Free profinite locally idempotent and locally commutative semigroups[1]

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Abstract

This paper is concerned with the structure of semigroups of implicit operations on the pseudovariety $\mathcal{L}\mathbf{Sl}$ of finite locally idempotent and locally commutative semigroups. We depart from a general result of Almeida and Weil to give two descriptions of these semigroups: the first in terms of infinite words, and the second in terms of infinite and bi-infinite words. We then derive some applications.

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1 Introduction

Recall that a *locally testable* language is a language that is a Boolean combination of languages of the form wA^*, A^*w and A^*wA^*, where A is a finite alphabet and $w \in A^+$. It is well known, as it was shown independently by Brzozowski and Simon [7] and McNaughton [16], that a language is locally testable if and only if its syntactic semigroup belongs to $\mathcal{L}\mathbf{Sl}$. A result of Medvedev [17] says that each rational language is an homomorphic image of a locally testable language. Moreover, a result of Chomsky and Schützenberger [8] shows that every context-free language can be characterized by *local* languages (which form a subclass of locally testable languages). Thus, because of its importance, the locally testable languages and the pseudovariety $\mathcal{L}\mathbf{Sl}$ have been very studied. We cite also, for instance, the work of authors like Kim, McCloskey, Trahtman and Zalcstein [13, 14, 25, 27, 28]. The work presented in this paper throws a new light on the subject.

A well known result of Reiterman [21] shows that pseudovarieties are defined by pseudoidentities, that is, by formal equalities between implicit operations. An implicit operation on a pseudovariety $\mathbf{V}$ over an alphabet A, is an element of a certain topological semigroup, called the free pro-$\mathbf{V}$ semigroup over A, denoted $\hat{F}_A(\mathbf{V})$, which plays the part of the free object of $\mathbf{V}$ over A. The theory of implicit operations proved to be an important tool in the study of pseudovarieties of semigroups and of the corresponding varieties of recognizable languages. It has been highly developed in the last few years principally by the work of Almeida [2, etc], but it has won many adepts as Azevedo, Selmi, Teixeira, Trotter, Volkov, Weil, Zeitoun, the author and others [4, 5, 6, 9, 10, 11, 23, 24, 26, 29].

The subpseudovarieties $\mathbf{V}$ of $\mathcal{D}\mathbf{S}$, the pseudovariety of all finite semigroups in which all regular elements lie in groups, enjoy good properties of factorization of their implicit

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operations. In fact, for those pseudovarieties, the semigroups $\hat{F}_A(\mathbf{V})$ have the *property of factorization*, i.e., each implicit operation on $\mathbf{V}$ can be factorized as a finite product of component projections and regular elements. This property of factorization combined with the characterization of the regular elements, which is known in many cases, has permitted to describe canonical factorizations for the elements of certain of those semigroups $\hat{F}_A(\mathbf{V})$. This is the case, for instance and restricting ourselves to aperiodic pseudovarieties, of $\mathbf{J}$ (Almeida [1]), $\mathbf{J} \cap \mathcal{L}\mathbf{Sl}$ (Selmi [23]), $\mathbf{R}$ (Almeida and Weil [5]), $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{J}$, $\mathbf{R} \cap \mathcal{L}\mathbf{J}$, $\mathcal{D}\mathbf{A} \cap \mathcal{L}\mathbf{Sl}$ and $\mathbf{R} \cap \mathcal{L}\mathbf{Sl}$ (Costa [10, 11]), where $\mathbf{J}$, $\mathbf{R}$, $\mathcal{L}\mathbf{J}$ and $\mathcal{D}\mathbf{A}$ are, respectively, the pseudovarieties of $\mathcal{J}$-trivial semigroups, of $\mathcal{R}$-trivial semigroups, of semigroups locally in $\mathbf{J}$ and of semigroups in which all regular elements are idempotents.

This paper is devoted to the study of implicit operations on $\mathcal{L}\mathbf{Sl}$. The pseudovariety $\mathcal{L}\mathbf{Sl}$ is not a subpseudovariety of $\mathcal{D}\mathbf{S}$, and the semigroups $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ do not satisfy the property of factorization. In fact, Proposition 12.3.1 in [2] provides an implicit operation on $\mathcal{L}\mathbf{Sl}$ which cannot be written as a finite product of component projections and regular elements. Because of this particularity, our study constitutes one of the very few examples until the moment which does not concentrate on the lattice of subpseudovarieties of $\mathcal{D}\mathbf{S}$.

Recall that $\mathcal{L}\mathbf{Sl} = \mathbf{Sl} * \mathbf{D}$, where $*$ denotes the operation of semidirect product of pseudovarieties of semigroups and $\mathbf{Sl}$ and $\mathbf{D}$ are, respectively, the pseudovarieties of semilattices (i.e. idempotent and commutative semigroups) and of semigroups S such that $Se = e$ for each idempotent e of S. Almeida and Weil [4] have obtained a general result (Theorem 2.4 below) which gives a description of the implicit operations on semidirect products of the form $\mathbf{Sl}*\mathbf{V}$, in terms of certain subgraphs of the Cayley graph of $\hat{F}_A(\mathbf{V})^1$. This result has permitted them, in particular, to describe the pseudovariety $\mathbf{Sl}*\mathbf{K}$ (which, in particular, they showed is equal to $\mathbf{Sl}*\mathbf{N}$) and its implicit operations, where $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$ and $\mathbf{K}$ is the pseudovariety of semigroups S such that $eS = e$ for each idempotent e of S. Since the pseudovariety $\mathbf{D}$ is the dual of $\mathbf{K}$, one might expect that a similar study would be possible to give the characterization of the semigroups $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. However, this is not the case since the graphs described in Theorem 2.4 are very simple for $\mathbf{K}$ and very complex for $\mathbf{D}$. Our work consists basically in the study of the graphs of Theorem 2.4 in the case $\mathbf{V} = \mathbf{D}$. We use the result of Almeida and Weil to give two alternative descriptions of the semigroups $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. The first in terms of infinite words, and the second in terms of bi-infinite words.

As a consequence of this work, we are able to give some useful information about $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. We give, in particular, a description of its idempotents and its regular elements. We show, furthermore, that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is "very large", more precisely we show that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ has uncountably many $\leq_{\mathcal{J}}$-incomparable elements. Moreover, we show that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is "very high", i.e., we show that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ has an uncountable ascending chain of $\mathcal{J}$-classes. This last fact reinforces our knowledge that the elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ do not enjoy good properties of factorization, and shows that the semigroups $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ are very complex.

As a particular case of our study, we will be interested in the subsemigroup of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ consisting of ω-words, i.e., elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ which are obtained by superposition of component projections and the unary operation $x \mapsto x^\omega$. We show that we can write each ω-word of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ in a certain form such that we can decide when two such forms represent the same element. Thus, the word problem for the subsemigroup of ω-words of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is decidable. We note that Almeida and Steinberg [3] have shown recently that, for many pseudovarieties $\mathbf{V}$, the subsemigroup of ω-words of $\hat{F}_A(\mathbf{V})$ constitutes a very important part of the semigroup $\hat{F}_A(\mathbf{V})$. In particular, the word problem for that subsemigroup is closely related to the hyperdecidability of the pseudovariety $\mathbf{V}$.

This paper is organized as follows. In section 2, we recall the main definitions and properties that we shall need in the sequel. In Section 3, we give two descriptions of the structure of the semigroups of implicit operations on $\mathcal{L}\mathbf{Sl}$. Sections 4 and 5 are devoted, respectively, to the characterization of the regular elements and Green's relations of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. In section 6, we show that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ admits an uncountable ascending chain of $\mathcal{J}$-classes. Finally, section 7 is devoted to the characterization of the subsemigroup of ω-words of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

2 Preliminaries

We begin by presenting basic definitions and notation concerning words. Next we review the main definitions and facts concerning free profinite semigroups. We then summarize some properties of the implicit operations on some subpseudovarieties of $\mathcal{L}\mathbf{Sl}$. We conclude by recalling the result of Almeida and Weil on the implicit operations on semidirect products of the form $\mathbf{Sl} * \mathbf{V}$.

2.1 Words

Let A be a finite non empty set, or *alphabet*. The elements of A are called *letters* and those of A^*, the free monoid on A, (finite) *words*. The identity of A^* is called the *empty word* and is denoted by 1. If $u = a_1 \cdots a_n$ ($a_i \in A$) is a word of A^+, the free semigroup on A, the number n is called the *length* of u and is denoted by $|u|$. The length of the empty word is 0. Moreover, if a is a letter of A, we denote by $|u|_a$ the number of occurrences of a in u.

We denote by $A^{\mathbb{N}}$ (resp. $A^{-\mathbb{N}}$, $A^{\mathbb{Z}}$) the set of sequences of letters of A indexed by $\mathbb{N}$ (resp. $-\mathbb{N}$, $\mathbb{Z}$). For each $n \in \mathbb{Z}$, we define a *shift* operator σ^n on $A^{\mathbb{Z}}$ by setting, for each $u = (u_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}$,

$$\sigma^n(u) = (u_{i+n})_{i\in\mathbb{Z}}.$$

Now, let $\sim$ be the equivalence on $A^{\mathbb{Z}}$ given by

$$u \sim v \quad \text{if and only if} \quad \exists n \in \mathbb{Z},\ v = \sigma^n(u).$$

The quotient set $A^{\mathbb{Z}}/\!\sim$ is represented by $A^{\tilde{\mathbb{Z}}}$. The elements of $A^{\tilde{\mathbb{Z}}}$ (resp. $A^{\mathbb{N}}$, $A^{-\mathbb{N}}$) are called *bi-infinite* (resp. *right-infinite*, *left-infinite*) words.

We denote by $u^{+\infty}$ (resp. $u^{-\infty}$) the right-infinite (resp. left-infinite) word obtained by repeating infinitely often the word $u \in A^+$. When it makes sense, we define *product* (of concatenation) of two words (possibly of different type) simply by juxtaposition. For instance, if $A = \{a, b\}$, the product of the left-infinite word $a^{-\infty}$ by the right-infinite word $ba^{+\infty}$ is

$$a^{-\infty}ba^{+\infty}$$

which represents the bi-infinite word over A which contains exactly one occurrence of the letter b. Note that the product of $ba^{+\infty}$ by $a^{-\infty}$ is not defined. If $u \in A^+$ we will usually denote by u^{∞} the bi-infinite word $u^{-\infty}u^{+\infty}$.

A word $w \in A^{\mathbb{N}}$ (resp. $w \in A^{-\mathbb{N}}$, $w \in A^{\tilde{\mathbb{Z}}}$) is said *ultimately periodic* if

$$w = yz^{+\infty} \text{ (resp. } w = x^{-\infty}y, w = x^{-\infty}yz^{+\infty})$$

for some $x, z \in A^+$ and $y \in A^*$. An ultimately periodic word $w \in A^{\mathbb{N}}$ (resp. $w \in A^{-\mathbb{N}}$, $w \in A^{\tilde{\mathbb{Z}}}$) which can be written in the form

$$w = x^{+\infty} \text{ (resp. } w = x^{-\infty}, w = x^{\infty})$$

for some $x \in A^+$, is said *periodic*.

A word $u \in A^*$ is a *prefix* (resp. *suffix*, *factor*) of a word x if there exist words y and z such that $x = uy$ (resp. $x = yu$, $x = yuz$). For each integer k we denote by $p_k(x)$ (resp. $s_k(x)$, $\mathrm{Fact}_k(x)$) the prefix (resp. suffix, set of factors) of x of length k, if it exists. For a finite or right-infinite (resp. left-infinite) word x we will denote by $\mathrm{Pref}(x)$ (resp. $\mathrm{Suff}(x)$) the set of all prefixes (resp. suffixes) of x.

We now introduce some notations for infinite and bi-infinite words which we will use frequently in the sequel. Let B be a subset of $A^{\tilde{\mathbb{Z}}} \cup A^{-\mathbb{N}}$. We denote

$$\overleftarrow{B} = \{w \in A^{-\mathbb{N}} \mid wu \in B \text{ for some word } u\}.$$

Symmetrically, for a subset B of $A^{\tilde{\mathbb{Z}}} \cup A^{\mathbb{N}}$ we let

$$\overrightarrow{B} = \{v \in A^{\mathbb{N}} \mid uv \in B \text{ for some word } u\}.$$

Finally, for a subset B of $A^{-\mathbb{N}}$, we let

$$\overleftrightarrow{B} = \{v \in A^{\tilde{\mathbb{Z}}} \mid \overleftarrow{\{v\}} \subseteq B\}.$$

For a word x, we write $\overleftarrow{x}$ (resp. $\overrightarrow{x}$, $\overleftrightarrow{x}$) instead of $\overleftarrow{\{x\}}$ (resp. $\overrightarrow{\{x\}}$, $\overleftrightarrow{\{x\}}$). For instance, if $B = \{a^{-\infty}b^na^m \in A^{-\mathbb{N}} \mid n, m \geq 1\}$, we have

$$\overleftarrow{B} = \{a^{-\infty}, a^{-\infty}b^na^m \mid n \geq 1, m \geq 0\} \qquad \text{and} \qquad \overleftrightarrow{B} = \{a^{-\infty}b^na^{+\infty} \mid n \geq 1\}.$$

2.2 Implicit operations

We assume the reader is familiar with the basic notions of finite semigroup theory (see, for instance, [12, 20] for an introduction to this theory). We will briefly recall some notions and notation concerning free profinite semigroups. For more details and proofs the reader is referred to [2, 4].

Recall that a *pseudovariety* of semigroups is a class of finite semigroups closed under taking subsemigroups, homomorphic images and finite direct products. Let $\mathbf{V}$ be a pseudovariety. A *profinite* (resp. *pro-*$\mathbf{V}$) semigroup is a projective limit of finite semigroups (resp. in $\mathbf{V}$). A topological semigroup is profinite (resp. pro-$\mathbf{V}$) if and only if it is compact and 0-dimensional (resp. and all its finite continuous homomorphic images are in $\mathbf{V}$). If A is an alphabet, we say that a profinite semigroup S is A*-generated* if there exists a mapping $\mu : A \to S$ such that the subsemigroup generated by $\mu(A)$ is dense in S. We denote by $\hat{F}_A(\mathbf{V})$ the projective limit of the A-generated elements of $\mathbf{V}$. The semigroup $\hat{F}_A(\mathbf{V})$ can also be viewed as the semigroup of A-ary implicit operations on $\mathbf{V}$. For this reason, the elements of $\hat{F}_A(\mathbf{V})$ are usually called (A*-ary*) *implicit operations* (*on* $\mathbf{V}$).

There exists a morphism $\iota : A^+ \to \hat{F}_A(\mathbf{V})$, called the *natural morphism* from A^+ into $\hat{F}_A(\mathbf{V})$, such that $\iota(A)$ generates a dense subsemigroup of $\hat{F}_A(\mathbf{V})$. We let ι_A be the restriction of ι to A. If $\mathbf{V}$ is not trivial ι_A is injective. In the sequel, we will often identify the isomorphic semigroups $\iota(A^+)$ and $A^+/\ker(\iota)$. In particular, when $\mathbf{V}$ satisfies no non trivial identity, we will view A^+ as a subsemigroup of $\hat{F}_A(\mathbf{V})$. The elements of $\iota(A^+)$ are called (A-*ary*) *explicit operations* (on $\mathbf{V}$).

If $\mathbf{W}$ is a subpseudovariety of $\mathbf{V}$, the identity of A induces a continuous onto homomorphism $\pi : \hat{F}_A(\mathbf{V}) \to \hat{F}_A(\mathbf{W})$, called the *canonical projection of* $\hat{F}_A(\mathbf{V})$ *onto* $\hat{F}_A(\mathbf{W})$. The image $\pi(x)$ of an element $x \in \hat{F}_A(\mathbf{V})$ is called the *restriction of* x *to* $\mathbf{W}$.

For each $x \in \hat{F}_A(\mathbf{V})$, the sequence $(x^{n!})_n$ converges in $\hat{F}_A(\mathbf{V})$. Its limit, denoted by x^ω, is the only idempotent in the topological closure of the subsemigroup generated by x.

2.3 Some important subpseudovarieties of $\mathcal{L}\mathbf{Sl}$

In this section we briefly review some facts concerning the most important pseudovarieties in this paper. We begin by recalling that, for a pseudovariety $\mathbf{V}$,

$$\mathcal{L}\mathbf{V} = \{S \in \mathbf{S} \mid eSe \in \mathbf{V} \text{ for all } e \in E(S)\}$$

is a pseudovariety of semigroups. Particularly important in this paper are the pseudovarieties $\mathcal{L}\mathbf{I}$ and $\mathcal{L}\mathbf{Sl}$, where $\mathbf{I}$ denotes the trivial pseudovariety. Recall that $\mathcal{L}\mathbf{Sl}$ is the pseudovariety defined by the pseudoidentities

$$x^\omega y x^\omega y x^\omega = x^\omega y x^\omega, \quad x^\omega y x^\omega z x^\omega = x^\omega z x^\omega y x^\omega.$$

We notice that $\mathbf{K}$, $\mathbf{D}$, $\mathcal{L}\mathbf{I}$ and $\mathcal{L}\mathbf{Sl}$ do not satisfy any non-trivial identity. In particular, we may identify the free semigroup A^+ with a subsemigroup of $\hat{F}_A(\mathbf{V})$, when $\mathbf{V} = \mathbf{K}$, $\mathbf{D}$, $\mathcal{L}\mathbf{I}$ or $\mathcal{L}\mathbf{Sl}$. Furthermore, we have (see [2]):

- $\hat{F}_A(\mathbf{K}) = A^+ \cup A^{\mathbb{N}}$ and the product in $\hat{F}_A(\mathbf{K})$ is extended from the product in A^+ by letting $ww' = w$ if $w \in A^{\mathbb{N}}$;
- $\hat{F}_A(\mathbf{D}) = A^+ \cup A^{-\mathbb{N}}$ and the product in $\hat{F}_A(\mathbf{D})$ is extended from the product in A^+ by letting $w'w = w$ if $w \in A^{-\mathbb{N}}$;
- $\hat{F}_A(\mathcal{L}\mathbf{I}) = A^+ \cup (A^{\mathbb{N}} \times A^{-\mathbb{N}})$ and the product in $\hat{F}_A(\mathcal{L}\mathbf{I})$ is given, for all $u, u' \in A^+$ and $(v, w), (v', w') \in A^{\mathbb{N}} \times A^{-\mathbb{N}}$, by:

$$\begin{array}{rcl} u \cdot u' & = & uu' \\ u \cdot (v, w) & = & (uv, w) \\ (v, w) \cdot u & = & (v, wu) \\ (v, w) \cdot (v', w') & = & (v, w'). \end{array}$$

Note that if $x = (u, v)$ is an element of $\hat{F}_A(\mathcal{L}\mathbf{I}) \setminus A^+$, then u (resp. v) is the restriction of x to $\mathbf{K}$ (resp. $\mathbf{D}$).

For an element $x \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$, denote by $\mathrm{Fact}(x)$ the set of all words $u \in A^+$ such that u is a factor of x, i.e., such that $x = yuz$ for some $y, z \in \hat{F}_A(\mathcal{L}\mathbf{Sl})^1$. Now, we recall that we dispose already of the following characterization of the implicit operations on $\mathcal{L}\mathbf{Sl}$ (see [2] or [10] for a proof).

Proposition 2.1 *Let A be a finite alphabet and let $x, y \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$. Then, $x = y$ if and only if* $\mathrm{Fact}(x) = \mathrm{Fact}(y)$ *and the restrictions of x and y to $\mathcal{L}\mathbf{I}$ coincide.* □

Now we recall a result (see [2, Corollary 5.6.2]) which presents an useful decomposition of the non explicit elements of the semigroups $\hat{F}_A(\mathbf{V})$.

Proposition 2.2 *Let $\mathbf{V}$ be a pseudovariety of semigroups and let x be a non explicit element of $\hat{F}_A(\mathbf{V})$. Then, there exist $y, z, w \in \hat{F}_A(\mathbf{V})$ such that $x = yz^\omega w$.* □

As an easy consequence, we derive the following property of the non explicit elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

Corollary 2.3 *If $x \in \hat{F}_A(\mathcal{L}\mathbf{Sl}) \setminus A^+$, then $x^\omega = x^2$.*

Proof. We know from Proposition 2.2 that $x = yz^\omega w$ for some $y, z, w \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$. Then, since $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ satisfies the pseudoidentity $a^\omega ba^\omega ba^\omega = a^\omega ba^\omega$ we deduce that

$$(x^2)^2 = (yz^\omega w)^4 = (yz^\omega w)^2 = x^2.$$

This shows that x^2 is idempotent, that is, $x^\omega = x^2$. □

2.4 Implicit operations on $\mathbf{Sl} * \mathbf{V}$

In this section, we recall the Theorem of Almeida and Weil on the implicit operations on semidirect products of the form $\mathbf{Sl} * \mathbf{V}$. This result appears in [4], to where the reader is referred for the proofs and more details. The notion of a *profinitely quasi-linear graph* that is used in that Theorem was claimed by Rhodes and Steinberg [22] to be not adequate in general. They proposed instead a refinement of that notion which is the notion of *profinite support* which we will use here. However, we will see that in the case $\mathbf{V} = \mathbf{D}$ in which we are interested, the graphs that we need are both profinite supports and profinitely quasi-linear.

A *profinite graph* is a projective limit of finite graphs. In particular, in a profinite graph, both the vertices and the edges constitute topological spaces which are compact and totally disconnected. Let $\Gamma = (V, E)$ be a profinite graph and let v and w be vertices of Γ. We say that w is *profinitely accessible from* v, and we write $v \preceq w$, if, in each finite continuous quotient of Γ, there exists a directed path from the image of v to the image of w. The classes of the equivalence relation associated to the quasi-order $\preceq$ are called the *profinitely strongly connected components* of Γ.

Let A be a profinite set and let M be an A-generated monoid, i.e., such that there exists a continuous mapping $\mu : A \to M$ such that the submonoid of M generated by $\mu(A)$ is dense. The *Cayley graph of M (associated to μ)* is the directed graph with vertex set M and edge set $M \times A$, where (m, a) is an edge from m to $m\,\mu(a)$. If $\mathbf{V}$ is a pseudovariety of monoids, then the Cayley graph of $\hat{F}_A(\mathbf{V})$ is written $\Gamma_A(\mathbf{V})$ (where the mapping from A to $\hat{F}_A(\mathbf{V})$ is the mapping ι_A). Notice that $\Gamma_A(\mathbf{V})$ is a profinite graph since it is the projective limit of the Cayley graphs of the A-generated elements of $\mathbf{V}$. If $\mathbf{V}$ is a pseudovariety of semigroups, we denote also $\Gamma_A(\mathbf{V})$ the Cayley graph of the monoid $\hat{F}_A(\mathbf{V})^1$.

Given $\Gamma = \Gamma_A(\mathbf{V})$ and two vertices $v, w \in \hat{F}_A(\mathbf{V})$ of Γ, one can verify that

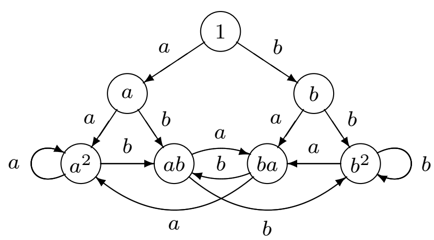
$$v \preceq w \qquad \text{if and only if} \qquad w \leq_{\mathcal{R}} v \text{ in } \hat{F}_A(\mathbf{V}), \tag{1}$$

and that, if $v = \lim_n v_n$ and $w = \lim_n w_n$ in $\hat{F}_A(\mathbf{V})$, and if $v_n \preceq w_n$ for all n, then $v \preceq w$ (i.e., $\preceq$ is a closed quasi-order).

Let Γ be a finite graph. A subgraph Δ of Γ is a *support* if there exists a directed path in Γ such that Δ is precisely the edges of that path. A *profinite support* is a closed subgraph Δ of a profinite graph Γ such that, in each finite continuous quotient of Γ the image of Δ is a support.

We say that a profinite graph Γ is *profinitely quasi-linear* if the quasi-order $\preceq$ is total. In that case Γ admits a unique $\preceq$-maximal (resp. $\preceq$-minimal) profinitely strongly connected component, denoted $\max(\Gamma)$ (resp. $\min(\Gamma)$). If Γ is finite this is equivalent to the condition that there exists a directed path in Γ which visits all the vertices. So, a profinite support is always quasi-linear, but the reverse is not true in general.

Example. The profinite Cayley graph $\Gamma_A(\mathbf{D})$, where A is a non trivial alphabet, is not a profinite support. For instance, the vertices a and b (where a and b are distinct letters of A) are $\preceq$-incomparable. Let indeed, for each integer k, $\mathbf{D}_k$ be the pseudovariety defined by the pseudoidentity $ba_1 \cdots a_k = a_1 \cdots a_k$, and remark that $\hat{F}_A(\mathbf{D}_k) \in \mathbf{D}$. The Cayley graph $\Delta = \Gamma_{\{a,b\}}(\mathbf{D}_2)$ is represented in the following figure.



In Δ there is no path with both the edge from 1 to a and the edge from 1 to b. So, Δ is not a support. Since Δ is a continuous image of $\Gamma_A(\mathbf{D})$, this shows that $\Gamma_A(\mathbf{D})$ is not a profinite support. ■

Notice that, for any profinite set A, $\hat{F}_A(\mathbf{Sl})$ is the semilattice $\bar{\mathcal{P}}(A)$ of closed subsets of A, a monoid under union. The announced result is the following.

Theorem 2.4 (Almeida & Weil) *Let* $\mathbf{V}$ *be a pseudovariety of monoids and let* A *be a profinite set. Then,* $\hat{F}_A(\mathbf{Sl}*\mathbf{V})$ *is isomorphic to the submonoid of* $\bar{\mathcal{P}}(\hat{F}_A(\mathbf{V})\times A)*\hat{F}_A(\mathbf{V})$ *which consists of all pairs* (G, x) *where* $x \in \hat{F}_A(\mathbf{V})$ *and where* G *is a profinite support of* $\Gamma_A(\mathbf{V})$ *such that* $1 \in \min(G)$ *and* $x \in \max(G)$. □

The same result holds if $\mathbf{V}$ is a pseudovariety of semigroups. As an application of this result, Almeida and Weil [4] have computed the pseudovariety $\mathbf{Sl} * \mathbf{K}$ and have described the corresponding free profinite semigroups (the reader may wish to consult their results in order to constitute an object of comparison with the case $\mathbf{Sl} * \mathbf{D}$).

3 The characterizations

In the sequel we fix a finite alphabet A, although the most part of the results are valid for any profinite set.

For the description of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$, we study the Cayley graph $\Gamma_A(\mathbf{D})$. We begin by noting that $\Gamma_A(\mathbf{D})$ is the graph with vertex set $A^* \cup A^{-\mathbb{N}}$ such that each vertex, distinct from the empty word, is the end of one and only one edge: if $x = ya$, with $a \in A$, then

- x is the end of the edge $y \xrightarrow{a} x$ if $x \neq y$;
- x is the end vertex of the loop $x \circlearrowright a$ if $x = y = a^{-\infty}$.

Now, notice that the vertex set of $\Gamma_A(\mathbf{K})$ is $A^* \cup A^{\mathbb{N}}$. If we compare the graphs $\Gamma_A(\mathbf{D})$ and $\Gamma_A(\mathbf{K})$ we verify that they coincide on the vertices in A^* (they both coincide with the tree of A^*), but they "behave" in a completely different way on the other vertices. While every profinitely strongly connected component of $\Gamma_A(\mathbf{K})$ is a singular set, in $\Gamma_A(\mathbf{D})$ the set $A^{-\mathbb{N}}$ forms a unique profinitely strongly connected component. In fact, for every $w \in A^{-\mathbb{N}}$ and $v \in A^* \cup A^{-\mathbb{N}}$, we have $w \leq_{\mathcal{R}} v$ in $\hat{F}_A(\mathbf{D})^1$ since $w = vw$. Therefore, we deduce from (1) that $v \preceq w$. So, the identification of the profinite supports G such that $1 \in \min(G)$ is easy in $\Gamma_A(\mathbf{K})$, while in the case of $\Gamma_A(\mathbf{D})$ we certainly cannot say the same. We now proceed to the study of these graphs.

Denote by $\mathcal{SG}$ (resp. $\mathcal{SG}_i$) the set of all (resp. infinite) profinitely quasi-linear closed subgraphs G of $\Gamma_A(\mathbf{D})$ such that $1 \in \min(G)$.

Proposition 3.1 *Let $G = (V, E) \in \mathcal{SG}$ and let $x, y \in V$ be such that $x = ya$ with $a \in A$. Then (y, a) is an edge of G. In particular, if $x = y = a^{-\infty}$, then the loop $(a^{-\infty}, a)$ is an edge of G.*

Proof. If $x \in A^+$ the assertion is trivial. Otherwise, the graph G is infinite and $x \in A^{-\mathbb{N}}$. For all $k \in \mathbb{N}$ let $\pi_k : \Gamma_A(\mathbf{D}) \to \Gamma_A(\mathbf{D}_k)$ be the canonical morphism, that is, for every $w \in A^* \cup A^{-\mathbb{N}}$,

$$\pi_k(w) = \begin{cases} w & \text{if } w \in A^{<k} \\ s_k(w) & \text{if } w \in A^{\geq k} \cup A^{-\mathbb{N}}. \end{cases}$$

Let now φ_k be the restriction of π_k to G and put $\Delta_k = \varphi_k(G)$. As Δ_k is a finite continuous quotient of G and $1 \preceq x$ in G, there exists a path in Δ_k from $\varphi_k(1) = 1$ to $\varphi_k(x) = s_k(x) = u_k a$, where $u_k = s_{k-1}(y)$. Clearly, the last edge of this path is $bu_k \xrightarrow{a} u_k a$, for some letter $b \in A$. So, for all $k \in \mathbb{N}$, there exists an edge (y_k, a) of G, such that $s_{k-1}(y_k) = u_k = s_{k-1}(y)$. Since the sequence $(y_k, a)_k$ converges to (y, a) in $\Gamma_A(\mathbf{D})$ and G is closed, we conclude that $(y, a) \in G$. $\square$

We deduce from this result that, if a word $w = \cdots w_{-3}w_{-2}w_{-1} \in A^{-\mathbb{N}}$ is a vertex of an element G of $\mathcal{SG}$, then all prefixes $\cdots w_{i-1}w_i$ ($i \in -\mathbb{N}$) of w are also vertices of G. In other words, $\overleftarrow{w} \subseteq G$.

Another consequence of this result is that an element of $\mathcal{SG}$ is completely defined by its vertices. In particular we deduce the following observation.

Corollary 3.2 *If G is a profinitely quasi-linear closed subgraph of $\Gamma_A(\mathbf{D})$ such that $1 \in \min(G)$, then G is a profinite support of $\Gamma_A(\mathbf{D})$. That is, each element of $\mathcal{SG}$ is a profinite support.* $\square$

In the sequel we identify a profinite support G of $\Gamma_A(\mathbf{D})$ (even if G is not such that $1 \in \min(G)$), containing all possible loops (i.e., if $a^{-\infty}$ is in G, with $a \in A$, then the loop labeled a is also in G) with the set of its vertices.

Thus, a profinite support G of $\Gamma_A(\mathbf{D})$ such that $1 \in \min(G)$ is either a finite path labeled $u \in A^+$, or it is a right-infinite path labeled $v \in A^{\mathbb{N}}$ with a certain set of added vertices in $A^{-\mathbb{N}}$. In this case $G \in \mathcal{SG}_i$, and we call *initial path of* G, denoted $\mathrm{ip}(G)$, the word v. Furthermore, we let $\hat{G} = G \setminus A^* = G \setminus \mathrm{Pref}(v)$. Notice that $\hat{G}$ is a profinite support of $\Gamma_A(\mathbf{D})$. As a result of the previous comments and of Theorem 2.4, we deduce the following characterization of the semigroups $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

Theorem 3.3 *Let A be a finite set. Then $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is the set*

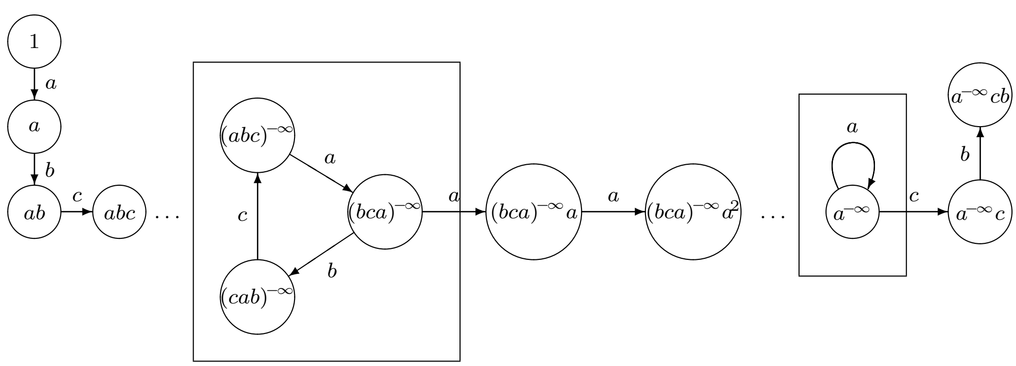
$$A^+ \cup \{(v, B, w) \in A^{\mathbb{N}} \times \mathcal{P}(A^{-\mathbb{N}}) \times A^{-\mathbb{N}} \mid \exists G \in \mathcal{SG}_i, v = \mathrm{ip}(G), B = \hat{G}, w \in \max(G)\}$$

and the product is given,— for all $u, u' \in A^+$, and all $(v, B, w), (v', B', w') \in A^{\mathbb{N}} \times \mathcal{P}(A^{-\mathbb{N}}) \times A^{-\mathbb{N}}$ such that $v = \mathrm{ip}(G), v' = \mathrm{ip}(G'), B = \hat{G}, B' = \hat{G}', w \in \max(G)$ and $w' \in \max(G')$ for some $G, G' \in \mathcal{SG}_i$ —, by

$$\begin{aligned} u \cdot u' &= uu' \\ u \cdot (v, B, w) &= (uv, B, w) \\ (v, B, w) \cdot u &= (v, B \cup \overleftarrow{wu}, wu) \\ (v, B, w) \cdot (v', B', w') &= (v, B \cup \overleftarrow{wv'} \cup B', w'). \end{aligned}$$ □

We remark that, in this theorem, the components v and w of an element $x = (v, B, w)$ of $\hat{F}_A(\mathcal{L}\mathbf{Sl}) \setminus A^+$ are, respectively, the restrictions of x to $\mathbf{K}$ and to $\mathbf{D}$. So, the characterization of x given by this theorem differs from that given by Proposition 2.1 only in the exchange of the set $\mathrm{Fact}(x)$ of finite words by a certain set B of infinite words of $A^{-\mathbb{N}}$. Therefore, at first sight Theorem 3.3 does not seem to be a great step forward from Proposition 2.1. Nevertheless it will allow us to obtain some useful and non trivial information about $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. It has also the merit of allowing a (graphical) visualization of the implicit operations on $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

Example 3.4 *Let a, b and c be three distinct letters of A. As one can convince oneself easily, the implicit operation $x = (abc)^\omega a^\omega cb$ on $\mathcal{L}\mathbf{Sl}$ is given by the following profinite support G of $\Gamma_A(\mathbf{D})$*



where $\max(G) = \{a^{-\infty}cb\}$. *So, with the notations of Theorem 3.3,*

$$(abc)^\omega a^\omega cb = ((abc)^{+\infty}, \overleftarrow{(abc)^{-\infty}a^{+\infty}} \cup \overleftarrow{a^{-\infty}cb}, a^{-\infty}cb).$$

We notice that, in the graph, each box is formed by the accumulation points, in $\hat{F}_A(\mathbf{D})$, of the sequence of vertices given, say, by the adjacent omission points $(\cdots)$. Remark that, in general each box obtained in this way is contained in a profinitely strongly connected component (in this graph, both boxes form profinitely strongly connected components, but that is not always the case). ■

We remark that the implicit operation x of this last example is determined by three types of paths in G:

- a *right-infinite* path (i.e., a sequence of consecutive edges indexed by $\mathbb{N}$) with beginning 1, labeled $(abc)^{+\infty}$;
- a *left-infinite* path (i.e., a sequence of consecutive edges indexed by $-\mathbb{N}$) which ends in $a^{-\infty}cb$, labeled $a^{-\infty}cb$;
- a set of *bi-infinite* paths (i.e., sequences of consecutive edges "infinite in both directions"), labeled $(abc)^{\infty}$, $(abc)^{-\infty}a^{+\infty}$ and a^{∞}.

In fact, this situation is general as it is shown in the following result.

Proposition 3.5 *Let $G \in \mathcal{SG}$ and $x \in G$, and suppose that* $\max(G) \neq \{x\}$. *Then, there exists $a \in A$ such that (x, a) is an edge of G.*

Proof. If $x \in A^+$ the assertion is trivial. Suppose, therefore, that $x \in A^{-\mathbb{N}}$ and let $y \in \max(G)$ be such that $x \neq y$. In particular, there exists an integer k such that $s_k(x) \neq s_k(y)$. For every integer $n \geq k$, let φ_n be the restriction to G of the canonical morphism from $\Gamma_A(\mathbf{D})$ to $\Gamma_A(\mathbf{D}_n)$, and put $\Delta_n = \varphi_n(G)$. Then, as $x \preceq y$ in G, there is a non empty path in Δ_n from $\varphi_n(x) = s_n(x)$ to $\varphi_n(y) = s_n(y)$. Suppose that $(s_n(x), a_n)$ is the first edge of that path. Thus, there is an edge (x_n, a_n) of G, such that $s_n(x_n) = s_n(x)$. Notice that, in particular, the sequence $(x_n)_{n \geq k}$ converges to x. Since the edge set of G is a compact topological space, the sequence $(x_n, a_n)_{n \geq k}$ admits a subsequence which converges, say to (x, a). Moreover, as G is closed we deduce that (x, a) is an edge of G. $\square$

Let G be an infinite element of $\mathcal{SG}$ and let $w \in A^{-\mathbb{N}}$ be a vertex of G. We have seen in Proposition 3.1 that all elements of $\overleftarrow{w}$ are vertices of G. This means that there exists a left-infinite path with end in w (and labeled w). Moreover, we deduce from Proposition 3.5 that, if $|\max(G)| \geq 2$, if $\max(G) = \{a^{-\infty}\}$ $(a \in A)$ or if $\max(G) = \{y\}$ and $w \notin \overleftarrow{y}$, then there exists a right-infinite path with beginning in w. As a consequence, there exists a bi-infinite path which passes in w, except in the case where $\max(G) = \{y\}$ and $w \in \overleftarrow{y}$ (this is the case, for instance, for $w = a^{-\infty}c$ and $y = a^{-\infty}cb$ in Example 3.4).

This shows that every infinite element G of $\mathcal{SG}$, with a fixed vertex $w \in \max(G)$, is a union of three types of paths:

- a right-infinite path with beginning in 1;
- a set of bi-infinite paths;
- a left-infinite path with end in w.

Thus, an element (v, B, w) of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ may be represented alternatively in the form

$$[v, C, w]$$

where C is the set of labels of all bi-infinite paths in B. We notice that

$$B = \overleftarrow{C} \cup \overleftarrow{w} \qquad \text{and} \qquad C = \overleftrightarrow{B}.$$

Moreover, if $x, y \in A^{\tilde{\mathbb{Z}}}$ are two elements of C we write $x \preceq y$ if, for all $x' \in \overleftarrow{x}$ and $y' \in \overleftarrow{y}$, we have $x' \preceq y'$ in B. Thus, in Example 3.4 we have $(abc)^{\infty} \preceq (abc)^{-\infty}a^{+\infty} \preceq a^{\infty}$ and $a^{\infty} \not\preceq (abc)^{-\infty}a^{+\infty} \not\preceq (abc)^{\infty}$. As one can verify, the relation $\preceq$ on C is a total quasi-order, and the associated equivalence relation admits a unique maximal (resp. minimal) class, denoted $\max(C)$ (resp. $\min(C)$).

We may therefore rewrite Theorem 3.3 in the following way. We remark that in this version the symmetry of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is more evident than in the last one.

Theorem 3.6 *Let A be a finite set. Then $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is the set*

$$A^+ \cup \{[v, B, w] \in A^{\mathbb{N}} \times \mathcal{P}(A^{\tilde{\mathbb{Z}}}) \times A^{-\mathbb{N}} \mid \exists G \in \mathcal{SG}_i, v = \mathrm{ip}(G), B = \overleftrightarrow{\hat{G}}, w \in \max(G)\}$$

and the product is given,— for all $u, u' \in A^+$, and all $[v, B, w], [v', B', w'] \in A^{\mathbb{N}} \times \mathcal{P}(A^{\tilde{\mathbb{Z}}}) \times A^{-\mathbb{N}}$ such that $v = \mathrm{ip}(G), v' = \mathrm{ip}(G'), w \in \max(G), w' \in \max(G'), B = \overleftrightarrow{\hat{G}}$ and $B' = \overleftrightarrow{\hat{G}'}$ for some $G, G' \in \mathcal{SG}_i$ —, by

$$\begin{aligned} u \cdot u' &= uu' \\ u \cdot [v, B, w] &= [uv, B, w] \\ [v, B, w] \cdot u &= [v, B, wu] \\ [v, B, w] \cdot [v', B', w'] &= [v, B \cup \{wv'\} \cup B', w']. \end{aligned} \qquad \square$$

In the sequel we will use freely the notation given by Theorems 3.3 and 3.6. Thus, the implicit operation $(abc)^{\omega}a^{\omega}cb$ of Example 3.4 may be represented, either by

$$((abc)^{+\infty}, \overleftarrow{(abc)^{-\infty}a^{+\infty}} \cup \overleftarrow{a^{-\infty}cb}, a^{-\infty}cb),$$

or by

$$[(abc)^{+\infty}, \{(abc)^{\infty}, (abc)^{-\infty}a^{+\infty}, a^{\infty}\}, a^{-\infty}cb].$$

4 The regular elements

In this section, we identify the regular elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. We begin by presenting some definitions.

Let $u \in A^{-\mathbb{N}} \cup A^{\mathbb{N}}$ be an infinite word on A. We let $F_{-\mathbb{N}}(u)$ by the set of accumulation points in $\hat{F}_A(\mathbf{D})$ of the sequence $(u'_n)_{n\in\mathbb{N}}$ which is:

- $(p_n(u))_{n\in\mathbb{N}}$, the sequence of prefixes of u, if $u \in A^{\mathbb{N}}$;
- $(s_n(u))_{n\in\mathbb{N}}$, the sequence of suffixes of u, if $u \in A^{-\mathbb{N}}$.

Notice that $F_{-\mathbb{N}}(u)$ is a subset of $A^{-\mathbb{N}}$. Moreover, a word $w \in A^{-\mathbb{N}}$ is in $F_{-\mathbb{N}}(u)$ if and only if there exists a subsequence of $(u'_n)_n$ which converges to w. If $v \in A^{\mathbb{N}}$, as $\preceq$ is a closed quasi-order, we deduce easily that the subgraph $G = \mathrm{Pref}(v) \cup F_{-\mathbb{N}}(v)$ of $\Gamma_A(\mathbf{D})$ is a profinite support such that $1 \in \min(G)$ and $\max(G) = F_{-\mathbb{N}}(v)$. One can also verify that, for all $w \in A^{-\mathbb{N}}$, $G = F_{-\mathbb{N}}(w) \cup \overleftarrow{w}$ is a profinite support of $\Gamma_A(\mathbf{D})$ such that $\min(G) = F_{-\mathbb{N}}(w)$ and $w \in \max(G)$. For a word $u \in A^{-\mathbb{N}} \cup A^{\mathbb{N}}$, we will write

$$F_{\tilde{\mathbb{Z}}}(u) = \overleftrightarrow{F_{-\mathbb{N}}(u)}$$

the set of labels of all bi-infinite paths in $F_{-\mathbb{N}}(u)$. Thus, for all $v \in A^{\mathbb{N}}$ and $w \in A^{-\mathbb{N}}$ such that $F_{-\mathbb{N}}(v) \cap F_{-\mathbb{N}}(w) \neq \emptyset$,

$$(v, F_{-\mathbb{N}}(v) \cup F_{-\mathbb{N}}(w) \cup \overleftarrow{w}, w) \text{ — or, alternatively, } [v, F_{\tilde{\mathbb{Z}}}(v) \cup F_{\tilde{\mathbb{Z}}}(w), w] \text{ —}$$

represents for sure an implicit operation on $\mathcal{L}\mathbf{Sl}$. Finally, notice that, if $[v, B, w]$ represents an element of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$, then $F_{\tilde{\mathbb{Z}}}(v) \subseteq \min(B)$ and $F_{\tilde{\mathbb{Z}}}(w) \subseteq \max(B)$.

Now, we are able to present the following description of the regular elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

Proposition 4.1 *Let $u \in A^+$ and $[v, B, w] \in \hat{F}_A(\mathcal{L}\mathbf{Sl}) \setminus A^+$. Then,*

(a) $u^{\omega} = [u^{-\infty}, u^{\infty}, u^{+\infty}]$;

(b) $[v, B, w]$ *is idempotent if and only if* $wv \in B$;

(c) $[v, B, w]$ *is regular if and only if* $v \in \overrightarrow{\max(B)}$ *and* $w \in \overleftarrow{B}$ *(which is equivalent to say that* $w \in \overleftarrow{\max(B)}$*). [Notice that the condition* $v \in \overrightarrow{\max(B)}$ *implies that* $B = \max(B)$*.]*

Proof. Statements (a) and (b) are evident. Put $x = [v, B, w]$. Then, x is regular if and only if there exists $y \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ such that $xyx = x$ and $yxy = y$. Now, we deduce that y is not explicit, say $y = [v', B', w']$, and, in particular, that $B = B'$. Moreover,

$$xyx = [v, B, w][v', B, w'][v, B, w] = [v, B \cup \{wv', w'v\}, w]$$

is equal to x if and only if $wv', w'v \in B$.

Therefore, x is regular if and only if there exists $y = [v', B, w'] \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ such that $wv', w'v \in B$ (in particular, $wv', w'v \in \max(B)$). That is, x is regular if and only if $v \in \overrightarrow{\max(B)}$ and $w \in \overleftarrow{B}$. Notice that, if $v' \in A^{\mathbb{N}}$ and $w' \in A^{-\mathbb{N}}$ are such that $wv', w'v \in \max(B)$, then $[v', B, w']$ represents an implicit operation on $\mathcal{L}\mathbf{Sl}$ since, in that case, $w' \in \overleftarrow{\max(B)}$, $\min(B) = \max(B) = B$ (as $v \in \overrightarrow{\max(B)}$ and $F_{\tilde{\mathbb{Z}}}(v) \subseteq \min(B)$), whence $F_{\tilde{\mathbb{Z}}}(v') \subseteq \min(B)$. $\square$

The most simple examples of idempotents of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ are the elements of the form $[v, A^{\tilde{\mathbb{Z}}}, w]$ for any $v \in A^{\mathbb{N}}$ and $w \in A^{-\mathbb{N}}$. In fact, the observation which follows is immediate.

Lemma 4.2 *The minimal ideal of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is the set*

$$K = \{[v, A^{\tilde{\mathbb{Z}}}, w] \mid v \in A^{\mathbb{N}}, w \in A^{-\mathbb{N}}\}.$$

The subsets of K of the form

$$K_w = \{[v, A^{\tilde{\mathbb{Z}}}, w] \mid v \in A^{\mathbb{N}}\} \quad \textit{and} \quad K_v = \{[v, A^{\tilde{\mathbb{Z}}}, w] \mid w \in A^{-\mathbb{N}}\}$$

where $w \in A^{-\mathbb{N}}$ and $v \in A^{\mathbb{N}}$, form, respectively, the minimal left and right ideals of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. $\square$

We give another example which will prove useful in the sequel.

Example 4.3 *Let $A = \{a, b\}$ and let $u = u_1u_2 \cdots \in A^{\mathbb{N}}$ where $(u_n)_{n\in\mathbb{N}}$ is the sequence of all words of A^+ in lexicographical order. That is,*

$$\begin{aligned} u &= a\, b\, a^2\, ab\, ba\, b^2\, a^3\, a^2b\, aba\, ab^2\, ba^2\, bab\, b^2a\, b^3 \cdots \\ &= aba^3b^2ab^2a^5baba^2b^3a^2bab^3ab^3 \cdots . \end{aligned}$$

In particular, one can verify easily that $F_{\tilde{\mathbb{Z}}}(u) = A^{\tilde{\mathbb{Z}}}$. Let now $v \in A^{\mathbb{N}}$ be the word obtained from u by the exchange of all factors a^i, with $i \geq 3$, by the word a^2. We have

$$v = aba^2b^2ab^2a^2baba^2b^3 \cdots$$

and, as one can verify easily,

$$F_{\tilde{\mathbb{Z}}}(v) = A^{\tilde{\mathbb{Z}}} \setminus A^{-\mathbb{N}}a^3A^{\mathbb{N}}.$$

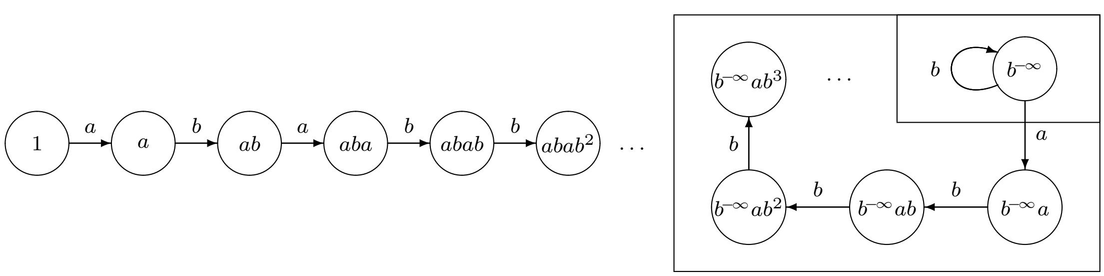
Moreover, $[v, F_{\tilde{\mathbb{Z}}}(v), b^{-\infty}]$ is an idempotent of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ since $b^{-\infty}v \in F_{\tilde{\mathbb{Z}}}(v)$. On the contrary, the element $x = [v, F_{\tilde{\mathbb{Z}}}(v), b^{-\infty}a^2]$ is not idempotent since

$$b^{-\infty}a^2v = b^{-\infty}\boldsymbol{a^3}ba^2b^2 \cdots$$

admits a^3 as a factor. Nevertheless, x is regular since $b^{-\infty}a^2 \in F_{-\mathbb{N}}(v)$ and $v \in \overrightarrow{F_{\tilde{\mathbb{Z}}}(v)}$ (remember that $\max(F_{\tilde{\mathbb{Z}}}(v)) = F_{\tilde{\mathbb{Z}}}(v)$). In fact, we have $xbx = [v, F_{\tilde{\mathbb{Z}}}(v) \cup \{b^{-\infty}a^2bv\}, b^{-\infty}a^2] = x$. ■

As an application of the results proved so far, we give an alternative proof to that given by Almeida [2, Proposition 12.3.1] of the fact that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ does not have the property of factorization. We use, only for comparison, the same sequence of Almeida's proof.

Example 4.4 *Consider the sequence $(u_n)_{n\in\mathbb{N}} = (abab^2ab^3 \cdots ab^na)_{n\in\mathbb{N}}$. Then, if we regard the terms u_n as elements of $\mathcal{SG}$, we verify without difficulties that the sequence $(u_n)_n$ converges to the following graph.*



Moreover, in $\hat{F}_A(\mathcal{L}\mathbf{I})$, the sequence converges to $(v, w) = (abab^2ab^3a \cdots, b^{-\infty}a)$. Thus, if we put $B = \{b^{\infty}, b^{-\infty}ab^{+\infty}\}$, we verify that the sequence converges to $x = [v, B, w]$ in $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. Suppose that x admits a factorization of the form

$$x = v_0x_1v_1 \cdots x_nv_n$$

where the v_i are explicit operations and the x_i are regular elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. In particular, x_1 is of the form $[z, C, t]$ with $v_0z = v$ and $C \subseteq B$. As x_1 is regular, $z \in \overrightarrow{C}$, from Proposition 4.1. But that is impossible since $C \subseteq B$ and $v \notin \overrightarrow{B}$ (and so $z \notin \overrightarrow{B}$ neither). This shows that x does not have the property of factorization. ■

5 The Green's relations

We are now able to characterize the Green's relations on $\hat{F}_A(\mathcal{L}\mathbf{Sl})$.

Proposition 5.1 *Let $x = [v, B, w] \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$. We have,*

$$
\begin{aligned}
R_x &= \begin{cases} \{[v, B, w'] \mid w' \in \overleftarrow{\max(B)}\} & \text{if } w \in \overleftarrow{B} \\ \{x\} & \text{otherwise} \end{cases} \\
L_x &= \begin{cases} \{[v', B, w] \mid v' \in \overrightarrow{\min(B)}\} & \text{if } v \in \overrightarrow{B} \\ \{x\} & \text{otherwise} \end{cases} \\
H_x &= \{x\} \\
J_x &= \begin{cases} \{[v', B, w'] \mid v' \in \overrightarrow{\min(B)}, w' \in \overleftarrow{\max(B)}\} & \text{if } v \in \overrightarrow{B} \text{ and } w \in \overleftarrow{B} \\ L_x & \text{if } v \in \overrightarrow{B} \text{ and } w \notin \overleftarrow{B} \\ R_x & \text{if } v \notin \overrightarrow{B} \text{ and } w \in \overleftarrow{B} \\ H_x & \text{if } v \notin \overrightarrow{B} \text{ and } w \notin \overleftarrow{B}. \end{cases}
\end{aligned}
$$

Proof. Notice first that if $[v', B', w'] \in J_x$ then $B' = B$. In fact, $[v', B', w'] \in J_x$ if and only if $[v, B, w] = y[v', B', w']z$ and $[v', B', w'] = y'[v, B, w]z'$ for some $y, y', z, z' \in \hat{F}_A(\mathcal{L}\mathbf{Sl})^1$. From the first equality, we deduce that $B' \subseteq B$, and from the second one that $B \subseteq B'$. As R_x, L_x and H_x are subsets of J_x, we deduce also that all the elements of R_x, L_x and H_x are of the form $[v', B, w']$.

We begin by proving the equality concerning R_x. Let $[v', B, w']$ be an element of R_x. Then,

$$[v, B, w] = [v', B, w']y \qquad \text{and} \qquad [v', B, w'] = [v, B, w]z$$

for some $y, z \in \hat{F}_A(\mathcal{L}\mathbf{Sl})^1$. In particular,

$$v = v', \qquad [v, B, w] = [v, B, w](zy)^\omega \qquad \text{and} \qquad [v', B, w'] = [v', B, w'](yz)^\omega.$$

Suppose that $w \neq w'$. Then $y, z \neq 1$ and so there exist in B two right-infinite paths with beginning, respectively, in w and w'. This shows that $w, w' \in \overleftarrow{\max(B)}$. Consequently, we deduce in particular that,

- $R_x = \{x\}$ if $w \notin \overleftarrow{B}$;
- $R_x \subseteq \{[v, B, w'] \mid w' \in \overleftarrow{\max(B)}\}$ if $w \in \overleftarrow{B}$.

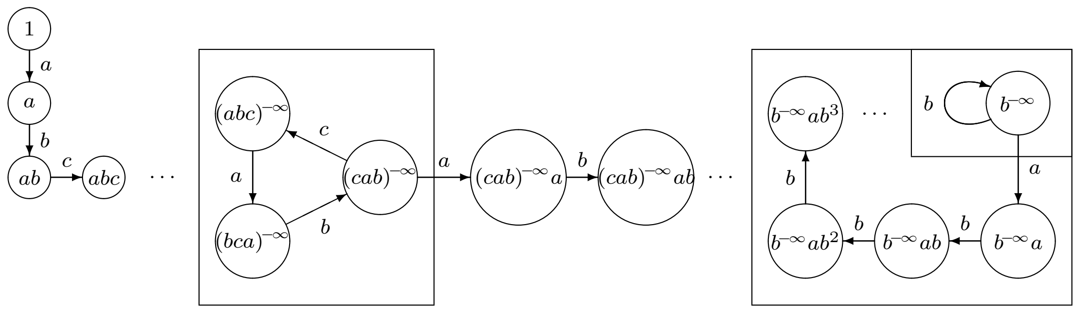
Suppose now that $w \in \overleftarrow{B}$ and let $w' \in \overleftarrow{\max(B)}$. In particular, $wz, w'z' \in \max(B)$ for some $z, z' \in \overrightarrow{\max(B)}$. Therefore,

$$[v, B, w] = [v, B, w'][z', \max(B), w] \qquad \text{and} \qquad [v, B, w'] = [v, B, w][z, \max(B), w'],$$

which shows that R_x contains the set $\{[v, B, w'] \mid w' \in \overleftarrow{\max(B)}\}$, and concludes the proof for R_x.

The proof for the relation $\mathcal{L}$ is analogous. Since $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is compact we have $\mathcal{J} = \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ and so the computation of J_x is a consequence of the previous calculations. To finish the proof it suffices to remark that $H_x = R_x \cap L_x = \{x\}$. □

Example 5.2 *Consider the limit* $x \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ *of the sequence* $(abab^2ab^3 \cdots ab^n a)_{n\in\mathbb{N}}$ *(cf. Example 4.4), and let* $y = (abc)^\omega x$*. The element of* $\mathcal{SG}$ *which corresponds to* y *is the following.*



Let $v \in A^{\mathbb{N}}$ *be the word* $abab^2ab^3a\cdots$*, and put* $C = \{(abc)^\infty, (abc)^{-\infty}v, b^\infty, b^{-\infty}ab^{+\infty}\}$*. Notice that* $\min(C) = \{(abc)^\infty\}$ *and* $\max(C) = \{b^\infty, b^{-\infty}ab^{+\infty}\}$*. Then,*

$$y = [(abc)^{+\infty}, C, b^{-\infty}a] \quad \textit{and} \quad J_y = \{[v', C, w'] \mid v' \in \overrightarrow{(abc)^\infty} \textit{ and } w' \in \overleftarrow{b^{-\infty}ab^{+\infty}}\},$$

which can be represented as follows.

$[(abc)^{+\infty}, C, b^{-\infty}]$	$[(abc)^{+\infty}, C, b^{-\infty}a]$	$[(abc)^{+\infty}, C, b^{-\infty}ab]$	$[(abc)^{+\infty}, C, b^{-\infty}ab^2]$	$\cdots$
$[(cab)^{+\infty}, C, b^{-\infty}]$	$[(cab)^{+\infty}, C, b^{-\infty}a]$	$[(cab)^{+\infty}, C, b^{-\infty}ab]$	$[(cab)^{+\infty}, C, b^{-\infty}ab^2]$	$\cdots$
$[(bca)^{+\infty}, C, b^{-\infty}]$	$[(bca)^{+\infty}, C, b^{-\infty}a]$	$[(bca)^{+\infty}, C, b^{-\infty}ab]$	$[(bca)^{+\infty}, C, b^{-\infty}ab^2]$	$\cdots$

■

We have seen in Proposition 5.1 that the $\mathcal{J}$-class of an element $[v, B, w] \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ is formed by elements of the form $[\_\,, B, \_\,]$. Let us now see how all the elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ "constructed with the same B" are related with respect to the relation $\leq_{\mathcal{J}}$.

Proposition 5.3 *Let* $[v, B, w], [v', B, w'] \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$*. Then,* $[v', B, w'] \leq_{\mathcal{J}} [v, B, w]$ *if and only if* $v \in \overrightarrow{B} \cup \overrightarrow{v'}$ *and* $w \in \overleftarrow{B} \cup \overleftarrow{w'}$*.*

Proof. Suppose that $[v', B, w'] \leq_{\mathcal{J}} [v, B, w]$. Then,

$$[v', B, w'] = x[v, B, w]y$$

for some $x, y \in \hat{F}_A(\mathcal{L}\mathbf{Sl})^1$. Suppose first that $x \in A^*$, whence $v' = xv$ and so $v \in \overrightarrow{v'}$. If $y \in A^*$, we have $w' = wy$ and so $w \in \overleftarrow{w'}$. If $y \notin A^*$, then $y = [z, B', w']$ for some $B' \subseteq B$ and $z \in A^{\mathbb{N}}$ such that $wz \in B$. Consequently $w \in \overleftarrow{B}$. The case $x \notin A^*$ is dual.

The reverse is immediate. For instance, if $v \in \overrightarrow{B}$ and $w \in \overleftarrow{w'}$, then there exist $z \in A^{-\mathbb{N}}$ and $u \in A^*$ such that $zv \in B$ (whence $zv \in \min(B)$) and $w' = wu$. Consequently, $[v', B, w'] = [v', \min(B), z][v, B, w]u$, which shows that $[v', B, w'] \leq_{\mathcal{J}} [v, B, w]$. □

We synthesize in the following corollary some observations which are immediate consequences of results of this section and of the previous one.

Corollary 5.4 *Consider the $\mathcal{J}$-class $J_B = \{[v, B, w] \mid v \in \overrightarrow{B}$ and $w \in \overleftarrow{B}\}$. Then,*

(a) J_B is $\leq_{\mathcal{J}}$-maximal among the $\mathcal{J}$-classes of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ constructed with B;

(b) J_B is a regular $\mathcal{J}$-class if and only if $\max(B) = B$. All the others $\mathcal{J}$-classes of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ constructed with B are irregular $\mathcal{R}$-classes and $\mathcal{L}$-classes. □

We can show by the way that $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is "very large".

Proposition 5.5 *There exist uncountably many elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ pairwise $\leq_{\mathcal{J}}$-incomparable.*

Proof. In fact, as we shall see, it suffices to consider elements constructed with a same B. Let $v = aba^2b^2ab^2a^2baba^2b^3 \cdots \in A^{\mathbb{N}}$ be the word of Example 4.3, and let $B = F_{\tilde{\mathbb{Z}}}(v) = A^{\tilde{\mathbb{Z}}} \setminus A^{-\mathbb{N}}a^3A^{\mathbb{N}}$. Notice that $\max(B) = B$, $\overrightarrow{B} = A^{\mathbb{N}} \setminus A^*a^3A^{\mathbb{N}}$ and $\overleftarrow{B} = A^{-\mathbb{N}} \setminus A^{-\mathbb{N}}a^3A^*$. Notice that in particular $\overleftarrow{B}$ is uncountable. Let now

$$C = \{[v, B, wba^3] \mid w \in \overleftarrow{B}\}.$$

We remark that the elements of C are well defined (though wba^3 does not belong to $\overleftarrow{B}$). We have $|C| = |\overleftarrow{B}| = 2^{\aleph_0}$. Moreover, for every pair x and y of distinct elements of C, x and y are $\leq_{\mathcal{J}}$-incomparable. This is a consequence of Proposition 5.3 and of the fact that, for all $w, w' \in \overleftarrow{B}$ with $w \neq w'$, we have $wba^3, w'ba^3 \notin \overleftarrow{B}$, $wba^3 \notin \overleftarrow{w'ba^3}$ and $w'ba^3 \notin \overleftarrow{wba^3}$. □

6 The height of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$

In this section we give an example of an uncountable ascending chain of $\mathcal{J}$-classes of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. This chain is constructed with the use of some notions of combinatorics on words, namely *sturmian words* (termed like this by Morse and Hedlund in [19],— to where the reader is referred for more details and omitted proofs), which we begin by recalling.

In this section, A denotes the two-letter alphabet $A = \{a, b\}$. For two words $u, v \in A^+$ with the same length, we define

$$\delta(u, v) = \big||u|_a - |v|_a\big| = \big||u|_b - |v|_b\big|.$$

A word $x \in A^{-\mathbb{N}} \cup A^{\mathbb{N}} \cup A^{\tilde{\mathbb{Z}}}$ is said *sturmian* if, for all $n \in \mathbb{N}$ and all $u, v \in \mathrm{Fact}_n(x)$, $\delta(u, v) \leq 1$. For instance, the bi-infinite words $a^{-\infty}ba^{+\infty}$ and a^{∞} are sturmian. Moreover, they are the only bi-infinite words in which the letter b does not appear an infinite number of times (to the left and to the right).

It is clear that, if $x \in A^{-\mathbb{N}} \cup A^{\tilde{\mathbb{Z}}}$ (resp. $x \in A^{\mathbb{N}} \cup A^{\tilde{\mathbb{Z}}}$) is a sturmian word, then all elements of $\overleftarrow{x}$ (resp. $\overrightarrow{x}$) are also sturmian words. Moreover (see [19]), if $x \in A^{-\mathbb{N}}$ (resp. $x \in A^{\mathbb{N}}$) is an infinite sturmian word, then there exists a bi-infinite sturmian word $y \in A^{\tilde{\mathbb{Z}}}$ such that $x \in \overleftarrow{y}$ (resp. $x \in \overrightarrow{y}$). That is, each infinite sturmian word may be "prolonged" into a bi-infinite sturmian word.

With each sturmian word x with at least two occurrences of the letter a (which is equivalent to say that x has an infinite number of occurrences of a), we associate a positive real number $\alpha(x)$, called its *frequency*, in the following way. First, for every integer n, a word of the form

$$ab^{k_1}ab^{k_2}\cdots b^{k_n}a$$

with $k_1, \ldots, k_n \geq 0$, is called an *n-chain*. That is, an *n-chain* is a word in $aA^* \cap A^*a$ with $n+1$ occurrences of a. Now, for all $n \in \mathbb{N}$, we fix an arbitrary n-chain $u = ab^{k_1}\cdots b^{k_n}a$ of x (i.e., such that u is a factor of x) and we put

$$b(x)_n = |u|_b = k_1 + k_2 + \cdots + k_n.$$

We notice that the number of b's in two n-chains of x differs of at most 1. Moreover, as it is shown in [19], the sequence $(\frac{b(x)_n}{n})_n$ tends to a finite limit, denoted $\alpha(x)$. For the sturmian words having at most one occurrence of the letter a, we put $\alpha(x) = +\infty$. Those words are called *special*.

Morse and Hedlund [19] have shown also that, for every real $r \geq 0$, there exists a sturmian word of frequency r. For instance,

- $\alpha(a^\infty) = \alpha(a^{-\infty}ba^{+\infty}) = 0$;
- $\alpha((ba^3)^\infty) = \alpha((ba^3)^{-\infty}a(ba^3)^{+\infty}) = \alpha((ba^3)^{-\infty}ba^2(ba^3)^{+\infty}) = \frac{1}{3}$;
- $\alpha((ab)^\infty) = \alpha((ab)^{-\infty}b(ab)^{+\infty}) = \alpha((ba)^{-\infty}a(ba)^{+\infty}) = 1$;
- $\alpha(\boldsymbol{f}) = \frac{\sqrt{5}-1}{2}$;

where

$$\boldsymbol{f} = abaababaabaababaababa\cdots \in A^{\mathbb{N}}$$

is the Fibonacci word, the most famous sturmian word. We recall that $\boldsymbol{f}$ is the limit of the sequence $(f_n)_{n\in\mathbb{N}}$ defined by recurrence by

$$f_1 = a, \qquad f_2 = ab \qquad \text{and} \qquad f_{k+1} = f_k f_{k-1}\ (k \geq 2).$$

Notice that $\boldsymbol{f} = abf_1f_2f_3\cdots$.

Moreover, a sturmian word has rational frequency if and only if it is ultimately periodic.

A word $u \in A^+$ which is factor of a sturmian word will be also called *sturmian*. For any sturmian n-chain u, we define

$$\begin{aligned}
\alpha'(u) &= \max\left\{\tfrac{|v|_b}{k} - \tfrac{1}{k} \mid k \leq n \text{ and } v \text{ is a } k\text{-chain of } u\right\},\\
\alpha''(u) &= \min\left\{\tfrac{|v|_b}{k} + \tfrac{1}{k} \mid k \leq n \text{ and } v \text{ is a } k\text{-chain of } u\right\}.
\end{aligned}$$

For instance,

- $\alpha'(aba) = 0$ and $\alpha''(aba) = 2$;
- $\alpha'(a^2ba^2ba^2) = \max\{-1, 0, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\} = \frac{1}{3}$ and $\alpha''(a^2ba^2ba^2) = \min\{1, 2, \frac{2}{3}, \frac{3}{4}, \frac{3}{5}\} = \frac{3}{5}$.

We group in the following theorem (constituted by Theorem 2.3 and Lemma 5.1 of [19]) some properties of sturmian words.

Theorem 6.1 *A (finite, infinite or bi-infinite) word x on A is a non special sturmian word if and only if there exists a real constant $\alpha \geq 0$ such that the number $b(x)_n$ of occurrences of b in n-chains of x satisfy one of the following conditions, for all n,*

(a) $n\alpha - 1 < b(x)_n \leq n\alpha + 1$;

(b) $n\alpha - 1 \leq b(x)_n < n\alpha + 1$.

If x is an infinite or bi-infinite sturmian word, α is its frequency.

Moreover, if x is a sturmian chain, one of the conditions (a) or (b) is verified if and only if $\alpha'(x) \leq \alpha \leq \alpha''(x)$, the equalities left and right being verified at most when $\alpha = \alpha'(x)$ and $\alpha = \alpha''(x)$, respectively. If x is a sturmian chain and its n-chains verify both (a) and (b) (whence $n\alpha - 1 < b(x)_n < n\alpha + 1$), then x is a chain of an infinite (resp. bi-infinite) sturmian word of frequency α. □

This theorem permits, in particular, to verify if a given sturmian chain can be a factor of a given infinite (or bi-infinite) sturmian word. For instance, the chain $a^2ba^2ba^2$ considered above is not factor of the Fibonacci word since $\frac{\sqrt{5}-1}{2} > \alpha''(a^2ba^2ba^2) = \frac{3}{5}$. On the contrary, for all real number $\frac{1}{3} < r < \frac{3}{5}$, $a^2ba^2ba^2$ is a factor of a sturmian word of frequency r.

It is also important to note the following observation.

Proposition 6.2 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of sturmian words of $A^{-\mathbb{N}}$ converging to x in $\hat{F}_A(\mathbf{D})$. Then, x is a sturmian word and $\alpha(x) = \lim_n \alpha(x_n)$.* □

Consider now the *Cantor set* $\mathcal{C}$, formed by the real numbers in the interval $[0,1]$ which do not use the number 1 in their development in base 3. That is, $\mathcal{C}$ is the set of all numbers of the form

$$\sum_{i=1}^{\infty} \frac{n_i}{3^i} \qquad \text{where } n_i \in \{0,2\} \text{ for all } i.$$

Notice that $\mathcal{C}$ is in bijection with the set of all sequences in $\{0,2\}$, whence $|\mathcal{C}| = 2^{\aleph_0}$.

Recall that the Cantor set can be geometrically described as follows. Divide the interval $[0,1]$ in three equal parts, and remove the open interval of the middle $]\frac{1}{3}, \frac{2}{3}[$. We eliminate in this way all real numbers in the interval $[0,1]$ which use $n_1 = 1$ in their development in base 3. At second step we remove the open interval of the middle of both the two intervals which remain, $[0, \frac{1}{3}]$ and $[\frac{1}{3}, 1]$, eliminating this way all numbers which use $n_2 = 1$ in their development in base 3. Iterating this process, at step k we remove the union I_k of the open intervals of the middle of the 2^{k-1} intervals which are left from step $k-1$. We obtain therefore

$$\mathcal{C} = [0,1] \setminus \bigcup_{k=1}^{\infty} I_k.$$

Let c and d be two elements of $\mathcal{C}$. We say that d is a *successor* of c (and that c is a *predecessor* of d), and we write $d = \mathrm{s}(c)$, if $c < d$ and it does not exist another element $e \in \mathcal{C}$ such that $c < e < d$. The subset of $\mathcal{C}$ formed by the elements having a successor will be denoted $\mathcal{C}_{\mathrm{s}}$. For instance,

- $\frac{1}{3} = \sum_{i=2}^{\infty} \frac{2}{3^i}$ has successor $\frac{2}{3}$, and $\frac{2}{3} + \frac{1}{3^3} = \frac{2}{3} + \sum_{i=4}^{\infty} \frac{2}{3^i}$ has successor $\frac{2}{3} + \frac{2}{3^3}$;
- $\frac{2}{3}$ does not have successor.

Observe the following property of the relation of successor in $\mathcal{C}$.

Remark. Let $c = \sum_{i=1}^{\infty} \frac{n_i}{3^i} \in \mathcal{C}$. Then, c is a successor if and only if there exists an integer k such that $n_k = 0$ and $n_i = 2$ for all $i > k$. In this case, $c = \sum_{i=1}^{k-1} \frac{n_i}{3^i} + \frac{1}{3^k}$ and its successor is $\mathrm{s}(c) = \sum_{i=1}^{k-1} \frac{n_i}{3^i} + \frac{2}{3^k}$. ■

Consider now, for each real number r in the interval $[0,1]$, the set $\mathcal{S}_r \subseteq A^{-\mathbb{N}}$ of all sturmian left-infinite words of frequency r. We remark that $\mathcal{S}_r$ is a closed subset of $\hat{F}_A(\mathbf{D})$ such that $\overleftarrow{\mathcal{S}_r} = \mathcal{S}_r$. In particular, we may regard $\mathcal{S}_r$ as a closed (and hence profinite) subgraph of $\Gamma_A(\mathbf{D})$. Moreover, we know from [19] that any two elements of $\mathcal{S}_r$ have an infinite number of common factors (if r is irrational they have the same factors). This implies that $\mathcal{S}_r$ is a (profinitely quasi-linear closed) subgraph of $\Gamma_A(\mathbf{D})$ with a unique profinitely strongly connected component.

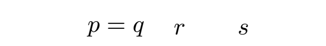
Let $r, s \in [0,1]$. We remark that, if $r \neq s$, then the sets $\mathcal{S}_r$ and $\mathcal{S}_s$ have only a finite number of common chains (and factors) since, otherwise, they would have a common element (obtained, for instance, using a sequence of common factors of increasing length). We may, therefore, define a mapping λ from $[0,1] \times [0,1]$ into $\mathbb{N} \cup \{\infty\}$ by letting

$$\lambda(r,s) = \begin{cases} \max\{k \in \mathbb{N} \mid \ \mathcal{S}_r \text{ and } \mathcal{S}_s \text{ have a common } k\text{-chain}\} & \text{if } r \neq s \\ \infty & \text{if } r = s. \end{cases}$$

We notice that, in particular, if $\lambda(r,s) = n$, then $\mathcal{S}_r$ and $\mathcal{S}_s$ have a common factor of length n, i.e., $\mathrm{Fact}_n(\mathcal{S}_r) \cap \mathrm{Fact}_n(\mathcal{S}_s) \neq \emptyset$. The mapping λ satisfies the following property.

Lemma 6.3 *Let $p, q, r, s \in [0,1]$ be such that $p \leq q < r \leq s$. Then, $\lambda(p,s) \leq \lambda(q,r)$.*

Proof. We will consider the case $p = q$ and $r \neq s$. The other ones are analogous.



Suppose that $\lambda(p,s) = n$. Hence, there exists a n-chain u, with n maximal, such that u is a (sturmian) chain of $\mathcal{S}_r$ and of $\mathcal{S}_s$. In particular, we deduce from Theorem 6.1 that $\alpha'(u) \leq p$ and that $s \leq \alpha''(u)$. Then, as $p < r < s$ we have $\alpha'(u) < r < \alpha''(u)$, and so, again by Theorem 6.1, we deduce that u is a chain of a sturmian word of frequency r. That is, $\lambda(p,s) \leq \lambda(p,r) = \lambda(q,r)$. □

Another observation which we will use is the following.

Lemma 6.4 *Let $r < s$ be two real numbers in the interval $[0,1]$, let $\varepsilon = s - r$ and let $n \in \mathbb{N}$. Then, there exists an integer $k \geq 1$ such that, for all $0 \leq i < k$,*

$$\lambda(r + \tfrac{i}{k}\,\varepsilon, r + \tfrac{i+1}{k}\,\varepsilon) \geq n.$$

Proof. The lemma is valid since, otherwise, we would find a sequence $([r_m, s_m])_{m\in\mathbb{N}}$ of subintervals of the interval $[r,s]$ such that $|s_m - r_m| \underset{m}{\longrightarrow} 0$, $[r_{m+1}, s_{m+1}] \subseteq [r_m, s_m]$ and $\lambda(r_m, s_m) < n$ for all m, which is clearly impossible by Theorem 6.1. $\square$

For each $c \in \mathcal{C}_{\mathrm{s}}$, let us fix a word $w_c \in \mathcal{S}_c$, and a word $v_c \in A^{\mathbb{N}}$ of frequency $\mathrm{s}(c)$. Finally, let

$$E = \Big(\bigcup_{c\in\mathcal{C}} \mathcal{S}_c\Big) \cup \{\overleftarrow{w_c v_c} \mid c \in \mathcal{C}_{\mathrm{s}}\}.$$

We may regard E as the subgraph of $\Gamma_A(\mathbf{D})$ formed by all left-infinite words of frequency in the Cantor set $\mathcal{C}$ with, moreover, right-infinite paths which "connect" each subset $\mathcal{S}_c$, with $c \in \mathcal{C}_{\mathrm{s}}$, to the subset $\mathcal{S}_{\mathrm{s}(c)}$.

Proposition 6.5 *The graph E is a profinitely quasi-linear closed subgraph of $\Gamma_A(\mathbf{D})$ with an uncountable number of profinitely strongly connected components.*

Proof. As one can easily verify, E is a closed subgraph of $\Gamma_A(\mathbf{D})$, whence it is profinite. Let us now show that E is quasi-linear. We know already that all elements of each $\mathcal{S}_c$ are in the same profinitely strongly connected component. To prove that E is quasi-linear it suffices therefore to show that, if $c < d$ are two elements of $\mathcal{C}$, then $\mathcal{S}_c \preceq \mathcal{S}_d$ in E.

Let therefore $c, d \in \mathcal{C}$ with $c < d$. Notice first that, if $c \in \mathcal{C}_{\mathrm{s}}$ and $d = \mathrm{s}(c)$, and since there exists a right-infinite path "from $\mathcal{S}_c$ to $\mathcal{S}_{\mathrm{s}(c)}$" (i.e., the initial vertex is in $\mathcal{S}_c$ and all points of accumulation of vertices of this path are in $\mathcal{S}_{\mathrm{s}(c)}$), then we deduce immediately that $\mathcal{S}_c \preceq \mathcal{S}_d$ in E.

For all $n \in \mathbb{N}$ let $\pi_n : \Gamma_A(\mathbf{D}) \to \Gamma_A(\mathbf{D}_n)$ be the canonical morphism. That is, for all $w \in A^* \cup A^{-\mathbb{N}}$,

$$\pi_n(w) = \begin{cases} w & \text{if } w \in A^{<n} \\ s_n(w) & \text{if } w \in A^{\geq n} \cup A^{-\mathbb{N}}. \end{cases}$$

Let now φ_n be the restriction of π_n to E, and put $E^n = \varphi_n(E)$ and $\mathcal{S}_r^n = \varphi_n(\mathcal{S}_r)$ $(r \in \mathcal{C})$. As E is the projective limit of the E^n (since $\Gamma_A(\mathbf{D})$ is the projective limit of the $\Gamma_A(\mathbf{D}_n)$), to show that $\mathcal{S}_c \preceq \mathcal{S}_d$ in E, it suffices to prove that, for all $n \in \mathbb{N}$, $\mathcal{S}_c^n \preceq \mathcal{S}_d^n$ in E^n (which we will denote by $\mathcal{S}_c \preceq_n \mathcal{S}_d$).

Let $n \in \mathbb{N}$ and let $\varepsilon = d - c$. From Lemma 6.4 there exists $k \in \mathbb{N}$ such that, for all $0 \leq i < k$,

$$\lambda(c + \tfrac{i}{k}\,\varepsilon, c + \tfrac{i+1}{k}\,\varepsilon) \geq n.$$

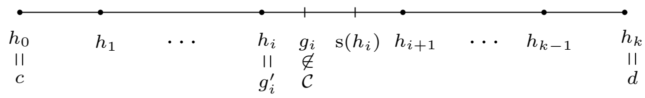
Put $g_i = c + \frac{i}{k}\,\varepsilon$ for all $0 \leq i \leq k$.

$$\begin{array}{cccccc} g_0 & g_1 & g_2 & \cdots & g_{k-1} & g_k \\ \parallel & \parallel & \parallel & & \parallel & \parallel \\ c & c+\frac{1}{k}\,\varepsilon & c+\frac{2}{k}\,\varepsilon & & c+\frac{k-1}{k}\,\varepsilon & d \end{array}$$

We have, $\lambda(g_i, g_{i+1}) \geq n$ for all $0 \leq i < k$. It can happen that one or more of the g_i's do not lie in $\mathcal{C}$. In that case, if $g_i \notin \mathcal{C}$, there exists an element g_i' of $\mathcal{C}$ which is the greatest of the elements of $\mathcal{C}$ which are lower than g_i. Notice that $g_i' \in \mathcal{C}_{\mathrm{s}}$.

For all $0 \leq i \leq k$, we let

$$h_i = \begin{cases} g_i & \text{if } g_i \in \mathcal{C} \\ g_i' & \text{otherwise.} \end{cases}$$



To show that $\mathcal{S}_c \preceq_n \mathcal{S}_d$, we prove that $\mathcal{S}_{h_i} \preceq_n \mathcal{S}_{h_{i+1}}$ for all $0 \le i < k$. Let $0 \le i < k$. We have four possibilities.

First case Suppose first that $g_i, g_{i+1} \in \mathcal{C}$ (whence, $h_i = g_i$ and $h_{i+1} = g_{i+1}$). As $\lambda(h_i, h_{i+1}) = \lambda(g_i, g_{i+1}) \ge n$, $\mathcal{S}_{h_i}$ and $\mathcal{S}_{h_{i+1}}$ have a common factor of length n, and so $\mathcal{S}^n_{h_i}$ and $\mathcal{S}^n_{h_{i+1}}$ have a common element. Consequently, it is clear that $\mathcal{S}^n_{h_i}$ and $\mathcal{S}^n_{h_{i+1}}$ are contained in the same strongly connected component of E^n, and, in particular, that $\mathcal{S}_{h_i} \preceq_n \mathcal{S}_{h_{i+1}}$.

Second case Suppose now that $g_i \in \mathcal{C}$ and $g_{i+1} \notin \mathcal{C}$ (whence, $h_i = g_i$ and $h_{i+1} = g'_{i+1}$). In this case, we have $g_i = h_i \le h_{i+1} < g_{i+1}$. From Lemma 6.3, $\lambda(h_i, h_{i+1}) \ge \lambda(g_i, g_{i+1}) \ge n$ and it follows, as in the last case, that $\mathcal{S}_{h_i} \preceq_n \mathcal{S}_{h_{i+1}}$.

Third case Suppose that $g_i \notin \mathcal{C}$ and $g_{i+1} \in \mathcal{C}$ (whence, $h_i = g'_i$ and $h_{i+1} = g_{i+1}$). Then, $h_i < g_i < \mathrm{s}(h_i) \le h_{i+1} = g_{i+1}$ and so $\lambda(\mathrm{s}(h_i), h_{i+1}) \ge \lambda(g_i, g_{i+1}) \ge n$. Thus $\mathcal{S}_{\mathrm{s}(h_i)} \preceq_n \mathcal{S}_{h_{i+1}}$, and since $\mathcal{S}_{h_i} \preceq_n \mathcal{S}_{\mathrm{s}(h_i)}$, we deduce by transitivity that $\mathcal{S}_{h_i} \preceq_n \mathcal{S}_{h_{i+1}}$.

Fourth case Suppose finally that $g_i, g_{i+1} \notin \mathcal{C}$ (whence, $h_i = g'_i$ and $h_{i+1} = g'_{i+1}$). If g_{i+1} is in the interval $[h_i, \mathrm{s}(h_i)[$ — so that $h_i = h_{i+1}$ — the conclusion is immediate. Otherwise, we have $h_i < g_i < \mathrm{s}(h_i) \le h_{i+1} < g_{i+1}$ and the sequel is analogous to the third case.

We have proved in this way that E is quasi-linear.

To finish the proof, it remains to show that E has an uncountable number of profinitely strongly connected components. For that, we will show that each $\mathcal{S}_c$, with $c \in \mathcal{C}$, forms a profinitely strongly connected component of E. The claim follows from the fact that $|\mathcal{C}| = 2^{\aleph_0}$.

We begin by noting that, if $c \in \mathcal{C}_{\mathrm{s}}$,— and as $\mathcal{C}$ does not contain any element of the interval $]c, \mathrm{s}(c)[$,— then $\mathcal{S}_{\mathrm{s}(c)} \not\preceq \mathcal{S}_c$ in E. In fact, let r be a real number in the interval $]c, \mathrm{s}(c)[$ and let k be an integer such that,

$$\text{for all } k\text{-chain } x \text{ of } \mathcal{S}_{\mathrm{s}(c)},\ \alpha'(x) > r. \tag{2}$$

Notice that one such k exists since the value $\alpha'(x)$ of a k-chain x of $\mathcal{S}_{\mathrm{s}(c)}$ tends to $\mathrm{s}(c)$ when k tends to infinity. Now let, for instance, $n = 12k$ and let u be an element of $\mathcal{S}^n_{\mathrm{s}(c)}$. Then $u \not\preceq_n \mathcal{S}^n_c$, i.e., there is no path in E^n from u to an element $u' \in \mathcal{S}^n_c$. In fact, suppose by contradiction that there exists a path

$$u_0 \xrightarrow{a_1} u_1 \xrightarrow{a_2} u_2 \quad \cdots \quad u_{p-1} \xrightarrow{a_p} u_p$$

in E^n with beginning vertex $u_0 = u$ and end vertex $u_p = u'$. By definition of E, for all $0 \le i \le p$, either

(a) u_i is sturmian, and so it is a factor of $\mathcal{S}_{g_i}$, for some $g_i \in \mathcal{C}$, or

(b) u_i is not sturmian, and so it is a factor of the word $w_{h_i}v_{h_i} \in A^{\tilde{\mathbb{Z}}}$, for some $h_i \in \mathcal{C}_{\mathrm{s}}$.

If every u_i is of first type, it is clear from the definitions of k and n that every g_i is $\geq \mathrm{s}(c)$, and so $u_p \notin \mathcal{S}_c^n$ which is impossible since $u_p = u'$. Otherwise, for all $0 \leq i \leq p$, put

$$u_i = u_{i,1}u_{i,2}u_{i,3}$$

with $|u_{i,1}| = |u_{i,2}| = |u_{i,3}| = \frac{n}{3} = 4k$. As we deal with factors of sturmian words of frequency ≤ 1, the factor b^2 appears at most one time in a factor of E. Therefore, each $u_{i,j}$ ($0 \leq i \leq p$, $1 \leq j \leq 3$) contains at least one k-chain $x_{i,j}$. Moreover, after (a) and (b), for some fixed i, at most one of the k-chains $x_{i,1}$, $x_{i,2}$ and $x_{i,3}$ is not sturmian. Furthermore,

- if $x_{i,3}$ is not sturmian, then x_i is of type (b), and $x_{i,1}$ and $x_{i,2}$ are sturmian. In particular, the word $x_{i,1}x_{i,2}$ is factor of a word w_{h_i} of frequency h_i.
- if $x_{i,2}$ is not sturmian (and so x_i is of type (b), and $x_{i,1}$ and $x_{i,3}$ are sturmian), then $x_{i,1}$ is factor of a word w_{h_i} and $x_{i,3}$ is factor of v_{h_i}. Consequently, if $h_i \geq \mathrm{s}(c)$, $x_{i,1}$ and $x_{i,3}$ are such that $\alpha'(x_{i,1}), \alpha'(x_{i,3}) > r$.
- if $x_{i,1}$ is not sturmian, then x_i is of type (b), and $x_{i,2}$ and $x_{i,3}$ are sturmian. In particular, the word $x_{i,2}x_{i,3}$ is factor of a word v_{h_i} of frequency $\mathrm{s}(h_i)$.

As a consequence of these three points and of the choice of k, it is clear that —, as we depart from $u = u_0$,— every k-chain x of the words u_i is such that $\alpha'(x) > r$. But that is impossible since $u' = u_p$ contains at least one k-chain y and so $\alpha'(y) \leq c < r$. We conclude that a path as that above does not exist, and so that $\mathcal{S}_{\mathrm{s}(c)} \not\preceq \mathcal{S}_c$ in E.

Let, now, c and d be two arbitrary elements of $\mathcal{C}$ with $c < d$. From the construction of $\mathcal{C}$, it is easy to verify that there exists $g \in \mathcal{C}_{\mathrm{s}}$ such that

$$c \leq g < \mathrm{s}(g) \leq d.$$

Consequently, $\mathcal{S}_d \not\preceq \mathcal{S}_c$ in E, since otherwise,— as $\mathcal{S}_c \preceq \mathcal{S}_g$ and $\mathcal{S}_{\mathrm{s}(g)} \preceq \mathcal{S}_d$,— we would have $\mathcal{S}_{\mathrm{s}(g)} \preceq \mathcal{S}_g$ which is false. This shows that, for all $c \in \mathcal{C}$, $\mathcal{S}_c$ is a profinitely strongly connected component of E, which concludes the proof. $\square$

This result provides the graph that we will use to construct the announced chain.

Theorem 6.6 *Let A be a non trivial finite alphabet. The semigroup $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ admits an uncountable ascending chain of $\mathcal{J}$-classes.*

Proof. Let a and b be two distinct letters of A, and let $G = \mathrm{Pref}(a^{+\infty}) \cup E$. Proposition 6.5 and Corollary 3.2 show immediately that G is a profinite support of $\Gamma_A(\mathbf{D})$, with an uncountable number of profinitely strongly connected components, such that $1 \in \min(G)$. For all $c \in \mathcal{C}$, let G_c be the subgraph of G defined by

$$\begin{aligned} G_c &= \{x \in G \mid x \preceq \mathcal{S}_c\} \\ &= \mathrm{Pref}(a^{+\infty}) \cup \Big(\bigcup_{\substack{d \in \mathcal{C} \\ d \leq c}} \mathcal{S}_d\Big) \cup \{\overleftarrow{w_d v_d} \mid d \in \mathcal{C}_{\mathrm{s}}, d < c\}. \end{aligned}$$

Then, G_c is a profinite support and $1 \in \min(G_c)$. Now, let x_c be the element of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ defined by

$$x_c = (a^{+\infty}, \hat{G}_c, z_c)$$

where $z_c \in \max(G_c) = \mathcal{S}_c$. It is clear that, for all $c, d \in \mathcal{C}$ with $c < d$, we have $x_d \leq_{\mathcal{J}} x_c$ (more precisely, $x_d \leq_{\mathcal{R}} x_c$) since

$$x_d = x_c(y, H, z_d)$$

where y is the label of an infinite path to the right with beginning in vertex z_c of $\mathcal{S}_c$ (whence y is a sturmian right-infinite word of frequency c) and $H = \big(G_d \setminus G_c\big) \cup \mathcal{S}_c$. On the contrary, $x_c \not\leq_{\mathcal{J}} x_d$ since $\hat{G}_d \not\subseteq \hat{G}_c$. The family $(x_c)_{c \in \mathcal{C}}$ forms therefore an ascending chain of elements of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$, from which the result follows. $\square$

7 The subsemigroup of ω-words

In this section we are interested in the subsemigroup of ω-words of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. Recall that an element x of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$ is an ω-word if we can obtain x from the component projections by the application of a finite number of times the operation of multiplication and the unary operation $x \mapsto x^\omega$. We prove that given two such words we can decide if they are equal or not. The result we want to prove is the following.

Theorem 7.1 *Each ω-word $x \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ can be written as a product*

$$x = u_0 x_1^\omega u_1 x_2^\omega \cdots x_k^\omega u_k$$

where $k \geq 0$, $u_0, \ldots, u_k \in A^*$, $u_0 \neq 1$ if $x = u_0$ and $x_1, \ldots, x_k \in A^+$.

Moreover, if $y = v_0 y_1^\omega v_1 y_2^\omega \cdots y_m^\omega v_m$ is another product of this type, then $x = y$ if and only if either $k = m = 0$ and $u_0 = v_0$, or $k, m \geq 1$, $u_0 x_1^{+\infty} = v_0 y_1^{+\infty}$, $x_k^{-\infty} u_k = y_m^{-\infty} v_m$ and the sets

$$I_x = \{x_i^\infty, x_j^{-\infty} u_j x_{j+1}^{+\infty} \mid 1 \leq i \leq k, 1 \leq j \leq k-1\}$$

and

$$I_y = \{y_i^\infty, y_j^{-\infty} v_j y_{j+1}^{+\infty} \mid 1 \leq i \leq m, 1 \leq j \leq m-1\}$$

are equal. Furthermore, the equality $x = y$ is effectively decidable.

As one recalls, (arbitrary) infinite and bi-infinite words play an important part in the description of the (arbitrary) implicit operations on $\mathcal{L}\mathbf{Sl}$. Now, as one can observe in the statement of Theorem 7.1, it suffices to consider *ultimately periodical* words to describe the ω-words of $\hat{F}_A(\mathcal{L}\mathbf{Sl})$. Before proving Theorem 7.1, we show how we can write each such word in a canonical form.

A word $u \in A^+$ is said *primitive* if it is not a power of another word, i.e., if

$$u = v^n, \ n \in \mathbb{N} \Rightarrow n = 1.$$

We say that a word u is a *conjugate* of another word v if there exist $x, y \in A^*$ such that

$$u = xy, \qquad v = yx.$$

We notice that, if u is a primitive word and v is a conjugate of u, then v is also primitive. Let an order be fixed for the letters of the alphabet A. A *Lyndon word* is a primitive word which is minimal in its conjugation class.

Consider an ultimately periodic word $w = w_1w_2w_3\cdots \in A^{\mathbb{N}}$ (with $w_k \in A$ for all k). Then $w = uv^{+\infty} \in A^{\mathbb{N}}$ for some $u \in A^*, v \in A^+$, and we may choose v to be a primitive word. Put $p = |v|$ and let i be the least integer such that, for all $n \geq i$, $w_n = w_{n+p}$. So, the word w can be written in the form

$$w = w_1w_2\cdots w_{i-1}(w_iw_{i+1}\cdots w_{i+p-1})^{+\infty}.$$

As one can show, this representation of w is unique (i.e., the values of i and p are unique), and it is called the *canonical form* of w. Notice that the choice of i implies that the letter w_{i-1} (when $i \geq 2$) is different from the letter w_{i+p-1}. Moreover, w is periodical if and only if $i = 1$.

Example. Consider the ultimately periodic word $w = ababab^2ab^2ab^2\cdots$. In this case $p = 3$ and we have,

$$\begin{aligned} w &= abab(ab^2)^{+\infty} \\ &= aba(bab)^{+\infty} \end{aligned}$$

and $s_1(aba) = a \neq b = s_1(bab)$. Therefore, this last representation of w is its canonical form. ∎

Naturally, each ultimately periodic word of $A^{-\mathbb{N}}$ admits a *canonical form* which can be symmetrically described. Let us now consider the bi-infinite case. Let $w = x^{-\infty}yz^{+\infty} \in A^{\tilde{\mathbb{Z}}}$, with $x, z \in A^+$ and $y \in A^*$, be an ultimately periodic word. We may suppose that x is a primitive word. Two cases may arise.

Suppose first that $w = x^{\infty}$. In the conjugation class of x there exists a unique Lyndon word v. In this case, we have

$$w = v^{\infty}$$

and this representation will be called the *canonical form* of w. We notice that, given w written in the form $w = x^{-\infty}yz^{+\infty}$, it is not difficult to verify if also $w = x^{\infty}$.

Now, suppose that w is not periodic. Then, there exist $u \in A^+$, $a \in A$ and $t \in A^{\mathbb{N}}$ such that $w = u^{-\infty}at$, u is a conjugate of x and the first letter of u is distinct from a. Now, if $t = rv^{+\infty}$ ($r \in A^*$, $v \in A^+$) is the canonical form of t, we have

$$w = u^{-\infty}arv^{+\infty}$$

and this representation will be called the *canonical form* of w.

Example. If $w_1 = (baba^2baba^2)^{-\infty}baba^2b(a^2ba^2ba^2b)^{+\infty}$ we deduce successively

$$\begin{aligned} w_1 &= ((\boldsymbol{baba^2})^2)^{-\infty}\boldsymbol{baba^2}b((a^2b)^3)^{+\infty} \\ &= (baba^2)^{-\infty}b(a^2b)^{+\infty} \\ &= (baba^2)^{-\infty}ba\boldsymbol{a}b(a^2b)^{+\infty} \\ &= (ba^2ba)^{-\infty}\boldsymbol{a}(ba^2)^{+\infty}. \end{aligned}$$

Now, if $w_2 = (b^2a)^{-\infty}b^2ab(babbab)^{+\infty}$ we have

$$\begin{aligned} w_2 &= (bab)^{-\infty}((bab)^2)^{+\infty} \\ &= (bab)^{\infty} \\ &= (ab^2)^{\infty}. \qquad \blacksquare \end{aligned}$$

We now present the proof of Theorem 7.1. In [9] the author has presented a proof based on Proposition 2.1 and not using Theorem 3.6. The proof presented here uses this result, which simplifies it very much.

Proof of Theorem 7.1. As x is a ω-word, we can write it on the form

$$x = w_0 z_1^{\omega} w_1 z_2^{\omega} \cdots z_m^{\omega} w_m$$

where $m \geq 0$, $w_0, \ldots, w_m \in A^*$, and $z_1, \ldots, z_m \in \hat{F}_A(\mathcal{L}\mathbf{Sl})$ are ω-words. From Corollary 2.3, we can replace each factor of x of the form z_i^{ω} by z_i^2 as soon as z_i is not explicit. