PSEUDOVARIETY JOINS INVOLVING $\mathcal{J}$-TRIVIAL SEMIGROUPS*

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J. Rhodes asked during the Chico Conference in 1986 for the calculation of joins of semigroup pseudovarieties. This paper proves that the join $\mathbf{J} \vee \mathbf{H}$ of the pseudovariety $\mathbf{J}$ of $\mathcal{J}$-trivial semigroups and of any 2-strongly decidable pseudovariety $\mathbf{V}$ of completely regular semigroups is decidable. This problem was proposed by the first author for $\mathbf{V} = \mathbf{G}$, the pseudovariety of finite groups.

1. Introduction

During the last thirty years, algorithmic questions coming from language theory have motivated an intensive study of semigroup pseudovarieties. Pseudovarieties (classes of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products) were introduced by Eilenberg [12] to formalize

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the correspondence between finite semigroups and rational languages. The main decision problem in this area is perhaps the *membership problem.* It is said to be *decidable* for a pseudovariety $\mathbf{V}$ if there exists an algorithm to test whether a given finite semigroup belongs to $\mathbf{V}$. We shall say in the affirmative case that $\mathbf{V}$ itself is *decidable.* Some advances in the knowledge of the behavior of operators on pseudovarieties towards this notion were accomplished — with great effort — during the last decade. Among these operators, the join is certainly the most disconcerting.

The join $\mathbf{V} \vee \mathbf{W}$ of $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. Since the intersection of two pseudovarieties is easily seen to be a pseudovariety, the set of all pseudovarieties forms a lattice, and it is obvious that if $\mathbf{V}$ and $\mathbf{W}$ are decidable, then so is $\mathbf{V} \cap \mathbf{W}$. A surprising result of Albert, Baldinger and Rhodes [1] states that $\mathbf{V} \vee \mathbf{W}$ might not be decidable even if $\mathbf{V}$ and $\mathbf{W}$ are. Rhodes [18] asked in 1986 during the Chico Conference for the calculation of certain joins, including the computation of the join of the pseudovariety of finite $\mathcal{J}$-trivial semigroups and of the pseudovariety $\mathbf{G}$ of finite groups. This specific question was also proposed in [4] by the first author. It was shown by Trotter and Volkov [21] that this join is not finitely based, but the question of its decidability was still open. In particular, our main result gives a positive answer to this question.

The proof is based on the notion of $\mathbf{V}$-pointlike set developed by Henckell [13]. Relational morphisms play a central role in the definition of this notion, which measures "how far from $\mathbf{V}$" a finite semigroup is. A *relational morphism* between two semigroups S and T is a relation τ from S to T such that $s\tau$ is never empty and satisfies $(s\tau)(s'\tau) \subseteq (ss')\tau$. A subset R of a finite semigroup S is said to be $\mathbf{V}$*-pointlike* if for every relational morphism τ from S into a semigroup T of $\mathbf{V}$, the intersection $\bigcap_{r\in R} r\tau$ is not empty. A pseudovariety is n*-strongly decidable* if there is an algorithm to test whether any subset of cardinality n of a given finite semigroup is $\mathbf{V}$-pointlike. It should be no surprise that if $\mathbf{V}$ is n-strongly decidable for some $n \geqslant 2$, then it is also decidable.

The result we shall prove is a little more general than the decidability of $\mathbf{J} \vee \mathbf{G}$: it deals with any 2-strongly decidable pseudovariety of completely regular semigroups, that is, semigroups whose $\mathcal{H}$-classes are groups. The theorem is as follows.

Theorem 1.1. *Let* $\mathbf{V}$ *be a 2-strongly decidable pseudovariety of completely regular semigroups. Then the join* $\mathbf{J} \vee \mathbf{V}$ *is also 2-strongly decidable.*

When dealing with $\mathbf{J}$, topological arguments have in general a combinatorial formulation. We have favoured the combinatorial approach, and the topological point of view will be frequently hidden behind compactness or denseness results. However, the proof is constructive, and most of the time, we work directly on automata to make the involved ideas visual. Much of the rest of this paper is now devoted to this proof. We shall first remind the reader of some points of the theory of finite semigroups in Sec. 2, where the terminology in use is also detailed. The proof of Theorem 1.1 occupies Sec. 3, and some applications of this theorem are listed in the final section.

2. Preliminaries

In this section, we list those basic results on rational languages, automata and finite semigroups which will be assumed without proof. We presuppose a familiarity with this material. Here is a selection of classical references the reader may refer to. The book of Hopcroft and Ullman [14] gives an introduction to the theory of languages and to decision problems. The books of Lallement [15] and Pin [16] both provide an overview of the links between languages and finite semigroups. Finally, Almeida [4] shows more recent developments and emphasizes the advantage of studying free profinite semigroups.

2.1. *Languages and finite automata*

We fix a finite alphabet $A = \{a_1, \ldots, a_s\}$. We denote by A^* (resp. by A^+) the free monoid (resp. the free semigroup) generated by A. By a *language*, we mean a subset of the free semigroup. By analogy with the number of elements $|X|$ of a finite set X, the length of a word $u \in A^*$ is denoted by $|u|$. The *content* $c(u)$ of u is the set of all letters occurring in u.

All automata we shall consider will be finite and deterministic. Given an automaton $\mathscr{A}$ recognizing a language on the alphabet A, the symbol $q_{\text{ini}}^{\mathscr{A}}$ denotes its initial state, $q_{\text{fin}}^{\mathscr{A}}$ one of its final states, and $Q_{\mathscr{A}}$ its set of states. The state obtained from state q after reading a word $u \in A^+$ is denoted by $q \cdot u$. For $p, q \in Q_{\mathscr{A}}$, we denote by $L_{\mathscr{A}}(p, q)$ the language recognized by the automaton deduced from $\mathscr{A}$ by taking p for initial state and q for unique final state. For $a \in A$, we denote a transition by $p \xrightarrow{a} q$, or by $p \longrightarrow q$ when there is no ambiguity; here $q = p \cdot a$. A *path* is a sequence of consecutive transitions. For $u \in A^+$, we denote by $p \overset{u}{\dashrightarrow} q$ the path labeled u going from p to q, but we may sometimes omit the label; here $q = p \cdot u$.

A *strongly connected component* of $\mathscr{A}$ is a maximal subset Q of $Q_{\mathscr{A}}$ such that there exists a (possibly empty) path between any two states of Q. Strongly connected components form a partition of the set of states. There is a natural order on the set of strongly connected components, defined by $Q_1 \preccurlyeq Q_2$ if there is a path from a state of Q_1 to a state of Q_2.

