ON UNIQUENESS AND DECAY OF SOLUTION FOR HIROTA EQUATION

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Abstract. We address the question of the uniqueness of solution to the initial value problem associated to the equation

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2\partial_x\overline{u} = 0, \quad x,t \in \mathbb{R},$$

and prove that a certain decay property of the difference $u_1 - u_2$ of two solutions u_1 and u_2 at two different instants of times $t = 0$ and $t = 1$, is sufficient to ensure that $u_1 = u_2$ for all the time.

1. Introduction

In this work we consider the following equation

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + i\gamma|u|^2 u + \delta|u|^2\partial_x u + \epsilon u^2\partial_x\overline{u} = 0, \quad x,t \in \mathbb{R}, \tag{1.1}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$, $\gamma, \delta, \epsilon \in \mathbb{C}$ and $u = u(x,t)$ is a complex valued function. Our main concern is to find a sufficient decay property satisfied by the difference of two different solutions at two different instants of time to prove the uniqueness of the solution to the initial value problem (IVP) associated to (1.1).

The equation (1.1), with the mixed structure of Korteweg-de Vries (KdV) and Schrödinger equations, was proposed by Hasegawa and Kodama in [8, 17] to describe the nonlinear propagation of pulses in optical fibers. This equation is also known as Hirota equation in the literature.

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Several aspects of this equation including well-posedness issues, solitary wave solutions, unique continuation property, have been studied by various authors recently, see for example [3], [4], [5], [18], [23] and references therein.

Study of unique continuation property (UCP) for certain models has drawn much attention of a considerable section of mathematicians in recent time, see for example [1], [5], [9] – [16], [19] – [22], [24], [25] and references therein. In particular, in [5] and [4] we addressed the UCP for the equation (1.1). In [5], we proved that if a sufficiently smooth solution u to the initial value problem associated to (1.1) is supported in a half line at two different instants of time then u vanishes identically. The precise statement of our result in [5] is the following.

Theorem 1.1. [5]. *Let $u \in C([t_1, t_2]; H^s) \cap C^1([t_1, t_2]; H^1)$, $s \geq 4$ be a solution of the equation (1.1) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If there exists $t_1 < t_2$ such that*

$$\operatorname{supp} u(\cdot, t_j) \subset (-\infty, a), \qquad j = 1, 2 \tag{1.2}$$

$$\text{or, } (\operatorname{supp} u(\cdot, t_j) \subset (b, \infty), \qquad j = 1, 2\,). \tag{1.3}$$

Then $u(t) = 0$ for all $t \in [t_1, t_2]$.

In our subsequent work [4], we obtained more general uniqueness property for the solution of the IVP associated to (1.1).

Theorem 1.2. [4]. *Let $u, v \in C([t_1, t_2]; H^s) \cap C^1([t_1, t_2]; H^1)$, $s \geq 4$ be solutions of the equation (1.1) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If there exists $b \in \mathbb{R}$ such that*

$$u(x,t) = v(x,t), \qquad (x,t) \in (b, \infty) \times \{t_1, t_2\}, \tag{1.4}$$

$$\text{or, } (u(x,t) = v(x,t), \qquad (x,t) \in (-\infty, b) \times \{t_1, t_2\}). \tag{1.5}$$

Then

$$u(t) = v(t) \quad \forall\, t \in [t_1, t_2].$$

Remark 1.1. *Theorem 1.1 is the special case of Theorem 1.2 when $v \equiv 0$.*

Motivation to obtain the above results is the following observation. Consider the IVP associated to the linear part of (1.1), i.e.,

$$\begin{cases} u_t + i\alpha u_{xx} + \beta u_{xxx} = 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.6}$$

If u and v are solutions to (1.6) then $w := u - v$ is also a solution to (1.6) with initial data $w(x,0) = u(x,0) - v(x,0) =: w_0(x)$. If w_0 is sufficiently smooth and has compact support, then using Paley-Wiener theorem it is easy to see (for detail see [5]) that $w \equiv 0$, i.e., $u \equiv v$. But the proof of the same property is not so simple when one considers the nonlinear terms as well, because in this case $w := u - v$ is no more a solution. To overcome this situation, we generalized and employed the techniques developed in the context of the generalized KdV equation by Kenig-Ponce-Vega in [13] and [14] to prove Theorems 1.1 and 1.2 .

Quite recently, Escauriaza, Kenig, Ponce and Vega in [7] introduced a new technique to obtain sufficient conditions on the behavior of the difference $u_1 - u_2$ of two solutions u_1 and u_2 of the generalized KdV equation at two different instants of time $t = 0$ and $t = 1$ that guarantees $u_1 \equiv u_2$. In [7], the authors obtained a sharp decay condition to guarantee the uniqueness of solution to the generalized KdV equation. So, there arise a natural question, whether one can find such a decay condition to get uniqueness property for a mixed equation of the KdV and Schrödinger type. In this work, we shall extend the approach in [7] to address a uniqueness question to the IVP associated to the Hirota equation (1.1) which has a mixed structure of the KdV and the Schrödinger equations. Our first main result of this work is the following.

Theorem 1.3. *Let $u_1, u_2 \in C([0,1]; H^3(\mathbb{R})) \cap L^2(|x|^2 dx))$, be strong solutions of the equation (1.1) with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$, $\beta \neq 0$. If, for any $a > 0$,*

$$u_1(\cdot,0) - u_2(\cdot,0), \quad u_1(\cdot,1) - u_2(\cdot,1) \in H^1(e^{ax_+^{3/2}} dx), \tag{1.7}$$

then

$$u_1 \equiv u_2.$$

To prove Theorem 1.3 we follow the techniques introduced in [7] by deriving some new estimates that are appropriate to work with the equation under consideration.

Using the gauge transformation

$$v(x,t) = e^{i\lambda x + i(\alpha\lambda^2 - 2\beta\lambda^3)t} u(2\alpha\lambda - 3\beta\lambda^2)t, t) \tag{1.8}$$

with $\lambda = \frac{\alpha}{3\beta}$, one can work for an equivalent equation for v without the term $i\alpha u_{xx}$ in the linear part. With this, it seems that, our result for the original equation also follows from the techniques in [7]. But it is not the case: because if we work on the transformed equation for v (without the Schrödinger term), in the beginning we need to suppose that

$$v_1(x,t_j) - v_2(x,t_j) \in H^1(e^{ax_+^{3/2}}) =: X_1, \qquad j = 1,2. \tag{1.9}$$

So, after undoing the transformation, for the original solution u, we need:

$$u_1(x,t_j) - u_2(x,t_j) \in H^1(e^{ax_+^{3/2}}) \Leftrightarrow v_1(x,t_j) - v_2(x,t_j) \in H^1(e^{a(x+\alpha^2/2\beta)_+^{3/2}}) =: X_2. \tag{1.10}$$

But (1.10) is not always true, because one can find function f for which $\|f\|_{X_1} < \infty$ but $\|f\|_{X_2} = \infty$ and vice versa.

Therefore, it is not possible to discard the Schrödinger term using gauge transform so as to apply the techniques from [7] directly in our case.

On the other hand, one may think of treating the term $i\alpha u_{xx}$ at par with nonlinear terms and apply the estimates from [7] directly. This is also not possible, because the term with α does not satisfy the necessary decay condition so as to use the estimates from our earlier works [4, 5]. This situation has been explained in our earlier work [4], Remark 3.10.

Although the idea and estimates are similar to the one introduced in [7], the presence of the Schrödinger term in the linear part creates obstacle to obtain such estimates, which can be seen more explicitly in the derivation of the lower estimates in Section 3. The proofs of several estimates that are crucial to prove the main results depend on the estimates obtained on our previous works [4] and [5], where the exponential decay property of the solution was necessary. As observed in [4] and [5], the presence of the third order derivative in (1.1) is fundamental to obtain the desired exponential decay property of the solution. So we will suppose $\beta \neq 0$ throughout this work. To be more precise, let us recall the following remark from [4].

Remark 1.2. *We can suppose $\beta > 0$. In fact, for $\alpha \neq 0$ we can suppose $\beta = |\alpha|/3$.*

If $\beta < 0$ we define $w(x,t) = u(-x,t)$ then w is a solution to the equation (1.1) with the coefficient of the third derivative is positive.

If $\beta > 0$ and $\alpha \neq 0$ we define $w(x,t) = u(\tilde{a}^{-1}x,t)$ with $\tilde{a} = |\alpha|/3\beta$, then w is a solution of the equation

$$w_t + i\alpha\tilde{a}^2 w_{xx} + \beta\tilde{a}^3 w_{xxx} + i\gamma|w|^2 w + \delta\tilde{a}|w|^2 w_x + \epsilon\tilde{a}w^2\bar{w}_x = 0,$$

and we have $\beta\tilde{a}^3 = |\alpha|\tilde{a}^2/3$.

As mentioned earlier, we are interested in finding a decay condition satisfied by the difference of two solutions at two different instants of time $t = 0$ and $t = 1$ that is sufficient to get the uniqueness of solution to the IVP associated to (1.1). Note that, while treating with the difference of two solutions, we need to address an equation with variable coefficients (see (4.2) below). Therefore, in the first instant, we consider a more general equation,

$$w_t + i\alpha w_{xx} + \beta w_{xxx} + a_2(x,t)w_{xx} + a_1(x,t)w_x + b_1(x,t)\bar{w}_x + a_0(x,t)w + b_0(x,t)\bar{w} = 0, \quad (1.11)$$

and prove the following result.

Theorem 1.4. *Assume that the coefficients in* (1.11) *satisfy that*

$$\begin{cases} a_0, b_0 \in L^{4/3}_{xt} \cap L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t, \\ a_1, b_1 \in L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t \cap L^{16/15}_x L^{16/3}_t, \\ a_2 \in L^{8/7}_x L^{8/3}_t \cap L^{16/15}_x L^{16/3}_t \cap L^1_x L^\infty_t. \end{cases} \quad (1.12)$$

and

$$\begin{cases} a_0, b_0, a_1, b_1, a_2, (a_0)_x, (b_0)_x, (a_1)_x, (b_1)_x, (a_2)_x, (a_2)_{xx}, (a_2)_{xxx}, (a_2)_t \in L^\infty(\mathbb{R}\times[0,1]), \\ a_2, (a_2)_t \in L^\infty_t([0,1]; L^1_x(\mathbb{R})). \end{cases} \quad (1.13)$$

If $w \in C\left([0,1]; H^2(\mathbb{R})\right) \cap L^2(|x|^2dx)\big)$ is a strong solution of (1.11) *with*

$$w(\cdot,0), w(\cdot,1) \in H^1(e^{ax_+^{3/2}}dx), \quad \forall a > 0,$$

then $w \equiv 0$.

