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Closed form micro-macro relationships for periodic masonry

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The use of theoretical models for providing a constitutive identification of masonry, starting from the individual properties of the phases (i.e. mortar and bricks), constitutes an attractive alternative to costly experimental investigations. In the case of brickwork with periodic texture, the latter issue is tackled resorting to the homogenization theory by Anthoine [1] and solved by means of the finite element method. In order to obtain closed form formulations many authors in literature made simplifying assumptions regarding either the masonry bond, Pande *et al.* [2], or the joints thickness, Cecchi & Sab [3]. The simplifications adopted turn out to introduce significant errors in the results when either a large difference in stiffness between the phases or a non negligible thickness of the joint are encountered.

In the present paper a homogenization procedure is presented, which takes into account the effect of the bond and the Poisson-type interaction between mortar and brick. Assuming a simplified kinematics for the phases belonging to the R.V.E., the so-called *localization* problem is solved by imposing the minimization of the average internal strain energy. Closed form formulations are then derived for the equivalent in-plane elastic constants of masonry. The expressions found are consistent with those obtained in literature in the limit cases in which masonry is tackled as a stratified medium or where the joints are treated as interfaces. The accuracy of the results is investigated by means of a comparison with finite element analysis. A parametric study, conducted varying the geometries and the mechanical properties of the phases, shows that the error introduced over a very wide range of values for the elastic properties is lower than 8%, meaning that the procedure is ready to be used for non-linear analysis.

Introduction

Aiming at predicting the behaviour of masonry structures, macro-modelling approaches, based on the definition of an equivalent homogeneous medium, are far more preferable than micro-modelling strategies, in which units and joints have to be reproduced separately. The latter approach in fact becomes not feasible in practical cases when the dimension of the structure is much bigger than the dimension of the unit.

When dealing with masonry showing a periodic texture, a constitutive description of the equivalent homogenous medium can be achieved resorting to the homogenization theory, by solving a boundary value problem (of localization) defined on a representative volume element (R.V.E.) of the material, Suquet [4]. In the specific case of running bond masonry, the problem was addressed by Anthoine [1] in a rigorous way and solved by means of the finite element method. The latter approach applied in a full three-dimensional fashion requires intensive numerical computations, Cecchi and Di Marco [5], so that the problem is usually tackled under plain conditions, by taking advantage from the geometrical aspect ratio of masonry walls.

In order to handle the problem analytically many authors in literature introduced simplifying assumptions regarding either the bond of the material, Pande *et al.* [2], following the so-called two-step homogenization approaches, or treating the joint as interfaces, Cecchi and Sab [3] and de Felice *et* *al.* [6]. These approaches provide analytical formulation for the overall elastic constants of masonry and allow to obtain the relations between microscopic and macroscopic stress and strain fields. The abovementioned relations result particularly attractive for non-linear problems within the framework of multi-scale analysis.

However, the simplifications commonly used in the literature are thought to introduce some errors in predicting the properties of the equivalent homogeneous continuum. Concerning the elastic moduli, quite good results are obtained when thin joints and small differences between the stiffnesses of mortar and brick are encountered; the error introduced by the models is expected to become larger in the case of thick joints and soft mortar, Zucchini and Lourenço [7]. Moreover, both the approaches fail in reproducing the microscopic stresses arising within the phases which strongly depend on the bond pattern, which is neglected in the multi-step approach, and on the Poisson-type interaction between mortar and brick, which is not accounted for when the joints are treated as interfaces. Furthermore, the plain stress assumption, which is commonly adopted, is in disagreement with the exact three-dimensional solution, as showed by Anthoine [1]. All these arguments suggest that the extension of the abovementioned models to the non-linear range may leads to erroneous estimation of the mechanical behaviour of masonry.

In the present paper the homogenization theory is applied rigorously to running bond masonry, i.e. in a three-dimensional fashion and by accounting for the actual thickness of the joints. A compatible approach is then followed by introducing a simplified kinematics with the purpose of obtaining, in analytical form, an upper bound estimation for the elastic constants of masonry and closed form micro-macro relationships. Finally, the results obtained within the elastic range are compared with the models proposed in literature and with the results of a standard finite element procedure.

