Baer-Levi semigroups of linear transformations

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Synopsis

Given an infinite-dimensional vector space $V$, we consider the semi-
group $GS(m, n)$ consisting of all injective linear $\alpha : V \to V$ for which
codim $\text{ran } \alpha = n$ where $\dim V = m \geq n \geq \aleph_0$. This is a linear ver-
sion of the well-known Baer-Levi semigroup $BL(p, q)$ defined on an
infinite set $X$ where $|X| = p \geq q \geq \aleph_0$. We show that, although
the basic properties of $GS(m, n)$ are the same as those of $BL(p, q)$,
the two semigroups are never isomorphic. We also determine all left
ideals of $GS(m, n)$ and some of its maximal subsemigroups: in this,
we follow previous work on $BL(p, q)$ by Sutov (1966) and Sullivan

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1. Introduction

Throughout this paper, $X$ is an infinite set with cardinal $p$, and $q$ is a cardinal such that $\aleph_0 \leq q \leq p$. Let $T(X)$ denote the semigroup under composition of all (total) transformations from $X$ to $X$. If $\alpha \in T(X)$, we write $\text{ran} \, \alpha$ for the range of $\alpha$ and define the rank of $\alpha$ to be $r(\alpha) = |\text{ran} \, \alpha|$. We also write $D(\alpha) = X \setminus X\alpha$, $d(\alpha) = |D(\alpha)|$, $C(\alpha) = \bigcup \{ y\alpha^{-1} : |y\alpha^{-1}| \geq 2 \}$, $c(\alpha) = |C(\alpha)|$, and refer to these cardinal numbers as the defect and the collapse of $\alpha$, respectively.

We now write $BL(p, q) = \{ \alpha \in T(X) : c(\alpha) = 0, d(\alpha) = q \}$ and call this the Baer-Levi semigroup on $X$: as shown in ([1] vol 2, section 8.1), it is a right simple, right cancellative semigroup without idempotents; and any semigroup with these properties can be embedded in some Baer-Levi semigroup. In addition, every automorphism $\varphi$ of $BL(p, q)$ is “inner”: that is, there exists $g \in G(X)$, the symmetric group on $X$, such that $\alpha \varphi = g \alpha g^{-1}$ for all $\alpha \in BL(p, q)$ [6].

In this paper, we examine a related semigroup defined as follows. Let $V$ be a vector space over a field $F$ and suppose $\dim V = p \geq \aleph_0$. To emphasize the analogy between our work and what has been done already for $BL(p, q)$, we let $T(V)$ denote the semigroup under composition of all linear transformations from $V$ to $V$: in other words, we use the ‘$V$’ in $T(V)$ to denote the fact that we are considering linear transformations. If $\alpha \in T(V)$, we write $\ker \alpha$ and $\text{ran} \, \alpha$ for the kernel and the range (image) of $\alpha$, and put $n(\alpha) = \dim \ker \alpha, \ r(\alpha) = \dim \text{ran} \, \alpha, \ d(\alpha) = \text{codim} \, \text{ran} \, \alpha$.

As usual, these are called the nullity, rank and defect of $\alpha$, respectively. For each cardinal $q$ such that $\aleph_0 \leq q \leq p$, we write $GS(p, q) = \{ \alpha \in T(V) : n(\alpha) = 0, d(\alpha) = q \}$ and call this the linear Baer-Levi semigroup on $V$. In section 2, we show this is indeed a semigroup with the same properties as $BL(p, q)$: this fact extends work by Lima [8] Proposition 4.1 on $GS(p, p)$. More importantly however, in section 3 we show these two types of Baer-Levi semigroups – one defined on sets, the other on vector spaces – are never isomorphic. In section 4, we transfer results of Sutov [11] and Sullivan [10] on the left ideals of $BL(p, q)$ to the vector space setting. Finally, in section 5 we initiate the study of maximal subsemigroups of $GS(p, q)$ by using ideas taken from [7].
2. Basic properties

In what follows, \( Y = A \cup B \) means \( Y \) is a disjoint union of \( A \) and \( B \), and we write \( \text{id}_Y \) for the identity transformation on \( Y \). We adopt the convention introduced in [1] vol 2, p 241: namely, if \( \alpha \in T(X) \) then we write
\[
\alpha = \left( \begin{array}{c} A_i \\ x_i \end{array} \right)
\]
and take as understood that the subscript \( i \) belongs to some (unmentioned) index set \( I \), that the abbreviation \( \{ x_i \} \) denotes \( \{ x_i : i \in I \} \), and that \( \text{ran} \alpha = \{ x_i \} \) and \( x_i\alpha^{-1} = A_i \).

A similar notation can be used for \( \alpha \in T(V) \) (see [9] p 125). That is, often it is necessary to construct some \( \alpha \in T(V) \) by first choosing a basis \( \{ e_i \} \) for \( V \) and some \( \{ u_i \} \subseteq V \), and then letting \( e_i\alpha = u_i \) for each \( i \in I \) and extending this action by linearity to the whole of \( V \). To abbreviate this process, we simply say, given \( \{ e_i \} \) and \( \{ u_i \} \) within context, that \( \alpha \in T(V) \) is defined by letting
\[
\alpha = \left( \begin{array}{c} e_i \\ u_i \end{array} \right).
\]

As usual, the subspace of \( V \) generated by a linearly independent subset \( \{ e_i \} \) of \( V \) is denoted by \( \langle e_i \rangle \); and, often when we write \( U = \langle e_i \rangle \), we will tacitly assume the set \( \{ e_i \} \) is a basis for the subspace \( U \). The following result is analogous to [1] vol 2, Theorem 8.2 (and to [8] Proposition 4.1 for the case \( p = q \)).

**Theorem 2.1.** If \( \dim V = p \geq q \geq \aleph_0 \) then \( GS(p, q) \) is a right cancellative, right simple semigroup without idempotents.

**Proof.** Assume \( \alpha, \beta \in GS(p, q) = S \) say, and let \( \text{ran} \alpha = \langle e_i \rangle \) and \( V = \langle e_i, e_j \rangle \), so \( |J| = q \). Then \( \{ e_i\beta \} \cup \{ e_j\beta \} \) is independent and generates \( \text{ran} \beta \), and \( \text{ran} \alpha\beta = \langle e_i\beta \rangle \).

Hence \( d(\alpha\beta) = q + q = q \), and clearly if \( \alpha, \beta \) are injective then \( \alpha\beta \) is also, so \( \alpha\beta \in S \).

Since elements of \( S \) are injective, the semigroup is right cancellative; also, if \( \varepsilon \in S \) is idempotent then \( (u\varepsilon)\varepsilon = (u)\varepsilon \) for all \( u \in V \) implies \( \varepsilon = \text{id}_V \), a contradiction. Suppose \( \alpha, \beta \in S \) and write \( V = \langle e_k \rangle \) and
\[
\alpha = \left( \begin{array}{c} e_k \\ x_k \end{array} \right), \quad \beta = \left( \begin{array}{c} e_k \\ y_k \end{array} \right).
\]

Now if \( V = \langle x_k, x_\ell \rangle = \langle y_k, y_\ell, y_m \rangle \) where \( |L| = |M| = q \) and we define
\[
\mu = \left( \begin{array}{cc} x_k & x_\ell \\ y_k & y_\ell \end{array} \right)
\]
then \( \mu \in S \) and \( \beta = \alpha\mu \), and we have shown \( GS(p, q) \) is right simple. \( \square \)
Clearly, before proceeding any further, it is important to decide whether any of the semigroups \(GS(m, n)\) are isomorphic to any of the \(BL(p, q)\) for appropriate cardinals \(m, n\) and \(p, q\) (this was not considered in [8]). This question can be answered in one of two ways: by showing the cardinals of \(BL(p, q)\) and \(GS(m, n)\) are different; or by finding some algebraic property of \(BL(p, q)\) that is not preserved under an isomorphism between it and \(GS(m, n)\). For their intrinsic interest, we now establish some results pertinent to the first approach. Something like the following appears in [3] Corollary 1.5.13 and Exercise 1.5.36, but for completeness we include a proof.

**Lemma 2.2.** If \(|X| = p \geq q\) and \(p \geq \aleph_0\) then the number of subsets of \(X\) with cardinal \(q\) equals \(p^q\). In fact, this is also the number of injective mappings from a set of cardinal \(q\) into a set of cardinal \(p\).

Proof. Let \(|A| = q, |B| = p\) and note that for each \(Y \subseteq B\) with cardinal \(q\), there is an injective map \(A \to B\) with range \(Y\). Hence the number \(k\) of \(Y \subseteq B\) with cardinal \(q\) is at most the number \(\ell\) of injective maps \(A \to B\), and clearly \(\ell \leq |B^A| = p^q\). Now each \(\alpha : A \to B\) is a subset of \(A \times B\) and \(|\alpha| = q\). Hence \(|B^A|\) is at most the number \(m\) of subsets of \(A \times B\) with cardinal \(q\). But \(q \times p = p\), so \(m = k\). Hence \(k = p^q\). Thus we have \(p^q = k \leq \ell \leq p^q\), and the result follows. \(\square\)

We can now determine the cardinal of \(BL(p, q)\). But first we need the order of \(G(X)\) where \(|X| = p \geq \aleph_0\). To find this, write \(X = A \cup B\) where \(|A| = |B| = p\) and note that for each \(Y \subseteq A\), there exists \(\pi \in G(X)\) which fixes \(Y\) pointwise and shifts all elements of \((A \setminus Y) \cup B\). Hence \(|G(X)| \geq 2^{|A|} = 2^p\) and of course \(|G(X)| \leq |T(X)| = 2^p\).

For clarity in what follows, we sometimes write \(BL(X, p, q)\) in place of \(BL(p, q)\), and similarly \(GS(V, m, n)\) instead of \(GS(m, n)\) (see Theorem 3.5 below).

**Theorem 2.3.** If \(|X| = p \geq q \geq \aleph_0\) then \(|BL(p, q)| = 2^p\).

Proof. Suppose \(q < p\). For each \(Y \subseteq X\) with cardinal \(q\), we know \(|X \setminus Y| = p\) and there exists a bijection \(\alpha : X \to X \setminus Y\), hence \(\alpha \in BL(p, q)\). In fact, the set of all such \(\alpha\) is in one-to-one correspondence with \(G(X \setminus Y)\). Therefore, since in this case \(p + q = p\), we have:

\[
|BL(p, q)| = \sum \{|G(X \setminus Y)| : Y \subseteq X, |Y| = q\} = 2^p \cdot p^q = p^p \cdot p^q = p^p = 2^p.
\]

To find the cardinal \(k\) of \(BL(p, p)\) when \(p > \aleph_0\), write \(X = Y \cup Z\) where \(|Y| = |Z| = p\) and fix \(\beta \in BL(Z, p, p)\). Then for \(\aleph_0 \leq q < p\) and each \(\alpha \in BL(Y, p, q)\), we have \(\alpha \cup \beta \in BL(X, p, p)\), so \(k \geq |BL(Y, p, q)| = 2^p\) and it follows that \(k = 2^p\).

Finally for \(p = \aleph_0\) we note that for each \(Y \subseteq X\) such that \(|Y| = |X \setminus Y| = \aleph_0\), there exists \(\alpha \in BL(p, p)\) such that \(\text{ran } \alpha = Y\), hence in this case \(|BL(p, p)|\) is at least the
number \( k \) of such subsets \( Y \) of \( X \). To calculate \( k \), note that \( \{ Y \subseteq X : |Y| = \aleph_0 \} \) equals
\[
\bigcup_n \{ Y \subseteq X : |Y| = \aleph_0, |X \setminus Y| = n < \aleph_0 \} \cup \{ Y \subseteq X : |Y| = |X \setminus Y| = \aleph_0 \}
\]
and, taking cardinals, we find by Lemma 2.2 that
\[
2^{\aleph_0} = \aleph_0^{\aleph_0} = \sum_{n<\aleph_0} \aleph_0^n + k = \aleph_0 + k.
\]
Hence \( k \) must equal \( 2^{\aleph_0} \). \( \square \)

To obtain analogous results for \( GS(p, q) \), we first recall [5] vol II, p 245: if \( V \) is a vector space over a field \( F \) and \( \dim V = p \geq \aleph_0 \) then \( |V| = p \times |F| \). Now let \( A \) be a basis for \( V \). Since each \( \alpha \in T(V) \) determines a unique map from \( A \) into \( V \), and conversely any map from \( A \) into \( V \) can be extended by linearity to a unique \( \alpha \in T(V) \), we have \( |T(V)| = |V|^p \). In fact, since \( p^p = 2^p \), we can deduce that
\[
|T(V)| = \begin{cases} 
2^p & \text{if } |F| \leq p, \\
|F|^p & \text{if } |F| > p.
\end{cases}
\]

**Lemma 2.4.** If \( V \) is a vector space with \( \dim V = p \geq q \) and \( p \geq \aleph_0 \), then the number of subspaces of \( V \) with dimension \( q \) equals \( |V|^q \). In fact, this is also the number of injective linear mappings from a vector space of dimension \( q \) into another with dimension \( p \) over the same field.