Once we get this theorem, the proof of the main theorem follows by proving that the variable coefficients involved in the equation in question satisfy the respective estimates.

Our next result is concerned with the existence of solution to the IVP associated to (1.1) that decays asymptotically in x. First, let us consider the IVP (1.6) associated to the linear part of (1.1). The solution to the IVP (1.6) is given by,

$$u(x,t) = \frac{1}{\sqrt[3]{3t}} G\Big(\frac{\cdot}{\sqrt[3]{3t}}\Big) * u_0(x) \tag{1.14}$$

where

$$G(x) = \int_{\mathbb{R}} e^{\frac{i8\pi^3}{3}\eta^3 + \frac{i\alpha t^{1/3} 4\pi^2}{\sqrt[3]{9}}\eta^2 + 2\pi i\eta x} d\eta. \tag{1.15}$$

With some easy calculations, one can obtain

$$|G(x)| = |Ai(x - 4\pi^2 B^2)|, \tag{1.16}$$

where Ai is the usual Airy function given by

$$Ai(x) = \int_{\mathbb{R}} e^{2\pi i x\xi + \frac{8}{3} i\pi^3 \xi^3} d\eta$$

and

$$B = \frac{\alpha t^{1/3}}{2\sqrt[3]{9\pi}}.$$

If $x > 8\pi^2 B^2$, we get

$$|Ai(x - 4\pi^2 B^2)| \leq Ce^{-C(x-4\pi^2 B^2)^{3/2}} \leq Ce^{-\frac{C}{2^{3/2}} x^{3/2}} \tag{1.17}$$

Therefore, from (1.16) and (1.17) we have, for any $t \in [0,1]$,

$$|G(x)| \leq Ce^{-\frac{C}{2^{3/2}} x^{3/2}}, \tag{1.18}$$

provided, $x > \frac{2\alpha^2}{3\sqrt[3]{3}}$.

The estimate shows that the decay condition in Theorem 1.3 is in accordance with the decay of the function G that describes the solution of the linear part.

In what follows, we show the existence of a local solution to the IVP associated to (1.1) that satisfies the similar decay property as the linear solution described above. More precisely, our second main theorem reads as follows.

Theorem 1.5. *There exists $u_0 \in \mathbb{S}(\mathbb{R})$, $u_0 \neq 0$ and $T > 0$ such that the IVP associated to* (1.1) *with data u_0 has a solution $u \in C([0,T] : \mathbb{S}(\mathbb{R}))$ which satisfies*

$$|u(x,t)| \leq ce^{-x^{3/2}/3}, \quad x > 1, \ t \in [0,T],$$

for some constant $c > 0$.

We organize this article in the following manner. In Sections 2 and 3 we prove some preliminary estimates (upper estimate and lower estimate) which play a vital role to prove our main theorem. In Section 4 we present a proof of a more general result, Theorem 1.4, and then the proofs of the main results of this work, Theorem 1.3 and Theorem 1.5. Before leaving this section, let us record some notations that are used throughout this work.

Notations: We use $\hat{f}(\xi)$ and $\hat{f}(\xi,\tau)$ to denote the Fourier transform defined by $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x)\, dx$, and $\hat{f}(\xi,\tau) = \frac{1}{2\pi} \int e^{-i(x\xi+t\tau)} f(x,t)\, dxdt$ respectively. We use $L^p_x L^q_t$ to denote mixed Lebesgue spaces. We write $A \lesssim B$ if there exists a constant $c > 0$ such that $A \leq cB$.

2. Upper estimates

This section is devoted to prove upper estimates that play crucial role in the proof of the main results. Let us first define the following operators

$$Hf = (\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3)f, \qquad H_m f = (\partial_t + e^{mx}(i\alpha\partial_x^2 + \beta\partial_x^3)e^{-mx})f. \tag{2.1}$$

By Remark 1.2, we can suppose that $\beta > 0$ and $|\alpha|/3 = \beta$. Also, let us define $v := e^{mx}u$, where u is a solution to (1.1). We begin with the following result.

Lemma 2.1. *The following estimate holds*

$$\|v\|_{L^\infty_t L^2_x} \leq C\Big(\|v(\cdot,0)\|_{L^2} + \|v(\cdot,1)\|_{L^2}\Big) + C\|H_m v\|_{L^1_t L^2_x}. \tag{2.2}$$

Proof. We have

$$H_m = \partial_t + e^{mx}(i\alpha\partial_x^2 + \beta\partial_x^3)e^{-mx} = \partial_t + i\alpha(e^{mx}\partial_x e^{-mx})^2 + \beta(e^{mx}\partial_x e^{-mx})^3. \tag{2.3}$$

Also using $(e^{mx}\partial_x e^{-mx})^j = (\partial_x - m)^j$, $j = 1,2,3$, we obtain

$$H_m f = \partial_t + \beta\partial_x^3 + (i\alpha - 3\beta m)\partial_x^2 + (3\beta m^2 - 2i\alpha m)\partial_x + i\alpha m^2 - \beta m^3)f. \tag{2.4}$$

The symbol of H_m is given by

$$\begin{aligned} &i\tau - i\beta\xi^3 - (i\alpha - 3\beta m)\xi^2 + (3\beta m^2 - 2i\alpha m)i\xi + i\alpha m^2 - \beta m^3 \\ &= i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2). \end{aligned} \tag{2.5}$$

Note that the real part of the symbol vanishes at

$$\xi_\pm = \frac{-\alpha \pm \sqrt{\alpha^2 + 3\beta^2 m^2}}{3\beta}. \tag{2.6}$$

As noted in [7], by an approximation argument, it suffices to prove (2.2) for $v \in C^\infty([0,1]; \mathcal{S}(\mathbb{R}))$ with $\hat{v}(\xi, t) = 0$ near $\xi_\pm$ for all $t \in [0,1]$.

Now, consider $f \in C^\infty([0,1]; \mathcal{S}(\mathbb{R}))$ with $f(x,t) = 0$ for t near 0 and 1 so that we can extend f as zero outside the strip $\mathbb{R} \times [0,1]$. Also suppose that $\hat{f}(\xi, t) = 0$ for ξ near $\xi_\pm$ for all $t \in \mathbb{R}$. For such a function f, define an operator T by

$$\widehat{Tf}(\xi,\tau) := \frac{\hat{f}(\xi,\tau)}{i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)}. \tag{2.7}$$

We claim that the operator T satisfies the estimate

$$\|Tf\|_{L_t^\infty L_x^2} \le C\|f\|_{L_t^1 L_x^2}, \tag{2.8}$$

which in turn implies (2.2).

To prove this, let us define $\eta_\varepsilon \in C^\infty(\mathbb{R})$, $\varepsilon \in (0, \frac{1}{4})$ such that

$$\eta_\varepsilon(t) = 1, \quad t \in [2\varepsilon, 1 - 2\varepsilon]; \qquad \operatorname{supp} \eta_\varepsilon \subset [\varepsilon, 1 - \varepsilon].$$

Define

$$v_\varepsilon = \eta_\varepsilon(t) v(x,t), \qquad f_\varepsilon(x,t) = H_m(v_\varepsilon)(x,t),$$

then, $v_\varepsilon = Tf_\varepsilon$. Now (2.8) gives,

$$\|v_\varepsilon\|_{L_t^\infty L_x^2} \le C\|H_m(v_\varepsilon)\|_{L_t^1 L_x^2} \le C\|\eta_\varepsilon(t) v\|_{L_t^1 L_x^2} + \|\eta_\varepsilon H_m(v)\|_{L_t^1 L_x^2}. \tag{2.9}$$

Letting $\varepsilon \to 0$, the left hand side of (2.9) converges to $\|v\|_{L^\infty_{[0,1]} L_x^2}$ and the limit in the right hand side is bounded by

$$C\big(\|v(\cdot,0)\|_{L^2} + \|v(\cdot,1)\|_{L^2}\big) + C\|H_m v\|_{L_t^1 L_x^2}.$$

Therefore, our task is to prove (2.8). As noted in [7], it is enough to prove that for $f(x,t) = f(x) \otimes \delta_{t_0}(t)$, with $\hat{f}(\xi) = 0$ near $\xi_\pm$, with $t_0 \in (0,1)$, one has

$$\|Tf\|_{L_t^\infty L_x^2} \leq C\|f\|_{L^2}, \tag{2.10}$$

where C is independent of t_0.

Let us recall the formulas

$$\Big(\frac{1}{\tau + ib}\Big)^\vee(t) = C \begin{cases} \chi_{(-\infty,0)}(t)e^{tb}, & b > 0 \\ \chi_{(0,\infty)}(t)e^{tb}, & b < 0, \end{cases} \tag{2.11}$$

so that for $a, b \in \mathbb{R}$,

$$\Big(\frac{e^{it_0\tau}}{\tau - a + ib}\Big)^\vee(t) = Ce^{ita} \begin{cases} \chi_{(-\infty,0)}(t-t_0)e^{(t-t_0)b}, & b > 0 \\ \chi_{(0,\infty)}(t-t_0)e^{(t-t_0)b}, & b < 0. \end{cases} \tag{2.12}$$

Hence,

$$\begin{aligned} \widehat{Tf}(\xi,\tau) &:= \frac{e^{it_0\tau}\hat{f}(\xi)}{i\{(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) + i(\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)\}} \\ &= -i\frac{e^{it_0\tau}\hat{f}(\xi)}{\tau - a(\xi) + ib(\xi)}. \end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), it is clear that the operator T acting on these functions becomes the one variable operator R given by,

$$\begin{aligned} \widehat{Rf}(\xi) = &\big(\chi_{\{b(\xi)>0\}}(\xi)e^{ita(\xi)}e^{(t-t_0)b(\xi)}\chi_{(-\infty,0)}(t-t_0)\big)\hat{f}(\xi) \\ &+ \big(\chi_{\{b(\xi)<0\}}(\xi)e^{ita(\xi)}e^{(t-t_0)b(\xi)}\chi_{(0,\infty)}(t-t_0)\big)\hat{f}(\xi), \end{aligned} \tag{2.14}$$

for which we need to establish that

$$\|Rf\|_{L_x^2} \leq C\|f\|_{L_x^2}, \tag{2.15}$$

with C independent of t_0 and m.