The path $p \dashrightarrow q$ is a *loop* if $p = q$. Two paths $\mathscr{P} = p \dashrightarrow q$ and $\mathscr{P}' = p' \dashrightarrow q'$ are *consecutive* if $q = p'$. In this case, we denote by $\mathscr{P}\mathscr{P}'$ the path obtained by concatenating $\mathscr{P}$ and $\mathscr{P}'$. The *content* $c(\mathscr{P})$ (resp. the *length* $|\mathscr{P}|$) of a path $\mathscr{P}$ is the content (resp. the length) of its label. A loop $\mathscr{L}$ is said to be *acceptable* if

$$\text{for any factorization } \mathscr{L} = p \overset{u_1}{\dashrightarrow} q \overset{u_2}{\dashrightarrow} q \overset{u_3}{\dashrightarrow} p, \text{ we have } c(u_2) \not\subseteq c(u_1u_3). \tag{1}$$

This term is used just for convenience in this paper. More precisely, it is introduced in order to have a *finite number* of "test loops".

Lemma 2.1. *In any deterministic finite automaton, there is a finite number of acceptable loops, and these loops can be determined.*

Proof. Assume that the number of acceptable loops is infinite. Since the automaton is finite, we can find a loop inside which a state q is repeated at least $r = |A| + 2$ times between the first and the last state:

$$p \xrightarrow{u_0} q \xrightarrow{u_1} q \quad \cdots \quad q \xrightarrow{u_{r-1}} q \xrightarrow{u_r} p.$$

Since the loop is acceptable, for $i \in [1, r-1]$, there is a letter in $c(u_i) \setminus \bigcup_{j \neq i} c(u_j)$, which makes at least $|A| + 1$ distinct letters. We have thus proved that the number of loops is finite, by proving that a state cannot be repeated more than r times inside the loop. This leads to a computable bound on the number of loops, and since it is easy to test whether a given loop is acceptable, we can determine all acceptable loops. □

A path $\mathscr{P}_1 \cdots \mathscr{P}_n$ is a *subpath* of $\mathscr{P}$ if there exist (possibly empty) paths $\mathscr{Q}_0, \ldots, \mathscr{Q}_n$, such that $\mathscr{P} = \mathscr{Q}_0 \mathscr{P}_1 \mathscr{Q}_1 \cdots \mathscr{P}_n \mathscr{Q}_n$. The path $\mathscr{P}$ *contains* $\mathscr{P}'$ if the sequence of transitions of $\mathscr{P}'$ is a subpath of $\mathscr{P}$. This relation is transitive.

Lemma 2.2. *Every loop $\mathscr{L}$ contains an acceptable loop which has the same content as $\mathscr{L}$.*

Proof. For $|\mathscr{L}| = 1$, the assertion is plainly true. Assume that it holds for any loop of length less than $n - 1$ and let $\mathscr{L}$ be a loop of length n. If it is acceptable, there is nothing to do. Otherwise, we can write $\mathscr{L} = p \xrightarrow{u_1} q \xrightarrow{u_2} q \xrightarrow{u_3} p$ with $c(u_2) \subseteq c(u_1 u_3)$. Therefore, $\mathscr{L}$ contains $\mathscr{L}' = p \xrightarrow{u_1} q \xrightarrow{u_3} p$, which has the same content as $\mathscr{L}$. Since $|\mathscr{L}'| < |\mathscr{L}|$, we can apply the induction hypothesis: $\mathscr{L}'$ contains an acceptable loop $\mathscr{L}''$ of content $c(\mathscr{L}') = c(\mathscr{L})$. Since $\mathscr{L}$ contains $\mathscr{L}'$, it contains also $\mathscr{L}''$. □

2.2. *Semigroups*

Given a semigroup S, we let S^1 be the semigroup S itself if it is a monoid, or the disjoint union $S \uplus \{1\}$ where 1 acts as a neutral element otherwise. Given an element t of a finite semigroup (resp. of a compact topological semigroup), the subsemigroup (resp. the closed subsemigroup) generated by t contains a unique idempotent, that we denote by t^ω. A *completely regular semigroup* is a semigroup whose $\mathscr{H}$-classes are groups. It is easy to check that S is completely regular if and only if $t^{\omega+1} = t$ for all t in S.

A *pseudovariety* is a class of finite semigroups closed under formation of finite direct products, subsemigroups and homomorphic images. For instance, the class of all finite groups forms a pseudovariety denoted by **G**. The pseudovariety **J** of all finite $\mathscr{J}$-trivial semigroups is the other pseudovariety we are interested in. We recall in Sec. 2.4 its main properties.

2.3. *Free profinite semigroups*

We now present without proofs some results of the theory of implicit operations. The connection with the theory of finite semigroups comes from an analogue of

Birkhoff's theorem identifying varieties of algebras as equational classes. This result is due to Reiterman [17] and states that pseudovarieties are exactly the classes of finite semigroups defined by sets of formal identities whose members are implicit operations. See [2, 4, 17] for details.

Let $\mathbf{V}$ be a pseudovariety. An *A-ary implicit operation* on $\mathbf{V}$ is a collection $(\pi_S)_{S\in\mathbf{V}}$ where $\pi_S : S^{|A|} \to S$ is a function such that for any morphism $\varphi : S \to T$ between members of $\mathbf{V}$, the following diagram commutes:

$$\begin{array}{ccc} S^{|A|} & \xrightarrow{\pi_S} & S \\ \Big\downarrow{\scriptstyle \varphi^{|A|}} & & \Big\downarrow{\scriptstyle \varphi} \\ T^{|A|} & \xrightarrow[\pi_T]{} & T \end{array}$$

The set of all A-ary implicit operations on $\mathbf{V}$, denoted by $\overline{\mathbf{F}}_A(\mathbf{V})$, is also called the $\mathbf{V}$-free profinite semigroup. It may be viewed in a natural way as a projective limit of elements of $\mathbf{V}$ and, as indicated below, endowed with the profinite topology.

We can associate to any word $u = a_{i_1} \cdots a_{i_k}$ of A^+ the collection of functions $(u_S)_{S\in\mathbf{V}}$ defined by $u_S(s_1, \ldots, s_{|A|}) = s_{i_1} \cdots s_{i_k}$. This clearly defines an implicit operation, which we simply denote by u. Such an operation, induced by a word, is said to be *explicit*. The set of A-ary explicit operations on $\mathbf{V}$ is denoted by $\mathbf{F}_A(\mathbf{V})$.

The multiplicative law on $\overline{\mathbf{F}}_A(\mathbf{V})$ defined by $(\pi_S) \cdot (\rho_S) = (\pi_S \cdot \rho_S)$ makes it a semigroup, and $\mathbf{F}_A(\mathbf{V})$ a subsemigroup of $\overline{\mathbf{F}}_A(\mathbf{V})$. We endow $\overline{\mathbf{F}}_A(\mathbf{V})$ with the initial topology for the evaluation morphisms

$$\begin{aligned} e_T : \overline{\mathbf{F}}_A(\mathbf{V}) &\longrightarrow T^{T^n} \\ (\pi_S)_{S\in\mathbf{V}} &\longmapsto \pi_T \end{aligned}$$

where T runs in $\mathbf{V}$, and where each finite semigroup T^{T^n} is endowed with the discrete topology. This topology makes $\overline{\mathbf{F}}_A(\mathbf{V})$ a compact and 0-dimensional topological semigroup in which $\mathbf{F}_A(\mathbf{V})$ is dense.

Assume that $\mathbf{V}$ contains another pseudovariety $\mathbf{W}$. If $\pi = (\pi_S)_{S\in\mathbf{V}}$ belongs to $\overline{\mathbf{F}}_A(\mathbf{V})$, then $(\pi_S)_{S\in\mathbf{W}}$ belongs to $\overline{\mathbf{F}}_A(\mathbf{W})$. This implicit operation on $\mathbf{W}$ is called the *projection* of π onto $\mathbf{W}$ and is denoted by $p_{\mathbf{W}}(\pi)$. The projection $p_{\mathbf{W}}$ is a continuous homomorphism. The case where $\mathbf{W} = \mathbf{Sl}$, the pseudovariety of finite semilattices, is of particular interest. Indeed, $\overline{\mathbf{F}}_A(\mathbf{Sl})$ may be viewed as the semigroup $(2^A \setminus \varnothing, \cup)$, and the projection onto $\mathbf{Sl}$ coincides with the content morphism on explicit operations. This is why we will write c instead of $p_{\mathbf{Sl}}$.

Given an implicit operation π on $\mathbf{V}$, it is easy to see that the sequence $(\pi^{k!})_{k\in\mathbb{N}}$ converges to an idempotent element of $\overline{\mathbf{F}}_A(\mathbf{V})$. This element is therefore the idempotent of the closed subsemigroup generated by π, that is, π^ω. Note that in pseudovarieties containing $\mathbf{Sl}$, we have $c(\pi^{k!}) = c(\pi)$. By continuity of c, we therefore have $c(\pi^\omega) = c(\pi)$.

The join of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. From Reiterman's theorem, we may deduce that if π and ρ belong to $\overline{\mathbf{F}}_A(\mathbf{V} \vee \mathbf{W})$, then $\pi = \rho$ if and only if $p_{\mathbf{V}}\,\pi = p_{\mathbf{V}}\,\rho$ and $p_{\mathbf{W}}\,\pi = p_{\mathbf{W}}\,\rho$.

In other terms, $p_{\mathbf{V}} \times p_{\mathbf{W}} : \overline{\mathrm{F}}_A(\mathbf{V} \vee \mathbf{W}) \to \overline{\mathrm{F}}_A(\mathbf{V}) \times \overline{\mathrm{F}}_A(\mathbf{W})$ is an embedding. Let now $\iota_{\mathbf{V}}$ be the canonical morphism from A^+ into $\overline{\mathrm{F}}_A(\mathbf{V})$ defined by $a_i\iota_{\mathbf{V}} = a_i$. We shall say for convenience that a sequence of words $(u_m)_{m\in\mathbb{N}}$ converges in $\mathbf{V}$ to an operation π if the sequence $(u_m\iota_{\mathbf{V}})_{m\in\mathbb{N}}$ converges to π.

Notice that $\overline{\mathrm{F}}_A(\mathbf{V} \vee \mathbf{W})$ is a closed subspace of $\overline{\mathrm{F}}_A(\mathbf{V}) \times \overline{\mathrm{F}}_A(\mathbf{W})$, so a sequence converges in $\mathbf{V} \vee \mathbf{W}$ if and only if it converges in both $\mathbf{V}$ and $\mathbf{W}$. Furthermore, it is easy to see that the limit of the sequence in $\mathbf{V} \vee \mathbf{W}$ only depends on its limits in $\mathbf{V}$ and $\mathbf{W}$.

For any language K over A, we denote by $\bar{K}_{\mathbf{V}}$ the closure of $K\iota_{\mathbf{V}}$ in $\overline{\mathrm{F}}_A(\mathbf{V})$. We shall say that $\bar{K}_{\mathbf{V}}$ is the $\mathbf{V}$-closure of K. The first author expressed in terms of closures of rational languages the notion of 2-strong decidability. He observed that the general definition of 2-strong decidability may be phrased as follows [5, Proposition 3.4]. A pseudovariety $\mathbf{V}$ is 2-strongly decidable if the following problem is decidable.

Data. Two rational languages K and L over a finite alphabet.

Question. Determine whether $\bar{K}_{\mathbf{V}}$ and $\bar{L}_{\mathbf{V}}$ are disjoint or not.

Eilenberg's correspondence between pseudovarieties and varieties of rational languages links the original definition and this new characterization. We shall also make use of the following easy yet important property. It is shown for instance in [5].

Proposition 2.3. *If* $\mathbf{V}$ *is* 2*-strongly decidable, then it is also decidable.*

As indicated in [5], this result may be verified with the following argument, which is easy to check: if L is a rational language, then the $\mathbf{V}$-closures of L and $A^+ \setminus L$ are disjoint if and only if L can be recognized by a semigroup of $\mathbf{V}$.

2.4. *An essential example: the pseudovariety* J

The historical importance of the pseudovariety $\mathbf{J}$ comes from a nice syntactic characterization of the piecewise testable languages. It was discovered by Simon [20] and led to one of the seminal papers of this theory. Furthermore, this pseudovariety is frequently encountered in many fundamental problems of the theory.

Let us say that a word $x = x_1 \cdots x_l$ is a *subword* of an implicit operation $\pi \in \overline{\mathrm{F}}_A(\mathbf{J})$ if π has a factorization of the form $\pi_0 x_1 \pi_1 \cdots x_l \pi_l$ where $\pi_i \in \overline{\mathrm{F}}_A(\mathbf{J})^1$. This definition coincides with the usual one on words when π is explicit. A language L of A^+ is *piecewise testable* when it is a boolean combination of languages of the form $A^*b_0A^* \cdots A^*b_nA^*$, with $b_0, \ldots, b_n \in A$. Simon's theorem states that piecewise testable languages are exactly those recognized by $\mathscr{J}$-trivial semigroups.

For π and ρ in $\overline{\mathrm{F}}_A(\mathbf{J})$ and for each natural ℓ, we write $\pi \sim_\ell \rho$ if π and ρ have the same subwords of length at most ℓ. Observe that $\sim_{\ell+1} \subseteq \sim_\ell$. The characterization of $\mathbf{J}$-recognizable languages has a natural interpretation in terms of implicit operations [3, 4]. The structure of $\overline{\mathrm{F}}_A(\mathbf{J})$ is summarized in the following theorem [4, Theorem 8.2.8], which is closely related to Simon's result.

Theorem 2.4. *Any idempotent implicit operation on* $\mathbf{J}$ *is of the form* u^ω, *where* u *is explicit. More generally, every implicit operation* π *on* $\mathbf{J}$ *has a factorization* $\pi = \pi_1 \cdots \pi_k$ *such that:*

(1) *Each factor* π_i *is either explicit or of the form* u_i^ω *where* u_i *is explicit.*
(2) *If* π_i *and* π_{i+1} *are idempotent, the sets* $c(\pi_i)$ *and* $c(\pi_{i+1})$ *are incomparable.*
(3) *Two consecutive factors* π_i *and* π_{i+1} *are not both explicit.*
(4) *If* π_i *is explicit and* π_{i+1} *idempotent, then the last letter of* π_i *is not in* $c(\pi_{i+1})$.
If π_i *is idempotent and* π_{i+1} *explicit, then the first letter of* π_{i+1} *is not in* $c(\pi_i)$.

Furthermore, if $\pi_1 \cdots \pi_k$ *is the factorization of* π *and if* $\rho_1 \cdots \rho_l$ *is the factorization of* ρ, *then the following conditions are equivalent:*

(i) $\pi = \rho$.
(ii) $k = l$, *and* $\pi_j = \rho_j$ *for* $1 \leqslant j \leqslant k$.
(iii) π *and* ρ *have the same subwords.*

The factorization of π satisfying the conditions (1) to (4) of Theorem 2.4 is its *canonical factorization.* Simple reduction rules can be used in order to obtain the canonical form of an implicit operation built from letters using multiplication and ω-powers (see [4, Sec. 8.2] for details). Let us list these rules, with which the reader should be acquainted.

(1) Eliminate parentheses concerning the application of the operation of multiplication.
(2) Substitute any occurrence of t^ω by u^ω, where u is the product of the letters that occur in t, say in increasing order of the indices.
(3) Absorb in factors of the form u^ω any adjacent factors in which only letters of u occur.

A direct consequence of Theorem 2.4 is that if u and v are two words with the same content, then the implicit operations u^ω and v^ω of $\overline{\mathrm{F}}_A(\mathbf{J})$ are equal. In the sequel, if $B = c(u)$, we will often denote this operation by B^ω. Another corollary is a characterization of converging sequences in $\overline{\mathrm{F}}_A(\mathbf{J})$.

Corollary 2.5. *Let* π *be an implicit operation on* $\mathbf{J}$. *A sequence* $(\pi_i)_{i\in\mathbb{N}}$ *converges to* π *in* $\overline{\mathrm{F}}_A(\mathbf{J})$ *if and only if for every* $\ell \in \mathbb{N}$, *there exists* $N \in \mathbb{N}$ *such that* $i > N$ *implies* $\pi \sim_\ell \pi_i$.

Define the *length* of a product $\pi = x_0 B_1^\omega x_1 \cdots B_n^\omega x_n$ of implicit operations on $\mathbf{J}$, where $x_i \in A^*$ and $\varnothing \neq B_i \subseteq A$, by $|\pi| = n + |x_0 \cdots x_n|$. The length $|\pi|$ of an implicit operation π is by definition the length of its canonical factorization. In this way, the prefix of length $k \leqslant n$ of a product $\pi_1 \cdots \pi_n$ where π_i is either a letter or an idempotent is by definition $\pi_1 \cdots \pi_k$. The suffix of length k of this product is defined symmetrically.

3. A Proof of the Main Theorem

In this section, we prove Theorem 1.1. For convenience, if K is a rational language, we shall write $\bar{K}$ instead of $\bar{K}_{\mathbf{J} \vee \mathbf{V}}$. More generally, whenever not explicitly stated otherwise, when referring to a topology, we mean the topology of $\overline{\mathrm{F}}_A(\mathbf{J} \vee \mathbf{V})$. What we have to show is that, given two rational languages K and L, and under the hypotheses of Theorem 1.1, we can decide whether the intersection $\bar{K} \cap \bar{L}$ is empty.

Consider deterministic automata $\mathscr{A}$ and $\mathscr{B}$ recognizing K and L respectively. The central argument of the algorithm deciding whether $\bar{K} \cap \bar{L}$ is empty is a reduction to a finite number of decidable questions. The two steps of the proof rely upon similar ideas.

(1) Proposition 3.2 below shows that if $\bar{K} \cap \bar{L}$ is not empty, then there exists an implicit operation π in this intersection such that the length of $p_{\mathbf{J}}(\pi)$ is smaller than an integer N (which we can compute) depending only on $\mathscr{A}$ and $\mathscr{B}$.
Since there is a finite number of idempotents in $\overline{\mathrm{F}}_A(\mathbf{J})$, there is also only a finite number of implicit operations in $\overline{\mathrm{F}}_A(\mathbf{J})$ whose length does not exceed N, which we can enumerate. Thus, this step translates the problem into a finite number of questions: to check that $\bar{K} \cap \bar{L}$ is not empty, it suffices to check that there exists an implicit operation $\pi_{\mathbf{J}} \in \overline{\mathrm{F}}_A(\mathbf{J})$ such that $|\pi_{\mathbf{J}}| < N$ which we can lift in $\bar{K} \cap \bar{L}$.

(2) Proposition 3.7 shows that the latter question, where $\pi_{\mathbf{J}}$ is given, is decidable.

The following property is shown in [7]. It attempts to capture combinatorial properties of converging sequences in $\overline{\mathrm{F}}_A(\mathbf{J})$. For a subset B of A and a natural number m, let $\mathrm{C}(B, m)$ be the set of products of m words of content B.

Lemma 3.1. *Let $x_0 B_1^{\omega} x_1 \cdots B_n^{\omega} x_n$ be the canonical factorization of an implicit operation π on $\overline{\mathrm{F}}_A(\mathbf{J})$ where each x_i is a (possibly empty) explicit operation. Let $m \geqslant |\pi|$, let $\ell > 2m$, and let w be a word such that $w \sim_{\ell} \pi$. Then, w has a factorization $x_0 y_1 x_1 \cdots y_n x_n$ such that for all $i \in [1, n]$, y_i belongs to $\mathrm{C}(B_i, m)$.*

We are now able to state and prove the first step of the proof.