Proof. Let \( k \) be the number of subspaces of \( V \) with dimension \( q \). Now, if a subspace \( U \) has dimension \( q \) then there is a basis \( A \subseteq U \) with \( |A| = q \), so \( k \) is at most the number \( |V|^q \) of subsets of \( V \) with cardinal \( q \). Now let \( U \) be any vector space with dimension \( q \). Note that each linear \( \alpha : U \to V \) can be regarded as a subspace of the vector space \( U \times V \). In fact, if \( A = \{ a_i \} \) is a basis for \( U \) then \( \{(a_i, a_i \alpha)\} \) is a basis for \( \alpha \subseteq U \times V \), hence \( \dim \alpha = q \). Therefore the number of linear \( U \to V \) is at most the number \( \ell \) of subspaces of \( U \times V \) with dimension \( q \). But \( \dim(U \times V) = q + p = p \) (since if \( \{ u_i \} \) is a basis for \( U \) and \( \{ v_j \} \) a basis for \( V \) then \( \{(u_i, 0)\} \cup \{(0, v_j)\} \) is a basis for \( U \times V \). Thus, \( U \times V \) and \( V \) have the same dimension, hence they are isomorphic, so \( \ell = k \). Also, if \( A \) is a basis for \( U \) then any map \( A \to V \) can be uniquely extended to a linear \( U \to V \); and any linear \( U \to V \) induces a unique map \( A \to V \). That is, the number of linear \( U \to V \) equals \( |V|^q \) and it follows that \( k = |V|^q \).

Finally, let \( U \) be a vector space with dimension \( q \) and \( V \) a vector space with dimension \( p \) over the same field. To find \( m \), the number of injective linear \( U \to V \), we follow
the corresponding argument in the proof of Lemma 2.2. That is, for each injective linear $U \to V$, there is an injective linear $U \to U \times V$ (for example, $U \to \{0\} \times V$); and conversely, since $q \times p = p$ and thus $U \times V$ is isomorphic to $V$, for each injective linear $U \to U \times V$, there is an injective linear $U \to V$. Now if $\alpha : U \to V$ is any linear map, let $\alpha' : U \to U \times V, u \mapsto (u, u\alpha)$, and note that $\alpha'$ is linear and injective. Hence the number $|V|^q$ of linear $U \to V$ is at most the number of injective linear $U \to U \times V$, and we have seen this equals $m$. It follows that $m = |V|^q$ as required.

**Theorem 2.5.** If $\dim V = p \geq q \geq \aleph_0$, then $|GS(p, q)| = |V|^p$.

Proof. Suppose $V = \langle v_i, v_j \rangle$ is a vector space over a field $F$ where $|I| = p$ and $|J| = q$, and let $W = \langle v_i \rangle$. Now, for each basis $A = \{a_i\}$ for $V$ and each $\alpha \in G(A)$, there exists an invertible linear $\alpha' : V \to V$ and an injective linear $\beta : V \to V, a_i \mapsto v_i$, and then $\alpha'\beta \in GS(p, q)$. In other words,

$$|GS(p, q)| \geq \sum \{|G(A)| : A \text{ is a basis for } V\}.$$  

But if $|F| \geq 3$ then, for all $k_i \in F^* = F \setminus \{0\}$, $\{k_ia_i\}$ is a basis for $V$, hence in this case the number of bases for $V$ is at least $|F^*|^p = |F|^p$. Thus

$$|GS(p, q)| \geq 2^p.|F|^p = (p.|F|)^p = |V|^p,$$

and equality follows.

Suppose now that $|F| = 2$. Let $\{e_i\}$ be a basis for $V$, so $|I| = p$. For each fixed $j \in I$, $\{e_j + e_i\}$ is a basis for $V$ and so the number of bases for $V$ is at least $p$. Hence

$$|GS(p, q)| \geq \sum \{|G(A)| : A \text{ is a basis for } V\} \geq p.2^p = (p.2)^p = |V|^p,$$

and then we also have equality in case $|F| = 2$. □

From Theorems 2.3 and 2.5 we deduce that $BL(p, q)$ is not isomorphic to $GS(m, n)$ when $|F| > 2^p$ and $m \geq p$. For, König’s Theorem states that if $\{r_i : i \in I\}$ and $\{s_i : i \in I\}$ are any sets of cardinals such that $r_i < s_i$ for each $i$ then $\sum_i r_i < \prod_i s_i$ ([3] Theorem 1.6.7). In particular, if $r_i = 2^p$ for each $i \in I$ and $|I| = p$ then $\sum_i r_i = p \times 2^p = 2^p$; and if $s_i = |F|$ for each $i$, then $\prod_i s_i = |F|^p$. So in this case

$$|GS(m, n)| = |V|^m \geq |V|^p = |F|^p > 2^p = |BL(p, q)|.$$  

To see that there are fields of any infinite order, we prove the following result for which we are unable to find a detailed reference.

**Lemma 2.6.** For each $k \geq \aleph_0$, there is a field $F$ such that $|F| = k$.  

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Proof. We begin by closely following [4] Exercise III.5.4. Namely, let $X$ be a non-empty set with cardinal $k \geq \aleph_0$, let $\mathbb{N}$ denote the set of non-negative integers, and suppose $\Phi$ is the set of all maps $\varphi : X \to \mathbb{N}$ such that $\varphi(x) \neq 0$ for at most a finite number of $x \in X$. Then $\Phi$ is an abelian monoid under the operation $\cdot$ defined by

$$(\varphi \cdot \psi)(x) = \varphi(x) + \psi(x).$$

We write $\varphi \cdot \psi = \varphi \psi$ when it is convenient to do so. For each $x \in X$ and $i \in \mathbb{N}$, we define $x^i \in \Phi$ by

$$x^i(y) = \begin{cases} i & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

If $\varphi \in \Phi$ and $x_1, \ldots, x_n$ are the only $y \in X$ such that $\varphi(y) \neq 0$, it can be shown that

$$\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n},$$

where $i_j = \varphi(x_j)$ for $j = 1, \ldots, n$. If $\mathbb{Q}$ is the field of rational numbers, we let $\mathbb{Q}[X]$ denote the set of all functions $f : \Phi \to \mathbb{Q}$ such that $f(\varphi) \neq 0$ for at most a finite number of $\varphi \in \Phi$. Then $\mathbb{Q}[X]$ is a commutative ring with identity under the operations:

$$(f + g)(\varphi) = f(\varphi) + g(\varphi),$$

$$(fg)(\varphi) = \sum f(\alpha)g(\beta),$$

where the summation is over all pairs $(\alpha, \beta)$ such that $\alpha \beta = \varphi$. If $\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n} \in \Phi$ and $r \in \mathbb{Q}$, we let $r \varphi$ denote the function $f : \Phi \to \mathbb{Q}$ defined by

$$f(\psi) = \begin{cases} r & \text{if } \psi = \varphi, \\ 0 & \text{if } \psi \neq \varphi. \end{cases}$$

Then every non-zero $f \in \mathbb{Q}[X]$ can be written as

$$f = \sum_{i=0}^{m} r_i x_1^{s_{i1}} x_2^{s_{i2}} \cdots x_n^{s_{in}},$$

(2.1)

where $r_i \in \mathbb{Q}$, $x_j \in X$ and $m, s_{ij} \in \mathbb{N}$ are all uniquely determined by $f$.

Now, as in [4] Theorem III.5.3, $\mathbb{Q}[X]$ is an integral domain, so we can form a field of ‘rational functions’ (compare [4] p 233, Example) thus:

$$\mathbb{Q}(X) = \{ f/g : f, g \in \mathbb{Q}[X], g \neq 0 \}.$$  

We assert that $|\mathbb{Q}(X)| = k$. To see this, first note that each polynomial $x1 \in \Phi \subseteq \mathbb{Q}[X]$ equals $x1/1 \in \mathbb{Q}(X)$, hence $|\mathbb{Q}(X)| \geq k$. On the other hand, using the map $f/g \mapsto (f, g)$, we have:

$$|\mathbb{Q}(X)| \leq |\mathbb{Q}[X] \times \mathbb{Q}[X]| = |\mathbb{Q}[X]|.$$
Now, by uniqueness, the number of polynomials in $\mathbb{Q}[X]$ with the form $r x_1^{s_1} x_2^{s_2} \cdots x_n^{s_n}$ is exactly

$$|\mathbb{Q}| \times k^{s_1} \times \cdots \times k^{s_n} = k.$$ 

Thus, to count all $f \in \mathbb{Q}[X]$ expressed as in (2.1) is equivalent to counting the number of subsets with cardinal $m < \aleph_0$ in a set with cardinal $k$, and by Lemma 2.2 this number equals $k^m = k$. It then follows that $|\mathbb{Q}(X)| = k$ as asserted. □

Of course, this discussion leaves open the question of whether $\text{BL}(p, q)$ and $\text{GS}(m, n)$ are isomorphic when the condition “$|F| > 2^p$ and $m \geq p$” does not hold. We consider this possibility in the next section.

3. Isomorphisms between Baer-Levi semigroups

In this section we aim to use algebraic conditions on $\text{BL}(p, q)$ to decide whether it is ever isomorphic to $\text{GS}(m, n)$. To do this, we first recall that Green’s $\mathcal{L}$ relation on $\text{BL}(p, q)$ equals the identity relation on $\text{BL}(p, q)$ and the $\mathcal{R}$ relation equals the universal relation on $\text{BL}(p, q)$. In addition, $\text{BL}(p, q)$ is not regular (since it contains no idempotents). In this situation, it can be useful to study Green’s $\ast$-relations instead. That is, following [2], if $S$ is any semigroup and $a, b \in S$, we say $a \mathcal{L}^* b$ if and only if

$$\text{for all } x, y \in S^1, ax = ay \text{ if and only if } bx = by,$$

and we define $\mathcal{R}^*$ on $S$ dually. Clearly these relations are equivalences on $S$. In fact, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ always, so $\mathcal{R}^*$ is universal on $\text{BL}(p, q)$. However the characterisation of $\mathcal{L}^*$ on $\text{BL}(p, q)$ is comparable with that of $\mathcal{L}$ on $T(X)$ [1] vol 1, Lemma 2.5: namely, from the next result, we deduce that $\alpha \mathcal{L}^* \beta$ on $\text{BL}(p, q)$ if and only if $\text{ran } \alpha = \text{ran } \beta$.

**Lemma 3.1.** If $\alpha, \beta \in \text{BL}(p, q)$ then the following are equivalent.

(a) $\text{ran } \beta \subseteq \text{ran } \alpha$,

(b) for each $\lambda, \mu \in \text{BL}(p, q)^1$, $\alpha \lambda = \alpha \mu$ implies $\beta \lambda = \beta \mu$,

(c) for each $\lambda \in \text{BL}(p, q)$, $\alpha \lambda = \alpha$ implies $\beta \lambda = \beta$.

**Proof.** Assume $\alpha, \beta \in \text{BL}(p, q)$ are such that $\text{ran } \beta \subseteq \text{ran } \alpha$. Then $\beta = \beta_1 \alpha$ for some $\beta_1 \in T(X)$. Let $\lambda, \mu \in \text{BL}(p, q)^1$. Then, $\alpha \lambda = \alpha \mu$ implies $\beta \lambda = (\beta_1 \alpha) \lambda = \beta_1 (\alpha \lambda) = \beta_1 (\alpha \mu) = (\beta_1 \alpha) \mu = \beta \mu$. Hence (a) implies (b). It is obvious that (b) implies (c). To prove (c) implies (a), assume that, for each $\lambda \in \text{BL}(p, q)$, $\alpha \lambda = \alpha$ implies $\beta \lambda = \beta$. Write $X = \{x_i\}$ and

$$\alpha = \left( \begin{array}{c} x_i \\ a_i \end{array} \right), \quad \beta = \left( \begin{array}{c} x_i \\ b_i \end{array} \right).$$

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If \( X = \{a_i\} \cup \{a_j\} = \{b_i\} \cup \{b_j\} \) where \(|J| = q\), write \( \{a_j\} = \{c_j\} \cup \{d_j\} \) and define \( \lambda = \begin{pmatrix} a_i & a_j \\ a_i & c_j \end{pmatrix} \), \( \mu = \begin{pmatrix} a_i & a_j \\ a_i & d_j \end{pmatrix} \).