Derivation of masonry elastic properties via homogenization

Let us consider a domain of reference consisting of a single-leaf masonry wall where the bricks, which have dimensions h_b and l_b, are arranged regularly so as to reproduce a running bond pattern. Two families of orthogonal mortar joints are then encountered: horizontal bed joints, which have thickness t_m^b and are continuous within the brickwork, and vertical head joints, which have thickness t_m^h and are staggered between adjacent courses. Furthermore let us denote as cross joints the mortar located at the intersection of the previous ones. The Representative Volume Element (R.V.E.) is defined as the lozenge having vertexes located in the centre of four adjacent bricks and extruded among the whole thickness T of the wall, Figure 1. The domain is periodic in the Oe_1e_2 plane and according to the previously defined R.V.E. the two directions of periodicity write:

$$i_1 = \frac{(l_b + t_m^h)}{2}\mathrm{e}_1 + (h_b + t_m^b)\mathrm{e}_2 \quad ; \quad i_2 = \frac{(l_b + t_m^h)}{2}\mathrm{e}_1 - (h_b + t_m^b)\mathrm{e}_2 \qquad (1)$$

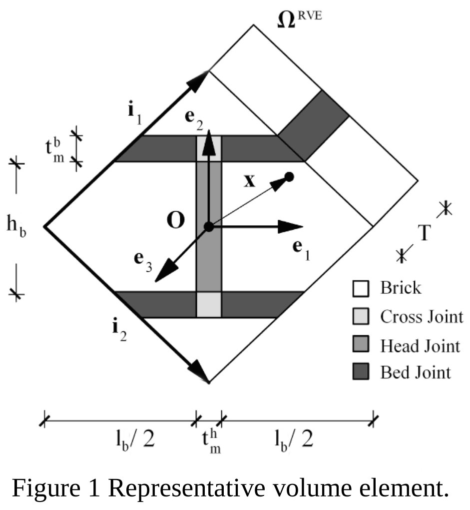

Figure 1 Representative volume element.

The phases are assumed to be elastic and isotropic, being their constitutive laws defined in terms of the stiffness tensors C_b and C_m, which are introduced, respectively, for brick and mortar:

$$\sigma(x) = \mathrm{C}(x):\varepsilon(x) \quad \text{where} \quad \mathrm{C}(x) = \begin{cases} \mathrm{C}_b & \forall x \in \Omega_b \\ \mathrm{C}_m & \forall x \in \Omega_m \end{cases} \tag{2}$$

where Ω_b and Ω_m denote the sub-domains of R.V.E. that pertains to mortar and brick, σ and ε the microscopic stress and strain fields. Let us consider the case in which the load acts in the Oe_1e_2 plane such that the components of the macroscopic stress Σ_{i3}, $i = 1,..,3$ are null in order to maintain the symmetry with respect to the middle plane of the wall. Accordingly, the macroscopic out-of-plane shear strains E_{13} and E_{23} vanish, whereas the axial strain E_{33} assumes a finite value.

Resorting to homogenization theory, Suquet [4], the in-plane behaviour of masonry viewed as a homogeneous medium can be derived by solving the well-known problem of localization, that is attached to Ω_{RVE} and reads:

$$\mathrm{div}\,\sigma(x) = 0 \tag{3}$$

$$\sigma(x) = \mathrm{C}(x):\varepsilon(x) \tag{4}$$

$$\varepsilon(x) = \mathrm{sym}\,\nabla u(x) \tag{5}$$

$$u(x) = \mathrm{E}.x + \hat{u}(x) \tag{6}$$

where $u(x)$ is the displacement field among Ω_{RVE} and $\hat{u}(x)$ is a periodic field that takes the same values on opposite sides of $\partial\Omega_{RVE}$, which correspond by periodicity. For a given macroscopic strain E, the solution of the localization problem amounts to a relation between microscopic and macroscopic strain fields that writes:

$$\varepsilon(x) = \mathrm{D}(x):\mathrm{E} \tag{7}$$

where $\mathrm{D}(x)$ is the fourth order tensor of strain localization. Finally, by introducing the homogenized stiffness tensor $\mathrm{C}^{\mathrm{hom}}$, the constitutive relation of masonry is given by:

$$\Sigma = \langle \sigma(x) \rangle_{\Omega_{RVE}} = \langle \mathrm{C}(x):\mathrm{D}(x) \rangle_{\Omega_{RVE}} :\mathrm{E} = \mathrm{C}^{\mathrm{hom}}:\mathrm{E} \tag{8}$$

where $<^*>$ is the averaging operator

From the definition of the tensor of strain localization and by virtue of Eqs. 2-8 the relation that stands between the macroscopic and microscopic stress fields reads:

$$\sigma(x,\Sigma) = \mathrm{C}(x):\mathrm{D}(x):\mathrm{C}^{\mathrm{hom}-1}:\Sigma = S(x):\Sigma \tag{9}$$

where $S(x)$ is the tensor of stress localization.

