crank and connecting rod and one clearance joint between connecting rod and slider; (d) All three revolute joints have clearance (multiple clearance joints)

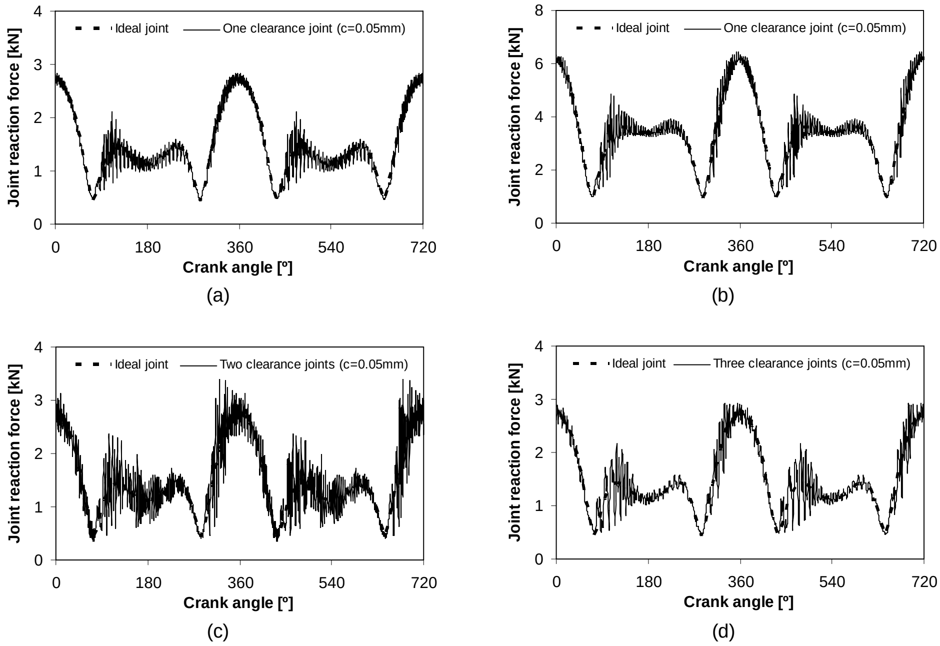

Figure 8: Joint reaction force for different number of clearance joints: (a) One clearance joint between connecting rod and slider; (b) One clearance joint between crank and connecting rod; (c) One clearance joint between crank and connecting rod and one clearance joint between connecting rod and slider; (d) All three revolute joints have clearance (multiple clearance joints)

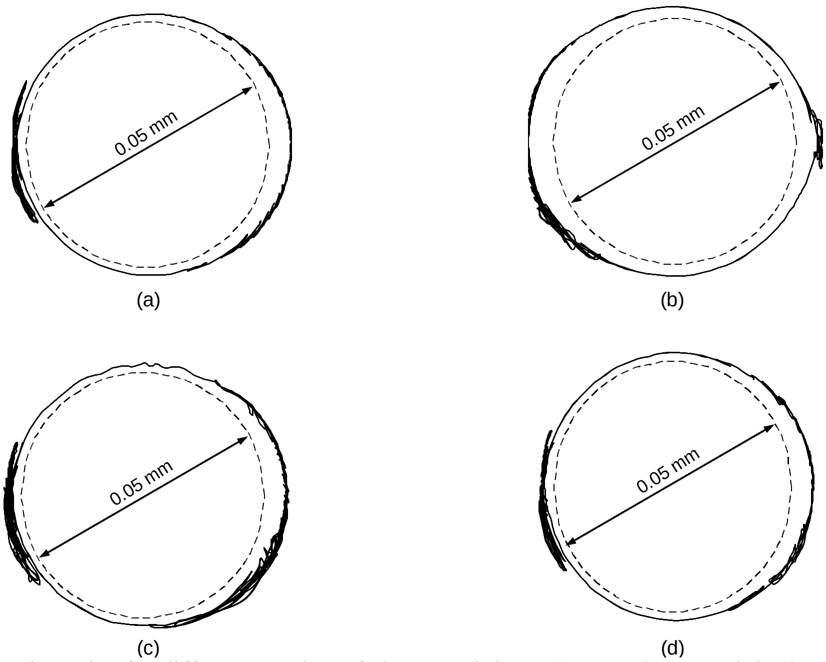

Figure 9: Journal center trajectories for different number of clearance joints: (a) One clearance joint between connecting rod and slider; (b) One clearance joint between crank and connecting rod; (c) One clearance joint between crank and connecting rod and one clearance joint between connecting rod and slider; (d) All three revolute joints have clearance (multiple clearance joints)