Then \( \lambda, \mu \in BL(p, q) \) and \( \alpha \lambda = \alpha = \alpha \mu \). Consequently \( \beta \lambda = \beta = \beta \mu \), and this implies \( \text{ran} \beta \subseteq \text{ran} \lambda = \{a_i\} \cup \{c_j\} \) and \( \text{ran} \beta \subseteq \text{ran} \mu = \{a_i\} \cup \{d_j\} \). Hence \( \text{ran} \beta \subseteq \{a_i\} = \text{ran} \alpha \), as required. \( \square \)

We now decide when \( BL(X, p, q) \) and \( BL(Y, m, n) \) are isomorphic: although the proof of the next result closely follows the arguments in [6], we provide all the details since similar ideas will be used later. However, first note that if \( \psi : A \to B \) is an order-isomorphism between two families of sets then \( (A_1 \cap A_2)\psi = A_1\psi \cap A_2\psi \) whenever \( A_1, A_2 \in A \) and \( A_1 \cap A_2 \in A \). This is because order-isomorphisms preserve infima.

**Theorem 3.2.** The semigroups \( BL(X, p, q) \) and \( BL(Y, m, n) \) are isomorphic if and only if \( p = m \) and \( q = n \). Moreover, for each isomorphism \( \theta \), there is a bijection \( h : X \to Y \) such that \( \alpha \theta = h^{-1} \alpha h \) for each \( \alpha \in BL(X, p, q) \).

Proof. Clearly, if the cardinals are equal as stated, then any bijection from \( X \) onto \( Y \) will induce an isomorphism between the semigroups. So we assume there is an isomorphism \( \theta : BL(X, p, q) \to BL(Y, m, n) \) and aim to find a bijection \( h : X \to Y \).

We begin by noting that Lemma 3.1 says: for \( \alpha_1, \alpha_2 \in BL(p, q) \), \( \text{ran} \alpha_1 \subseteq \text{ran} \alpha_2 \) if and only if for each \( \beta \) such that \( \alpha_2 \beta = \alpha_2 \), we have \( \alpha_1 \beta = \alpha_1 \). Since \( \theta \) is an isomorphism, it follows that \( \text{ran} \alpha_1 = \text{ran} \alpha_2 \) if and only if \( \text{ran}(\alpha_1 \theta) = \text{ran}(\alpha_2 \theta) \).

Hence, if \( B(X, q) \) is the family of all subsets \( A \) of \( X \) such that \(|A| = p \) and \(|X \setminus A| = q \), and \( B(Y, n) \) the family of all subsets \( B \) of \( Y \) such that \(|B| = m \) and \(|Y \setminus B| = n \), then \( \psi_\theta : B(X, q) \to B(Y, n) \), defined by letting \( A \psi_\theta = \text{ran}(\alpha \theta) \) where \( \alpha \in BL(p, q) \) is such that \( \text{ran} \alpha = A \), is a well-defined order-isomorphism of \( B(X, q) \) onto \( B(Y, n) \).

Next we show that every order-isomorphism \( \psi \) of \( B(X, q) \) onto \( B(Y, n) \) is induced by a bijection of \( X \) onto \( Y \). Let \( A \in B(X, q) \) and \( x \in X \setminus A \). We write \( A \cup \{x\} \) as \( A \cup x \).

Clearly, \( A \cup x \in B(X, q) \) and \( A \cup x \) covers \( A \). Hence \( (A \cup x)\psi \) covers \( A \psi \), that is, \( (A \cup x)\psi = A \psi \cup y \) for some \( y \in Y \setminus A \psi \). Write \( y = xh_A \). We proceed to show that \( xh_{A_1} = xh_{A_2} \) for all \( A_1, A_2 \in B(X, q) \) not containing \( x \). Let \( A_1, A_2 \in B(X, q) \) with \( x \notin A_1 \cup A_2 \). If \( A_1 \cap A_2 \in B(X, q) \), then

\[
\begin{align*}
(A_1 \psi \cap A_2 \psi) \cup xh_{A_1 \cap A_2} &= (A_1 \cap A_2) \psi \cup xh_{A_1 \cap A_2} \\
&= ((A_1 \cap A_2) \cup x) \psi \\
&= ((A_1 \cup x) \cap (A_2 \cup x)) \psi \quad \text{(3.1)} \\
&= (A_1 \cup x) \psi \cap (A_2 \cup x) \psi \\
&= (A_1 \psi \cup xh_{A_1}) \cap (A_2 \psi \cup xh_{A_2}) .
\end{align*}
\]
Finally, we prove that, for each $x$ be such that $x \in A_1 \psi$ and so $((A_1 \cap A_2) \cup x) \psi \subseteq A_1 \psi$ by (3.1). Since $\psi$ preserves order, $(A_1 \cap A_2) \cup x \subseteq A_1$ and this implies $x \in A_1$, a contradiction. Therefore, $xh_{A_2} \notin A_1 \psi$. Similarly, we conclude that $xh_{A_1} \notin A_2 \psi$ and hence $\{xh_{A_1}, xh_{A_2}\} = \{x(A_1 \cap A_2) \cup x\}$ with $\beta, \gamma \in y$. Thus $A_1 = xh_{A_2} = xh_{A_1 \cap A_2}$. On the other hand, if $A_1 \cap A_2 \notin B(X, q)$ then, since $|X \setminus (A_1 \cap A_2)| = q$, we have $|A_1 \cap A_2| \neq p$ and thus $p$ must equal $q$. In addition, $|A_1| = |A_1 \setminus A_2| = p = |A_2 \setminus A_1| = |A_2|$. We write $A_2 \setminus A_1$ as the disjoint union of two sets $M$ and $N$, with $|M| = |N| = p$ and let $A_3 = (A_1 \setminus A_2) \cup M$. By construction, both $M$ and $A_3$ belong to $B(X, q)$. Moreover, $x \notin A_1 \cup A_3$, $A_1 \cap A_3 \in B(X, q)$ and $x \notin A_2 \cup A_3$, $A_2 \cap A_3 \in B(X, q)$. From the first case, we may conclude that $xh_{A_1} = xh_{A_2} = xh_{A_3}$.

We now define $h : X \to Y$ as follows: $xh = xh_A$, where $A \in B(X, q)$ satisfies $x \notin A$. The foregoing argument shows $h$ is well-defined. Suppose $xh_1 = xh_2$ for $xh_1, xh_2 \in X$ and take $A \in B(X, q)$ with $xh_1, xh_2 \in X \setminus A$. Then $(A \cup xh_1) \psi = A \psi \cup xh_A = \psi xh_A \psi = (A \cup xh_2) \psi$ and hence $A \cup xh_1 = A \cup xh_2$ since $\psi$ is one-to-one. Therefore $x_1 = x_2$ and thus $h$ is one-to-one. In order to show that $h$ is onto, let $y \in Y$ and $B \in B(Y, n)$, with $y \in B$. Let $A_1, A_2 \in B(X, q)$ be such that $A_1 \psi = B \setminus y$ and $A_2 \psi = B$. Then $A_2$ covers $A_1$ and so there exists $x \in X \setminus A_1$ such that $A_2 = A_1 \cup x$. Thus $B = (B \setminus y) \cup xh_{A_1}$ and $y = xh_{A_1}$. Hence $h$ is a bijection and $|X| = |Y|$.

Next we show that $\psi$ is induced by $h$, that is, $A \psi = Ah$ for each $A \in B(X, q)$. Let $y \in Ah$. Then there exists $x \in A$ with $y = xh$. Since $A \setminus x \in B(X, q)$ and $A$ covers $A \setminus x$, we have $A \psi = (A \setminus x) \psi \cup xh_{A \setminus x}$ which equals $(A \setminus x) \psi \cup A_{\psi, x}$ by the definition of $h$. Hence $y \in A \psi$. Conversely, if $y \in A \psi$ then $A \psi$ covers $A \psi \setminus y$. Let $A_1 \in B(X, q)$ be such that $A \psi \setminus y = A_1 \psi$. Then, $A_1$ covers $A_1$ since $\psi$ preserves order, and so there exists $x \in X \setminus A_1$ with $A = A_1 \cup x$. Thus $A \psi = (A \psi \setminus y) \cup xh_\psi$ (again by definition of $h$) and hence $y = xh \in Ah$. Therefore $A \psi = Ah$.

Finally, we prove that, for each $\alpha \in BL(p, q)$, $\alpha \theta = h_\theta^{-1} \alpha h_\theta$ where $h_\theta$ is the bijection corresponding to the order-isomorphism $\psi_\theta$. Let $\alpha \in BL(p, q), x_1 \in X$ and $x_2 = x_1 \alpha$. We may choose $A_1, A_2$ in $B(X, q)$ such that $A_1 \subseteq A_2$ and $A_2 \setminus A_1 = \{x_1\}$, together with $\beta, \gamma \in BL(X, q)$ such that $\text{ran } \beta = A_1$ and $\text{ran } \gamma = A_2$. Now $\text{ran } \gamma \setminus \text{ran } \beta = \{x_1\}$ and so

$$\text{ran } ((\gamma \alpha) \theta) \setminus \text{ran } ((\beta \alpha) \theta) = \text{ran } ((\gamma \theta) (\alpha \theta)) \setminus \text{ran } ((\beta \theta) (\alpha \theta))$$

$$= (\text{ran } (\gamma \theta) \setminus \text{ran } (\beta \theta)) (\alpha \theta)$$

$$= (A_{\psi_\theta} \setminus A_{\psi_\theta} (\alpha \theta))$$

$$= \{x_1 h_\theta \} \alpha \theta.$$
On the other hand, \( \text{ran}(\gamma \alpha) \setminus \text{ran}(\beta \alpha) = (A_2 \setminus A_1) \alpha = \{x_2\} \) and so

\[
\text{ran}(({\gamma \alpha} \theta) \setminus \text{ran}(({\beta \alpha} \theta)) = (\text{ran}(\gamma \alpha)) \psi_\theta \setminus (\text{ran}(\beta \alpha)) \psi_\theta
= \text{ran}(\gamma \alpha) h \setminus \text{ran}(\beta \alpha) h
= \{x_2h_\theta\}.
\]

Thus \( x_1h_\theta \alpha \theta = x_2h_\theta = x_1 \alpha \theta \) for all \( x_1 \in X \) and so \( \alpha \theta = h_\theta^{-1} \alpha h_\theta \). Finally, since \( \alpha \theta \in BL(Y, n) \) implies that \( |Y \setminus Y \alpha \theta| = n \) and, on the other hand, \( |Y \setminus Y h^{-1} \alpha h| = |(X \setminus X \alpha) h| = q \) for any bijection \( h : X \to Y \), we also have \( q = n \).

We now use a similar argument to show that \( BL(X, p, q) \) is never isomorphic to \( GS(V, m, m) \). For this, we need a result for \( GS(m, n) \) which is analogous to Lemma 3.1 (its proof uses the well-known characterisation of Green’s \( L \)-relation on \( T(V) \): see [1] vol 1, p 57, Exercise 6).

**Lemma 3.3.** If \( \alpha, \beta \in GS(m, n) \) then the following are equivalent.

- (a) \( \text{ran} \beta \subseteq \text{ran} \alpha \),
- (b) for each \( \lambda, \mu \in GS(m, n)^1 \), \( \alpha \lambda = \alpha \mu \) implies \( \beta \lambda = \beta \mu \),
- (c) for each \( \lambda \in GS(m, n) \), \( \alpha \lambda = \alpha \) implies \( \beta \lambda = \beta \).

**Proof.** Let \( \alpha, \beta \in GS(m, n) \) be such that \( \text{ran} \beta \subseteq \text{ran} \alpha \). Since \( \alpha, \beta \in T(V) \), there is some \( \beta_1 \in T(V) \) such that \( \beta = \beta_1 \alpha \). Let \( \lambda, \mu \in GS(m, n)^1 \). Then, \( \alpha \lambda = \alpha \mu \) implies \( \beta \lambda = (\beta_1 \alpha) \lambda = \beta_1 (\alpha \lambda) = \beta_1 (\alpha \mu) = (\beta_1 \alpha) \mu = \beta \mu \). Therefore (a) implies (b). Clearly (b) implies (c). Now assume (c) holds and write \( V = \langle e_i \rangle \). It follows that \( \text{ran} \alpha = \langle e_i \alpha \rangle \) where \( \{e_i \alpha\} \) is linearly independent since \( \alpha \) is one-to-one, and \( V = \langle e_i \alpha, e_j \rangle \) with \( |J| = n \) since \( d(\alpha) = n \). Write \( \{e_j\} = \{u_j\} \cup \{v_j\} \) and define \( \lambda, \mu \in T(V) \) as follows:

\[
\lambda = \begin{pmatrix} e_i \alpha & e_j \\ e_i \alpha & u_j \end{pmatrix}, \quad \mu = \begin{pmatrix} e_i \alpha & e_j \\ e_i \alpha & v_j \end{pmatrix}.
\]

Then \( \lambda, \mu \in GS(m, n) \) and \( \alpha \lambda = \alpha = \alpha \mu \). Hence \( \beta \lambda = \beta = \beta \mu \), so \( \text{ran} \beta \subseteq \text{ran} \lambda = \langle e_i \alpha, u_j \rangle \) and \( \text{ran} \beta \subseteq \text{ran} \mu = \langle e_i \alpha, v_j \rangle \). Now, if \( w \in \text{ran} \beta \) then \( w = \sum x_i(e_i \alpha) + \sum y_j u_j \) and \( w = \sum a_i(e_i \alpha) + \sum b_j v_j \) for some scalars \( x_i, y_j \) and \( a_i, b_j \); hence, by linear independence, \( y_j = b_j = 0 \) for each \( j \). Thus, \( \text{ran} \beta \subseteq \langle e_i \alpha \rangle = \text{ran} \alpha \), as required for (a).

Next we need [9] Lemma 6 which we quote below for convenience: as observed by Lima [8] p 433, this result highlights an essential difference between sets and vector spaces. For, if \( X = A \cup B \) where \( |A| = |B| = p \) and \( A \cap B = \emptyset \), then there is no \( C \subseteq X \) such that \( |C| = p \) and \( C \cap A = \emptyset = C \cap B \).
Lemma 3.4. If \( \dim V = p \geq \aleph_0 \) and \( U_1, U_2 \) are subspaces of \( V \) with codimension \( p \) in \( V \) then there is a subspace \( W \) of \( V \) such that \( \dim W = p \) and \( W \cap U_1 = \{0\} = W \cap U_2 \).

Theorem 3.5. The semigroups \( BL(X, p, q) \) and \( GS(V, m, m) \) are not isomorphic for any (infinite) cardinals \( p, q \) and \( m \), with \( q \leq p \).

Proof. Suppose \( \phi \) is an isomorphism from \( BL(X, p, q) \) onto \( GS(V, m, m) \). Then, from Lemmas 3.1 and 3.3 we have

\[
\text{ran } \alpha \subseteq \text{ran } \beta \quad \text{if and only if} \quad \text{ran}(\alpha \phi) \subseteq \text{ran}(\beta \phi). \quad (3.2)
\]

Let \( B(X, p, q) \) denote the family of all \( A \subseteq X \) such that \( |A| = p \) and \( |X \setminus A| = q \) and let \( G(V, m, m) \) denote the family of all subspaces \( U \) of \( V \) such that \( \dim U = m \) and \( \text{codim} U = m \). We observe that \( \phi \) gives rise in a natural way to a mapping \( \varphi \) from \( B(X, p, q) \) into \( G(V, m, m) \): for each \( A \in B(X, p, q) \), let \( A \varphi = \text{ran}(\alpha \phi) \) for some \( \alpha \in BL(X, p, q) \) such that \( \text{ran} \alpha = A \). From (3.2), we readily deduce that \( \varphi \) is a well-defined order-isomorphism of \( B(X, p, q) \) onto \( G(V, m, m) \).

Let \( A_1, A_2 \in B(X, p, q) \) and write \( X = A_1 \cup B_1 = A_2 \cup B_2 \) where \( |A_i| = p \) and \( |B_i| = q \) for \( i = 1, 2 \). Then \( A_1 \varphi, A_2 \varphi \) are elements of \( G(V, m, m) \), and hence \( \text{codim}(A_1 \varphi) = \dim V = \text{codim}(A_2 \varphi) \). By Lemma 3.4, there is a subspace \( W \) of \( V \) such that \( \dim W = m \) and \( W \cap A_1 \varphi = \{0\} = W \cap A_2 \varphi \). Let \( \{w_i\} \) be a basis for \( W \) and \( \{a_i\} \) a basis for \( A_1 \varphi \). Since \( W \cap A_1 \varphi = \{0\} \), it follows that \( \{w_i\} \cup \{a_i\} \) is linearly independent. Hence, it can be expanded to a basis \( \{w_i, a_i, v_k\} \) for \( V \), and so \( \text{codim} W = |I| + |K| = m \).

Thus, \( W \in G(V, m, m) \) and, since \( \varphi \) is onto, there is a subset \( C \) of \( X \) in \( B(X, p, q) \) such that \( W = C \varphi \). We have \( C = C \cap X = (C \cap A_1) \cup (C \cap B_1) \). Since \( |C| = p \) and \( |C \cap B_1| \leq q \), it follows that \( |C \cap A_1| = p \) when \( q < p \). Moreover, \( X = (C \cap A_1) \cup (C \cap B_1) \cup (X \setminus C) \) and so \( |X \setminus (C \cap A_1)| = q \). Therefore, \( C \cap A_1 \in B(X, p, q) \) if \( q < p \). Since \( C \cap A_1 \subseteq C \) and \( C \cap A_1 \subseteq A_1 \) and \( \varphi \) preserves order, we have \( (C \cap A_1) \varphi \subseteq W \cap A_1 \varphi = \{0\} \), which contradicts the fact that \( (C \cap A_1) \varphi \) belongs to \( G(V, m, m) \). On the other hand, if \( q = p \) then either \( |C \cap A_1| = p \) or \( |C \cap B_1| = p \). Without loss of generality, suppose \( |C \cap A_1| = p \) and write \( C \cap A_1 = Y \cup Z \) where \( |Y| = p = |Z| \). Then \( C = Y \cup Z \cup (C \cap B_1) \) and so \( |X \setminus Y| \geq |Z \cup (C \cap B_1)| = p \).

Therefore, \( Y \in B(X, p, p) \). Since \( Y \subseteq C \), \( Y \subseteq A_1 \) and \( \varphi \) preserves order, we have \( Y \varphi \subseteq W \cap A_1 \varphi = \{0\} \), which contradicts the fact that \( Y \varphi \in G(V, m, m) \). \( \Box \)

To obtain useful algebraic conditions on \( BL(p, q) \) when \( q < p \), we first observe that it contains a copy of \( BL(q, q) \): namely, if \( Y \subseteq X \) has cardinal \( q \), we let

\[
B(Y) = \{ \alpha \in BL(p, q) : Y \alpha \subseteq Y, \; \alpha \mid (X \setminus Y) = \text{id}_{X \setminus Y} \}
\]
which is clearly non-empty and isomorphic to $BL(Y, q, q)$. For each $\alpha \in BL(p, q)$, we define the shift of $\alpha$ to be

$$S(\alpha) = \{ x \in X : x\alpha \neq x \}, \quad s(\alpha) = |S(\alpha)|$$

and write

$$F(\alpha) = X \setminus S(\alpha) = \{ x \in X : x\alpha = x \}.$$ 

Note that $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$, so $s(\alpha\beta) \leq s(\alpha) + s(\beta)$ always. Clearly, $\lambda\alpha = \lambda$ in $BL(p, q)$ if and only if ran $\lambda \subseteq \text{Fix} \alpha$. Also if $\alpha \in BL(p, q)$ then $s(\alpha) = q$ if and only if $\lambda\alpha = \lambda$ for some $\lambda \in BL(p, q)$. For, we know $X \setminus \text{ran} \alpha \subseteq S(\alpha)$, so $s(\alpha) \geq q$ always. If $\lambda\alpha = \lambda$ for some $\lambda \in BL(p, q)$ then $\text{ran} \lambda \subseteq F(\alpha)$, so $S(\alpha) \subseteq \text{ran} \lambda$ and hence $s(\alpha) \leq q$; conversely, if $s(\alpha) = q < p$ then $|X| = |F(\alpha)|$ and any bijection $\lambda : X \to F(\alpha)$ satisfies $\lambda\alpha = \lambda$ and belongs to $BL(p, q)$. Thus, we have an algebraic characterisation for the elements of the semigroup:

$$\Lambda(q) = \{ \alpha \in BL(p, q) : s(\alpha) = q \}. \quad (3.3)$$

Next we define an equivalence $\sim$ on $\Lambda(q)$ by:

$$\alpha \sim \beta \quad \text{if and only if} \quad S(\alpha) = S(\beta).$$

Surprisingly, this has an algebraic characterisation which is similar to Lemma 3.1(c). Here it is also worth recalling [1] vol 2, Lemma 8.3: namely, the equation $xy = y$ cannot occur in any right simple, right cancellative semigroup without idempotents.

**Lemma 3.6.** If $\alpha, \beta \in \Lambda(q)$ then the following are equivalent.

(a) $S(\beta) \subseteq S(\alpha)$,

(b) for each $\lambda \in BL(p, q)$, $\lambda\alpha = \lambda$ implies $\lambda\beta = \lambda$.

**Proof.** Suppose $S(\beta) \subseteq S(\alpha)$. Let $\lambda \in BL(p, q)$ be such that $\lambda\alpha = \lambda$. Then ran $\lambda \subseteq F(\alpha)$ and since $S(\beta) \subseteq S(\alpha)$ it follows that ran $\lambda \subseteq F(\beta)$. Therefore, since $x\lambda \in \text{ran} \lambda$ for each $x$ in $X$, we have $x(\lambda\beta) = (x\lambda)\beta = x\lambda$, and hence $\lambda\beta = \lambda$. Conversely, assume (b) holds. If $F(\alpha) = \{e_i\}$ and $S(\alpha) = \{x_j\}$, write $\{e_i\} = \{f_i\} \cup \{f_j\}$ and

$$\alpha = \begin{pmatrix} f_i & f_j & x_j & x_j\alpha \end{pmatrix}.$$ 

Define

$$\lambda = \begin{pmatrix} e_i & x_j \\ f_i & f_j \end{pmatrix}.$$ 

Then $\lambda\alpha = \lambda$ and $\lambda \in BL(p, q)$ since $d(\lambda) = q = s(\alpha)$. Hence $\lambda\beta = \lambda$ and $F(\alpha) = \text{ran} \lambda \subseteq F(\beta)$. Thus $S(\beta) \subseteq S(\alpha)$. \qed
If we fix some $\beta \in \Lambda(q)$ and put $S(\beta) = Y$ then $F(\beta) = X \setminus Y$ and we have:

$$B(Y) = \{ \alpha \in \Lambda(q) : S(\alpha) \subseteq Y \},$$

and this is the set of all $\alpha \in BL(p,q)$ such that $\mu \alpha = \mu$ for some $\mu \in BL(p,q)$ and, for each $\lambda \in BL(p,q)$, $\lambda \beta = \lambda$ implies $\lambda \alpha = \lambda$. In other words, we have an algebraic description of each $BL(q,q)$ inside $BL(p,q)$ when $q < p$.

The aim now is to use this description to show that $BL(p,q)$ cannot be isomorphic to any $GS(m,n)$ when $p > q$. However, for this we need to identify a subset of $GS(m,n)$ which will correspond to some $B(Y)$ in $BL(p,q)$ under an isomorphism.

We start by defining, for each $\alpha \in T(V)$,

$$\text{Fix}(\alpha) = \{ u \in V : u\alpha = u \}.$$

Since this is a subspace of $V$, we can let $s(\alpha) = \text{codim} \text{Fix}(\alpha)$, and we call this the shift of $\alpha \in T(V)$. It can be shown that $s(\alpha \beta) \leq s(\alpha) + s(\beta)$: see [9] Lemma 5.

Hence, by analogy with $\Lambda(q)$ in $BL(p,q)$, if $m > n$ then there exists a subsemigroup of $GS(m,n)$ defined by:

$$\Sigma(n) = \{ \alpha \in GS(m,n) : s(\alpha) = n \}.$$

Furthermore, we can characterise $\Sigma(n)$ algebraically as follows: given $\alpha \in GS(m,n)$,

$$s(\alpha) = n \text{ if and only if } \lambda \alpha = \lambda \text{ for some } \lambda \in GS(m,n). \quad (3.4)$$

For, $\text{Fix}(\alpha) \subseteq \text{ran} \alpha$ implies $n = d(\alpha) \leq s(\alpha)$. If $\lambda \alpha = \lambda$ for some $\lambda \in GS(m,n)$ then $\text{ran} \lambda \subseteq \text{Fix}(\alpha)$ and this implies $s(\alpha) \leq d(\lambda) = n$; conversely, if $s(\alpha) = n < m$ then $\dim V = \dim \text{Fix}(\alpha)$ and any linear bijection $\lambda : V \rightarrow \text{Fix}(\alpha)$ satisfies $\lambda \alpha = \lambda$ and belongs to $GS(m,n)$.

Next we define an equivalence $\approx$ on $\Sigma(n)$ by

$$\alpha \approx \beta \text{ if and only if } \text{Fix}(\alpha) = \text{Fix}(\beta).$$

Its algebraic characterization is analogous to that of the equivalence $\sim$ defined on the subsemigroup $\Lambda(q)$ of $BL(p,q)$.

**Lemma 3.7.** If $\alpha, \beta \in \Sigma(n)$ then the following conditions are equivalent.

(a) $\text{Fix}(\alpha) \subseteq \text{Fix}(\beta)$,

(b) for each $\lambda \in GS(m,n)$, $\lambda \alpha = \lambda$ implies $\lambda \beta = \lambda$.
Proof. Assume \( \text{Fix}(\alpha) \subseteq \text{Fix}(\beta) \) and let \( \lambda \in \text{GS}(m, n) \) be such that \( \lambda \alpha = \lambda \). Then \( \text{ran} \lambda \subseteq \text{Fix}(\alpha) \) and so \( \text{ran} \lambda \subseteq \text{Fix}(\beta) \). Therefore, \( \lambda \beta = \lambda \). Conversely, suppose \( \{e_i\} = \{f_i\} \cup \{f_j\} \) is a basis for \( \text{Fix}(\alpha) \), where \( |I| = m > n = |J| \) since \( \alpha \in \Sigma(n) \).

Expand \( \{e_i\} \) to a basis \( \{e_i, v_j\} \) for \( V \) and note that

\[
\alpha = \begin{pmatrix} f_i & f_j & v_j \end{pmatrix}.
\]

Define \( \lambda \in T(V) \) by

\[
\lambda = \begin{pmatrix} e_i & v_j \end{pmatrix}.
\]

Then \( \lambda \alpha = \lambda \) and \( \lambda \in \text{GS}(m, n) \) since \( d(\lambda) = n = s(\alpha) \). Hence \( \lambda \beta = \lambda \) and so \( \text{Fix}(\alpha) = \text{ran} \lambda \subseteq \text{Fix}(\beta) \). \( \square \)

One candidate for a linear version of \( B(Y) \), the copy of \( BL(Y, q, q) \) in \( BL(p, q) \), can be defined as follows. If \( U \) is a subspace of \( V \) with dimension \( m \) and codimension \( n \) and if \( W \) is a complement of \( U \) in \( V \), then we let

\[
G(U, W) = \{ \alpha \in \text{GS}(m, n) : W\alpha \subseteq W, U \subseteq \text{Fix}(\alpha) \}
\]

which is clearly non-empty and isomorphic to \( \text{GS}(W, n, n) \). Unfortunately, whereas the complement of a subset \( Y \) in \( X \) is unique, this is not true for a complement of a subspace \( U \) in \( V \). Therefore, we now fix some \( \beta \in \Sigma(n) \) and put \( \text{Fix}(\beta) = U \) and \( V = U \oplus W \), so we have

\[
G(U, W) \subsetneq G(U) = \{ \alpha \in \Sigma(n) : U \subseteq \text{Fix}(\alpha) \}.
\]

Note that \( G(U) \) is the set of all \( \alpha \in \text{GS}(m, n) \) such that \( \mu \alpha = \mu \) for some \( \mu \) in \( \text{GS}(m, n) \) and, for each \( \lambda \in \text{GS}(m, n) \), \( \lambda \beta = \lambda \) implies \( \lambda \alpha = \lambda \); that is, \( G(U) \) has the same characteristics as \( B(Y) \) in \( BL(p, q) \). Note also that the above containment is ‘proper’.

For, if \( \{u_i\} \) is a basis for \( U \) and \( \{w_j\} \) a basis for \( W \) then \( V = \langle u_i, w_j \rangle \). Write \( \{u_i\} = \{v_i\} \cup \{v_j\} \) (possible since \( |J| = n \leq m = |I| \) by the choice of \( U \) and \( W \)) and also write \( \{v_j + w_j\} = \{x_j\} \cup \{y_j\} \). Then \( \{v_i\} \cup \{v_j\} \cup \{v_j + w_j\} \) is a basis for \( V \) and

\[
\alpha = \begin{pmatrix} u_i & w_j \\ u_i & x_j \end{pmatrix}
\]

is an element of \( G(U) \) (note that \( w_j \alpha \neq w_j \) for each \( j \)) and it does not belong to \( G(U, W) \) since \( W\alpha \cap W = \{0\} \).

To proceed further, we require two technical results whose purpose will become apparent in the proof of Theorem 3.10.

**Lemma 3.8.** For each vector space \( W \) with dimension \( n \geq \aleph_0 \), there exists \( \alpha \in \text{GS}(W, n, n) \) which fixes exactly one element of \( W \), namely 0.
Proof. Consider a basis for $W$ of the form:

$$\{w_{1k}\} \cup \{w_{2k}\} \cup \ldots.$$  

That is, $W = \langle w_{ik} \rangle$ where $|I| = \aleph_0$ and $|K| = n$. Define $\alpha \in T(W)$ by

$$\alpha = \left( \begin{array}{ccc} w_{1k} & \cdots & w_{ik} \\ w_{2k} & \cdots & w_{i+1,k} \end{array} \right).$$

Then $d(\alpha) = n$, so $\alpha \in GS(W, n, n)$. Now each $v \in W$ can be written as

$$v = \sum_k x_{i_1,k} w_{i_1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k}$$  \hspace{1cm} (3.5)

where the $x_{i_j,k}$ are scalars, each sum is over a finite (and possibly different) index set and we can assume $i_1 < i_2 < \ldots < i_r$. Therefore, if $v\alpha = v$, we have:

$$\sum_k x_{i_1,k} w_{i_1,k} + \sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k}$$

$$= \sum_k x_{i_1,k} w_{i_1+1,k} + \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \hspace{1cm} (3.6)$$

Since all the $w_{i_j,k}$ are linearly independent, and $w_{i_1,k}$ does not appear on the right of this equation, we deduce that $x_{i_1,k} = 0$ for all $k$. Then (3.6) reduces to

$$\sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} = \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \hspace{1cm} (3.7)$$

Again, $w_{i_2,k}$ appears nowhere on the right of this new equation, so $x_{i_2,k} = 0$ for all $k$. In like manner, all coefficients in (3.5) equal 0, hence $v = 0$ as required. \hfill $\Box$

**Lemma 3.9.** Let $V$ be a vector space of dimension $m$ and $U$ a subspace of $V$ with dimension $m$ and codimension $n$. If $W_1, W_2$ are subspaces of $V$ with codimension $n$ which contain $U$ and satisfy $\dim(W_1/U) = n = \dim(W_2/U)$, then there exists a subspace $L$ of $V$ with codimension $n$ in $V$ which properly contains $U$ such that $L \cap W_1 = U = L \cap W_2$.

Proof. Let $W_1, W_2$ be subspaces of $V$ such that $U \subseteq W_1$, $U \subseteq W_2$, $\text{codim}(W_1) = n = \text{codim}(W_2)$ and $\dim(W_1/U) = n = \dim(W_2/U)$. Recall that $\dim(V/U)$ equals the codimension of $U$ in $V$ and that there is a natural (linear) isomorphism between $V/W_i$ and $(V/U)/(W_i/U)$ for $i = 1, 2$. Hence, $W_i/U$ has codimension $n$ in $V/U$. By Lemma 3.4, there exists a subspace $L/U$ of $V/U$ such that $\dim(L/U) = n$ and $L/U \cap W_1/U = \{U\} = L/U \cap W_2/U$. Since $\dim(L/U) = n$, $U$ is properly contained in $L$. Moreover, since $L/U \cap W_1/U = \{U\}$,

$$n = \dim(W_1/U) \leq \text{codim}(L/U) \leq \dim(V/U) = n,$$
and so codim(L) = n. From L/U ∩ W1/U = {U} = L/U ∩ W2/U, we may conclude that L ∩ W1 = U = L ∩ W2.

\[ \square \]

**Theorem 3.10.** The semigroups BL(X, p, q) and GS(V, m, n) are not isomorphic for any infinite cardinals p, q, m, n with q < p and n < m.

Proof. Suppose \( \phi \) is an isomorphism from BL(X, p, q) onto GS(V, m, n). Let \( Y \subseteq X \) be such that \( |Y| = q \) and let \( \beta \in BL(p, q) \) be such that \( S(\beta) = Y \). Then, \( \beta \phi \in GS(m, n) \). Moreover, \( s(\beta \phi) = n \), since \( s(\beta) = q \) and so there exist \( \mu \in BL(p, q) \) and \( \mu \phi \in GS(m, n) \) such that \( \mu \beta = \mu \) and \( (\mu \phi)(\beta \phi) = \mu \phi \). Hence, \( \dim \text{Fix}(\beta \phi) = m \). Let \( U = \text{Fix}(\beta \phi) \) and \( V = U \oplus W \). Let \( B \) be the family of all subsets of \( Y \) with cardinal \( q \) and let \( G \) be the family of all subspaces of \( V \) with codimension \( n \) which contain \( U \). Consider \( \varphi \) defined as follows: given \( B \in B \), let \( B \varphi = \text{Fix}(\alpha \phi) \), where \( \alpha \in B(Y) \) is such that \( S(\alpha) = B \). We assert that \( \varphi \) is an anti-isomorphism from \( B \) onto \( G \).

Let \( B = \{ b_j \} \cup \{ e_j \} \cup \{ d_j \} \in B \), with \( |J| = q \) and write \( \{ d_j \} = \{ e_j \} \cup \{ f_j \} \). Write \( X = \{ x_i \} \cup B \) and define \( \alpha \in T(X) \) by

\[
\alpha = \begin{pmatrix}
  x_i & b_j & c_j & d_j \\
  x_i & e_j & b_j & c_j
\end{pmatrix}.
\]

Then \( c(\alpha) = 0, d(\alpha) = q \) and \( S(\alpha) = B \). Hence \( \alpha \in \Lambda(q) \) and, by the characterisations discussed at (3.3) and (3.4), we have \( \alpha \phi \in \Sigma(n) \). Also, since \( S(\alpha) \subseteq Y \), Lemmas 3.6 and 3.7 imply \( U \subseteq \text{Fix}(\alpha \phi) \). Therefore, \( \text{Fix}(\alpha \phi) \in G \). If \( B_1, B_2 \in B \) and \( \alpha_1, \alpha_2 \in B(Y) \) are such that \( S(\alpha_1) = B_1 \) and \( S(\alpha_2) = B_2 \), then

\[
B_1 \subseteq B_2 \iff S(\alpha_1) \subseteq S(\alpha_2)
\]

\[
\iff \lambda \alpha_2 = \lambda \quad \text{implies} \quad \lambda \alpha_1 = \lambda \quad \text{for all} \quad \lambda \in BL(p, q)
\]

\[
\iff \mu(\alpha_2 \phi) = \mu \quad \text{implies} \quad \mu(\alpha_1 \phi) = \lambda \quad \text{for all} \quad \mu \in GS(m, n)
\]

\[
\iff \text{Fix}(\alpha_2 \phi) \subseteq \text{Fix}(\alpha_1 \phi)
\]

\[
\iff B_2 \varphi \subseteq B_1 \varphi.
\]

Thus, \( \varphi \) is a well-defined one-to-one mapping which inverts order. To show that \( \varphi \) is onto, we will use Lemma 3.8. Let \( G = \{ e_i \} \in G \). Then codim \( G = n \) and \( U \subseteq G \).

