

# A nilpotent-generated semigroup associated with a semigroup of full transformations

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## Synopsis

Let  $X$  be a set with infinite regular cardinality  $\mathbf{m}$  and let  $\mathcal{T}(X)$  be the semigroup of all self-maps of  $X$ . The semigroup  $Q_{\mathbf{m}}$  of 'balanced' elements of  $\mathcal{T}(X)$  plays an important role in the study by Howie [3, 5, 6] of idempotent-generated subsemigroups of  $\mathcal{T}(X)$ , as does the subset  $S_{\mathbf{m}}$  of 'stable' elements, which is a subsemigroup of  $Q_{\mathbf{m}}$  if and only if  $\mathbf{m}$  is a regular cardinal. The principal factor  $P_{\mathbf{m}}$  of  $Q_{\mathbf{m}}$ , corresponding to the maximum  $\mathcal{J}$ -class  $J_{\mathbf{m}}$ , contains  $S_{\mathbf{m}}$  and has been shown in [7] to have a number of interesting properties.

Let  $N_2$  be the set of all nilpotent elements of index 2 in  $P_{\mathbf{m}}$ . Then the subsemigroup  $\langle N_2 \rangle$  of  $P_{\mathbf{m}}$  generated by  $N_2$  consists exactly of the elements in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . Moreover  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  has 2-nilpotent-depth 3, in the sense that  $N_2 \cup N_2^2 \subset P_{\mathbf{m}} \setminus S_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3$ .

## 1. Introduction and background

Let  $X$  be a set with infinite cardinality  $\mathbf{m}$  and let  $\mathcal{T}(X)$  be the full transformation semigroup of  $X$ . (For this and other undefined semigroup notions see [4].) In his study of idempotent-generated subsemigroups of  $\mathcal{T}(X)$ , Howie [3, 5] made use of the following subsets associated with an element  $\alpha$  of  $\mathcal{T}(X)$ :

$$S(\alpha) = \{x \in X : x\alpha \neq x\}, \quad Z(\alpha) = X \setminus X\alpha,$$
$$C(\alpha) = \bigcup \{y\alpha^{-1} : y \in X\alpha \text{ and } |y\alpha^{-1}| \geq 2\}.$$

The cardinal numbers  $|S(\alpha)|$ ,  $|Z(\alpha)|$  and  $|C(\alpha)|$  are called, respectively, the *shift*, the *defect* and the *collapse* of  $\alpha$ , and the subsemigroup

$$Q_{\mathbf{m}} = \{\alpha \in T(X) : |S(\alpha)| = |Z(\alpha)| = |C(\alpha)| = \mathbf{m}\}$$

of *balanced* elements of weight  $\mathbf{m}$  is an important part of the subsemigroup of  $\mathcal{T}(X)$  generated by the idempotents.

The principal factor  $P_{\mathbf{m}}$  associated with the maximum  $\mathcal{J}$ -class  $J_{\mathbf{m}}$  in  $Q_{\mathbf{m}}$  is the Rees quotient  $Q_{\mathbf{m}}/I_{\mathbf{m}}$ , where  $I_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| < \mathbf{m}\}$ . It is usually convenient to think of it as  $J_{\mathbf{m}} \cup \{0\}$ , where  $J_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| = \mathbf{m}\}$ , and where the product of two elements of  $J_{\mathbf{m}}$  is taken to be zero if it falls in  $I_{\mathbf{m}}$ .

The importance of cardinals in semigroup theory was demonstrated by Preston

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[9] when he showed that the set

$$M = \{\alpha \in \mathcal{T}(X) : |X\alpha| = \mathbf{m} \text{ and } (\forall y \in X\alpha) |y\alpha^{-1}| < \mathbf{m}\}$$

is a subsemigroup of  $\mathcal{T}(X)$  if and only if  $\mathbf{m}$  is a *regular* cardinal, i.e. if and only if  $|\Lambda| < \mathbf{m}$  and  $\mathbf{m}_\lambda < \mathbf{m}$  for all  $\lambda$  in  $\Lambda$  together imply that

$$\sum_{\lambda \in \Lambda} \mathbf{m}_\lambda < \mathbf{m}.$$

The same equivalence holds good for the set  $S_{\mathbf{m}} = M \cap Q_{\mathbf{m}}$  considered in [6] by Howie, who also showed that, in the case where  $\mathbf{m}$  is regular,  $S_{\mathbf{m}}$  is a regular, bisimple, idempotent-generated semigroup.