Aiming at obtaining an analytical solution of the localization problem the R.V.E. is divided in sub-domains corresponding to bricks, head, bed and cross joints, Figure 2, and the following simplifying assumptions are made concerning the displacement field $u(x)$, $x \in \Omega_{RVE}$.

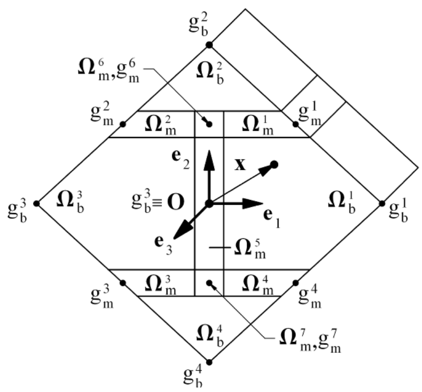

Figure 2 Division of Ω_{RVE} in brick (Ω_b^i)and mortar (Ω_m^i) sub-domains.

Bricks, head and bed joints are supposed to undergo an affine displacement:

$$u(x) = u(g_b^i) + \mathrm{H}_b^i.(x - g_b^i) \quad \forall x \in \Omega_b^i \quad i = 1,..,4 \tag{10}$$

$$u(x) = u(g_m^i) + \mathrm{H}_m^i.(x - g_m^i) \quad \forall x \in \Omega_m^i \quad i = 1,..,5 \tag{11}$$

where g_b^i and g_m^i are the centres of each sub-domain as depicted in Figure 2 , H_b^i and H_m^i are second-order tensors representing the gradient of the displacement within Ω_b^i and Ω_m^i, respectively. The displacement filed within the cross joints is described by a second-order polynomial function where the linear term depends upon H_m^i , $i = 6,7$ and the bi-linear term is defined introducing three symmetric second order tensors B_w^i with $w = 1,..,3$:

$$u(x) = u(g_m^i) + \mathrm{H}_m^i.(x - g_m^i) + \sum_{w=1}^{3} (x - g_m^i)^T.\mathrm{B}_w^i.(x - g_m^i)\, e_w \quad \forall x \in \Omega_m^i \quad i = 6,7 \tag{12}$$

By introducing the previously given field $u(x)$, the localization problem reduces to an algebraic problem whose variables are the 135 components of tensors $\mathrm{H}_b^i, \mathrm{H}_m^i$ and B_w^i.By imposing the periodicity conditions, it is straightforward to demonstrate that:

$$\mathrm{H}_b^i = \mathrm{H}_b \quad i = 1,..,4 \tag{13}$$

$$\mathrm{H}_m^1 = \mathrm{H}_m^3 \tag{14}$$

$$\mathrm{H}_m^2 = \mathrm{H}_m^4 \tag{15}$$

Moreover, by imposing the compatibility conditions of the displacement filed at the boundary of each sub-domain, the variables of the problem can be further reduced to the sole components of tensor H_b. Therefore, an upper bound estimation of the macroscopic strain energy density for the equivalent homogeneous continuum is obtained as:

$$\Psi(\mathrm{E}) = \frac{1}{\Omega_{RVE}} \min_{\mathrm{H}_b} \psi[\mathrm{E}, \mathrm{H}_b] \tag{16}$$

where

$$\psi[\mathrm{E}, \mathrm{H}_b] = \int_{\Omega_{RVE}} \sigma(x, \mathrm{E}, \mathrm{H}_b).\varepsilon(x, \mathrm{E}, \mathrm{H}_b) \cdot d\Omega \tag{17}$$

The equivalent homogeneous continuum behaves as an orthotropic medium whose elastic tensor $\mathrm{C}^{\mathrm{hom}}$ is derived from Eq. 16 by differentiation with respect to E and is defined by four constants Y_{11}^{hom} , Y_{22}^{hom} , μ_{12}^{hom} and ν_{12}^{hom}, namely the elastic modulus in the direction parallel and normal to the bed joints, the shear modulus and the Poisson ratio.
The proposed procedure has been implemented in an algebraic manipulator and analytical formulations that express the tensors $\mathrm{C}^{\mathrm{hom}}$, $D(x)$ and $S(x)$ as a function of mortar and brick geometrical and mechanical properties have been obtained. Owing to their complexity the expressions derived are not reported here.

Comparison with models proposed in literature

The solution of the localization problem obtained in the present work is now studied in the limit conditions where the joints reduce to interfaces, i.e. the joints thicknesses t_m^b and t_m^h reduce to zero. Two cases are then discussed: that of a perfectly cohesive interface, which assures the continuity of the displacement field between adjacent bricks, and that in which the interface keeps a finite stiffness that allows a displacement jump between adjacent bricks to take place, see also Cecchi and Sab [3].

Clearly, when both head and bed joints reduce to perfect cohesive interfaces, masonry behaves as an homogenous material showing the same properties of the brick:

$$\lim_{t_m^h, t_m^b \to 0} C^{\text{hom}} = C^b \tag{18}$$

In the case where the sole head joint reduces to a perfectly cohesive interface, masonry becomes a stratified material made by alternating horizontal layers of mortar and brick. The resulting elastic constants are reported below:

$$\frac{1}{Y_{11}^{\text{hom}}} = \frac{(h_b + t_m^b)[t_m^b\ \mu_m(\lambda_b + 2\mu_b)(\lambda_m + \mu_m)] + h_b\,\mu_b(\lambda_b + \mu_b)(\lambda_m + 2\mu_m)]}{(h_b\mu_b + t_m^b\mu_m\ )\ [h_b\mu_b(3\lambda_b + 2\mu_b)(\lambda_m + 2\mu_m) + t_m^b\mu_m(3\lambda_m + 2\mu_m)(\lambda_b + 2\mu_b)]} \tag{19}$$

$$\frac{1}{Y_{22}^{\text{hom}}} = \frac{(t_m^b\ )^2(\lambda_b + 2\mu_b)(\lambda_m + \mu_m) + h_b^2(\lambda_b + \mu_b)(\lambda_m + 2\mu_m) + h_b t_m^b[2(\lambda_b\lambda_m + \mu_b^2 + \mu_m^2) + 3(\lambda_b\mu_b + \lambda_m\mu_m)]}{(h_b + t_m^b\ )\ [h_b\mu_b(3\lambda_b + 2\mu_b)(\lambda_m + 2\mu_m) + t_m^b\mu_m(3\lambda_m + 2\mu_m)(\lambda_b + 2\mu_b)]} \tag{20}$$

$$\frac{\nu_{12}^{\text{hom}}}{Y_{11}^{\text{hom}}} = -\frac{t_m^b\ \lambda_m(\lambda_b + 2\mu_b) + h_b\lambda_b(\lambda_m + 2\mu_m)}{2\ [h_b\mu_b(3\lambda_b + 2\mu_b)(\lambda_m + 2\mu_m) + t_m^b\mu_m(3\lambda_m + 2\mu_m)(\lambda_b + 2\mu_b)]} \tag{21}$$

$$\frac{1}{\mu_{12}^{\text{hom}}} = \frac{t_m^b}{h_b + t_m^b}\frac{1}{\mu_m} + \frac{h_b}{h_b + t_m^b}\frac{1}{\mu_b} \tag{22}$$

where λ_m, μ_m and λ_b, μ_b are the Lamé constants of mortar and brick, respectively. The expressions obtained result in close agreement with the formulation originally proposed by Salomon [8] for stratified materials and then applied by Pande *et al.* [2] to the case of masonry.

Finally, when the whole set of joints reduces to elastic interfaces, masonry elastic constants reads:

$$\frac{1}{Y_{11}^{\text{hom}}} = \frac{4(h_b + t_m^b)}{4(h_b + t_m^b)(l_b + t_m^h)K_n^h + (l_b + t_m^h)^2 K_t^b} + \frac{(\lambda_b + 2\mu_b)}{(3\lambda_b\mu_b + 2\mu_b^2)} \tag{23}$$

$$\frac{1}{Y_{22}^{\text{hom}}} = \frac{1}{(h_b + t_m^b)K_n^b} + \frac{(\lambda_b + 2\mu_b)}{(3\lambda_b\mu_b + 2\mu_b^2)} \tag{24}$$

$$\frac{\nu_{12}^{\text{hom}}}{Y_{11}^{\text{hom}}} = -\frac{\lambda_b}{6\lambda_b\mu_b + 4\mu_b^2} \tag{25}$$