Write \( V = G \oplus H \), with \( H = \{ f_j \} \) and define \( \varepsilon \in T(V) \) by

\[
\varepsilon = \begin{pmatrix}
  e_i & f_j \\
  e_i & f_j \alpha
\end{pmatrix},
\]

where \( \alpha \in GS(H, n, n) \) fixes exactly one element of \( H \), namely 0. Now, \( \varepsilon \in GS(V, m, n) \) and \( \text{Fix}(\varepsilon) = G \). For, if \( v = \sum a_i e_i + \sum b_j f_j \), then \( v \varepsilon = v \) if and only if \( \alpha \) fixes the element \( \sum b_j f_j \in H \). But the latter happens if and only if \( \sum b_j f_j = 0 \) in which case
We have before that \( S = \delta \). Let \( B = S(\delta) \). Since \( \text{Fix}(\delta) = U \subseteq G = \text{Fix}(\delta) \), we conclude as before that \( S(\delta) \subseteq S(\beta) = Y \). That is, \( B \in \mathcal{B} \) and \( B \varphi = G \).

We now show that, for subspaces \( W_1 = B_1 \varphi, W_2 = B_2 \varphi \) of \( V \) in \( \mathcal{G} \) with \( W_1 \cap W_2 = U \), we have \( B_1 \cup B_2 = Y \). Since \( \varphi \) inverts order, \( (B_1 \cup B_2) \varphi \) is a subset of \( B_1 \varphi \cap B_2 \varphi = W_1 \cap W_2 = U = Y \varphi \) (the last equation holds since \( Y \) is the greatest element of \( \mathcal{B} \) and \( U \) is the least element of \( \mathcal{G} \)). Hence, \( Y \subseteq B_1 \cup B_2 \) and so \( B_1 \cup B_2 = Y \).

Next we use the above results to produce a contradiction. Let \( B_1, B_2 \in \mathcal{B} \) be such that \( B_1 \cup B_2 = Y \). Then, \( B_1 \varphi = W_1 = \langle u_i, v_k \rangle \) and \( B_2 \varphi = W_2 = \langle u_i, w_\ell \rangle \), where \( U = \langle u_i \rangle \). Since \( \text{codim} \, W_1 = n = \text{codim} \, W_2 \), we can choose bases \( \{x_j\} \cup \{y_j\} \) and \( \{s_j\} \cup \{t_j\} \) for complements of \( W_1 \) and \( W_2 \), respectively, where \( |J| = n \). Then

\[
V = \langle u_i, v_k, x_j, y_j \rangle = \langle u_i, w_\ell, s_j, t_j \rangle.
\]

Let \( W'_1 = \langle u_i, v_k, x_j \rangle \) and \( W'_2 = \langle u_i, w_\ell, s_j \rangle \). Then \( W'_1, W'_2 \in \mathcal{G} \) and \( \text{dim}(W'_1/U) = n = \text{dim}(W'_2/U) \). By Lemma 3.9, there exists an element \( L \neq U \) in \( \mathcal{G} \) such that \( L \cap W'_1 = U = L \cap W'_2 \). Since \( W_1 \subseteq W'_1 \) and \( W_2 \subseteq W'_2 \), we have \( L \cap W_1 = U = L \cap W_2 \). Also, since \( \varphi \) is onto, there exists \( B \in \mathcal{B} \) such that \( B \varphi = L \). Therefore, \( B \varphi \cap B_1 \varphi = U = B \varphi \cap B_2 \varphi \), which implies that \( B \cup B_1 = Y = B \cup B_2 \). Thus, \( B_1, B_2 \subseteq B \) and \( Y = B \). Hence \( U = L \), a contradiction. \( \square \)

Next we show that \( BL(p,p) \) and \( GS(m,n) \), with \( n < m \), are not isomorphic. We recall that \( BL(X,p,p) \) is embeddable in \( BL(Y,r,p) \), with \( X \subseteq Y \) and \( p < r \), and consider the semigroup

\[
S = \{ \alpha \in BL(Y,r,p) : S(\alpha) \subseteq X \}.
\]

For each \( \alpha \in S \), \( s(\alpha) = p \) since \( D(\alpha) \subseteq S(\alpha) \subseteq X \). Let

\[
T = \{ \alpha \in S : |X \cap F(\alpha)| = p \}
\]

which is easily seen to be non-empty. If \( \alpha \in T \), write \( X = \{x_j\} = \{s_j\} \cup \{t_j\} \), where \( S(\alpha) = \{s_j\} \) and \( X \cap F(\alpha) = \{t_j\} \). Write \( Y = \{y_i\} \cup \{x_j\} \) and \( \{t_j\} = \{u_j\} \cup \{v_j\} \), with \( \{v_j\} = \{a_j\} \cup \{b_j\} \). Define

\[
\lambda = \begin{pmatrix}
  y_i & u_j & v_j & s_j \\
  y_i & a_j & u_j & b_j
\end{pmatrix}.
\]

Then \( \lambda \in S \) and \( \lambda \alpha = \lambda \). On the other hand, let \( \alpha \in S \) be such that \( \lambda \alpha = \lambda \) for some \( \lambda \in S \). Since \( \lambda \in S \), we have \( S(\lambda) \subseteq X \). Hence \( Y \setminus X \subseteq F(\lambda) \). We also have
ran(λ) ⊆ F(α) since λα = λ. Hence Xλ ⊆ X ∩ F(α) and so |X ∩ F(α)| = p. Thus, we have an algebraic characterisation for the elements of the set T.

However, T is not a semigroup. To see this, let $X = A \cup B \cup C$, each with cardinal $p$, and let $B = B_1 \cup B_2$, $C = C_1 \cup C_2$, also each with cardinal $p$. Suppose $α ∈ S$ fixes both $Y$ and $A$ pointwise, and maps $B$ onto $C$ and $C$ onto $B_1$. Also, let $β ∈ S$ fix both $Y$ and $B$ pointwise, and map $A$ onto $C_1$ and $C$ onto $A$. Then $F(αβ) = Y$ and $|X ∩ F(αβ)| = 0$. Hence $α, β ∈ T$ but $αβ ∉ T$.

**Theorem 3.11.** The semigroups $BL(X, p, p)$ and $GS(V, m, n)$ are not isomorphic for any infinite cardinals $p, m, n$ with $n < m$.

**Proof.** Suppose $BL(X, p, p)$ is isomorphic to $GS(V, m, n)$. Let $Y$ be a set with cardinal $r > p$ such that $Y ⊆ X$. Then, $BL(X, p, p)$ is isomorphic to a subset of $BL(Y, r, p)$ — namely, $S = \{α ∈ BL(Y, r, p) : S(α) ⊆ X\}$ — and there is an isomorphism $φ$ from $S$ onto $GS(V, m, n)$. Let $T = \{α ∈ S : |X ∩ F(α)| = p\}$. Clearly $φ$ induces a one-to-one mapping from $T$ onto $Σ(n)$. For, $α ∈ T$ if and only if $λα = λ$ for some $λ ∈ S$, which in turn is equivalent to saying: $μ(αφ) = μ$ for some $μ ∈ GS(V, m, n)$ (even though $T$ is not a semigroup). But $Σ(n)$ is a subsemigroup of $GS(V, m, n)$ and $φ$ is an isomorphism, hence $Σ(n)φ^{-1} = T$ must be a subsemigroup of $S$, contradicting our earlier remark. □

Since we have now shown that $BL(p, q)$ and $GS(m, n)$ are never isomorphic, it is worth observing the following result.

**Theorem 3.12.** Any right simple, right cancellative semigroup $S$ without idempotents can be embedded in some $GS(m, m)$.

**Proof.** Let $|S| = m$ and write $S^1 = \{a_i\}$, with $|I| = m$. Note that $S$ is infinite, since $S$ has no idempotents. Let $F$ be any field and let $F_i$ be a copy of $F$ for each $i ∈ I$.

As in [4] p182, Remark (c), we let $V$ be the vector space $∑F_i$ over $F$ whose basis can be identified in a natural way with $\{a_i\}$: that is, $∑F_i$ is the set of all $(r_i)_{i ∈ I}$ where $r_i ∈ F_i$ and at most finitely many $r_i$ are non-zero. Since $S$ is right cancellative, the extended right regular representation of $S$ is a faithful representation of $S$ as a semigroup of one-to-one mappings of $S^1$ into itself. Let $x ∈ S$. Then $x$ is represented by $ρ_x : S^1 → S^1, a_i ↦ a_ix$, which is a one-to-one mapping of the basis $\{a_i\}$ into itself. Hence $ρ_x$ can be extended by linearity to a one-to-one linear map $V → V$.

Moreover, since $S$ is infinite, [1] vol 2, Lemma 8.4 implies that

$$|S^1| = |S| = |S \setminus Sx| = |S^1 \setminus (x \cup Sx)| = |S^1 \setminus S^1 ρ_x|.$$

Therefore, codim $ρ_x = |S| = m$ and hence $ρ_x ∈ GS(V, m, m)$. The faithfulness of the extended right regular representation implies that $S$ is embedded in $GS(V, m, m)$. □
4. Left ideals of $GS(m, n)$

In this section we transfer results of Sutov [11] and Sullivan [10] on the left ideals of $BL(p, q)$ to the linear Baer-Levi semigroup on $V$. By analogy with their work, the most natural way to do this is to show that the left ideals of $GS(m, n)$ are precisely the subsets $L$ of $GS(m, n)$ which satisfy the condition:

$$(\alpha \in L, \beta \in GS(m, n), \text{ran} \beta \subseteq \text{ran} \alpha, \dim(\text{ran} \alpha/\text{ran} \beta) = n) \implies \beta \in L.$$

Although this result is valid, to obtain more information about the left ideals of $GS(m, n)$ we proceed as follows.

If $Y$ is a non-empty subset of $GS(m, n)$, we let $L_Y^+ = Y \cup L_Y$, where

$$L_Y = \{\beta \in GS(m, n) : \text{ran} \beta \subseteq \text{ran} \alpha, \dim(\text{ran} \alpha/\text{ran} \beta) = n \text{ for some } \alpha \in Y\}.$$

To show $L_Y$ is non-empty, choose any $\alpha \in Y$. Suppose $\{e_i\}$ is a basis for $V$ and write $e_i\alpha = a_i$ for each $i$. Since $\alpha$ is one-to-one, $\{a_i\}$ is linearly independent and so it can be expanded into a basis $\{a_i\} \cup \{b_j\}$ for $V$. Note that $|J| = d(\alpha) = n \leq m$. Therefore we can write $\{a_i\} = \{c_i\} \cup \{d_j\}$ and define

$$\beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

This is in $GS(m, n)$ since $\beta$ is one-to-one and $d(\beta) = \dim \langle d_j, b_j \rangle = n$. We have $\text{ran} \beta \subseteq \text{ran} \alpha$ and $\dim(\text{ran} \alpha/\text{ran} \beta) = \dim \langle d_j \rangle = n$. Hence $\beta \in L_Y$ and so $L_Y$ is non-empty.

**Theorem 4.1.** If $Y$ is a non-empty subset of $GS(m, n)$, then $L_Y^+$ is a left ideal of $GS(m, n)$. Conversely, if $I$ is a left ideal of $GS(m, n)$, then $I = L_Y^+$.

**Proof.** Suppose $Y$ is a non-empty subset of $GS(m, n)$ and let $\alpha \in L_Y^+$ and $\beta \in GS(m, n)$. Then $\beta \alpha \in GS(m, n)$ and $\text{ran}(\beta \alpha) \subseteq \text{ran} \alpha$. Suppose $\{e_i\}$ is a basis for $V$. Since $\beta$ is one-to-one, $\{e_i \beta\}$ is a basis for $\text{ran} \beta$, which can be expanded into another basis $\{e_i \beta, e_j\}$ for $V$, with $|J| = d(\beta) = n$. Then $\text{ran} \alpha = \langle e_i \beta \alpha, e_j \alpha \rangle$. On the other hand, $\text{ran}(\beta \alpha) = \langle e_i \beta \alpha \rangle$ and so $\dim(\text{ran} \alpha/\text{ran}(\beta \alpha)) = \dim \langle e_j \alpha \rangle = n$. If $\alpha \in Y$, then $\beta \alpha \in L_Y$. If not, then $\alpha \in L_Y$ and so $\text{ran} \alpha \subseteq \text{ran} \gamma$ and $\dim(\text{ran} \gamma/\text{ran} \alpha) = n$ for some $\gamma \in Y$. Thus $\text{ran}(\beta \alpha) \subseteq \text{ran} \alpha \subseteq \text{ran} \gamma$ and $n = \dim(\text{ran} \gamma/\text{ran} \alpha) \leq \dim(\text{ran} \gamma/\text{ran}(\beta \alpha)) \leq d(\beta \alpha) = n$. Therefore $\beta \alpha \in L_Y$. In other words, we have shown that $L_Y^+$ is a left ideal of $GS(m, n)$.