In an analogous manner Marques [7] showed that  $P_{\mathbf{m}}$  is a regular, 0-bisimple, idempotent-generated semigroup.

The semigroup  $P_{\mathbf{m}}$  has a zero element and is easily seen to contain nilpotent elements. If  $\mathbf{m}$  is regular then (see Theorem 2.17) the subsemigroup  $\langle N \rangle$  generated by the nilpotent elements is  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . In fact one requires only the set

$$N_2 = \{\alpha \in P_{\mathbf{m}} : \alpha^2 = 0, \alpha \in 0\}$$

of nilpotents of index 2, and

$$N_2 \cup N_2^2 \subset P_{\mathbf{m}} \setminus S_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3.$$

## 2. Regular cardinals

Throughout this section  $X$  will be a set with infinite cardinal  $\mathbf{m}$ , and  $\mathbf{m}$  will be regular. Hence, as remarked in the last section, the set

$$S_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| = \mathbf{m} \text{ and } (\forall y \in X\alpha) |y\alpha^{-1}| < \mathbf{m}\} \quad (2.1)$$

is a subsemigroup of  $P_{\mathbf{m}} = J_{\mathbf{m}} \cup \{0\}$ , where

$$J_{\mathbf{m}} = \{\alpha \in Q_{\mathbf{m}} : |X\alpha| = \mathbf{m}\}. \quad (2.2)$$

We also have the following proposition:

PROPOSITION 2.1. *The set  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  is a regular subsemigroup of  $P_{\mathbf{m}}$ .*

*Proof.* Let  $\alpha \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . Then the equivalence relation  $\ker \alpha (= \{(x, y) : x\alpha = y\alpha\})$  on  $X$  has at least one class  $y\alpha^{-1}$  ( $y \in X\alpha$ ) with cardinality  $\mathbf{m}$ . For all  $\beta$  in  $P_{\mathbf{m}}$  we have  $\ker(\alpha\beta) \supseteq \ker \alpha$  and so certainly  $\ker(\alpha\beta)$  has at least one class of cardinality  $\mathbf{m}$ . Thus  $\alpha\beta$  is in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . Notice that we have actually shown that  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  is a right ideal of  $P_{\mathbf{m}}$ . It is not hard to see that the left ideal property does not hold.

To see that the semigroup is regular, consider the  $\mathbf{m}$  sets  $y\alpha^{-1}$ , where  $y \in X\alpha$ , and choose an element  $x_y$  in each  $y\alpha^{-1}$ . Define  $\xi \in \mathcal{T}(X)$  by the rule that

$$\begin{aligned} y\xi &= x_y & (y \in X\alpha), \\ x\xi &= x_0, & (x \in Z(\alpha)), \end{aligned}$$

where  $x_0$  is some arbitrarily chosen fixed element of  $X$ . Then  $\alpha\xi\alpha = \alpha$ . Moreover,

$$S(\xi) \supseteq Z(\alpha) \setminus \{x_0\}, \quad C(\xi) \supseteq Z(\alpha), \quad X\xi \supseteq \{x_y : y \in X\alpha\},$$

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from which it follows that  $|S(\xi)| = |C(\xi)| = |X\xi| = \mathbf{m}$ . Also, at least one of the sets  $y\alpha^{-1}$  is of cardinality  $\mathbf{m}$ ; hence  $Z(\xi)$  contains  $y\alpha^{-1} \setminus \{x_y\}$  and so  $|Z(\xi)| = \mathbf{m}$ . Finally,  $x_0\xi^{-1} = Z(\alpha)$ , of cardinality  $\mathbf{m}$ , and we conclude that  $\xi \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ .

We know that  $S_{\mathbf{m}}$  is a subsemigroup of  $P_{\mathbf{m}}$  not containing the zero element of  $P_{\mathbf{m}}$ . It follows that no element  $\alpha$  of  $S_{\mathbf{m}}$  can be nilpotent, since  $\alpha^k \in S_{\mathbf{m}}$  for every  $k$  and so cannot be zero. Thus  $N$ , the set of nilpotents in  $P_{\mathbf{m}}$ , is contained in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ , and it now follows from Proposition 2.1 that  $\langle N \rangle$ , the smallest subsemigroup of  $P_{\mathbf{m}}$  containing  $N$ , is also contained in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ ; that is

$$\langle N \rangle \subseteq P_{\mathbf{m}} \setminus S_{\mathbf{m}}. \quad (2.3)$$

We now aim to show that  $P_{\mathbf{m}} \setminus S_{\mathbf{m}} \subseteq \langle N_2 \rangle$  ( $\subseteq \langle N \rangle$ ), from which our first main result will follow. First, we give an alternative characterisation of the set

$$N_2 = \{\eta \in Q_{\mathbf{m}}: |X\eta| = \mathbf{m} \text{ and } |X\eta^2| < \mathbf{m}\}.$$

PROPOSITION 2.2. *Let  $\alpha$  be a non-zero element of  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ , and let*

$$A = \cup \{y\alpha^{-1}: y \in X\alpha, |y\alpha^{-1}| < \mathbf{m}\}.$$

*Then  $\alpha \in N_2$  if and only if (i)  $|X\alpha \cap A| < \mathbf{m}$  and (ii)  $|\{y \in X\alpha: |y\alpha^{-1}| = \mathbf{m} \text{ and } X\alpha \cap y\alpha^{-1} \neq \emptyset\}| < \mathbf{m}$ .*

*Proof.* Let

$$B = \cup \{y\alpha^{-1}: y \in X\alpha, |y\alpha^{-1}| = \mathbf{m}\}. \quad (2.4)$$

Since  $X$  is the disjoint union of  $A$  and  $B$ , it follows that

$$X\alpha = (X\alpha \cap A) \cup (X\alpha \cap B), \quad X\alpha^2 = (X\alpha \cap A)\alpha \cup (X\alpha \cap B)\alpha. \quad (2.5)$$

Suppose now that  $\alpha$  satisfies (i) and (ii).

From (i) it follows that  $|(X\alpha \cap A)\alpha| < \mathbf{m}$ . Also

$$\begin{aligned} y \in (X\alpha \cap B)\alpha &\Rightarrow y\alpha^{-1} \cap X\alpha \cap B \neq \emptyset \\ &\Rightarrow |y\alpha^{-1}| = \mathbf{m} \text{ and } y\alpha^{-1} \cap X\alpha \neq \emptyset, \end{aligned} \quad (2.6)$$

and so from (ii) it follows that  $|(X\alpha \cap B)\alpha| < \mathbf{m}$ . Hence  $|X\alpha^2| < \mathbf{m}$  and so  $\alpha \in N_2$  as required.

Conversely, let  $\alpha \in N_2$ . By (2.5) it follows that both  $|(X\alpha \cap A)\alpha|$  and  $|(X\alpha \cap B)\alpha|$  are less than  $\mathbf{m}$ . Now

$$\begin{aligned} y \in (X\alpha \cap A)\alpha &\Rightarrow y\alpha^{-1} \cap X\alpha \cap A \neq \emptyset \\ &\Rightarrow |y\alpha^{-1}| < \mathbf{m}. \end{aligned}$$

Hence by the regularity of  $\mathbf{m}$  the set

$$X\alpha \cap A \subseteq \cup \{y\alpha^{-1}: y \in (X\alpha \cap A)\alpha\}$$

has cardinality less than  $\mathbf{m}$ . To show (ii) we simply observe that the sequence of implications (2.6) is reversible.

Notice that from this proof it follows that for every  $\alpha$  in  $N_2$

$$|X\alpha \cap B| = \mathbf{m}. \quad (2.7)$$

Also,

$$(X\alpha \cap B)\alpha = \{y \in X\alpha: |y\alpha^{-1}| = \mathbf{m} \text{ and } X\alpha \cap y\alpha^{-1} \neq \emptyset\}.$$

Notice now that for each non-zero  $\alpha$  in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  the set  $Y(\alpha)$  defined by

$$Y(\alpha) = \{y \in X\alpha : |y\alpha^{-1}| = \mathbf{m}\} \quad (2.8)$$

is non-empty.

**PROPOSITION 2.3.** *Let  $\alpha$  be a non-zero element of  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . If there exists  $y$  in  $Y(\alpha)$  such that  $|X\alpha \cap y\alpha^{-1}| < \mathbf{m}$  then  $\alpha \in N_2^2$ .*

*Proof.* Suppose that  $|X\alpha \cap y_0\alpha^{-1}| < \mathbf{m}$ , where  $y_0 \in Y(\alpha)$ . Let  $\theta$  be a bijection from  $X\alpha$  onto  $y_0\alpha^{-1}$  and define  $\eta_1 = \alpha\theta$ . Then  $\ker \eta_1 = \ker \alpha$  and  $X\eta_1 = y_0\alpha^{-1}$ , from which it follows that  $\eta_1 \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . Also  $X\eta_1^2 = \{y_0\theta\}$  and so certainly  $\eta_1 \in N_2$ .

Next, if  $x \in X\eta_1 = y_0\alpha^{-1}$ , we define

$$x\eta_2 = x\theta^{-1} (\in X\alpha).$$

Otherwise we define  $x\eta_2 = a$ , where  $a$  is some arbitrarily chosen element in  $X\alpha$ . Then  $\eta_2 \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ . Also

$$\begin{aligned} X\eta_2^2 &= (X\alpha)\eta_2 \\ &= (X\alpha \cap y_0\alpha^{-1})\eta_2 \cup (X\alpha \setminus y_0\alpha^{-1})\eta_2 \\ &= (X\alpha \cap y_0\alpha^{-1})\eta_2 \cup \{a\} \end{aligned}$$

and so  $|X\eta_2^2| < \mathbf{m}$ . Thus  $\eta_2 \in N_2$ .

It is now clear that  $\eta_1\eta_2 = \alpha$ , and so  $\alpha \in N_2^2$  as required.

The converse of Proposition 2.3 is not true, as the next proposition makes clear.

**PROPOSITION 2.4.** *Let  $\alpha$  be a non-zero element of  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  and suppose that  $|X\alpha \cap y\alpha^{-1}| = \mathbf{m}$  for all  $y$  in  $Y(\alpha)$ . Then  $\alpha \in N_2^2$  if and only if there exists  $z$  in  $Y(\alpha)$  for which  $|z\alpha^{-1} \setminus X\alpha| = \mathbf{m}$ .*

*Proof.* Suppose first that there is an element  $z_0$  in  $Y(\alpha)$  such that  $|z_0\alpha^{-1} \setminus X\alpha| = \mathbf{m}$ . We may write  $z_0\alpha^{-1} \setminus X\alpha$  as a disjoint union  $A \cup B$ , where  $|A| = |B| = \mathbf{m}$ . Let  $\theta$  be a bijection from  $X\alpha$  onto  $A$ , and define  $\eta_i = \alpha\theta$ . Then  $\eta_i \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ ,  $|X\eta_i| = |A| = \mathbf{m}$ ,  $X\eta_i^2 = \{z_0\theta\}$ ; hence  $\eta_i \in N_2$ .

Then by analogy with Proposition 2.3 we define  $\eta_2: X \rightarrow X$  by

$$x\eta_2 = \begin{cases} x\theta^{-1} & (x \in A) \\ t & (x \notin A) \end{cases}$$

where  $t$  is an arbitrarily chosen element of  $X\alpha$ . Then  $\eta_2 \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  and  $X\eta_2^2 = \{t\}$ . Hence  $\eta_2 \in N_2$ , and it is clear that  $\eta_1\eta_2 = \alpha$ .