Proposition 3.2. *If $\bar{K} \cap \bar{L}$ is not empty, then there exists an integer N depending only on $\mathscr{A}$ and $\mathscr{B}$ which we can compute and an implicit operation $\pi \in \bar{K} \cap \bar{L}$ such that $|p_{\mathbf{J}}(\pi)| \leqslant N$.*

Proof. Let $\rho \in \bar{K} \cap \bar{L}$ be an implicit operation such that $|p_{\mathbf{J}}(\rho)|$ is minimal, and set $\pi = p_{\mathbf{J}}(\rho) = x_0 B_1^{\omega} x_1 \cdots B_n^{\omega} x_n$. Take two sequences of words $u_m \in K$ and $v_m \in L$ converging to ρ in $\overline{\mathrm{F}}_A(\mathbf{J} \vee \mathbf{V})$. Then u_m and v_m converge to π in $\overline{\mathrm{F}}_A(\mathbf{J})$. From Corollary 2.5 and Lemma 3.1 we may assume that both u_m and v_m have factorizations of the form

$$\begin{aligned} u_m &= x_0 y_{1,m} x_1 \cdots x_{n-1} y_{n,m} x_n \\ v_m &= x_0 z_{1,m} x_1 \cdots x_{n-1} z_{n,m} x_n \end{aligned} \qquad \text{with } y_{k,m}, z_{k,m} \in \mathrm{C}(B_k, m).$$

For $0 \leqslant k \leqslant |\pi|$, let $\pi = \pi_k \pi'_k$ where π_k is the prefix of length k of π and π'_k its suffix of length $|\pi| - k$. Let $u_{k,m}$ (resp. $v_{k,m}$) be the prefix of u_m (resp. of v_m) obtained by replacing each B_i^{ω} by $y_{i,m}$ (resp. by $z_{i,m}$) in π_k, and let $u_m = u_{k,m} u'_{k,m}$ (resp. $v_m = v_{k,m} v'_{k,m}$).

The number of states in $\mathcal{A}$ (resp. in $\mathcal{B}$) is finite. Hence, extracting subsequences if necessary, we may assume that for every $k \in [0, |\pi|]$, the states

$$p_k = q_{\text{ini}}^{\mathcal{A}} \cdot u_{k,m}, \qquad q_k = q_{\text{ini}}^{\mathcal{B}} \cdot v_{k,m}$$

do not depend on m. Moreover, the compactness of $\overline{\mathrm{F}}_A(\mathbf{J} \vee \mathbf{V})$ allows us to assume that each sequence $(y_{i,m})_{m \in \mathbb{N}}$ and each sequence $(z_{i,m})_{m \in \mathbb{N}}$ converges in both $\mathbf{J}$ and $\mathbf{V}$. (Their convergence in $\mathbf{J}$ was already known.)

Let $n_{\mathcal{A}}$ (resp. $n_{\mathcal{B}}$) be the maximum number of states in a strongly connected component of $\mathcal{A}$ (resp. of $\mathcal{B}$) and let $N_1 = 2n_{\mathcal{A}} n_{\mathcal{B}} + 1$. Let N_2 be the maximal number of strongly connected components in any chain of strongly connected components in $\mathcal{A}$ and $\mathcal{B}$. Assume first that there exists no j such that

$$\begin{aligned} &p_{j+1}, \ldots, p_{j+N_1} \text{ belong to the same strongly connected component of } \mathcal{A}; \\ &q_{j+1}, \ldots, q_{j+N_1} \text{ belong to the same strongly connected component of } \mathcal{B}. \end{aligned} \qquad (2)$$

Let $p\gamma_{\mathcal{A}}$ (resp. $p\gamma_{\mathcal{B}}$) be the strongly connected component of $\mathcal{A}$ (resp. of $\mathcal{B}$) which contains the state p. Then it would be impossible for N_1 consecutive pairs of the form $(p_{j+1}\gamma_{\mathcal{A}}, q_{j+1}\gamma_{\mathcal{B}}), \ldots, (p_{j+N_1}\gamma_{\mathcal{A}}, q_{j+N_1}\gamma_{\mathcal{B}})$ to be identical. Therefore, the length of the sequence (p_k), and hence the length of π, would be bounded by $2N_1N_2$, and this would conclude the proof of Proposition 3.2.

Let us now prove that (2) is impossible. Assume that it holds. By the choice of N_1, there would exist $j_1, j_2, j_3 \in [j+1, j+N_1]$ such that $p_{j_1} = p_{j_2} = p_{j_3}$ and $q_{j_1} = q_{j_2} = q_{j_3}$. This is depicted in Fig. 1, where the central state of the automaton $\mathcal{A}$ (resp. $\mathcal{B}$) is p_{j_1} (resp. q_{j_1}).

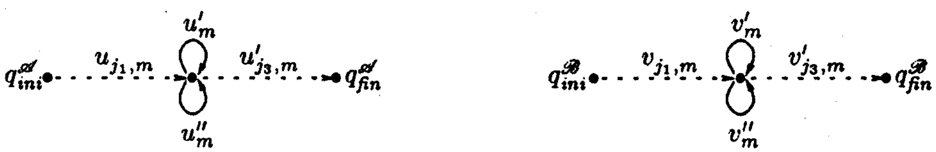

Fig. 1. The paths labeled u_m in $\mathcal{A}$ and v_m in $\mathcal{B}$.

Let u'_m be the word labeling the path between p_{j_1} and p_{j_2} (so that $u_{j_1,m} \cdot u'_m = u_{j_2,m}$), and let u''_m be the word labeling the path between p_{j_2} and p_{j_3} (so that

$u_{j_2,m} \cdot u''_m = u_{j_3,m}$). Similarly, let v'_m be the word labeling the path between q_{j_1} and q_{j_2}. We have $v_{j_1,m} \cdot v'_m = v_{j_2,m}$. Let v''_m be the word labeling the path between q_{j_2} and q_{j_3}. We have $v_{j_2,m} \cdot v''_m = v_{j_3,m}$. Let $U_m = u_{j_1,m} \cdot (u'_m u''_m)^{m!+1} \cdot u'_{j_3,m}$ and $V_m = v_{j_1,m} \cdot (v'_m v''_m)^{m!+1} \cdot v'_{j_3,m}$. We have:

Fact 3.3. (a) U_m belongs to K and V_m belongs to L.

(b) The words $u_{j,m}$, $u'_{j,m}$, u'_m and u''_m are concatenations of $y_{i,m}$'s and of letters of the x_i's in an order which does not depend on m.

(c) The word $v_{j,m}$ (resp. $v'_{j,m}$, resp. v'_m, resp. v''_m) is obtained from $u_{j,m}$ (resp. from $u'_{j,m}$, resp. from u'_m, resp. from u''_m) by substituting each $y_{i,m}$ by the corresponding $z_{i,m}$.

We claim that the sequences $(U_m)_{m\in\mathbb{N}}$ and $(V_m)_{m\in\mathbb{N}}$ converge to a common point in $\mathbf{J} \vee \mathbf{V}$. To show this, it is plainly sufficient to prove that U_m and V_m converge to a common point in $\mathbf{V}$ and to a common point in $\mathbf{J}$.