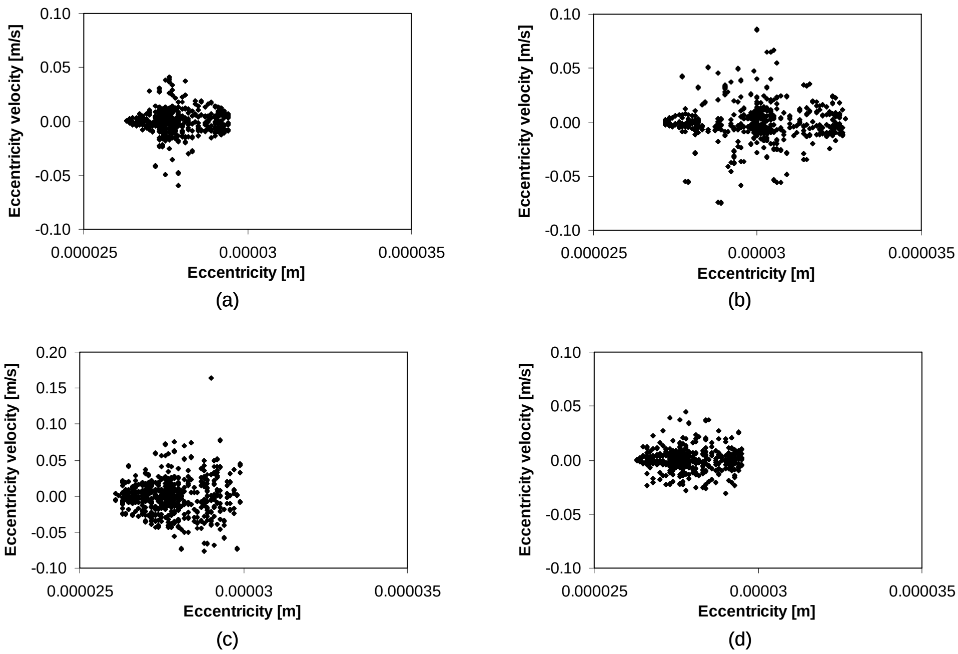

Figure 10: Poincaré maps for different number of clearance joints: (a) One clearance joint between connecting rod and slider; (b) One clearance joint between crank and connecting rod; (c) One clearance joint between crank and connecting rod and one clearance joint between connecting rod and slider; (d) All three revolute joints have clearance (multiple clearance joints)

Finally, it would be of interest to analyze the computation time consumed in each simulation performed in the previous paragraphs. Figure 11 shows the computation time utilized for different diametric clearance size and for different number of joint modeled as clearance joints. From Figure 11, it can be observed that the computation time decreases when the number of clearance joints in the system increases, because a higher number of contact-impact events take place. Secondly, the computation time slightly increases when the clearance size also increases, as Fig. 11a shows. The computational simulations were performed in a Pentium 4 (3.2GHz) computer. For this purpose, a FORTRAN code named MUBODYNA (acronym for Multibody Dynamics) was utilized. This program has been developed for the dynamic analysis of general planar multibody systems [75].

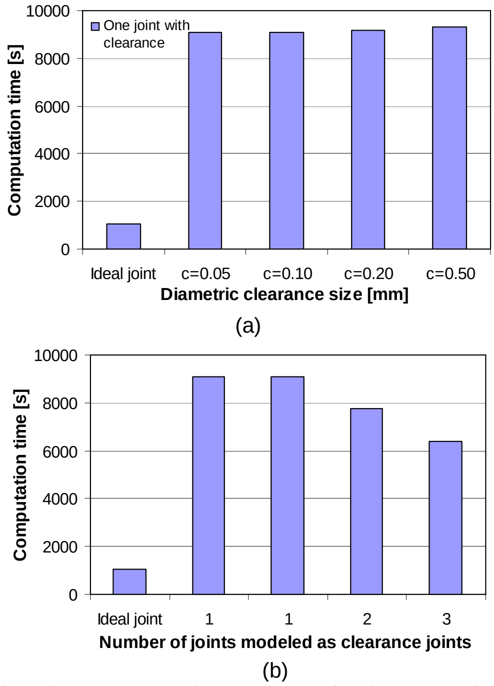

Figure 11: Variation of the computation time consumed: (a) Diametric clearance size; (b) Number of joints with clearance

The application of the proposed methodology to a more complex multibody system is represented by the simulation of a planar quick-return mechanism. This mechanism is made of six rigid bodies, one ideal revolute joint between the ground and crank, two perfect translational joints and four revolute clearance joints, as illustrated in Fig. 12. Due to the existence of the four revolute clearance joints, the system has a total of nine degrees-of-freedom. The acceleration due to gravity is taken to be in the negative *Y* direction and the system is defined as moving the *XY* plane. The set of data adopted for the model is listed in Table 3.

Table 3: Geometric and inertia data of the quick-return mechanism

Body Nr.	Length [mm]	Mass [kg]	Moment of inertia [kgm^2]
2	100	0.20	0.010
3	500	1.00	0.100
4	-	0.50	0.025
5	120	0.24	0.012
6	-	0.50	0.025

The quick-return mechanism, which produces a slow cutting stroke of a tool and a rapid return stroke, is driven by a rotational motor attached to the crank rotating with a constant angular velocity of 3 rad/s. The remaining initial conditions, necessary to start the dynamic analysis, are obtained from the kinematic simulation of the quick-return mechanism, in which all the joints are modeled as ideal joints. The parameters used for the dynamic simulation are listed in Table 4. A radial clearance of 0.5 mm is used in all clearance joints.

Table 4: Parameters used in the dynamic simulation for the quick-return mechanism

Bearing radius	10.0 mm
Journal radius	9.5 mm
Restitution coefficient	0.9
Young's modulus	207 GPa
Poisson's ratio	0.3
Integration step	0.0001 s
Integration algorithm	Runge-Kutta - 4th order

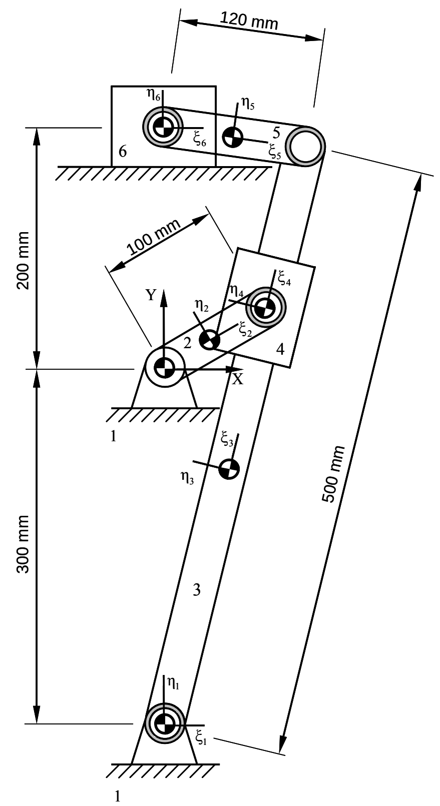

Figure 12: Quick-return mechanism with four revolute clearance joints

Each clearance joint is modeled as two colliding bodies, being the journal freely to move inside the bearing boundaries. The occurrence of contact within the clearance joints is determined by evaluating the relative penetration at any time during the numerical solution of the system dynamics as

$$\delta_i = e_i - c_i \qquad (i=1, \ldots, 4) \tag{14}$$

The computational modeling of the contact between journal and bearing, within the clearance revolute joints, uses continuous contact force model with the energy dissipation in form of hysteresis damping, given by Eq. (10). The performance of the quick-return mechanism is quantified by plotting the *X* position, velocity and acceleration components of the slider, represented by body 6. This allows comparing the results obtained with the current model to the dynamic behavior of the mechanism with ideal joints simulated with the same conditions. Figure 13 shows how the revolute clearance joints influence the position, velocity and acceleration of the slider. The horizontal line segments in the velocity-time curve correspond to constant slider velocity, which means that there is no contact between the journal and bearing surfaces while such velocity is constant. Sudden changes in the velocity are clearly associated with the impacts within the clearance joints, which are quite visible by the step shaped curve of the velocity diagram. Smooth changes in the velocity can also be observed indicating that the journal and bearing surfaces are in permanent or continuous contact, that is, the journal follows the bearing wall. These phenomena are also quite visible in the plots of the slider acceleration illustrated in Fig. 13c.

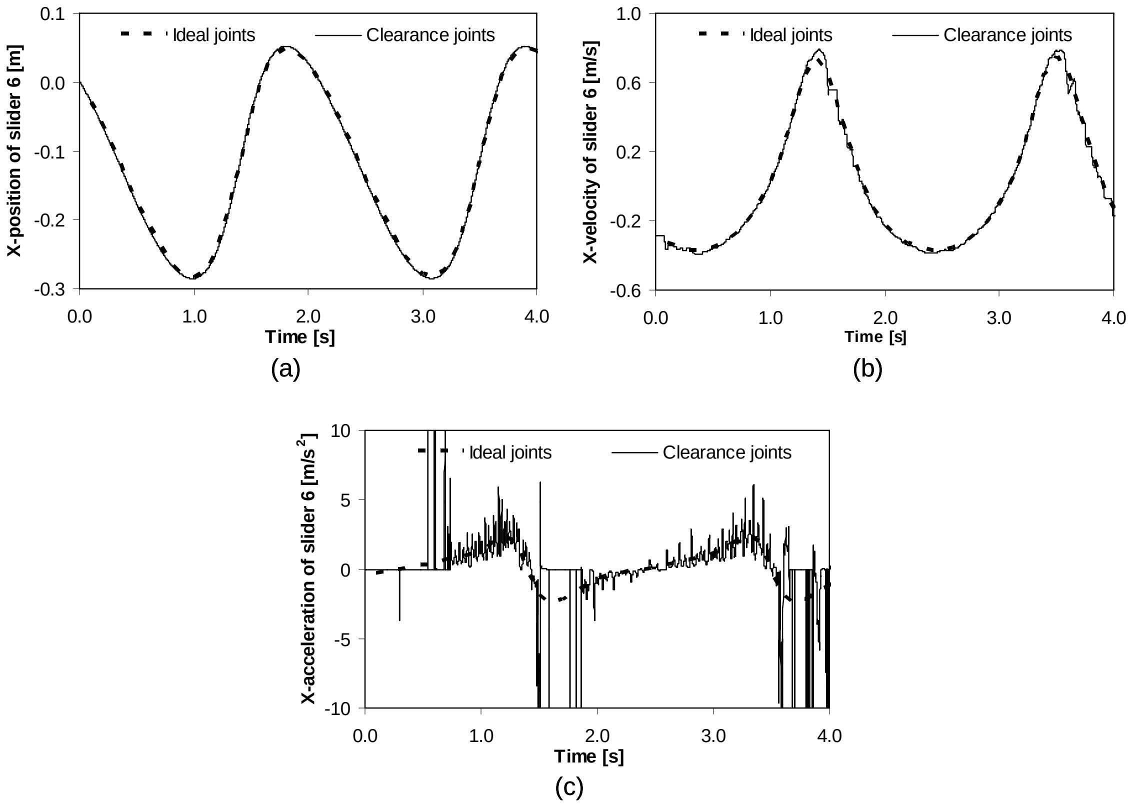

Figure 13: (a) *X*-position of the slider, represented by body 6; (b) *X*-velocity of the slider; (c) *X*-acceleration of the slider

In order to better understand what occurs during the dynamic simulation of the quick-return mechanism, consider Fig. 14a, in which the journal center orbits inside the bearing boundaries between the journal and bearing in the revolute clearance joint that connects the crank and slider represented by body 4. The different types of relative motion between the journal and bearing are observed, namely, the free flight motion, the impact followed by rebound and the permanent or continuous contact motion. The relative indentation is also visible by the points plotted outside the clearance circle of Fig. 14a. A point is plotted for each integration time step. The point density in Fig. 14a is quite high when the journal contacts the bearing, which means that the step size is small. When the journal is in free flight motion the time step is increased and, consequently, the dots plotted in the Fig. 14a are further apart. The slider acceleration and velocity are used to plot the Poincaré map of Fig. 14b. From the analysis of the Poincaré map of Fig.14b, it can be concluded that the system's response tends to have an attractor in the measure that the plots form a closed curve. This fact can be explained by the low crank speed of the driving link.

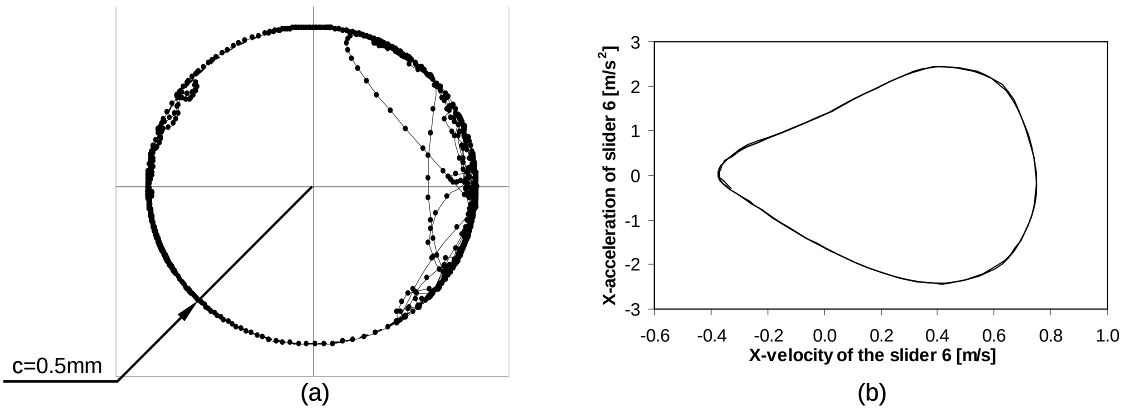

Figure 14: (a) Journal center orbit inside the bearing boundaries; (b) Poincaré map of the slider 6

CONCLUSIONS

A general methodology for modeling and simulating of multiple revolute clearance joints in planar rigid multibody systems has been presented and discussed throughout in this work. The equations of motion that govern the dynamic response of the general multibody systems incorporate the contact forces due to the collisions of the bodies that constitute the clearance joints. A continuous contact force model provides the intra-joint contact-impact forces that develop during the normal operations of the mechanisms. A suitable model for the revolute clearance joints is embedded into the multibody systems methodology. The effect of multiple clearance joints is particularly studied in this paper. In general, the presented methodology is straightforward and general from computational view point.

From the numerical simulations performed, it can be concluded that there are considerable changes in joint reaction forces and on the crank moments due to the effect of clearances. These changes are clearly associated with the peaks observed in the slider acceleration curves for both mechanisms analyzed. It is also important to note that the magnifications of forces and moments in the slider-crank mechanism as affected by revolute clearance joints essentially depend on the clearance size and number of joints modeled as clearance joints. From the main results obtained in this work, it can be also observed that multibody systems with clearance joints are known as nonlinear dynamic systems which, under certain conditions, exhibit a chaotic response. As for multiple clearances, the number of clearance joints is increased; it seems that the system is exposed to oscillations, until the system becomes kinematic-joint force. Once the system becomes mechanical-joint force, a reduction in the chaotic behavior is observed in the response of the system.

From the results presented here, it is found that the dynamics of the revolute clearance joint in mechanical systems is sensitive to the clearance size and operating conditions. With a small change in one of these parameters, the response of the system can shift from chaotic to periodic and vice-versa.

Finally, it should be stated that the results presented in this paper represent an upper bound of the joint reaction forces and crank moments due to the existence of clearance joints, since the elasticity of bodies and lubrication action in the joints were not included in the analysis. These phenomena seem to have an affect and might reduce the joint reaction forces.

ACKNOWLEDGMENT

The support of *Fundação para a Ciência e a Tecnologia* (FCT) through the project, PTDC/EME-PME/099764/2008, titled BIOJOINTS - Development of advanced biological joint models for human locomotion biomechanics is gratefully acknowledged.

REFERENCES

[1] Nikravesh, P.E., 1988, *Computer Aided Analysis of Mechanical Systems*. Prentice Hall, Englewood Cliffs, New Jersey.

[2] Haug, E.J., 1989, *Computer-Aided Kinematics and Dynamics of Mechanical Systems - Volume I: Basic Methods*. Allyn and Bacon, Boston, Massachusetts, (1989)

[3] Shabana, A.A.: Dynamics of Multibody Systems, John Wiley and Sons, New York.

[4] Nikravesh, P.E., 2008, *Planar Multibody Dynamics: Formulation, Programming, and Applications*. CCR Press, London.

[5] McCarthy, J.M., 2000, *Geometric Design of Linkages: Systems and Controls*, Springer, New York.

[6] Flores, P., Ambrósio, J., Claro, J.C.P., Lankarani, H.M., 2006, "Dynamics of Multibody Systems with Spherical Clearance Joints". Journal of Computational and Nonlinear Dynamics, **1**(3), 240-247.

[7] Ravn, P., Shivaswamy, S., Alshaer, B.J., Lankarani, H.M., 2000, "Joint Clearances With Lubricated Long Bearings in Multibody Mechanical Systems", Journal of Mechanical Design, **122**, pp. 484-488.

[8] Ambrósio, J., Veríssimo, P., 2009, "Sensitivity of a vehicle ride to the suspension bushing characteristics". Journal of Mechanical Science and Technology, **23**, 1075-1082.

[9] Park, J., Nikravesh, P.E., 1997, "Effect of Steering-Housing Rubber Bushings on The Handling of a Vehicle". SAE Paper 970103, SAE, Warrendale, Pennsylvania.

[10] Lankarani, H.M., Nikravesh, P.E., 1990, "A Contact Force Model With Hysteresis Damping for Impact Analysis of Multibody Systems". Journal of Mechanical Design, **112**, 369-376.

[11] Pfeiffer. F., Glocker C., 1996, *Multibody dynamics with unilateral constraints*, John Wiley and Sons, New York.

[12] Ambrósio, J., Veríssimo, P., 2009, "Improved bushing models for general multibody systems and vehicle dynamics". Multibody System Dynamics, **22**, 341-365.

[13] Tian, Q., Zhang, Y., Chen, L., Flores, P., 2009, "Dynamics of spatial flexible multibody systems with clearance and lubricated spherical joints". Computers and Structures, **87**(13-14), pp. 913-929.

[14] Orden, J.C.G., 2005, "Analysis of Joint Clearances in Multibody Systems". Multibody System Dynamics, **13**(4), pp. 401-420.

[15] Megahed, S.M., Haroun, A.F., 2012, “Analysis of the Dynamic Behavioral Performance of Mechanical Systems With Multi–Clearance Joints”, Journal of Computational and Nonlinear Dynamics, **7**, pp. 0110021-10.

[16] Bauchau, O.A., Rodriguez, J., 2002, “Modelling of Joints with Clearance in Flexible Multibody Systems”. International Journal of Solids and Structures, **39**, 41-63.

[17] Innocenti, C., 2002, “Kinematic Clearance Sensitivity Analysis of Spatial Structures With Revolute Joints”. Journal of Mechanical Design, **124**, 52-57.

[18] Flores, P., Ambrósio, J., Claro, J.C.P., Lankarani, H.M., 2008, “Translational Joints With Clearance in Rigid Multibody Systems”, Journal of Computational and Nonlinear Dynamics, **3**(1), pp. 0110071-10.

[19] Farahanchi, F., Shaw, S., 1994, “Chaotic and Periodic Dynamics of a Slider-Crank Mechanism with Slider Clearance”. Journal of Sound and Vibration, **177**(3), 307-324.

[20] Rhee, J., Akay, A., 1996, Dynamic Response of a Revolute Joint with Clearance. Mechanism and Machine Theory, **31**, 121-124.

[21] Ryan, R.R., 1990, *ADAMS-Multibody System Analysis Software, Multibody Systems Handbook*, Berlin, Springer-Verlag.

[22] Smith, R.C., Haug, E.J., 1990, *DADS-Dynamic Analysis and Design System, Multibody Systems Handbook*, Berlin, Springer-Verlag.

[23] Pfeiffer, F., 2003, “The idea of complementarity in multibody dynamics”. Archive of Applied Mechanics, **72**(11-12), 807-816.

[24] Gonthier, Y., McPhee, J., Lange, C., Piedboeuf, J-C., 2004, “A regularized contact model with asymmetric damping and dwell-time dependent friction”. Multi-body System Dynamics, **11**, 209-233.

[25] Tasora, A., Anitescu, M., 2010, “A Convex Complementarity Approach for Simulating Large Granular Flows”. Journal of Nonlinear and Computational Dynamics, **5**(3), 031004, 10p.

[26] Najafabadi, S.A.M., Kövecses, J., Angeles, J., 2008, “Generalization of the Energetic Coefficient of Restitution for Contacts in Multibody Systems”. Journal of Nonlinear and Computational Dynamics, **3**(4), 041008, 14p.

[27] Wriggers, P., 2006, *Computational contact mechanics*. Second Edition. Springer-Verlag Berlin.

[28] Ebrahimi, S., Eberhard, P., 2006, “A linear complementarity formulation on position level for frictionless impact of planar deformable bodies”. ZAMM Z. Angew. Math. Mech., **86**(10), 807-817.

[29] Liu, C-S., Zhang, K., Yang, R., 2007, “The FEM analysis and approximate model for cylindrical joints with clearances”. Mechanism and Machine Theory, **42**, 183-197.

[30] Wu, S-H., Tsai, S-J., 2009, “Contact stress analysis of skew conical involute gear drives in approximate line contact”. Mechanism and Machine Theory, **44**(9), 1658-1676.

[31] Wojtyra, M., 2009, “Joint reactions in rigid body mechanisms with dependent constraints”. Mechanism and Machine Theory, **44**(12), 2265-2278.

[32] Ebrahimi, S., Kövecses, J., 2010, “Unit homogenization for estimation of inertial parameters of multibody mechanical systems”. Mechanism and Machine Theory, **45**(3), 438-453.

[33] Dopico, D., Luaces, A., Gonzalez, M., Cuadrado, J. 2010, “Dealing with multiple contacts in a human-in-the-loop-application”. Multibody System Dynamics, **25**, 167-183.

[34] Machado, M., Flores, P., Claro, J.C.P., Ambrósio, J., Silva, M., Completo, A., Lankarani, H.M., 2010, "Development of a planar multibody model of the human knee joint". Nonlinear Dynamics, **60**(3), 459-478.

[35] Tasora, A., Negrut, D., Anitescu, A., 2008, "Large-scale parallel multi-body dynamics with frictional contact on the graphical processing unit". Proceedings of the Institution of Mechanical Engineers, Part-K Journal of Multi-body Dynamics, **222**, 315-326.

[36] Hirschkorn, M., McPhee, J., Birkett, S., 2006, "Dynamic modeling and experimental testing of a piano action mechanism". Journal of Computational and Nonlinear Dynamics, **1**(1), 47-55.

[37] Gilardi, G., Sharf, I., 2002, "Literature survey of contact dynamics modeling". Mechanism and Machine Theory, **37**, 1213-1239.

[38] Hippmann, G., 2004, "An algorithm for compliant contact between complexly shaped bodies". Multibody System Dynamics, **12**, 345-362.

[39] Flores, P., Ambrósio, J., 2010, "On the contact detection for contact-impact analysis in multibody systems". Multibody System Dynamics, **24**(1), 103-122.

[40] Khulief, Y.A., Shabana, A.A., 1987, "A continuous force model for the impact analysis of flexible multibody systems". Mechanism and Machine Theory, **22**, 213-224.

[41] Lankarani, H.M., Nikravesh, P.E., 1994, "Continuous contact force models for impact analysis in multibody systems". Nonlinear Dynamics, **5**, 193-207.

[42] Dias, J.M.P., Pereira, M.S., 1995, "Dynamics of flexible mechanical systems with con-tact-impact and plastic deformations". Nonlinear Dynamics, **8**, 491-512.

[43] Hunt, K.H., Crossley, F.R.E., 1975, "Coefficient of restitution interpreted as damping in vibroimpact". Journal of Applied Mechanics, **7**, 440-445.

[44] Oden, J.T., Martins, J.A.C., 1985, "Models and computational methods for dynamic friction phenomena". Computer Methods in Applied Mechanics and Engineering, **52**, 527-634.

[45] Feeny, B., Guran, A., Hinrichs, N., Popp, K., 1998, "A historical review on dry friction and stick-slip phenomena". Applied Mechanics Review, **51**, 321-341.

[46] Anitescu, M., Potra, F.A., Stewart, D.E., 1999, "Time-stepping for three-dimensional rigid body dynamics". Computer Methods in Applied Mechanics and Engineering, **177**(3), 183-197.

[47] Pang, J-S., Stewart, D.E., 2008, "Differential variational inequalities". Math. Program. **113**, 345-424.

[48] Glocker, C., Studer, C., 2005, "Formulation and preparation for numerical evaluation of linear complementary systems in dynamics". Multibody System Dynamics, **13**, 447-463.

[49] Förg, M., Pfeiffer, F., Ulbrich, H., 2005, "Simulation of unilateral constrained systems with many bodies". Multibody System Dynamics, **14**(2), 137-154.

[50] Acary, V., Brogliato, B., 2008, *Numerical methods for nonsmooth dynamical systems: Applications in mechanics and electronics*. Lecture Notes in Applied and Computational Mechanics Vol. 35, Berlin Heidelberg New-York, Springer-Verlag.

[51] Studer, C., Leine, R.I., Glocker, C., 2008, "Step size adjustment and extrapolation for time-stepping schemes in non-smooth dynamics". International Journal for Numerical Methods in Engineering, **76**(11), 1747-1781.