To prove the converse implication, we need the following simple lemma concerning regular semigroups with zero:

**LEMMA 2.5.** *Let  $S$  be a regular semigroup with zero. Let  $a = zb$ , where  $a, b \neq 0$  and  $z$  is a nilpotent of index 2. Then there exists a nilpotent  $z_1$  of index 2 in  $S$  such that  $a = z_1b$  and  $z_1\mathcal{R}a$ .*

*Proof.* Let  $z_1 = aa'z$ , where  $a'$  is an inverse of  $a$ . Then

$$z_1b = aa'zb = aa'a = a,$$

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and it is clear that  $z_1 \mathcal{R} a$ . Also

$$z_1^2 = aa' zaa' z = aa' z^2 ba' z = 0.$$

We now return to the proof of Proposition 2.4. Let  $\alpha = \eta_1 \eta_2 \in N_2^2$ , with  $|X\alpha \cap y\alpha^{-1}| = \mathbf{m}$  for all  $y$  in  $Y(\alpha)$ . By the lemma we may assume that  $\ker \eta_1 = \ker \alpha$ . Now assume by way of contradiction that  $|z\alpha^{-1} \setminus X\alpha| < \mathbf{m}$  for every  $z$  in  $Y(\alpha)$ .

Consider the set

$$D = \{y \in X\eta_1 : |y\eta_1^{-1}| = \mathbf{m} \text{ and } X\eta_1 \cap y\eta_1^{-1} \neq \emptyset\}.$$

By Proposition 2.2 this set has cardinality less than  $\mathbf{m}$ . On the other hand, in the notation of (2.4),

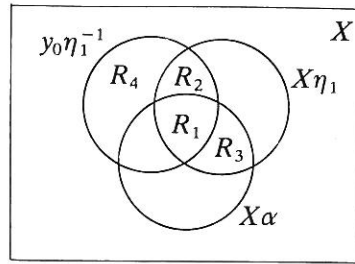
$$\bigcup \{X\eta_1 \cap y\eta_1^{-1} : y \in D\} = X\eta_1 \cap B$$

and so, by (2.7), has cardinality  $\mathbf{m}$ . Hence, by the regularity of  $\mathbf{m}$ ,

$$|X\eta_1 \cap y_0\eta_1^{-1}| = \mathbf{m}$$

for some  $y_0$  in  $D$ .

We illustrate the situation in the Venn diagram below:



We have just established that  $|R_1 \cup R_2| = \mathbf{m}$  and we are assuming that  $R_2 \cup R_4$  ( $= (y_0\eta_2)\alpha^{-1} \setminus X\alpha$ ) has cardinality less than  $\mathbf{m}$ . Hence  $|R_2| < \mathbf{m}$ . Hence certainly  $|R_1 \cup R_3| = \mathbf{m}$ ; that is,

$$|X\alpha \cap X\eta_1| = \mathbf{m}.$$

Now, from the fact that  $\ker \eta_1 = \ker \alpha$  it follows that  $\eta_2$  must map  $X\eta_1$  in a one-one fashion. Hence

$$|(X\alpha \cap X\eta_1)\eta_2| = \mathbf{m}.$$

But  $X\eta_2 \supseteq X\alpha$  and so  $X\eta_2^2 \supseteq X\alpha\eta_2 \supseteq (X\alpha \cap X\alpha \cap X\eta_1)\eta_2$ . It follows that  $|X\eta_2^2| = \mathbf{m}$ , contrary to the assumption that  $\eta_2 \in N_2$ .

This completes the proof of Proposition 2.4.

Propositions 2.3 and 2.4 give a characterisation of those elements of  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  that can be expressed as products of 2 nilpotents of index 2. The remaining elements can be expressed as products of length 3:

**PROPOSITION 2.6.** *Let  $\alpha$  be a non-zero element in  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ , and suppose that  $|X\alpha \cap y\alpha^{-1}| = \mathbf{m}$  and  $|y\alpha^{-1} \setminus X\alpha| < \mathbf{m}$  for all  $y$  in  $Y(\alpha)$ . Then there exist  $\eta_1, \eta_2, \eta_3$  in  $N_2$  such that  $\alpha = \eta_1 \eta_2 \eta_3$ .*

*Proof.* Choose  $y_0$  in  $Y(\alpha)$  and express  $y_0\alpha^{-1}$  as a disjoint union  $A \cup B$ , where  $|A| = |B| = \mathbf{m}$ . Let  $\theta: X\alpha \rightarrow A$  be a bijection, and define  $\eta_1: X \rightarrow X$  by  $\eta_1 = \alpha\theta$ . Then  $\eta_1 \in N_2$ .