Since $(y_{i,m})_{m\in\mathbb{N}}$ converges in $\mathbf{V}$, so does $(u'_m u''_m)$ by Fact (b). Since $\mathbf{V}$ is a pseudovariety of completely regular semigroups, $(u'_m u''_m)^{\omega+1} = u'_m u''_m$ in $\overline{\mathrm{F}}_A(\mathbf{V})$. Hence, $(u'_m u''_m)^{m!+1}$ converges to the same limit as $u'_m u''_m$ in $\mathbf{V}$. Therefore, (U_m) and (u_m) converge to the same limit in $\mathbf{V}$, that is, to $p_{\mathbf{V}}(\rho)$; similarly, (V_m) also converges to $p_{\mathbf{V}}(\rho)$ in $\mathbf{V}$.

To show that $(U_m)_{m\in\mathbb{N}}$ and $(V_m)_{m\in\mathbb{N}}$ converge to a common point in $\mathbf{J}$, it is enough to show that $c(u'_m u''_m) = c(v'_m v''_m)$. Using Fact (c) and the equality $c(y_{i,m}) = c(z_{i,m})$, this is indeed true, and we have proved that (U_m) and (V_m) have a common limit ζ in $\mathbf{J} \vee \mathbf{V}$.

To obtain $p_{\mathbf{J}}(\zeta)$, we have replaced in $\pi = p_{\mathbf{J}}(\rho)$ the limit in $\mathbf{J}$ of $u'_m u''_m$ by the limit in $\mathbf{J}$ of $(u'_m u''_m)^\omega$. The limit in $\mathbf{J}$ of $u'_m u''_m$ was of length at least 2, since we started from the canonical factorization; the limit in $\mathbf{J}$ of $(u'_m u''_m)^\omega$ has length 1. Hence $|p_{\mathbf{J}}(\zeta)| < |\pi| = |p_{\mathbf{J}}(\rho)|$, a contradiction of the minimality of $|p_{\mathbf{J}}(\rho)|$. So (2) is impossible, as required. □

To prove the second proposition, we first need a definition. Let $\mathscr{A}$ be an automaton. We say that $\mathfrak{S} = \langle \underline{\mathrm{P}}, \mathfrak{L}, \underline{\mathrm{X}} \rangle$ is a *sequence of stages of length n in* $\mathscr{A}$ if

- $\underline{\mathrm{P}} = ((\overrightarrow{p}_i)_{0\leqslant i\leqslant n}, (p_i)_{1\leqslant i\leqslant n}, (\overleftarrow{p}_i)_{1\leqslant i\leqslant n+1})$ is a finite sequence of states such that $\overrightarrow{p}_0$ is the initial state, $\overleftarrow{p}_{n+1}$ is final, and for all $i \in [1, n]$, there exists a path $\overleftarrow{p}_i \dashrightarrow p_i \dashrightarrow \overrightarrow{p}_i$.
- $\mathfrak{L} = (\mathscr{L}_1, \ldots, \mathscr{L}_n)$ is a finite sequence of acceptable loops such that the first state of $\mathscr{L}_i$ is p_i.
- $\underline{\mathrm{X}} = (x_0, \ldots, x_n)$ is a finite sequence of words of A^* such that $\overrightarrow{p}_i \cdot x_i = \overleftarrow{p}_{i+1}$.

We use the following convention: when the sequence of loops is denoted by $\mathfrak{L}$ (resp. by $\mathfrak{K}$), the corresponding acceptable loops are denoted by $\mathscr{L}_1, \ldots, \mathscr{L}_n$ (resp. by $\mathscr{K}_1, \ldots, \mathscr{K}_n$).

From Lemma 2.1, we immediately deduce the following result.

Lemma 3.4. *Let* $\underline{\mathrm{X}} = (x_0, \ldots, x_n)$ *be a finite sequence of words of* A^*. *In any deterministic finite automaton, there is a finite number of sequences of stages of length n of the form* $\mathfrak{S} = \langle \underline{\mathrm{P}}, \mathfrak{L}, \underline{\mathrm{X}} \rangle$, *and these sequences can be determined.*

If $\mathfrak{S} = \langle \underline{P}, \mathfrak{L}, \underline{X} \rangle$ is given as above, we say that u is $\mathfrak{S}$-*compliant*, and we write $u \in \mathrm{Comp}(\mathfrak{S})$, if u has a factorization $x_0(\overleftarrow{u}_1 u_1 \overrightarrow{u}_1)x_1 \cdots x_{n-1}(\overleftarrow{u}_n u_n \overrightarrow{u}_n)x_n$ that labels the following path in $\mathscr{A}$:

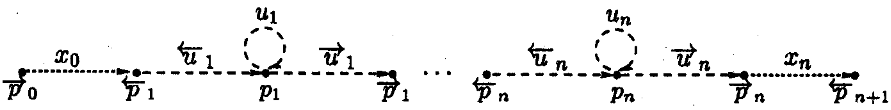

Fig. 2. A $\mathfrak{S}$-compliant labeling.

such that for $i \in [1,n]$, $c(\overleftarrow{u}_i \overrightarrow{u}_i) \subseteq c(u_i) = c(\mathscr{L}_i)$ and such that the loop $p_i \overset{u_i}{\dashrightarrow} p_i$ contains $\mathscr{L}_i$. Observe that any word of $\mathrm{Comp}(\mathfrak{S})$ is accepted by $\mathscr{A}$ by definition.

Lemma 3.5. *Let $\mathscr{A}$ be an automaton recognizing a language K, and let $\mathfrak{S}$ be a sequence of stages of length n in $\mathscr{A}$. Then the language* $\mathrm{Comp}(\mathfrak{S})$ *is rational and we can compute an automaton recognizing it.*

Proof. Let $b_1 \cdots b_k$ be the word labeling $\mathscr{L}_i$ and $B_i = c(\mathscr{L}_i)$. We drop the double indices for clarity. Let $q_0 = p_i$, and $q_j = p_i \cdot b_1 \cdots b_j$ for $j \geqslant 1$. Since $\mathscr{L}_i$ is a loop, $q_k = q_0$. The set of words of content B_i labeling a loop $p_i \dashrightarrow p_i$ which contains $\mathscr{L}_i$ is clearly $L_i = B_i^* \cap L_{\mathscr{A}}(q_0, q_0) b_1 L_{\mathscr{A}}(q_1, q_1) \cdots L_{\mathscr{A}}(q_{k-1}, q_{k-1}) b_k L_{\mathscr{A}}(q_k, q_k)$, and thus is rational. Let $X_i = \{x_i\} \cap L_{\mathscr{A}}(\overrightarrow{p}_i, \overleftarrow{p}_{i+1})$, $\overleftarrow{L}_i = B_i^* \cap L_{\mathscr{A}}(\overleftarrow{p}_i, p_i)$ and finally $\overrightarrow{L}_i = B_i^* \cap L_{\mathscr{A}}(p_i, \overrightarrow{p}_i)$. By definition, we have $\mathrm{Comp}(\mathfrak{S}) = X_0(\overleftarrow{L}_1 L_1 \overrightarrow{L}_1)X_1 \cdots X_{n-1}(\overleftarrow{L}_n L_n \overrightarrow{L}_n)X_n$, which is a rational expression of $\mathrm{Comp}(\mathfrak{S})$. □

Lemma 3.6. *Let $\mathscr{A}$ be an automaton recognizing a language K. Let $\pi = x_0 B_1^\omega x_1 \cdots B_n^\omega x_n$ be canonical on* $\mathbf{J}$. *Then, there is a sequence of words $u_m \in K$ converging to π if and only if there exists a sequence of stages $\mathfrak{S} = \langle \underline{P}, \mathfrak{L}, \underline{X} \rangle$ of length n in $\mathscr{A}$ with $X = \{x_0, \ldots, x_n\}$ and $c(\mathscr{L}_i) = B_i$ such that* $\mathrm{Comp}(\mathfrak{S}) \neq \varnothing$.

Furthermore, if there is a sequence of words $u_m \in K$ converging to π, then it is possible to extract a subsequence of u_m whose elements belong to $\mathrm{Comp}(\mathfrak{S})$.

Proof. Assume that $\mathrm{Comp}(\mathfrak{S})$ is not empty. Then it contains a word of the form

$$x_0(\overleftarrow{u}_1 u_1 \overrightarrow{u}_1)x_1 \cdots x_{n-1}(\overleftarrow{u}_n u_n \overrightarrow{u}_n)x_n.$$

By definition, the words

$$x_0(\overleftarrow{u}_1 u_1^{m!} \overrightarrow{u}_1)x_1 \cdots x_{n-1}(\overleftarrow{u}_n u_n^{m!} \overrightarrow{u}_n)x_n$$

are also in $\mathrm{Comp}(\mathfrak{S})$ and their sequence converges to π, since $(\overleftarrow{u}_i u_i^{m!} \overrightarrow{u}_i)_{m \in \mathbb{N}}$ converges to B_i^ω in $\mathbf{J}$.

Conversely, let $u_m \in K$ be a sequence converging to π. From Corollary 2.5 and Lemma 3.1 we may assume that u_m has a factorization of the form $x_0 y_{1,m} x_1 \cdots x_{n-1} y_{n,m} x_n$ with $y_{k,m} \in \mathrm{C}(B_k, m)$. Since the number of states is finite, there exists a subsequence $u_{m\varphi}$ of u_m such that the states $\overleftarrow{p}_i = q_{\text{ini}}^{\mathscr{A}} \cdot x_0 y_{1,m\varphi} x_1 \cdots y_{i-1,m\varphi} x_{i-1}$ and $\overrightarrow{p}_i = q_{\text{ini}}^{\mathscr{A}} \cdot x_0 y_{1,m\varphi} x_1 \cdots y_{i-1,m\varphi} x_{i-1} y_{i,m\varphi}$ do not depend on m. Since $\mathrm{C}(B_k, m)$ is the set of words that are products of m words of content B_k, we may extract this subsequence so that the path labeled by $y_{i,m\varphi}$ contains a loop $\mathscr{L}_{i,m\varphi}$, say around state p_i. By Lemma 2.2, the loop $\mathscr{L}_{i,m\varphi}$ itself contains an acceptable loop having the same content as $\mathscr{L}_{i,m\varphi}$. After extracting, we may assume that this acceptable loop $\mathscr{L}_i$ does not depend on m, since by Lemma 2.1 there is a finite number of such loops. We have extracted a subsequence of u_m taking its values in $\mathrm{Comp}(\mathfrak{S})$, as required. In particular, $\mathrm{Comp}(\mathfrak{S})$ is not empty. □

Finally we state the last proposition. As seen before, it concludes the proof of the theorem.

Proposition 3.7. *Let $\pi_{\mathbf{J}} \in \overline{\mathrm{F}}_A(\mathbf{J})$ be an implicit operation, and let K and L be two rational languages of A^+. Then it is decidable whether there exists $\pi \in \bar{K} \cap \bar{L}$ such that $p_{\mathbf{J}}(\pi) = \pi_{\mathbf{J}}$.*

Proof. Let $\pi_{\mathbf{J}} = x_0 B_1^{\omega} x_1 \cdots B_n^{\omega} x_n$ and set $\underline{X} = (x_0, \ldots, x_n)$. Assume that there are sequences $u_m \in K$ and $v_m \in L$ converging to $\pi \in \bar{K} \cap \bar{L}$ such that $p_{\mathbf{J}}(\pi) = \pi_{\mathbf{J}}$. Apply Lemma 3.6: there exists a sequence of stages of the form $\mathfrak{S} = \langle \underline{P}, \mathfrak{K}, \underline{X} \rangle$ in $\mathscr{A}$ and another one $\mathfrak{T} = \langle \underline{Q}, \mathfrak{L}, \underline{X} \rangle$ in $\mathscr{B}$ such that $c(\mathscr{K}_i) = c(\mathscr{L}_i) = B_i$ and subsequences of u_m and v_m belonging to $K' = \mathrm{Comp}(\mathfrak{S})$ and $L' = \mathrm{Comp}(\mathfrak{T})$ respectively. Therefore, π belongs to $\bar{K}' \cap \bar{L}'$. Conversely, if π belongs to $\bar{K}' \cap \bar{L}'$, then it belongs also to $\bar{K} \cap \bar{L}$ since $K' \subseteq K$ and $L' \subseteq L$.

There is a finite number of sequences of stages of length n in both $\mathscr{A}$ and $\mathscr{B}$ by Lemma 3.4, and we can compute the associated languages K' and L' by Lemma 3.5. We have to decide whether there exists such a choice of sequences of stages such that $\bar{K}' \cap \bar{L}'$ is not empty. We compute all possible pairs (K', L'), and we shall show that for a given pair, we can decide whether $\bar{K}' \cap \bar{L}'$ is empty. Let $\underline{P} = ((\overrightarrow{p}_i)_{0 \leqslant i \leqslant n}, (p_i)_{1 \leqslant i \leqslant n}, (\overleftarrow{p}_i)_{1 \leqslant i \leqslant n+1})$ and $\underline{Q} = ((\overrightarrow{q}_i)_{0 \leqslant i \leqslant n}, (q_i)_{1 \leqslant i \leqslant n}, (\overleftarrow{q}_i)_{1 \leqslant i \leqslant n+1})$ be the sequences of states of $\mathfrak{S}$ and $\mathfrak{T}$ corresponding to K' and L' respectively.