But, looking at the multiplier in (2.14), the estimate (2.15) holds true and this completes the proof. □

Our next result deals with the crucial upper estimate and reads as follows.

Lemma 2.2. *There exists $k \in \mathbb{Z}^+$ such that if $u \in C^\infty([0,1]; C_0^\infty(\mathbb{R}))$, then for any $m \geq 1$, the following estimate holds:*

$$\begin{aligned}\|e^{mx}u\|_{L^8_{xt}} + \|e^{mx}\partial_x u\|_{L^{16}_x L^{16/5}_t} + \|e^{mx}\partial_x^2 u\|_{L^\infty_x L^2_t} \\ \leq Cm^{2k}\Big(\|J(e^{mx}u(\cdot,0))\|_{L^2} + \|J(e^{mx}u(\cdot,1))\|_{L^2}\Big) \\ + C\Big(\|e^{mx}Hu\|_{L^{8/7}_{xt}} + \|e^{mx}Hu\|_{L^{16/15}_x L^{16/11}_t} + \|e^{mx}Hu\|_{L^1_x L^2_t}\Big),\end{aligned} \tag{2.16}$$

where $\widehat{Jg}(\xi) := (1+|\xi^2\|)^{1/2}\hat{g}(\xi)$ and $\|\cdot\|_{L^p_t}$ are restricted in $[0,1]$.

Proof. As noted in the beginning of this section, by Remark 1.2, we can suppose that $\beta > 0$ and $|\alpha|/3 = \beta$. Let us define

$$v = e^{mx}u \in C^\infty([0,1]; \mathcal{S}(\mathbb{R})), \tag{2.17}$$

then the estimate (2.16) can be written as

$$\begin{aligned}\|v\|_{L^8_{xt}} + \|e^{mx}\partial_x e^{-mx}v\|_{L^{16}_x L^{15/5}_t} + \|e^{mx}\partial_x^2 e^{-mx}\|_{L^\infty_x L^2_t} \\ \leq Cm^{2k}\Big(\|Jv(\cdot,0)\|_{L^2} + \|Jv(\cdot,1)\|_{L^2}\Big) \\ + C\Big(\|H_m v\|_{L^{8/7}_{xt}} + \|H_m v\|_{L^{16/15}_x L^{16/11}_t} + \|H_m v\|_{L^1_x L^2_t}\Big).\end{aligned} \tag{2.18}$$

The estimate (2.18) will hold true if we can prove the following set of estimates

$$\|v\|_{L^8_{xt}} \leq C\Big(\|v(\cdot,0)\|_{L^2} + \|v(\cdot,1)\|_{L^2}\Big), \tag{2.19}$$

$$\|v\|_{L^{16}_x L^{16/5}_t} \leq Cm^k\Big(\|J^{1/2}v(\cdot,0)\|_{L^2} + \|J^{1/2}v(\cdot,1)\|_{L^2}\Big) + C\|H_m v\|_{L^{16/15}_x L^{16/11}_t} \tag{2.20}$$

and

$$\|e^{mx}\partial_x^2 e^{-mx}v\|_{L^\infty_x L^2_t} \leq Cm^{2k}\Big(J\|v(\cdot,0)\|_{L^2} + \|Jv(\cdot,1)\|_{L^2}\Big) + C\|H_m v\|_{L^1_x L^2_t}. \tag{2.21}$$

We start by proving the estimate (2.19): As in Lemma 2.1, it is enough to prove (2.19) for $v \in C^\infty([0,1] : \mathcal{S}(\mathbb{R}))$ such that $\hat{v}(\xi,t) = 0$ near $\xi_\pm$. Suppose that $f \in C^\infty([0,1] : \mathcal{S}(\mathbb{R}))$ with $f(x,t) = 0$ for t near 0 and 1, so we can extend f to 0 outside the strip $\mathbb{R} \times [0,1]$. Also suppose that $\hat{f}(\xi,t) = 0$ for ξ near $\xi_\pm$ for all $t \in \mathbb{R}$. We will show that for the operator T defined in (2.7)

$$\|Tf\|_{L^8_{xt}} \leq C\|f\|_{L^{8/7}_{xt}} \tag{2.22}$$

and

$$\|Tf\|_{L^8_{xt}} \leq C\|f\|_{L^1_t L^2_x}, \tag{2.23}$$

for $f \in \mathcal{S}(\mathbb{R}^2)$ with $\hat{f}(\xi,t)=0$ for ξ near $\xi_\pm$ for all $t \in \mathbb{R}$.

The estimate (2.22) is proved in [4]. To get (2.23) we restrict to consider $f(x,t)=f(x)\otimes\delta_{t_0}(t)$, and reduce the case to show that the operator R defined in (2.14) satisfies

$$\|Rf\|_{L^8_{xt}} \leq C\|f\|_{L^2}, \tag{2.24}$$

with C independent of m and t_0. But this is done in [4]

Now we show that estimates (2.22) and (2.23) imply the estimate (2.19). For this, consider

$$v_\varepsilon(x,t)=\eta_\varepsilon(t)v(x,t), \qquad H_m(v_\varepsilon)=\eta'_\varepsilon(t)v+\eta_\varepsilon H_m(v)=f_1(x,t)+f_2(x,t). \tag{2.25}$$

Suppose,

$$v_1(x,t)=Tf_1(x,t), \qquad v_2(x,t)=Tf_2(x,t), \tag{2.26}$$

where both make sense because of our assumption on v. Then,

$$v_\varepsilon(x,t)=v_1(x,t)+v_2(x,t), \tag{2.27}$$

since both sides are in l^2_{xt} and have the same Fourier transform. Hence, from (2.22) and (2.23) it follows that

$$\begin{aligned}\|v_\varepsilon\|_{L^8_{xt}} &\leq \|v_1\|_{L^8_{xt}}+\|v_2\|_{L^8_{xt}} \leq C\|f_1\|_{L^1_tL^2_x}+C\|f_2\|_{L^{8/7}_{xt}}\\ &\leq C\|\eta'_\varepsilon(t)v\|_{L^1_tL^2_x}+C\|\eta_\varepsilon(t)H_mv\|_{L^{8/7}_{xt}}.\end{aligned} \tag{2.28}$$

Now, letting $\varepsilon\to 0$ we get the required estimate (2.19).

Next, we prove the estimate (2.21): As earlier, here too we make our usual assumptions on $\hat{v}(\xi,\tau)$. For $f\in\mathcal{S}(\mathbb{R}^2)$ with $\hat{f}(\xi,t)=0$ near $\xi_\pm$ for all $t\in\mathbb{R}$ we define

$$\begin{aligned}\widehat{T_2f}(\xi,\tau) &:= (i\xi-m)^2\widehat{Tf}(\xi,\tau)\\ &= \frac{(i\xi-m)^2\hat{f}(\xi,\tau)}{i(\tau-\beta\xi^3-\alpha\xi^2+3\beta m^2\xi+\alpha m^2)-(\beta m^3-2\alpha m\xi-3\beta m\xi^2)}.\end{aligned} \tag{2.29}$$

Let

$$\tilde{T_2}f(x,t)=\chi_{[0,1]}T_2f(x,t). \tag{2.30}$$

We will show that

$$\|\tilde{T_2}f\|_{L^\infty_xL^2_t} \leq C\|f\|_{L^1_xL^2_t}, \tag{2.31}$$

$$\|\tilde{T_2}f\|_{L^\infty_xL^2_t} \leq Cm^2\|Jf\|_{L^1_tL^2_x}. \tag{2.32}$$

Before proving (2.31) and (2.32), we show that these estimates imply (2.21). Using the notations introduced in (2.25), (2.26) and (2.27), the estimates (2.31) and (2.32) yield

$$\begin{aligned}\|(\partial_x - m)^2 v_\varepsilon\|_{L^\infty_x L^2_t} &\leq \|\chi_{[0,1]}(\partial_x - m)^2 v_1\|_{L^\infty_x L^2_t} + \|\chi_{[0,1]}(\partial_x - m)^2 v_2\|_{L^\infty_x L^2_t} \\ &\leq \|\tilde{T}_2 f_1\|_{L^\infty_x L^2_t} + \|\tilde{T}_2 f_2\|_{L^\infty_x L^2_t} \\ &\leq Cm^2 \|Jf_1\|_{L^1_t L^2_x} + C\|f_2\|_{L^1_x L^2_t} \\ &\leq Cm^2 \|\eta_\varepsilon(t) Jv\|_{L^1_t L^2_x} + C\|\eta_\varepsilon(t) H_m v\|_{L^1_x L^2_t}.\end{aligned} \tag{2.33}$$

Now in the limit as $\varepsilon \to 0$ we get (2.21). So, to complete the proof of (2.21) it is enough to prove (2.31) and (2.32).

With minor modification from the argument in [4], we get

$$\|T_2 f\|_{L^\infty_x L^2_t} \leq C\|f\|_{L^1_x L^2_t}, \tag{2.34}$$

which in turn implies (2.31).

Now we move to prove (2.32). Let $\theta_r \in C_0^\infty(\mathbb{R})$ with $\theta_r(x) = 1$ for $|x| \leq 3r$ and *supp* $\theta_r \subset \{|x| \leq 4r\}$ and consider

$$\begin{aligned}\widehat{T_2 f}(\xi,\tau) &= \frac{\theta_m(\xi)(i\xi - m)^2 \hat{f}(\xi,\tau)}{i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)} \\ &\quad + \frac{(1-\theta_m(\xi))(i\xi - m)^2 \hat{f}(\xi,\tau)}{i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)} \\ &= \widehat{T_{2,1} f}(\xi,\tau) + \widehat{T_{2,2} f}(\xi,\tau).\end{aligned} \tag{2.35}$$

Let $\tilde{T}_{2,1} := \chi_{[0,1]} T_{2,1}$. From the Sobolev lemma we obtain

$$\|\tilde{T}_{2,1} f\|_{L^\infty_x L^2_t} \leq C\|J\tilde{T}_{2,1} f\|_{L^2_x L^2_t} = C\|J\tilde{T}_{2,1} f\|_{L^2_t L^2_x} \leq C\|JT_{2,1}\|_{L^\infty_t L^2_x}. \tag{2.36}$$

Now suppose,

$$\hat{g_1}(\xi,\tau) = \theta_m(\xi)(1+|\xi|^2)^{1/2}(i\xi - m)^2 \hat{f}(\xi,\tau),$$

so that

$$JT_{2,1} f(x,t) = Tg_1(x,t)$$

and therefore from (2.8) and (2.36) it follows that

$$\|\tilde{T}_{2,1} f\|_{L^\infty_x L^2_t} \leq C\|g_1\|_{L^2_t L^2_x} = Cm^2 \|Jf\|_{L^1_t L^2_x}. \tag{2.37}$$

To complete (2.32), it is enough to prove

$$\|T_{2,2}f\|_{L_x^\infty L_t^2} \leq C\|Jf\|_{L_t^1 L_x^2}. \tag{2.38}$$

Here too, with the similar argument as in [7], we reduce the case to consider functions of the form $f(x,t) = f(x) \otimes \delta_{t_0}(t)$ so that we just need to bound the operator

$$\widehat{R_{2,2}}(\xi,t) = (1-\theta_m(\xi))(i\xi - m)^2 \chi_{\{b(\xi)<0\}}(\xi) e^{ita(\xi)} e^{(t-t_0)b(\xi)} \chi_{(0,\infty)}(t-t_0)\hat{f}(\xi), \tag{2.39}$$

as

$$\|R_{2,2}f\|_{L_x^\infty L_t^2} \leq C\|Jf\|_{L_x^2}, \tag{2.40}$$

with C independent of m and t_0.