$$\frac{1}{\mu_{12}^{\text{hom}}} = \frac{1}{(h_b + t_m^b)K_t^b} + \frac{4(h_b + t_m^b)}{(l_b + t_m^h)^2 K_t^h + 4(h_b + t_m^b)(l_b + t_m^h)K_n^b} + \frac{1}{\mu_b} \tag{26}$$

where K_n^b, K_t^b and K_n^h, K_t^h are the normal and tangential stiffness of bed and head interfaces, respectively. The expressions given above are in agreement with those obtained by de Felice *et al.* [6] for plain stress condition.

Comparison with finite element analysis

The localization problem was implemented in a standard finite element code with the purpose to test the accuracy of the proposed model. The R.V.E. adopted has two orthogonal planes of symmetry and is defined by a frame of two orthogonal vectors, Figure 3a. The abovementioned properties allow to model a sole portion of the whole domain and to reduce the periodicity conditions to ordinary Dirichlet conditions, see Anthoine [1]. Elastic analyses were performed varying the joint thickness from 2.5 to 20 mm and reducing the Young's modulus of mortar so as to obtain Y_b / Y_m ratios ranging from 1 to 1000. In real cases, the elastic stiffness ratio between brick and mortar typically reaches values closer to the lower bound of the abovementioned range. However, in the case of inelastic behaviour, assuming that the modulus Y_m is a secant/tangent approximation of the actual stiffness of mortar, the ratio Y_b / Y_m can reach values of 1000 or higher as a result of the degradation of the mortar joint. The Poisson coefficients of mortar and brick have been assumed equal to 0.25 and 0.20, respectively.