Since $\mathbf{V}$ is supposed to be 2-strongly decidable, we can decide whether $\bar{K}'_{\mathbf{V}} \cap \bar{L}'_{\mathbf{V}}$ is empty. If so, then $\bar{K}' \cap \bar{L}'$ is also empty. If not, we claim that $\bar{K}' \cap \bar{L}'$ is not empty. Indeed, there are in this case sequences u_m of K' and v_m of L' converging to $\pi_{\mathbf{V}}$ in $\bar{K}'_{\mathbf{V}} \cap \bar{L}'_{\mathbf{V}}$. In view of the definition of a compliant word, we may write u_m and v_m as

$$u_m = x_0 \overleftarrow{y}_{1,m} y_{1,m} \overrightarrow{y}_{1,m} x_1 \cdots x_{n-1} \overleftarrow{y}_{n,m} y_{n,m} \overrightarrow{y}_{n,m} x_n$$
$$v_m = x_0 \overleftarrow{z}_{1,m} z_{1,m} \overrightarrow{z}_{1,m} x_1 \cdots x_{n-1} \overleftarrow{z}_{n,m} z_{n,m} \overrightarrow{z}_{n,m} x_n$$

where $\overleftarrow{y}_{i,m} y_{i,m} \overrightarrow{y}_{i,m}$ (resp. $\overleftarrow{z}_{i,m} z_{i,m} \overrightarrow{z}_{i,m}$) labels the path from $\overleftarrow{p}_i$ to $\overrightarrow{p}_i$ (resp. from $\overleftarrow{q}_i$ to $\overrightarrow{q}_i$).

By construction of $\mathrm{Comp}(\mathfrak{S})$ and $\mathrm{Comp}(\mathfrak{T})$, both $c(y_{i,m})$ and $c(z_{i,m})$ are contained in B_i. Furthermore, $y_{i,m}$ (resp. $z_{i,m}$) labels a loop containing $\mathscr{K}_i$ (resp. $\mathscr{L}_i$), and since $c(\mathcal{K}_i) = c(\mathcal{L}_i) = B_i$, we have $c(y_{i,m}) = c(z_{i,m}) = B_i$ exactly. Let

$$\tilde{u}_m = x_0[\overleftarrow{y}_{1,m}(y_{1,m})^{m!+1}\overrightarrow{y}_{1,m}]x_1 \cdots [\overleftarrow{y}_{n,m}(y_{n,m})^{m!+1}\overrightarrow{y}_{n,m}]x_n$$
$$\tilde{v}_m = x_0[\overleftarrow{z}_{1,m}(z_{1,m})^{m!+1}\overrightarrow{z}_{1,m}]x_1 \cdots [\overleftarrow{z}_{n,m}(z_{n,m})^{m!+1}\overrightarrow{z}_{n,m}]x_n.$$

By definition again, these words stay in K' and L' respectively. (In Fig. 2, reading these words corresponds to iterating the reading of the loops.) Now, the limit in $\overline{\mathrm{F}}_A(\mathbf{V})$ of $\tilde{u}_m$ (resp. of $\tilde{v}_m$) is the same as the limit of u_m (resp. of v_m) in $\overline{\mathrm{F}}_A(\mathbf{V})$. Indeed, $\mathbf{V}$ is a pseudovariety of completely regular semigroups, so for any $u \in A^*$, $u^{m!+1}$ converges to $u^{\omega+1} = u$. On $\mathbf{J}$, each factor $[\overleftarrow{y}_{i,m}(y_{i,m})^{m!+1}\overrightarrow{y}_{i,m}]$ converges to B_i^ω since $c(\overleftarrow{y}_{i,m}\overrightarrow{y}_{i,m}) \subseteq c(y_{i,m}) = B_i$, so $\tilde{u}_m$ converges to $x_0 B_1^\omega x_1 \cdots B_n^\omega x_n$. In the same way, $\tilde{v}_m$ converges to $x_0 B_1^\omega x_1 \cdots B_n^\omega x_n$. So $\tilde{u}_m$ and $\tilde{v}_m$ converge to a common point in $\mathbf{J} \vee \mathbf{V}$, and $\bar{K}' \cap \bar{L}'$ is not empty, as claimed. □

4. Applications

The first corollary of Theorem 1.1 is the answer to Rhodes's question. Take for $\mathbf{V}$ the pseudovariety of finite groups. The fact that it is 2-strongly decidable comes from Ash's proof [8] of the type II conjecture, and was also provided by Ribes and Zalesskiĭ [19].

Corollary 4.1. *The pseudovariety* $\mathbf{J} \vee \mathbf{G}$ *is decidable.*

Recall that Reiterman's theorem gives a characterization of pseudovarieties as classes of finite semigroups defined by sets of pseudoidentities (formal identities whose members are implicit operations). A pseudovariety is finitely based if it can be defined by a finite set of pseudoidentities. Trotter and Volkov [21] have proved that the pseudovariety $\mathbf{J} \vee \mathbf{G}$ is not finitely based. Corollary 4.1 does not provide any basis for $\mathbf{J} \vee \mathbf{G}$. The second author [9] proposed as a possible basis the following set of pseudoidentities, a guess which was repeated by the third author [22]:

$$(xy)^\omega = (yx)^\omega, \qquad x^{\omega+1}y_1^\omega \cdots y_p^\omega x^\omega = x^\omega y_1^\omega \cdots y_p^\omega x^{\omega+1}, \quad p > 0.$$

After a more detailed study of this basis, we now think that the pseudovariety defined by those pseudoidentities strictly contains $\mathbf{J} \vee \mathbf{G}$.

The second author [10] studied the joins of $\mathbf{J}$ with permutative pseudovarieties. In particular, he gave a pseudoidentity basis for the join $\mathbf{J} \vee \mathbf{Ab}$, where $\mathbf{Ab}$ is the pseudovariety of finite abelian groups, which proves that $\mathbf{J} \vee \mathbf{Ab}$ is decidable. Theorem 1.1 may also take advantage of a result due to Delgado [11], which states that $\mathbf{Ab}$ is also strongly decidable. We thus get the following corollary:

Corollary 4.2. *The pseudovariety* $\mathbf{J} \vee \mathbf{Ab}$ *is decidable.*

Let us conclude with a final application. The first author proved that the pseudovariety $\mathbf{CS}$ of completely simple semigroups is hyperdecidable. This is also a consequence of a more general theorem proved by Silva and the first author [6]. Since a completely simple semigroup is completely regular, we have:

Corollary 4.3. *The pseudovariety* $\mathbf{J} \vee \mathbf{CS}$ *is decidable.*

Finally, note that Theorem 1.1 could be stated by replacing 2-strong decidability by n-strong decidability. Since the arguments are essentially the same, we chose to formulate them only for 2-strong decidability.

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