Let us write

$$R_{2,2}f(x,t) = \int e^{ix\xi}(1-\theta_m(\xi))(i\xi-m)^2 \chi_{\{b(\xi)<0\}}(\xi) e^{ita(\xi)} e^{(t-t_0)b(\xi)} \chi_{(0,\infty)}(t-t_0)\hat{f}(\xi)d\xi \tag{2.41}$$

and recall that $a(\xi) = \beta\xi^3 + \alpha\xi^2 - 3\beta m^2\xi - \alpha m^2$. Now, making change of variable $\lambda = a(\xi)$ we get $d\lambda = (3\beta\xi^2 + 2\alpha\xi - 3\beta m^2)d\xi$.

From the definition of $\theta_m(\cdot)$, the domain of integration in (2.41) is equal to $\{|\xi| \geq 3m\}$ where $|3\beta\xi^2 + 2\alpha\xi - 3\beta m^2| \cong |\xi|^2$ in fact if $\alpha \neq 0$, by Remark 1.2

$$\begin{aligned}
|3\beta\xi^2 + 2\alpha\xi - 3\beta m^2| =& |\alpha|\,|\xi^2 \pm 2\xi - m^2| \\
\geq& |\alpha|(|\xi|^2 - m^2 - 2|\xi|) \\
\geq& |\alpha|((8/9)|\xi|^2 - 2|\xi|) = |\alpha|\,|\xi|\{(8/9)|\xi| - 2\} \geq |\alpha||\xi^2|/9,
\end{aligned}$$

and the transformation is one-to-one since $a'(\xi) = |\alpha|(\xi^2 \pm 2\xi - m^2) \gtrsim \xi^2$.

Thus we have $\xi = \xi(\lambda)$ and

$$\begin{aligned}
R_{2,2}f(x,t) &= \int e^{it\lambda} \frac{e^{ix\xi}(1-\theta_m(\xi))(i\xi-m)^2}{3\beta\xi^2 + 2\alpha\xi - 3\beta m^2} \chi_{\{b(\xi)<0\}}(\xi) e^{(t-t_0)b(\xi)} \chi_{(0,\infty)}(t-t_0)\hat{f}(\xi)d\lambda \\
&= \int e^{it\lambda}\hat{g_2}(\lambda)\psi(\lambda,t)d\lambda,
\end{aligned} \tag{2.42}$$

with

$$\hat{g_2}(\lambda) = \frac{e^{ix\xi}(1-\theta_m(\xi))(i\xi-m)^2}{3\beta\xi^2 + 2\alpha\xi - 3\beta m^2}\hat{f}(\xi),$$

$$\psi(\lambda,t) = \chi_{\{b(\xi)<0\}}(\xi) e^{(t-t_0)b(\xi)} \chi_{(0,\infty)}(t-t_0).$$

Observe that,

$$|\psi(\lambda,t)| \leq C, \qquad \forall\, (\lambda,t) \in \mathbb{R}^2$$

and

$$\int |\partial_t \psi(\lambda,t)| dt \leq C \qquad \forall\, \lambda \in \mathbb{R}.$$

Therefore, using the result in [6] and taking adjoint we get,

$$\begin{aligned} \| \int e^{it\lambda} \hat{g_2}(\lambda)\psi(\lambda,t) d\lambda \|_{L^2_t} &\leq C\|\hat{g_2}\|_{L^2} \\ &\leq C\Big(\int \frac{|e^{x\xi}(1-\theta_m(\xi))(i\xi-m)^2 \hat{f}(\xi)}{|3\beta\xi^2+2\alpha\xi-3\beta m^2||3\beta\xi^2+2\alpha\xi-3\beta m^2|} d\xi \Big)^{1/2} \\ &\leq C\Big(\int \frac{|1-\theta_m(\xi)|^2|\xi^2+m^2|^2|\hat{f}(\xi)|^2}{|3\beta\xi^2+2\alpha\xi-3\beta m^2|} d\xi \Big)^{1/2} \\ &\leq C\|Jf\|_{L^2} \end{aligned} \tag{2.43}$$

which is (2.32).

Finally, we supply a proof of the estimate (2.20): At this point too, let us make the usual assumptions on v and $\hat{v}$. For $f \in \mathcal{S}(\mathbb{R}^2)$ with $\hat{f}(\xi,t) = 0$ near $\xi_\pm$ for all $t \in \mathbb{R}$, we define using (2.7)

$$\widehat{T_1 f}(\xi,\tau) = (i\xi - m)\widehat{Tf}(\xi,\tau) = \frac{(i\xi-m)\hat{f}(\xi,\tau)}{i(\tau-\beta\xi^3-\alpha\xi^2+3\beta m^2\xi+\alpha m^2)-(\beta m^3-2\alpha m\xi-3\beta m\xi^2)}. \tag{2.44}$$

Now define,

$$\tilde{T}_1 f(x,t) = \chi_{[0,1]}(t) T_1 f(x,t). \tag{2.45}$$

We claim that

$$\|\tilde{T}_1 f\|_{L^{16}_x L^{16/5}_t} \leq C\|f\|_{L^{16/15}_x L^{16/11}_t} \tag{2.46}$$

and

$$\|\tilde{T}_1 f\|_{L^{16}_x L^{16/5}_t} \leq Cm\|J^{1/2} f\|_{L^1_t L^2_x}. \tag{2.47}$$

As earlier, the estimate (2.20) easily follows from the estimates (2.46) and (2.47). Let us recall, in [4] it was proved that

$$\|T_1 f\|_{L^{16}_x L^{16/5}_t} \leq C\|f\|_{L^{16/15}_x L^{16/11}_t}, \tag{2.48}$$

which implies (2.46). To obtain (2.47) we write T_1 in the following way

$$\widehat{T_1 f}(\xi,\tau) = \frac{\theta_m(\xi)(i\xi - m)\hat{f}(\xi,\tau)}{i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)} + \frac{(1-\theta_m(\xi))(i\xi - m)\hat{f}(\xi,\tau)}{i(\tau - \beta\xi^3 - \alpha\xi^2 + 3\beta m^2\xi + \alpha m^2) - (\beta m^3 - 2\alpha m\xi - 3\beta m\xi^2)} \widehat{T_{1,1} f}(\xi,\tau) + \widehat{T_{1,2} f}(\xi,\tau). \tag{2.49}$$

Let $\tilde{T}_{1,1} = \chi_{[0,1]}(t) T_{1,1}$. Now from (2.37) we have

$$\|\tilde{T}_{2,1} f\|_{L_x^\infty L_t^2} \leq C m^2 \|Jf\|_{L_t^1 L_x^2} \tag{2.50}$$

and from (2.23) we get

$$\|\tilde{T}_{0,1} f\|_{L_{xt}^8} \leq C\|f\|_{L_t^1 L_x^2}. \tag{2.51}$$

Hence, using the interpolation argument based on the Littlewood-Paley decomposition as in [14] we obtain

$$\|\tilde{T}_{1,1}\|_{L_x^{16} L_t^{16/5}} \leq Cm\|J^{1/2} f\|_{L_t^1 L_x^2}. \tag{2.52}$$

Finally we interpolate between

$$\|\tilde{T}_{0,2} f\|_{L_{xt}^8} \leq C\|f\|_{L_t^1 L_x^2}, \tag{2.53}$$

which follows from (2.22), with (2.37) to get

$$\|\tilde{T}_{1,2} f\|_{L_x^{16} L_t^{16/5}} \leq Cm\|J^{1/2} f\|_{L_t^1 L_x^2}, \tag{2.54}$$

and this yields (2.47). □

In an analogous manner, as it has been worked out in [7], the above result holds for a larger class of functions, for example:

$$u \in C([0,1]; H^{k+3}(e^{\beta x}dx) \cap H^{k+3}(\mathbb{R})) \cap C^1([0,1]; H^k(e^{\beta x}dx) \cap H^k(\mathbb{R})),$$

with $k \in \mathbb{Z}$, $k \geq 1$ and for all $\beta > 0$.