[52] Brogliato, B., Ten Dam, A.A., Paoli, L., Genot, F., Abadie, M., 2002, "Numerical simulations of finite dimensional multibody nonsmooth mechanical systems". Applied Mechanics, **55**, 107-150.

[53] Glocker, C., Pfeiffer, F., 1993, "Complementarity problems in multibody systems with planar friction". Archive of Applied Mechanics, **63**(7), 452-463.

[54] Pang, J., Trinkle, J.C., 1996, "Complementarity formulations and existence of solutions of dynamic multi-rigid-body contact problems with Coulomb friction". Mathematical Programming, **73**(2), 199-226.

[55] Trinkle, J.C., Tzitzouris, J.A., Pang, J.S., 2001, "Dynamic multi-rigid-body systems with concurrent distributed contacts". Philosophical Transactions: Mathematical, Physical and Engineering Sciences, **359**, 2575-2593.

[56] Signorini, A., 1933, *Sopra alcune questioni di elastostatica*. Atti della Societa Italian per il Progresso della Scienza.

[57] Moreau, J.J., 1979, "Application of convex analysis to some problems of dry friction". Trends in Applications of Pure Mathematics to Mechanics, (H. Zorski, Ed.) London, U.K.: Pitman, **2**, 263-280.

[58] Panagiotopoulos, P.D., 1985, *Inequality problems in mechanics and applications. Convex and Nonconvex Energy Functionals*. Birkhäuser-Verlag., Basel, Boston, Stuttgart.

[59] Kwak, B.M., 1991, "Complementarity problem formulation of three-dimensional frictional contact". Journal of Applied Mechanics, **58**, 134-140.

[60] Anitescu, M., Potra, F.A., 1997, "Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems". Nonlinear Dynamics, **14**(3), 231-247.

[61] Brogliato, B., 2003, "Some perspectives on the analysis and control of complementarity systems". IEEE Transactions on Automatic Control, **48**(6), 918-935.

[62] Timoshenko, S.P., Goodier, J.N., 1970, *Theory of Elasticity*, McGraw Hill, New York.

[63] Hertz, H., 1896, *On the contact of solids - On the contact of rigid elastic solids and on hardness*, (Translated by D. E. Jones and G. A. Schott), Miscellaneous Papers, MacMillan and Co. Ltd., London, 146-183.

[64] Erkaya S., Uzmay I., 2009, "Investigation on effect of joint clearance on dynamics of four-bar mechanism". Nonlinear Dynamics, **58**(1-2), 179-198.

[65] Flores, P., 2010, "A parametric study on the dynamic response of planar multibody systems with multiple clearance joints", Nonlinear Dynamics, **61**(4), pp. 633-653.

[66] Zukas, J.A., Nicholas, T., Greszczuk, L.B., Curran, D. R., 1982, *Impact Dynamics*, John Wiley and Sons, New York.

[67] Goldsmith, W., 1960, *Impact - The Theory and Physical Behaviour of Colliding Solids*, Edward Arnold Ltd, London.

[68] Greenwood, D.T., 1965, *Principles of Dynamics*. Prentice Hall, Englewood Cliffs, New Jersey.

[69] Flores, P., 2004, *Dynamic Analysis of Mechanical Systems with Imperfect Kinematic Joints*. Ph.D. Dissertation, Mechanical Engineering Department, University of Minho, Guimarães, Portugal.

[70] Baumgarte, J., 1972, "Stabilization of constraints and integrals of motion in dynamical systems", Computer Methods in Applied Mechanics and Engineering, **1**, pp. 1-16.

[71] Flores, P., Machado, M., Seabra, E., Silva, M.T., 2011, "A parametric study on the Baumgarte stabilization method for forward dynamics of constrained multibody systems", Journal of Computational and Nonlinear Dynamics, **6**(1), pp. 011019-9.

[72] Shampine, L., Gordon, M., 1975, *Computer solution of ordinary differential equations: the initial value problem*, Freeman, San Francisco, California.

[73] ANSY, Y14.5M-1994, 1994, *Dimensional and Tolerancing*. ASME, New York.

[74] Wiggins, S., 1990, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer Verlag, New York.

[75] Flores, P., 2010, *MUBODYNA – A FORTRAN program for dynamic analysis of planar multibody systems*. University of Minho, Guimarães, Portugal.