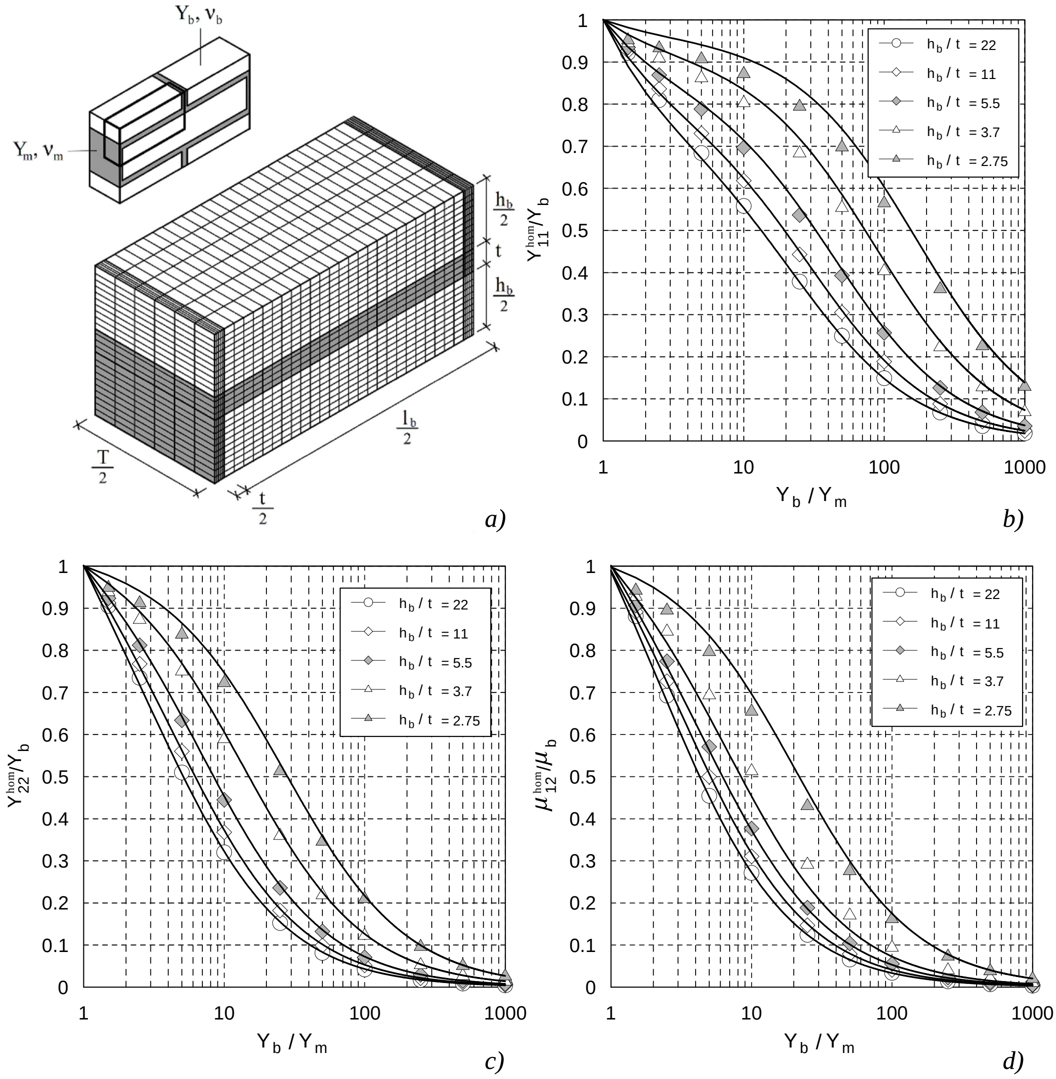

Figure 3 Finite element mesh defined on one-eight of the total volume of the R.V.E. adopted (a); comparison of the model with the numerical results obtained for the elastic modulus in horizontal direction (b), in vertical direction (c) and for the shear modulus (d).

For the whole sets of analysis performed a good agreement is found between the analytical estimate given above and the numerical F.E. results. The errors provided by the analytical prediction of the macroscopic elastic modules in vertical and horizontal directions and of the in-plane shear modulus of masonry are always lower than 8%. The Poisson coefficients of masonry display higher discrepancies that may reach the magnitude of 40% for low values of mortar stiffness and high thickness of the joints; these errors, however, are thought to be not relevant for structural analysis.

A further issue consists in evaluating the capability of the model in reproducing microscopic stress (or strain) fields that develop within the R.V.E. for a given macroscopic stress. For this purpose finite element analysis were conducted applying an unitary macroscopic vertical stress to the R.V.E. considering the joint 10 mm thick and varying the ratio Y_b/Y_m from 1.5 to 1000. Despite the fact that the stress concentration that develops within the phases is not accounted for in the model, since it provides a piece-wise constant stress field, a good agreement is found in predicting the average stresses in the mortar joints and in the bricks, as shown in Figure 4.