Now we want to extend the estimates in (2.16) in Lemma 2.2 to solutions with variable coefficients

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + a_2(x,t)\partial_x^2 u + a_1(x,t)\partial_x u + b_1(x,t)\partial_x \bar{u} + a_0(x,t)u + b_0(x,t)\bar{u} = g. \tag{2.55}$$

Let us introduce the notation

$$H_a u := \partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + a_2(x,t)\partial_x^2 u + a_1(x,t)\partial_x u + b_1(x,t)\partial_x \bar{u} + a_0(x,t)u + b_0(x,t)\bar{u}, \quad (2.56)$$

and suppose that multiplication by $a_0(x,t)$ and $b_0(x,t)$ map

$$L^8_{xt} \to L^{8/7}_{xt}, \qquad L^8_{xt} \to L^{16/15}_x L^{16/11}_t, \qquad L^8_{xt} \to L^1_x L^2_t, \quad (2.57)$$

multiplication by $a_1(x,t)$ and $b_1(x,t)$ map

$$L^{16}_x L^{16/5}_t \to L^{8/7}_{xt}, \qquad L^{16}_x L^{16/5}_t \to L^{16/15}_x L^{16/11}_t, \qquad L^{16}_x L^{16/5}_t \to L^1_x L^2_t. \quad (2.58)$$

and multiplication by $a_2(x,t)$ maps

$$L^\infty_x L^2_t \to L^{8/7}_{xt}, \qquad L^\infty_x L^2_t \to L^{16/15}_x L^{16/11}_t, \qquad L^\infty_x L^2_t \to L^1_x L^2_t. \quad (2.59)$$

To guarantee that the coefficients satisfy these conditions, it is enough to consider,

$$\begin{aligned} &a_0, b_0 \in L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t \cap L^{16/15}_x L^{16/3}_t, \\ &a_1, b_1 \in L^{4/3}_{xt} \cap L^{16/13}_x L^{16/9}_t \cap L^{8/7}_x L^{8/3}_t \\ &a_2 \in L^{8/7}_x L^{8/3}_t \cap L^{16/15}_x L^{16/3}_t \cap L^1_x L^\infty_t, \end{aligned} \quad (2.60)$$

Also, if we assume that, if the coefficients satisfy

$$\begin{aligned} &a_0, b_0, a_1, b_1, a_2, \partial_x a_0, \partial_x b_0, \partial_x a_1, \partial_x b_1, \partial_x a_2, \partial_x^2 a_2, \partial_x^3 a_2, \partial_t a_2 \in L^\infty(\mathbb{R}\times[0,1]), \\ &a_2, \partial_t a_2 \in L^\infty_t([0,1]; L^1_x(\mathbb{R})). \end{aligned} \quad (2.61)$$

with small norms in (2.60), then Lemma 2.2 holds for H_a instead of H. In fact we have the following result.

Lemma 2.3. *Suppose that the coefficients a_0, b_0, a_1, b_1, a_2 satisfy* (2.60) *and* (2.61) *with small norms in the spaces in* (2.60). *There exists $k \in \mathbb{Z}^+$ such that if $u \in C^\infty([0,1]; C_0^\infty(\mathbb{R}))$, then for any $m \geq 10\|a_2\|_{L^\infty(\mathbb{R}\times[0,1])}$*

$$\begin{aligned} &\|e^{mx}u\|_{L^8_{xt}} + \|e^{mx}\partial_x u\|_{L^{16}_x L^{16/5}_t} + \|e^{mx}\partial_x^2 u\|_{L^\infty_x L^2_t} \\ &\qquad \leq Cm^{2k}\Big(\|J(e^{mx}u(\cdot,0))\|_{L^2} + \|J(e^{mx}u(\cdot,1))\|_{L^2}\Big) \\ &\qquad\quad + C\Big(\|e^{mx}H_a u\|_{L^{8/7}_{xt}} + \|e^{mx}H_a u\|_{L^{16/15}_x L^{16/11}_t} + \|e^{mx}H_a u\|_{L^1_x L^2_t}\Big), \end{aligned} \quad (2.62)$$

Proof. Let us define

$$\|\!|f|\!\|_1 := \|e^{mx}f\|_{L^8_{xt}} + \|e^{mx}\partial_x f\|_{L^{16}_x L^{16/5}_t} + \|e^{mx}\partial_x^2 f\|_{L^\infty_x L^2_t}$$

$$\|\!|f|\!\|_2 := \|f\|_{L^{8/7}_{xt}} + \|f\|_{L^{16/15}_x L^{16/11}_t} + \|f\|_{L^1_x L^2_t}$$

From Lemma 2.2 we have

$$\begin{aligned}
\|\!|u|\!\|_1 &\le Cm^{2k}\Big(\|J(e^{mx}u(\cdot,0))\|_{L^2} + \|J(e^{mx}u(\cdot,1))\|_{L^2}\Big) + C\|\!|e^{mx}Hu|\!\|_2 \\
&\le Cm^{2k}\Big(\|J(e^{mx}u(\cdot,0))\|_{L^2} + \|J(e^{mx}u(\cdot,1))\|_{L^2}\Big) + C\|\!|e^{mx}H_a u|\!\|_2 \\
&\quad + \|\!|e^{mx}(a_1\partial_x u + a_2\partial_x\bar{u} + a_3 u + a_4\bar{u})|\!\|_2 \\
&\le Cm^{2k}\Big(\|J(e^{mx}u(\cdot,0))\|_{L^2} + \|J(e^{mx}u(\cdot,1))\|_{L^2}\Big) + C\|\!|e^{mx}H_a u|\!\|_2 + \frac{1}{2}\|\!|u|\!\|_1,
\end{aligned} \tag{2.63}$$

which gives the desired result. □

One can extend this result to a boarder class of solutions as in [7].

Theorem 2.4. *Let the coefficients a_0, b_0, a_1, b_1, a_2 satisfy the conditions in* (2.60) *and* (2.61). *If $u = u(x,t)$ is a solution of*

$$\partial_t u + i\alpha\partial_x^2 u + \beta\partial_x^3 u + a_2(x,t)\partial_x^2 u + a_1(x,t)\partial_x u + b_1(x,t)\partial_x\bar{u} + a_0(x,t)u + b_0(x,t)\bar{u} = 0, \tag{2.64}$$

with $u \in C([0,1]; H^1(\mathbb{R}))$ satisfying that

$$u(\cdot,0),\ u(\cdot,1) \in H^1(e^{ax_+^l})$$

for some $l > 1$ and $a > 0$, then there exist c_0 and $R_0 > 0$ sufficiently large such that for $R \ge R_0$

$$\|u\|_{L^2(\{R<x<R+1\}\times(0,1))} + \|\partial_x u\|_{L^2(\{R<x<R+1\}\times(0,1))} + \|\partial_x^2 u\|_{L^2(\{R<x<R+1\}\times(0,1))} \le c_0 e^{-aR^l/4^l}. \tag{2.65}$$

Proof. Choose R so large that in the x-interval (R,∞), the coefficients a_0, b_0, a_1, b_1, a_2 satisfy the conditions in (2.60) and (2.61) with small norm in corresponding spaces in (2.60).

Let $\mu \in C^\infty(\mathbb{R})$ with $\mu(x) = 0$ if $x < 1$ and $\mu(x) = 1$ in $x > 2$.

For $\mu_R(x) = \mu(x/R)$, define

$$u_R(x,t) = \mu_R(x)u(x,t),$$

so that $u_R(x,t)$ satisfies the equation

$$\partial_t u_R+\beta\partial_x^3 u_R+i\alpha\partial_x^2 u_R+a_2(x,t)\partial_x^2 u_R+a_1(x,t)\partial_x u_R+b_1(x,t)\partial_x \bar{u_R}+a_0(x,t)u_R+b_0(x,t)\bar{u_R}=F_R \tag{2.66}$$

where,

$$\begin{aligned} F_R = \beta\frac{1}{R^3}\mu_R'''u + 3\beta\frac{1}{R^2}\mu_R''\partial_x u + 3\beta\frac{1}{R}\mu_R'\partial_x^2 u + i\alpha\frac{1}{R^2}\mu_R''u + 2i\alpha\frac{1}{R}\mu_R'\partial_x u \\ + a_2(x,t)\frac{1}{R^2}\mu_R''u + 2a_2(x,t)\frac{1}{R}\mu_R'\partial_x u + a_1(x,t)\frac{1}{R}\mu_R'u + b_1(x,t)\frac{1}{R}\mu_R'\bar{u}. \end{aligned} \tag{2.67}$$

Note that, supp $F_R \subset \{x : R < x < 2R\}$. Let us choose, $m = \frac{a}{2}R^{l-1}$. Now, we cam use Lemma 2.3 to u_R with

$$H_{a\tilde{\mu}_R} = \partial_t + \beta\partial_x^3 + i\alpha\partial_x^2 + \tilde{\mu}_R a_2(x,t)\partial_x^2 + \tilde{\mu}_R a_1(x,t)\partial_x + \tilde{\mu}_R b_1(x,t)\partial_x + \tilde{\mu}_R a_0(x,t) + \tilde{\mu}_R b_0(x,t),$$

where $\tilde{\mu}_R(x)\mu_R(x) = \mu_R(x)$, which assures that the coefficients $\tilde{\mu}_R(x)a_j(x,t)$, $j = 0,1,2$ and $\tilde{\mu}_R(x)b_j(x,t)$, $j = 0,1$ have small norms in the corresponding spaces in (2.60) for $R > R_0$. Therefore, applying (2.62) for R large, we get

$$|\!|\!| u_R |\!|\!|_1 \le cm^{2k}\Big(\|J(e^{mx}u_R(\cdot,0))\|_{L^2} + \|J(e^{mx}u_R(\cdot,1))\|_{L^2}\Big) + |\!|\!| e^{mx}F_R |\!|\!|_2. \tag{2.68}$$

With the argument similar to the one in [7], the first two terms in the right hand side of (2.68) are bounded by $c_{a,l}$.

Now we move to bound the last term in (2.68).

Recall that supp $F_R \subset \{x : R < x < 2R\}$. Now, the combination of Hölder and Minkowski's integral inequality yield,

$$\begin{aligned} |\!|\!| e^{mx}F_R |\!|\!|_2 &= \|e^{mx}F_R\|_{L^{8/7}_{xt}} + \|e^{mx}F_R\|_{L^{16/15}_x L^{16/11}_t} + \|e^{mx}F_R\|_{L^1_x L^2_t} \\ &\le ce^{aR^{l-1}R}\|(|u| + |\partial_x u| + |\partial_x^2|)\chi_{\{x:R<x<2R\}}\|_{L^\infty_t L^2_x} \\ &\le c'e^{aR^{l-1}R}. \end{aligned} \tag{2.69}$$

Hence, from (2.68) we obtain,

$$\|e^{mx}u\|_{L^8_{\{x>4R\}}L^8_t} + \|e^{mx}\partial_x u\|_{L^{16}_{\{x>4R\}}L^{16/5}_t} + \|e^{mx}\partial_x^2 u\|_{L^\infty_{\{x>4R\}}L^2_t} \le c_{a,l} + c'e^{aR^l}. \tag{2.70}$$

Once again, using Hölder inequality in (2.70) we get, for sufficiently large R

$$\|u\|_{L^2_{(\{4R<x<4R+1\}\times(0,1))}} + \|\partial_x u\|_{L^2_{(\{4R<x<4R+1\}\times(0,1))}} + \|\partial_x^2 u\|_{L^2_{(\{4R<x<4R+1\}\times(0,1))}} \le c_{a,l}e^{-aR^l}. \tag{2.71}$$

Replacing $4R$ by R' we obtain,

$$\|u\|_{L^2_{(\{R'<x<R'+1\}\times(0,1))}} + \|\partial_x u\|_{L^2_{(\{R'<x<R'+1\}\times(0,1))}} + \|\partial_x^2 u\|_{L^2_{(\{R'<x<R'+1\}\times(0,1))}} \leq c_{a,l} e^{-a(R'/4)^l}, \tag{2.72}$$

which yields the required estimate (2.65). □

3. Lower estimates

This section is concerned with lower estimates that play fundamental role in the proof of the main result of this work. Let us begin with following lemma.

Lemma 3.1. *Assume that φ is a real smooth function with compact support in $[0,1]$ and $\beta \neq 0$. Then, there exist $c > 0$ and $M = M(\|\varphi'\|_\infty; \|\varphi''\|_\infty) > 0$ such that the inequality*

$$\begin{aligned} &\frac{a^{5/2}}{R^3} \left\| e^{a(\frac{x}{R}+\varphi(t))^2} \left(\frac{x}{R}+\varphi(t)\right)^2 g \right\|_{L^2(dxdt)} + \frac{a^{3/2}}{R^2} \left\| e^{a(\frac{x}{R}+\varphi(t))^2} \left(\frac{x}{R}+\varphi(t)\right)^2 \partial_x g \right\|_{L^2(dxdt)} \\ &+ \frac{a^{1/2}}{R} \left\| e^{a(\frac{x}{R}+\varphi(t))^2} \partial_x^2 g \right\|_{L^2(dxdt)} \leq c(\beta) \left\| e^{a(\frac{x}{R}+\varphi(t))^2} (\partial_t + i\alpha\partial_x^2 + \beta\partial_x^3) g \right\|_{L^2(dxdt)} \end{aligned} \tag{3.1}$$

holds, for $R \geq \alpha^2/\beta^{1/3}$, a such that $a^2 \geq MR^3$, and $g \in C_0^\infty(\mathbb{R}^2)$ supported in

$$\left\{(x,t) \in \mathbb{R}^2 : \left|\frac{x}{R} + \varphi(t)\right| \geq 1\right\}.$$

Proof. Initially we consider the case when $\beta = 1$. In a similar way as in [7], we define a function $f(x,t) = e^{a\theta(x,t)} g(x,t)$ with $\theta(x,t) = (\frac{x}{R} + \varphi(t))^2$ and the expression

$$e^{a\theta(x,t)}(\partial_t + i\alpha\partial_x^2 + \partial_x^3)\left(e^{-a\theta(x,t)} f(x,t)\right) = (S_a + S_{a,\alpha})f + (A_a + A_{a,\alpha})f, \tag{3.2}$$

where

$$S_a f = -3a(\theta_x f_x)_x - a^3\theta_x^3 f - a\theta_{xxx} f - a\theta_t f; \quad S_{a,\alpha} f = -i\alpha a\theta_{xx} f - 2i\alpha a\theta_x f_x,$$

and

$$A_a f = f_t + f_{xxx} + 3a^2\theta_x^2 f_x + 3a^2\theta_x\theta_{xx} f; \quad A_{a,\alpha} f = i\alpha a^2\theta_x^2 f + i\alpha f_{xx}.$$

We have $S_a^* = S_a$, $S_{a,\alpha}^* = S_{a,\alpha}$, $A_a^* = -A_a$ and $A_{a,\alpha}^* = -A_{a,\alpha}$, and therefore,

$$\begin{aligned} &\left\| e^{a(\frac{x}{R}+\varphi(t))^2} (\partial_t + i\alpha\partial_x^2 + \partial_x^3) g \right\|^2_{L^2(dxdt)} = \|(S_a + S_{a,\alpha})f + (A_a + A_{a,\alpha})f\|^2_{L^2(dxdt)} \\ &\geq \langle \{(S_a A_a - A_a S_a) + (S_a A_{a,\alpha} - A_{a,\alpha} S_a) + (S_{a,\alpha} A_a - A_a S_{a,\alpha}) + (S_{a,\alpha} A_{a,\alpha} - A_{a,\alpha} S_{a,\alpha})\} f, f \rangle . \end{aligned} \tag{3.3}$$

We find that

$$\begin{aligned}(S_aA_a - A_aS_a)f =& \ [S_a; A_a]f \\ =& \ 9a(\theta_{xx}f_{xx})_{xx} + ((6a\theta_{xt} - 18a^3\theta_x^2\theta_{xx})f_x)_x + (-3a^3\theta_{xx}^3 + a\theta_{tt} + 6a^3\theta_x^2\theta_{xt} + 9a^5\theta_x^4\theta_{xx})f,\end{aligned} \tag{3.4}$$

$$[S_a; A_{a,\alpha}]f = [S_{a,\alpha}; A_a]f = \ i6a\alpha\theta_{xx}f_{xxx} - i6a^3\alpha\theta_x\theta_{xx}^2 f - i6a^3\alpha\theta_x^2\theta_{xx}f_x + i2a\alpha\theta_{xt}f_x, \tag{3.5}$$

and

$$[S_{a,\alpha}; A_{a,\alpha}]f = \ -4a\alpha^2\theta_{xx}f_{xx}. \tag{3.6}$$

In [7] it was proved that, if $a^2 \geq (\|\varphi'\|_\infty + \|\varphi''\|_\infty^{1/2} + 1)R^3$, then

$$\begin{aligned}\langle [S_a; A_a]f, f\rangle \geq& \frac{18a}{R^2}\iint |f_{xx}|^2\,dxdt + \frac{132a^3}{R^4}\int\int\left(\frac{x}{R} + \varphi(t)\right)^2|f_x|^2\,dxdt \\ &+ \frac{216a^5}{R^6}\int\int\left(\frac{x}{R} + \varphi(t)\right)^4|f|^2\,dxdt.\end{aligned} \tag{3.7}$$

From (3.5) and (3.7) one has that

$$\begin{aligned}&\langle [S_a; A_{a,\alpha}]f + [S_{a,\alpha}; A_a]f + [S_{a,\alpha}; A_{a,\alpha}]f, f\rangle \\ &\quad = i12a\alpha\iint \theta_{xx}f_{xxx}\bar{f} - i12a^3\alpha\iint \theta_x\theta_{xx}^2|f|^2 - i12a^3\alpha\iint \theta_x^2\theta_{xx}f_x\bar{f} \\ &\qquad + i4a\alpha\iint \theta_{xt}f_x\bar{f} + 4a\alpha^2\iint \theta_{xx}|f_x|^2 \\ &\quad =: iJ_1 + iJ_2 + iJ_3 + iJ_4 + J_5.\end{aligned} \tag{3.8}$$

Integrating by parts we observe that $iJ_1, iJ_2 + iJ_3, iJ_4 \in \mathbb{R}$. Since $\theta(x,t) = (\frac{x}{R} + \varphi(t))^2$ we have that

$$|iJ_1| \leq \frac{24a|\alpha|}{R^2}\left|\iint f_{xx}\bar{f}_x dxdt\right| \leq \frac{12a}{R^2}\iint |f_{xx}|^2 dxdt + \frac{12\alpha^2 a}{R^2}\iint |f_x|^2 dxdt. \tag{3.9}$$

Similarly

$$\begin{aligned}|iJ_2 + iJ_3| = |\Im J_3| \leq |J_3| \leq& \ 96\iint\left(\frac{a^{5/2}(\frac{x}{R} + \varphi(t))^2|f|}{R^3}\right)\left(\frac{|\alpha|a^{1/2}|\bar{f}_x|}{R}\right)dxdt \\ \leq& \ \frac{48a^5}{R^6}\int\int\left(\frac{x}{R} + \varphi(t)\right)^4|f|^2\,dxdt + \frac{48\alpha^2 a}{R^2}\iint |f_x|^2 dxdt.\end{aligned} \tag{3.10}$$

As $a^2 \geq \|\varphi'\|_{L^\infty} R^3$ and $(\frac{x}{R}+\varphi(t))^4 > 1$, we get

$$\begin{aligned} |iJ_4| &\leq 8 \iint \left(a^{1/2}|\varphi' f|\right)\left(\frac{a^{1/2}|\alpha||\bar{f_x}|}{R}\right) dxdt \\ &\leq 4a \iint |\varphi' f|^2 dxdt + \frac{4\alpha^2 a}{R^2}\iint |f_x|^2 dxdt \\ &\leq \frac{4a^5}{R^6}\int\int \left(\frac{x}{R}+\varphi(t)\right)^4 |f|^2\, dxdt + \frac{4\alpha^2 a}{R^2}\iint |f_x|^2 dxdt. \end{aligned} \tag{3.11}$$

Now $R \geq \alpha^2$ and $a^2 \geq R^3$ imply that

$$\frac{\alpha^2 a}{R^2}\iint |f_x|^2 dxdt \leq \frac{a^3}{R^4}\int\int \left(\frac{x}{R}+\varphi(t)\right)^2 |f_x|^2\, dxdt. \tag{3.12}$$

Combining (3.9)-(3.12) we obtain

$$\begin{aligned} iJ_1 + iJ_2 + iJ_3 + iJ_4 + J_5 \geq \frac{-12a}{R^2}\iint |f_{xx}|^2\, dxdt - \frac{64a^3}{R^4}\int\int \left(\frac{x}{R}+\varphi(t)\right)^2 |f_x|^2\, dxdt \\ - \frac{52a^5}{R^6}\int\int \left(\frac{x}{R}+\varphi(t)\right)^4 |f|^2\, dxdt. \end{aligned}$$

This inequality, (3.3) and (3.8) yield

$$\begin{aligned} \left\| e^{a(\frac{x}{R}+\varphi(t))^2}(\partial_t + i\alpha\partial_x^2 + \partial_x^3)g \right\|^2_{L^2(dxdt)} &\geq \langle [S_a; A_a]f + [S_a; A_{a,\alpha}]f + [S_{a,\alpha}; A_a]f + [S_{a,\alpha}; A_{a,\alpha}]f, f\rangle \\ &\geq \frac{6a}{R^2}\iint |f_{xx}|^2\, dxdt + \frac{68a^3}{R^4}\int\int \left(\frac{x}{R}+\varphi(t)\right)^2 |f_x|^2\, dxdt \\ &\quad + \frac{164a^5}{R^6}\int\int \left(\frac{x}{R}+\varphi(t)\right)^4 |f|^2\, dxdt, \end{aligned}$$

which concludes the proof of the Lemma when $\beta = 1$.