Suppose $I$ is a left ideal of $GS(m, n)$. We assert that $I = L_Y^+$. Let $\beta \in I$. Then there exists $\alpha \in I$ such that $\text{ran} \beta \subseteq \text{ran} \alpha$ and $\dim(\text{ran} \alpha/\text{ran} \beta) = n$. If $\{e_i\}$ is a
basis for $V$ then $\text{ran } \beta = \langle e_i \beta \rangle$ and, since $\text{ran } \beta \subseteq \text{ran } \alpha$, $\text{ran } \alpha = \langle e_i \beta, e_j \rangle$ for some linearly independent set $\{e_i \beta, e_j\}$. Moreover, $|J| = n$ since $\dim(\text{ran } \alpha / \text{ran } \beta) = n$. Since $\alpha$ is one-to-one and $e_i \beta, e_j \in \text{ran } \alpha$, we can choose unique $f_i$ and $f_j$ in $V$ such that $f_i \alpha = e_i \beta$ and $f_j \alpha = e_j$. Then $\{f_i\} \cup \{f_j\}$ is a basis for $V$ since $\alpha$ is one-to-one and $\{e_i \beta, e_j\}$ is a basis for $\text{ran } \alpha$. Thus, we have

$$\alpha = \begin{pmatrix} f_i \\ e_i \beta \\ e_j \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ e_i \beta \end{pmatrix}.\]$$

Define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then $\gamma \in GS(m,n)$ and $\beta = \gamma \alpha$. Since $I$ is a left ideal, it follows that $\beta \in I$. Therefore, $L_I \subseteq I$ and so $L_I^+ = I$. □

**Remark 4.2.** The left ideals of $GS(m,n)$ do not form a chain under $\subseteq$. For, suppose $\{e_i\}$ is a basis for $V_i$, let $\alpha \in GS(m,n)$ and write $e_i \alpha = a_i$ for each $i$. We can expand $\{a_i\}$ into a basis $\{a_i\} \cup \{b_j\}$ for $V$, with $|J| = n$. Let $|K| < n$ and write $\{e_i\} = \{f_i\} \cup \{f_k\}$ and $\{b_j\} = \{c_k\} \cup \{d_j\}$. Define

$$\beta = \begin{pmatrix} f_i \\ f_k \\ a_i \\ c_k \end{pmatrix}.$$

Then $\alpha \notin L_{\{\beta\}}^+$ and $\beta \notin L_{\{\alpha\}}^+$. Thus $L_{\{\alpha\}}^+ \nsubseteq L_{\{\beta\}}^+$ and $L_{\{\beta\}}^+ \nsubseteq L_{\{\alpha\}}^+$.

The next result determines when one left ideal of $GS(m,n)$ is contained in another.

**Theorem 4.3.** Let $A, B$ be non-empty subsets of $GS(m,n)$. Then $L_A^+ \subseteq L_B^+$ if and only if $A \setminus B \subseteq L_B$.

Proof. If $L_A^+ \subseteq L_B^+$, then $A \subseteq B \cup L_B$ and so $A \setminus B \subseteq L_B$. Suppose now that the latter happens and let $\alpha \in L_A^+$. Then $\alpha \in A$ or $\alpha \in L_A$. If $\alpha \in A \cap B$, then $\alpha \in B$. If $\alpha \in A \setminus B$, then $\alpha \in L_B$. On the other hand, if $\alpha \in L_A$, then there exists $\beta \in A$ such that $\text{ran } \alpha \subseteq \text{ran } \beta$ and $\dim(\text{ran } \beta / \text{ran } \alpha) = n$. If $\beta \in B$, then $\alpha \in L_B$. If not, then $\beta \in A \setminus B \subseteq L_B$ and so there exists $\gamma \in B$ such that $\text{ran } \beta \subseteq \text{ran } \gamma$ and $\dim(\text{ran } \gamma / \text{ran } \beta) = n$. Therefore $\text{ran } \alpha \subseteq \text{ran } \gamma$ and $n \geq \dim(\text{ran } \gamma / \text{ran } \alpha) \geq \dim(\text{ran } \beta / \text{ran } \alpha) = n$ and hence $\alpha \in L_B$. Thus we have shown that $\alpha \in L_B^+$ and the result follows. □

Hence $A \subseteq B$ implies $L_A^+ \subseteq L_B^+$, but not conversely. For, suppose $\{e_i\}$ is a basis for $V$ and write $\{e_i\} = \{a_i\} \cup \{b_j\}$ and $\{a_i\} = \{c_i\} \cup \{c_j\}$, with $|J| = n$. Define

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$
in $T(V)$. Since $\alpha, \beta$ are one-to-one and $d(\alpha) = \dim(b_j) = n = \dim(b_j, c_j) = d(\beta)$, $\alpha$ and $\beta$ are elements of $GS(m, n)$. If $A = \{\beta\}$ and $B = \{\alpha\}$ then $L_A^+ \subseteq L_B^+$ but $A \not\subseteq B$.

**Corollary 4.4.** Let $A, B$ be non-empty subsets of $GS(m, n)$. Then $L_A^+ \cup L_B^+ = L_{A \cup B}^+$.

**Proof.** Since $A, B \subseteq A \cup B$, we have $L_A^+ \cup L_B^+ \subseteq L_{A \cup B}^+$. Let $\gamma \in L_{A \cup B}^+$. Then $\gamma \in A \cup B$, and so $\gamma \in A$ or $\gamma \in B$, or $\gamma \in L_{A \cup B} \cap L_{A \cup B}^+$. If the latter happens, then there exists $\alpha \in A \cup B$ such that $\ran \gamma \subseteq \ran \alpha$ and $\dim(\ran \alpha/\ran \gamma) = n$. Hence $\gamma \in L_A \cup L_B$. Therefore $\gamma \in L_A^+ \cup L_B^+$ and the result follows. \hfill $\Box$

A similar result does not hold for the intersection of two non-empty subsets of $GS(m, n)$. That is, there are non-empty subsets $A, B$ of $GS(m, n)$ whose intersection is also non-empty but $L_A^+ \cap L_B^+ \not\subseteq L_A^+ \cap L_B^+$. To see this, suppose $\{e_i\}$ is a basis for $V$ and write $\{e_i\} = \{a_i\} \cup \{b_j\} \cup \{c_j\} \cup \{d_j\}$, with $|J| = n$. Since $n \leq m$, we can also write $\{a_i\} \cup \{b_j\} = \{x_i\}$, $\{a_i\} \cup \{b_j\} \cup \{c_j\} = \{y_i\}$ and $\{a_i\} \cup \{d_j\} = \{z_i\}$. Now define

\[
\alpha = \left( \begin{array}{c} e_i \\ x_i \end{array} \right), \quad \beta = \left( \begin{array}{c} e_i \\ y_i \end{array} \right), \quad \gamma = \left( \begin{array}{c} e_i \\ z_i \end{array} \right)
\]

in $T(V)$. It is easy to see that $\alpha, \beta, \gamma \in GS(m, n)$ and $\ran \alpha \subseteq \ran \beta$, $\dim(\ran \beta/\ran \alpha) = n$ and $\ran \alpha \not\subseteq \ran \gamma$. Let $A = \{\alpha, \gamma\}$ and $B = \{\beta, \gamma\}$. Then $A \cap B = \{\gamma\}$. Since $\alpha \in A$ and $\alpha \in L_B$, it follows that $\alpha \in L_A^+ \cap L_B^+$. On the other hand, $\alpha \not\in \gamma$ and $\alpha \not\in \{\gamma\}$. Hence $\alpha \not\in L_{A \cap B}^+$. In addition, the correspondence $A \mapsto L_A^+$ is not one-to-one. For example, if $C = \{\alpha, \beta\}$ and $D = \{\beta\}$ where $\alpha, \beta$ are the linear transformations defined in the last paragraph, then $L_C^+ = L_D^+$. To see this, let $\delta \in GS(m, n)$ be such that $\ran \delta \subseteq \ran \alpha$ and $\dim(\ran \alpha/\ran \delta) = n$. Then $\ran \delta \subseteq \ran \alpha \subseteq \ran \beta$ and

\[
n = \dim(\ran \alpha/\ran \delta) \leq \dim(\ran \beta/\ran \delta) \leq d(\delta) = n.
\]

That is, if $\delta \in L_C^+$ then $\delta = \beta$ or $(\ran \delta \subseteq \ran \beta$ and $\dim(\ran \beta/\ran \delta) = n$) (by the definition of $\alpha$ and $\beta$, this covers the possibility that $\delta = \alpha$). Hence $\delta \in L_D^+$, and clearly $L_D^+ \subseteq L_C^+$, so we have equality as stated.

Note that by [1] vol 2, p 85, Exercise 3, if $S$ is a right simple semigroup without idempotents and if $S = Sx \cup \{x\}$ then $x$ belongs to (at least) two distinct principal left ideals $L_1$ and $L_2$, hence $S$ is contained in both of these and so $L_1 = L_2$, a contradiction. That is, $GS(m, n)$ is not a principal left ideal of itself.

To decide when other left ideals of $GS(m, n)$ are principal, we first observe that the principal left ideal generated by $\alpha \in GS(m, n)$ is $L_{\{\alpha\}}^+$. For, clearly $GS(m, n)^1 \alpha \subseteq L_{\{\alpha\}}^+$ since $\alpha \in L_{\{\alpha\}}^+$ and $L_{\{\alpha\}}^+$ is a left ideal of $GS(m, n)$. Conversely, the argument
Corollary 4.5. Let $A$ be a non-empty subset of $GS(m, n)$ and $\alpha \in GS(m, n)$. Then $L_A^+ = L_{\{\alpha\}}^+$ if and only if $\alpha \in L_A^+$ and $A \setminus \{\alpha\} \subseteq L_{\{\alpha\}}$.

In effect, the following result determines when left ideals are proper.

Theorem 4.6. Let $A$ be a non-empty subset of $GS(m, n)$. Then $L_A^+ = GS(m, n)$ if and only if for each $\alpha \in GS(m, n)$ there exists $\lambda \in A$ such that $\text{ran } \alpha \subseteq \text{ran } \lambda$.

Proof. Suppose the latter condition holds for a non-empty $A \subseteq GS(m, n)$. Let $\{e_i\}$ be a basis for $V$, suppose $\beta \in GS(m, n)$ and write $e_i \beta = b_i$ for each $i$. We can expand $\{b_i\}$ into a basis for $V$, say $\{b_i\} \cup \{b_j\}$. Write $\{b_j\} = \{c_j\} \cup \{d_j\}$ and let $\{c_j\} = \{b_i\} \cup \{c_j\}$. Define

$$\gamma = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$ 

Then $\gamma \in GS(m, n)$ and so there exists $\lambda \in A$ such that $\text{ran } \gamma \subseteq \text{ran } \lambda$. Hence $\text{ran } \beta \subseteq \text{ran } \gamma \subseteq \text{ran } \lambda$ and $n \geq \text{dim}(\text{ran } \lambda / \text{ran } \beta) \geq \text{dim}(\text{ran } \gamma / \text{ran } \beta) = n$. Therefore $\beta \in L_A \subseteq L_A^+$. Thus $GS(m, n) \subseteq L_A^+$ and equality follows. Conversely, if there exists $\alpha \in GS(m, n)$ such that $\text{ran } \alpha \nsubseteq \text{ran } \lambda$ for all $\lambda \in A$, then clearly $\alpha \notin L_A^+$ and hence $L_A^+$ is a proper subset of $GS(m, n)$. \qed

To see that $A$ may not equal $GS(m, n)$ in the above result, fix $\alpha \in GS(m, n) = G$ say, and write $\beta = \gamma \alpha$ for some fixed $\gamma \in G$. Put $A = G \setminus \{\beta\}$ and recall (see before Lemma 3.6) that $\alpha \neq \gamma \alpha$ in $G$, so $\alpha \in A$. Clearly $G = GA \cup A$. Also, if $\mu \in G$ then either $\mu \in A$ or $\mu = \gamma' \lambda$ for some $\lambda \in A$, and in each case $\text{ran } \mu \subseteq \text{ran } \lambda$ for some $\lambda \in A$. Hence, by the Theorem, $L_A^+ = G$ where $A \nsubseteq G$.

It is easy to see that $GS(m, n)$ has no minimal left ideals. For, by [1] vol 2, p 85, Exercise 4, if $S$ is any right simple semigroup without idempotents then $Sba$ is a proper subset of $Sa$ for each $a, b \in S$. But if $L$ is a minimal left ideal of $S$ and $x, y \in L$ then $Syx = L = Sx$ by minimality, hence $S$ cannot contain any minimal left ideals. However, it is not as easy to see that $GS(m, n)$ has no maximal left ideals.

Theorem 4.7. The semigroup $GS(m, n)$ has no maximal (proper) left ideals.

Proof. From Theorem 4.6, $L_A^+$ is a proper left ideal if and only if there exists some $\alpha$ in $GS(m, n)$ such that $\text{ran } \alpha \nsubseteq \text{ran } \lambda$ for all $\lambda \in A$. 23
Let $L^+_V$ be a proper left ideal of $GS(m, n)$. Then there exists $\alpha \in GS(m, n)$ such that $\text{ran} \alpha \not\subseteq \text{ran} \lambda$ for all $\lambda \in Y$. Let $Z = Y \cup \{\alpha\}$. Then $L^+_V \subseteq L^+_Z$. Obviously $\alpha \notin L^+_V$ and so $L^+_V \not\subseteq L^+_Z$. We assert that $L^+_Z \not\subseteq GS(m, n)$.

Write $e_i \alpha = a_i$ where $\{e_i\}$ is a basis for $V$, and expand $\{a_i\}$ into a basis for $V$, say $\{a_i\} \cup \{a_j\}$. Write $\{a_j\} = \{b_j\} \cup \{c_j\}$ and let $\{b_i\} = \{a_i\} \cup \{b_j\}$. Define

$$\beta = \left(\begin{array}{c} e_i \\ b_i \end{array}\right) \in GS(m, n).$$

Then $\text{ran} \alpha \subseteq \text{ran} \beta$ and so $\beta \not\subseteq Y$. Since $\alpha \neq \beta$, we have $\beta \not\in Z$. Suppose $\beta \in L_Z$. Then $\text{ran} \beta \subseteq \text{ran} \gamma$ and $\text{dim(ran} \gamma / \text{ran} \beta) = n$ for some $\gamma \in Z$. If $\gamma = \alpha$, then $\text{ran} \beta \subseteq \text{ran} \alpha$, a contradiction. Then $\gamma \in Y$, but $\text{ran} \alpha \subseteq \text{ran} \beta \subseteq \text{ran} \gamma$, which contradicts our condition on $\alpha$ and $Y$. Therefore, $\beta \notin L^+_Z$ and hence $L^+_Z \not\subseteq GS(m, n)$.

In other words, given any proper left ideal $A$, we can find a strictly larger proper left ideal that contains $A$. Hence there are no maximal left ideals of $GS(m, n)$.

\[\Box\]

5. Maximal subsemigroups of $GS(m, n)$

In this section, we show that any subspace $U \neq \{0\}$ of $V$ with codimension at least $n$ gives rise to a maximal subsemigroup of $GS(m, n)$: here, our work closely follows that in [7].

Let $U \neq \{0\}$ be a subspace of $V$ with $\text{codim}(U) \geq n$ and define

$$M_U = \{\alpha \in GS(m, n) : U \not\subseteq \text{ran} \alpha \text{ or } (U \alpha \subseteq U \text{ or } \text{dim}(V\alpha/U) < n)\}.$$

**Theorem 5.1.** For each subspace $U \neq \{0\}$ of $V$ with $\text{codim}(U) \geq n$, $M_U$ is a maximal subsemigroup of $GS(m, n)$.

Proof. We first show that $M_U$ is a subsemigroup of $GS(m, n)$. Let $\alpha, \beta \in M_U$. Since $\alpha, \beta \in GS(m, n)$, it follows that $\alpha \beta \in GS(m, n)$. If $U \not\subseteq \text{ran}(\alpha \beta)$ then $\alpha \beta \in M_U$. If $U \subseteq \text{ran}(\alpha \beta)$ then $U \subseteq \text{ran} \beta$. Hence $U \beta \subseteq U \text{ or } \text{dim}(\text{ran} \beta/U) < n$. If the latter holds then $\text{dim(\text{ran}(\alpha \beta)/U) \leq \dim(\text{ran} \beta/U) < n}$ and so $\alpha \beta \in M_U$. If $U \beta \subseteq U$ then $U \beta \subseteq \text{ran}(\alpha \beta)$ and so $U \subseteq \text{ran} \alpha$. Thus, $U \alpha \subseteq U \text{ or } \text{dim(\text{ran} \alpha/U) < n}$ since $\alpha \in M_U$. Suppose $U \alpha \subseteq U$. Then $U \alpha \beta \subseteq U \beta \subseteq U \text{ and therefore } \alpha \beta \in M_U$. If $\text{dim(\text{ran} \alpha/U) < n}$, write $U = \langle u_i \rangle$ and so $U \beta = \langle u_i \beta \rangle$. Hence $U = \langle u_i \beta, u_j \rangle$ for some linearly independent set $\{u_i \beta\} \cup \{u_j\}$, and likewise $\text{ran} \alpha = \langle u_i, w_r \rangle$ and $\text{ran}(\alpha \beta) = \langle u_i \beta, w_r, \beta \rangle$. On the other hand, since $U = \langle u_i \beta, u_j \rangle \subseteq \text{ran}(\alpha \beta)$, we have $\text{ran}(\alpha \beta) = \langle u_i \beta, u_j, w_s \rangle$. Hence $|R| = |J| + |S|$. Thus,

$$\text{dim(\text{ran}(\alpha \beta)/U) = |S| \leq |R| < n}.$$

Therefore, $\alpha \beta \in M_U$ and $M_U$ is a subsemigroup of $GS(m, n)$.
In order to prove the maximality of $M_\alpha$, we show that a subsemigroup $M$ of $GS(m, n)$ properly containing $M_\alpha$ necessarily is $GS(m, n)$ itself. Let $M$ be a subsemigroup of $GS(m, n)$ satisfying these conditions. Let $\gamma \in M \setminus M_\alpha$ and $\alpha \in GS(m, n) \setminus M_\alpha$. Since $\gamma, \alpha \notin M_\alpha$, we know that $U \subseteq \text{ran } \gamma$, $U \not\subseteq U$, $\dim(\text{ran } \gamma/U) \geq n$ and $U \subseteq \text{ran } \alpha$, $U \alpha \not\subseteq U$, $\dim(\text{ran } \alpha/U) \geq n$. If $U\alpha^{-1} = \langle a_i \rangle$ and $U\gamma^{-1} = \langle b_j \rangle$, then $U = \langle a_i \alpha \rangle = \langle b_j \gamma \rangle$ and $\{a_i \alpha\}, \{b_j \gamma\}$ are bases for $U$, since $\alpha$ and $\gamma$ are one-to-one. Therefore $|I| = |J|$ and we can write $U\gamma^{-1} = \langle b_i \rangle$ and $U = \langle a_i \alpha \rangle = \langle b_i \gamma \rangle$. Since $U\alpha^{-1}$ is a subspace of $V$, we can expand $\{a_i \}$ into a basis for $V$, say $\{a_i \} \cup \{e_k \}$. Then $\alpha \gamma = \langle a_i \alpha, e_k \alpha \rangle$ where $\{a_i \alpha, e_k \alpha \}$ is linearly independent. Hence $\text{codim}(U\alpha^{-1}) = |K| = \dim(\text{ran } \alpha/U)$. Since $\alpha \gamma = \langle a_i \alpha, e_k \alpha \rangle$ and $\alpha \gamma \subseteq V$, we can expand $\{a_i \alpha\} \cup \{e_k \alpha\}$ into a basis for $V$, say $\{a_i \alpha, e_k \alpha, e_\ell \}$ with $|L| = n$ and so $\text{codim } U = |K| + n = |K|$. Analogously we can expand $\{b_i \}$ into a basis for $V$, say $\{b_i, f_\ell \}$, and $\gamma \alpha$ is spanned by the linearly independent set $\{b_i \beta, f_\ell \gamma \}$. Hence

$$\text{codim}(U\gamma^{-1}) = |R| = \dim(\text{ran } \gamma/U) \geq n.$$  

We can expand $\{b_i \beta, f_\ell \gamma \}$ into a basis for $V$, say $\{b_i \beta, f_\ell \gamma, f_\ell \}$. Hence $d(\gamma) = n = |S|$ and, since $|L| = n$, this means we can write $\{f_\ell \}$ instead of $\{f_s \}$. Moreover $\text{codim } U = |R| = |K|$. Therefore, we can also write $\{f_\ell \}$ and $\{f_\ell \gamma \}$ instead of $\{f_\ell \}$ and $\{f_\ell \gamma \}$, respectively.

Since $U\gamma \not\subseteq U$, there exists $u \in U$ such that $u\gamma \not\subseteq U$. It follows that $\{b_i, u\}$ and $\{b_i \beta, u\gamma \}$ are linearly independent. We can expand these sets into bases for $V$ and for $\text{ran } \gamma$, respectively, say $\{b_i, u, h_k \}$ and $\{b_i \beta, u\gamma, g_k \}$ (note that $|K| = \text{codim}(U\gamma^{-1}) = \dim(u, h_k)$ and $|K| = \dim(\text{ran } \gamma/U) = \dim(u\gamma, g_k)$). We can also expand $\{b_i \beta, u\gamma, g_k \}$ into a basis $\{b_i \beta, u\gamma, g_k, e_\ell \}$ for $V$, where $|T| = d(\gamma) = n = |L|$. Write $\{g_\ell \}$ instead of $\{g_k \}$ and let $W = \langle u\gamma, g_k, g_\ell \rangle$. Then $W$ is a complement of $U$ in $V$. We have $\langle u \rangle \subseteq U \cap W\gamma^{-1}$. Also $\langle u \rangle \subseteq \langle u, h_k \rangle$, which is a complement of $U\gamma^{-1}$ in $V$. Since $|K| = \dim(\text{ran } \gamma/U) \geq n = |L|$, we may write $\{h_k \} = \{e_k \} \cup \{d_\ell \}$. Define

$$\beta = \begin{pmatrix} a_i \\ b_i \\ c_k \end{pmatrix}.$$  

Since $u \in U$ and $u \not\in \text{ran } \beta$, it follows that $U \not\subseteq \text{ran } \beta$ and so $\beta \in M_\alpha$. Write $\{u \} \cup \{d_\ell \} = \{c_\ell \}$ and $c_\ell \gamma = z_\ell$ for each $\ell$. Then

$$\gamma = \begin{pmatrix} b_i \\ c_k \\ c_\ell \end{pmatrix}.$$  

Let $\langle w_\ell \rangle$ be a complement of $\text{ran } \gamma$ in $V$. As in the second paragraph above, let $\{e_\ell \}$ be a basis for a complement of $\text{ran } \alpha$ in $V$ and write $\{e_\ell \} = \{x_\ell \} \cup \{y_\ell \}$. Now write $\{z_\ell \} \cup \{w_\ell \} = \{v_\ell \}$ and define

$$\delta = \begin{pmatrix} b_i \gamma \\ a_i \alpha \\ e_k \alpha \\ x_\ell \\ v_\ell \end{pmatrix}.$$  

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Since \( U = \langle a_i \alpha \rangle \subseteq \text{ran} \delta \) and \( U \delta = \langle b_i \gamma \delta = \langle a_i \alpha \rangle = U \), it follows that \( \delta \in M_U \). Since \( \beta \gamma \delta = \alpha \), we have \( \alpha \in M_U \). Therefore, \( M = GS(m,n) \) and hence \( M_U \) is maximal.

\( \square \)

References