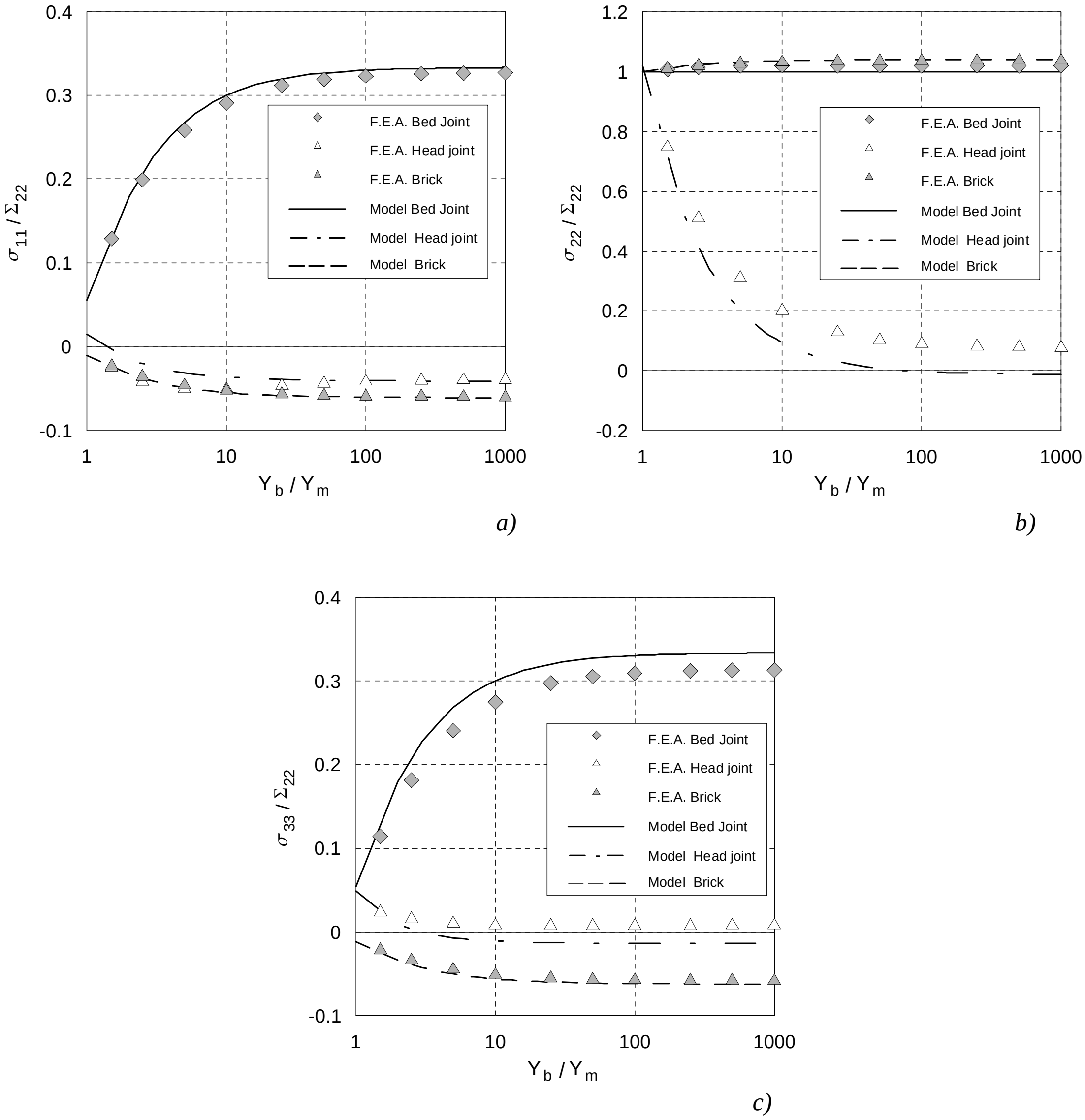

Figure 4 Comparison between the phase-average stresses evaluated from finite element analysis and the ones predicted by the model for an unitary macroscopic vertical stress Σ_{22}.

Conclusion

A homogenization procedure for running bond masonry was presented. The procedure relies upon a simplifying kinematics defined within the R.V.E. and provides analytically the elastic properties of masonry as a function of the geometry and of the individual properties of mortar and brick.

The formulation includes, as a particular case, previous models proposed in literature, in which masonry is regarded either as a stratified material or as a system of blocks connected by interfaces.

By comparison the results with F.E. analysis ,the errors introduced by the model are low from an engineering view point, even when large differences between mortar and brick stiffness are considered or when thick joints are taken into account.

On the basis of the results obtained, the proposed formulation seems able to reproduce the essential feature of masonry behaviour and thus it constitutes a promising tool, which can be adopted in the framework of multi-scale analysis of masonry structures.

References

[1] A. Anthoine. Derivation of the in-plane elastic characteristic of masonry through homogenization theory, *International Journal of Solids and Structures*, 32:2, pp. 137-163, 1995.

[2] G.N. Pande, J.X. Liang and J. Middleton. (1989).Equivalent elastic moduli for brick masonry, *Computer Geotechnics*, 8:5, pp.243-265, 1989.

[3] A. Cecchi and K. Sab. A multi-parameter homogenization study for modeling elastic masonry, *European Journal of Mechanics A/Solids*, 21, pp.249-268, 2002.

[4] P.M. Suquet. Elements of homogenization for inelastic solid mechanics, In *Homogenization Techniques for Composite Media*, Sanchez-Palencia E. and Zaoui A. (eds). Springer: Berlin, pp. 193-279, 1987.

[5] A. Cecchi and R. Di Marco. Homogenization of masonry walls with a computational oriented procedure. Rigid or elastic block?, *European Journal of Mechanics* **-** *A/Solids*, 19:3, pp. 535-546, 2000.

[6] G. de Felice, A. Amorosi and M. Malena. Elasto-plastic analysis of block structures through a homogenization method, *International Journal for Numerical and Analytical Methods in Geomechanics*, 34:3, pp. 221-247, 2010.

[7] A. Zucchini and P.B. Lourenço. A micro-mechanical model for the homogenization of masonry, *International Journal of Solids and Structures*, 39:12, pp. 3233-3255, 2002.

[8] M.D.G. Salomon. Elastic Moduli of a stratified rock mass. *International Journal of Rock Mechanics and Mining Science*, 5, pp. 519-527, 1968.