Now if $\beta > 0$, $\beta \neq 1$ (see Remark 1.2) we use the case $\beta = 1$ with $\alpha := \alpha/\beta^{2/3}$ and $g(x,t) = g(\beta^{1/3}x, t)$. Finally, we perform a change of variable $x := \beta^{1/3}x$ to obtain (3.1).

□

In an analogous manner as in [7], we have the following result.

Lemma 3.2. *Assume that φ is a real smooth function with compact support in $[0,1]$ and that a_0, a_1, b_0, b_1 are complex functions in $L^\infty(\mathbb{R}^2)$. Then there exist $c > 0$,*

$$R_0 = R_0(\|\varphi'\|_\infty; \|\varphi''\|_\infty; \|a_0\|_\infty; \|a_1\|_\infty) > 1, \text{ and } M = M(\|\varphi'\|_\infty; \|\varphi''\|_\infty) > 0$$

such that the inequality

$$\frac{a^{5/2}}{R^3}\left\|e^{a(\frac{x}{R}+\varphi(t))^2}\left(\frac{x}{R}+\varphi(t)\right)^2 g\right\|_{L^2(dxdt)}+\frac{a^{3/2}}{R^2}\left\|e^{a(\frac{x}{R}+\varphi(t))^2}\left(\frac{x}{R}+\varphi(t)\right)^2\partial_x g\right\|_{L^2(dxdt)}$$
$$\leq c\left\|e^{a(\frac{x}{R}+\varphi(t))^2}(g_t+i\alpha g_{xx}+\beta g_{xxx}+a_1(x,t)g_x+b_1(x,t)\bar{g}_x+a_0(x,t)g+b_0(x,t)\bar{g})\right\|_{L^2(dxdt)} \tag{3.13}$$

holds, for $R\geq R_0+\alpha^2$*, a such that* $a\geq MR^{3/2}$*, and* $g\in C_0^\infty(\mathbb{R}^2)$ *supported in*

$$\left\{(x,t)\in\mathbb{R}^2:\left|\frac{x}{R}+\varphi(t)\right|\geq 1\right\}.$$

Theorem 3.3. *Let* $u\in C([0,1];H^3(\mathbb{R}))$ *be a solution of*

$$u_t+i\alpha u_{xx}+\beta u_{xxx}+a_2(x,t)u_{xx}+a_1(x,t)u_x+b_1(x,t)\bar{u}_x+a_0(x,t)u+b_0(x,t)\bar{u}=0, \tag{3.14}$$

with $b_0,b_1,a_0,a_1,a_2,(a_2)_x,(a_2)_{xx}\in L^\infty(\mathbb{R}^2)$ *and* $a_2,(a_2)_t\in L_t^\infty(\mathbb{R}:L_x^1(\mathbb{R}))$*. If*

$$\int_{\mathbb{R}}\int_0^1(|u|^2+|u_x|^2+|u_{xx}|^2)(x,t)dxdt\leq A^2, \tag{3.15}$$

and

$$\int_{3/8}^{5/8}\int_0^1 u^2(x,t)dxdt\geq 1, \tag{3.16}$$

then there exist constants $R_0,c_0,c_1>0$ *depending on*

$$A,\|b_0\|_\infty,\|b_1\|_\infty,\|a_0\|_\infty,\|a_1\|_\infty,\|a_2\|_\infty,\|\partial_x a_2\|_\infty,\|\partial_x^2 a_2\|_\infty,\|a_2\|_{L_t^\infty L_x^1},\|\partial_t a_2\|_{L_t^\infty L_x^1}$$

such that for $R\geq R_0$

$$\delta(R)=\delta_u(R)=\left(\int_0^1\int_{R-1}^R(|u|^2+|u_x|^2+|u_{xx}|^2)(x,t)dxdt\right)^{1/2}\geq c_0e^{-c_1R^{3/2}}. \tag{3.17}$$

Proof. Considering the gauge transformation

$$v(x,t)=u(x,t)e^{1/(3\beta)\int_0^x a_2(s,t)ds}, \tag{3.18}$$

the equation for $v = v(x,t)$ can be written as

$$\begin{aligned}
&v_t - \left(\frac{1}{3\beta}\int_0^x \partial_t a_2(s,t)ds\right)v + i\alpha\left(v_{xx} - \frac{2}{3\beta}a_2 v_x + \left(-\frac{1}{3\beta}\partial_x a_2 + \frac{1}{9\beta^2}a_2^2\right)v\right)\\
&\quad + \beta\left(v_{xxx} - \frac{a_2}{\beta}v_{xx} + \left(-\frac{1}{3\beta}\partial_x a_2 + \frac{1}{9\beta^2}a_2^2\right)v_x + \left(-\frac{a_2^3}{27\beta^3} + \frac{a_2}{3\beta^2}\partial_x a_2 - \frac{1}{3\beta}\partial_x^2 a_2\right)v\right)\\
&\quad + a_2 v_{xx} - \frac{2a_2^2}{3\beta}v_x + \left(-\frac{a_2\partial_x a_2}{3\beta} + \frac{a_2^3}{9\beta^2}\right)v + a_1 v_x - \frac{a_2 a_1}{3\beta}v + a_0 v - \frac{\bar{a}_2 b_1}{3\beta}\bar{v}_x + \left(b_0 - \frac{\bar{a}_2 b_1}{3\beta}\right)\bar{v}\\
&= v_t + i\alpha v_{xx} + \beta v_{xxx} + \tilde{a_1}(x,t)v_x + \tilde{a_0}(x,t)v + \tilde{b_1}(x,t)\bar{v}_x + \tilde{b_0}(x,t)\bar{v} = 0. \qquad (3.19)
\end{aligned}$$

where $\tilde{a_0}, \tilde{a_1}, \tilde{b_0}, \tilde{b_1}$ are complex functions in $L^\infty(\mathbb{R}^2)$.

As in [7], we define the functions $\theta_R(x) = 1$ if $x < R-1$, $\theta_R(x) = 0$ if $x > R$, $\mu(x) = 1$ if $x > 2$, $\mu(x) = 0$ if $x < 1$ and $\varphi(t) = 3$ if $t \in [3/8, 5/8]$, $\varphi(t) = 0$ if $t \in [0, 1/4] \cup [3/4, 1]$, $0 \le \theta_R, \mu \le 1$, $\theta_R, \mu \in C^\infty(\mathbb{R})$, $0 \le \varphi \le 3$ and $\varphi \in C_0^\infty(\mathbb{R})$ and the function

$$g(x,t) = \theta_R(x)\mu\left(\frac{x}{R} + \varphi(t)\right)v(x,t), \quad (x,t) \in \mathbb{R}\times[0,1],$$

so that g has support on $(-2R, R)\times(0,1)$ and can be assumed to satisfy the hypothesis of Lemma 3.1.

Using (3.14) one has that

$$\begin{aligned}
&g_t + i\alpha g_{xx} + \beta g_{xxx} + \tilde{a}_1 g_x + \tilde{b}_1\bar{g}_x + \tilde{a}_0 g + \tilde{b}_0\bar{g}\\
&= \mu\left(\frac{x}{R} + \varphi(t)\right)\left(i\alpha\theta_R^{(2)}v + 2i\alpha\theta_R^{(1)}v_x + \beta\theta_R^{(3)}v + 3\theta_R^{(2)}v_x + 3\beta\theta_R^{(1)}v_{xx} + \tilde{a}_1\theta_R^{(1)}v + \tilde{b}_1\theta_R^{(1)}\bar{v}\right)\\
&\quad + \theta_R(x)\left(\mu^{(1)}\left(\varphi^{(1)} + \frac{\tilde{a}_1}{R}\right)v + i\alpha\frac{\mu^{(2)}}{R^2}v + 2i\alpha\frac{\mu^{(1)}}{R}v_x + \beta\frac{\mu^{(3)}}{R^3}v + 3\beta\frac{\mu^{(2)}}{R^2}v_x + 3\beta\frac{\mu^{(1)}}{R}v_{xx}\right)\\
&\quad + 2i\alpha\theta_R^{(1)}\frac{\mu^{(1)}}{R}v + 3\beta\theta_R^{(2)}\frac{\mu^{(1)}}{R}v + 3\beta\theta_R^{(1)}\frac{\mu^{(2)}}{R^2}v + 6\beta\theta_R^{(1)}\frac{\mu^{(1)}}{R}v_x + \tilde{b}_1\theta_R\frac{\mu^{(1)}}{R}\bar{v}.
\end{aligned} \qquad (3.20)$$

The remaining part of the proof follows as in [7]. In fact, using the definitions of θ_R, μ, φ and (3.16) we get

$$\frac{a^{5/2}}{R^3}\left\| e^{a(\frac{x}{R}+\varphi(t))^2}\left(\frac{x}{R} + \varphi(t)\right)^2 g\right\|_{L^2(dxdt)} \ge c\frac{a^{5/2}}{R^3}e^{9a}. \qquad (3.21)$$

On the other hand, we observe that the first term in the right-hand side of (3.20) is supported in $[R-1, R]\times[0,1]$ where $e^{a(x/R+\varphi(t))^2} \le e^{16a}$, and in the remaining terms in the right-hand

side of (3.20) we have $e^{a(x/R+\varphi(t))^2} \leq e^{4a}$. Thus (3.13) and (3.15) imply that for $a \geq M_1 R^{3/2}$ (M_1 as in Lemma 3.2),

$$\begin{aligned}
&\frac{a^{5/2}}{R^3}\left\| e^{a(\frac{x}{R}+\varphi(t))^2}\left(\frac{x}{R}+\varphi(t)\right)^2 g\right\|_{L^2(dxdt)} \\
&\quad \leq c\left\| e^{a(\frac{x}{R}+\varphi(t))^2}(g_t + i\alpha g_{xx} + \beta g_{xxx} + a_1(x,t)g_x + b_1(x,t)\bar{g}_x + a_0(x,t)g + b_0(x,t)\bar{g})\right\|_{L^2(dxdt)} \\
&\quad \leq c_1 e^{16a}\delta_v(R) + c_1 e^{4a} A. \qquad (3.22)
\end{aligned}$$

Combining (3.21) and (3.22) it follows that

$$c\frac{a^{5/2}}{R^3}e^{9a} \leq c_1 e^{16a}\delta_v(R) + c_1 e^{4a} A, \quad \forall a \geq M_1 R^{3/2}.$$

In particular, for $a = M_1 R^{3/2}$ with R sufficiently large we obtain

$$\delta_v(R) \geq c_0 e^{-c_1 R^{3/2}}.$$

By the hypothesis on the coefficients a_0, a_1, a_2 and the definitions (3.17), (3.18) we conclude that

$$\delta_u(R) \sim \delta_v(R) \geq c_0 e^{-c_1 R^{3/2}}.$$

□

4. Proof of the Main Results

This section is devoted to provide proofs of the main results of this work. First, let us begin with the proof of the Theorem 1.4.