Next, let

$$P = Z(\alpha) \setminus y_0\alpha^{-1}, \quad Q = Z(\alpha) \cap y_0\alpha^{-1} \quad (= y_0\alpha^{-1} \setminus X\alpha);$$

then  $Z(\alpha) = P \cup Q$ ,  $|P| = \mathbf{m}$ ,  $|Q| < \mathbf{m}$ . Define  $\eta_2$  as a bijection from  $X\eta_1$  onto  $P$ , and complete the definition by specifying that

$$(X \setminus X\eta_1)\eta_2 = a,$$

where  $a$  is arbitrarily chosen in  $P$ . Then  $\eta_2 \in N_2$ . In fact  $X\eta_2^2 = P\eta_2 = \{a\}$ , since

$$P \subseteq X \setminus y_0\alpha^{-1} \subseteq X \setminus A = X \setminus X\eta_1.$$

Finally, define  $\eta_3$  for each  $x$  in  $X\eta_1\eta_2$  by

$$x\eta_3 = x\eta_2^{-1}\theta^{-1}$$

and complete the definition by putting

$$(X \setminus X\eta_1\eta_2)\eta_3 = b,$$

where  $b$  is chosen arbitrarily in  $X\alpha$ . Then

$$X\eta_3 = X\alpha \subseteq X \setminus P = X \setminus X\eta_1\eta_2$$

and so  $X\eta_3^2 = \{b\}$ . Since it is clear that  $\eta_3 \in P_{\mathbf{m}} \setminus S_{\mathbf{m}}$ , we thus have  $\eta_3 \in N_2$ . It is easily seen that  $\eta_1\eta_2\eta_3 = \alpha$ .

We have now shown that every non-zero element of  $P_{\mathbf{m}} \setminus S_{\mathbf{m}}$  belongs to  $\langle N_2 \rangle$ . It is evident that  $0 = \eta^2 \in N_2^2 \subseteq \langle N_2 \rangle$ , where  $\eta$  is any element of  $N_2$ . Thus we have the following result:

**THEOREM 2.7.** *Let  $X$  be a set with infinite cardinality  $\mathbf{m}$ , where  $\mathbf{m}$  is regular, and let  $P_{\mathbf{m}}$ ,  $S_{\mathbf{m}}$  be defined by equations (2.1) and (2.2). Let  $N$  be the set of nilpotents in  $P_{\mathbf{m}}$ , and let  $N_2$  be the set of nilpotents of index 2. Then  $\langle N \rangle$ , the subsemigroup of  $P_{\mathbf{m}}$  generated by  $N$ , is equal to  $\bar{T}$ . Moreover,*

$$N_2 \cup N_2^2 \subset P_{\mathbf{m}} \setminus S_{\mathbf{m}} = N_2 \cup N_2^2 \cup N_2^3.$$

*Remark.* It is not hard to see that the restriction of regularity is necessary. If  $\mathbf{m}$  is irregular, then the subset  $S_{\mathbf{m}}$  (no longer a semigroup) contains nilpotent elements. To see this, write  $X = A \cup B$ , where  $|A| = |B| = \mathbf{m}$  and  $A \cap B = \emptyset$ , and suppose that

$$A = \bigcup \{A_\lambda : \lambda \in \Lambda\},$$

where  $|\Lambda| < \mathbf{m}$  and  $|A_\lambda| < \mathbf{m}$  for all  $\lambda$ . Let  $\theta: \Lambda \cup B \rightarrow A$  be a bijection and define  $\eta: X \rightarrow X$  by

$$x\eta = \begin{cases} \lambda\theta & \text{if } x \in A_\lambda \quad (\lambda \in \Lambda) \\ x\theta & \text{if } x \in B \end{cases}.$$

Then the  $(\ker \eta)$ -classes are  $A_\lambda$  ( $\lambda \in \Lambda$ ) and  $\{b\}$  ( $b \in B$ ) and so  $\eta \in S_{\mathbf{m}}$ . However  $X\eta^2 = \Lambda\theta$ , of cardinality less than  $\mathbf{m}$ , and so  $\eta$  is a nilpotent of index 2 in  $P_{\mathbf{m}}$ .

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