Proof of Theorem 1.4. If $u \neq 0$, we can suppose that u satisfies the hypothesis of Theorem 3.3 and therefore

$$\delta_u(R) \geq c_0 e^{-c_1 R^{3/2}}, \qquad (4.1)$$

and apply Theorem 2.4 with $l = 3/2$, $a \gg 8c_1$, c_1 as above we have

$$\delta_u(R) \leq c e^{-aR^{3/2}/8},$$

which is a contradiction with (4.1) for R sufficiently large. □

Now we are position to supply proof if the first main result of this work.

Proof of Theorem 1.3. Let u_1, u_2 be strong solutions of the equation (1.1), then their difference $w = u_1 - u_2$ satisfies the following equation

$$\partial_t w + i\alpha\partial_x^2 w + \beta\partial_x^3 w + \delta a_1(x,t)\partial_x w + \epsilon b_1(x,t)\partial_x \bar{w} + a_0(x,t)w + b_0(x,t)\bar{w} = 0, \tag{4.2}$$

where $a_1(x,t) = |u_1|^2$, $b_1(x,t) = u_1^2$, $a_0(x,t) = i\gamma(|u_1|^2 + |u_2|^2) + \delta\bar{u}_2\partial_x u_2 + \epsilon(u_1+u_2)\partial_x\bar{u}_2$ and $b_0(x,t) = i\gamma u_1 u_2 + \delta u_1\partial_x u_2$.

To conclude the proof of the theorem, it is sufficient to prove that w, a_0, a_1, b_0, b_1 satisfy the hypotheses of Theorem 1.4. As in [7], this is a consequence of the estimates

$$\|vu\|_{L_x^p L_t^p} \leq \|v\|_{L_x^\infty L_t^\infty}\|u\|_{L_x^p L_t^p}. \tag{4.3}$$

For the sake of completeness, we present the proofs of the estimates correcting some mistakes present in [7].

We need show that

$$a_0, b_0 \in L_{xt}^{4/3} \cap L_x^{16/13}L_t^{16/9} \cap L_x^{8/7}L_t^{8/3}, \quad a_1, b_1 \in L_x^{16/13}L_t^{16/9} \cap L_x^{8/7}L_t^{8/3} \cap L_x^{16/15}L_t^{16/3}.$$

We will prove the estimates only for b_0 and b_1, because those for a_0 and a_1 are similar. Using the hypothesis

$$u_j \in C([0,1] : H^3 \cap L^2(|x|^2 dx)), \quad j = 1,2, \tag{4.4}$$

we have (see [7])

$$|x|u_j, |x|^{2/3}(u_j)_x, |x|^{1/3}(u_j)_{xx}, (u_j)_{xxx} \in L^\infty([0,1], L_x^2), \quad j = 1,2 \tag{4.5}$$

$$u_j, |x|^{2/3}u_j \in L^\infty([0,1], L_x^\infty), \quad j = 1,2. \tag{4.6}$$

Thus, (4.3), (4.6) and Holders inequality yield

$$\|u_1 u_2\|_{L_{xt}^{4/3}} \leq c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} \|\langle x\rangle^{1/2^+} u_2\|_{L_x^2} \leq c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} (\|u_2\|_{L_x^2} + \|xu_2\|_{L_x^2}).$$

Similarly

$$\|u_1(u_2)_x\|_{L_{xt}^{4/3}} \leq c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} \|\langle x\rangle^{1/2^+}(u_2)_x\|_{L_x^2} \leq c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} (\|u_2\|_{L_x^2} + \||x|^{2/3}(u_2)_x\|_{L_x^2}).$$

Now we will prove that $b_0 \in L_x^{16/13} L_t^{16/9}$. We have,

$$\begin{aligned}
\|u_1 u_2\|_{L_x^{16/13} L_t^{16/9}} \leq & c\|u_1\|_{L_{x,t}^\infty} \left(\int \frac{\langle x\rangle^p}{\langle x\rangle^p} \int \left(|u_2|^{16/9} dt \right)^{9/13} dx \right)^{13/16}, \quad 4/13 < p < 15/13, \\
\leq & c\|u_1\|_{L_{x,t}^\infty} \left(\int\int \langle x\rangle^{13p/9} |u_2|^{16/9} dx dt \right)^{9/16}, \\
\leq & c\|u_1\|_{L_{x,t}^\infty} \left(\int \|\langle x\rangle u_2\|_{L_x^2}^{16/9} dt \right)^{9/16} \\
\leq & c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} (\|u_2\|_{L_x^2} + \|x u_2\|_{L_x^2}).
\end{aligned}$$

Analogously, using $4/13 < p < 35/39$, we get

$$\|u_1 (u_2)_x\|_{L_x^{16/13} L_t^{16/9}} \leq c\|u_1\|_{L_{x,t}^\infty} \sup_{t\in[0,1]} (\|u_2\|_{L_x^2} + \||x|^{2/3} (u_2)_x\|_{L_x^2}).$$

Now we will prove that $b_0 \in L_x^{8/7} L_t^{8/3}$. Similarly as in [7], we get

$$\begin{aligned}
\|u_1 u_2\|_{L_x^{8/7} L_t^{8/3}} \leq & c\|u_1\|_{L_{x,t}^\infty} \left(\int\int \langle x\rangle^{4^+/3} |u_2|^{8/3} dx dt \right)^{3/8} \\
\leq & c\|u_1\|_{L_{x,t}^\infty} \|\langle x\rangle^{3/2\epsilon} u_2\|_{L_{x,t}^\infty}^{1/4} \sup_{t\in[0,1]} \|\langle x\rangle^{2/3} u_2\|_{L_{x,t}^2}^{3/4}, \quad 0 < \epsilon \ll 1,
\end{aligned}$$

and we have similar estimate for $\|u_1 (u_2)_x\|_{L_x^{8/7} L_t^{8/3}}$.

Finally, it is sufficient to prove that $b_1 = u_1^2 \in L_x^{16/15} L_t^{16/3}$. In fact (see [7])

$$\|u_1^2\|_{L_x^{16/15} L_t^{16/3}} \leq c\|u_1\|_{L_{x,t}^\infty} \|\langle x\rangle u_1\|_{L_{x,t}^2}^{3/8} \|\langle x\rangle^{2/3} u_1\|_{L_{x,t}^\infty}^{5/8}.$$

Using (4.4)-(4.6) we conclude the proof of the theorem. □

In what follows, we provide the proof of the second main result about the decay property of the solution to the Hirota equation.

Proof of Theorem 1.5. The proof of this theorem is very similar to the proof of Theorem 1.4 in [7]. For the sake of clarity, we provide a brief idea pointing out the differences that arise in our case.

Let $\psi \in C_0^\infty(\mathbb{R})$, $\psi \geq 0$, $\operatorname{supp}\psi \subset (-\delta, \delta)$, $\delta \in (0, 1/8)$ and $\int \psi(x) dx = 1$.

We consider the IVP

$$\begin{cases} u_t + i\alpha u_{xx} + \beta u_{xxx} + F(u) = 0, \quad (x,t) \in \mathbb{R} \times [0, \Delta T] \\ u(x,0) = u_0(x) = \varepsilon V(1)\phi = \varepsilon S_1 * \psi, \end{cases} \tag{4.7}$$

where $F(u) = i\gamma|u|^2 u + \delta|u|^2 \partial_x u + \epsilon u^2 \partial_x \overline{u}$, and $\varepsilon, \Delta T$ are sufficiently small.

Without loss of generality we can suppose $\beta = 1$ (Remark 1.2).

Let $V(t)u_0$ be the solution of IVP (4.7) when $\beta = 1$ and let $V_\alpha(t)u_0$ be the solution of IVP (4.7) when $\beta = 0$.

Let us consider $\phi = V_{-\alpha}\psi$, thus $\psi = V_\alpha \phi$, then

$$V(t)\phi = S_t * V_\alpha \phi = S_t * \psi,$$

where $S_t * f(x) = \frac{1}{\sqrt[3]{3t}} Ai(\frac{\cdot}{\sqrt[3]{3t}}) * f(x)$ and $Ai(x)$ is the Airy function.

The solution to the IVP (4.7) (see [7] and [12]) is obtained by iterating

$$\Phi(u^n)(t) = u^{n+1}(t) = \varepsilon S_{1+t} * \psi + \int_0^t S_{t-t'} * F(u^n)(t')dt',$$

$n = 1, 2, \cdots$ in the ball

$$|||w|||_{T,s,k} \leq 2\varepsilon(\|S_1 * \psi\|_{H^s} + \|x^k S_1 * \psi\|_{L^2}), \tag{4.8}$$

where

$$|||w|||_{T,s,k} = \sup_{[0,T]}(\|w(t)\|_{H^s} + \|x^k w(t)\|_{L^2}) + \|w(t)\|_{L^2_x L^\infty_t([0,T])} + \|\partial_x^{s+1} w(t)\|_{L^\infty_x L^2_t([0,T])}. \tag{4.9}$$

The sequence $\{u^n\}$ converges in the norm given by (4.9), for $T > 0$ sufficiently small, inside the ball defined in (4.8).

Using the induction principle, the integral equation and properties of $S_t * \psi$ (Airy function), $t \in [1, 1 + \Delta T]$, $\Delta T > 0$ small enough (see [7]) we obtain

$$|F^n(x,t)| \leq c\varepsilon^3 \begin{cases} e^{-x^{3/2}}, \quad \text{if} \quad x > 1/2, \\ 1, \quad \text{if} \quad |x| \leq 1/2, \\ 1/(1+x^2)^{2k}, \quad \text{if} \quad x \in \mathbb{R}. \end{cases} \tag{4.10}$$

This inequality, properties of Airy function, a limit process and the same argument as in [7] for ε sufficiently small yield the desired result. □

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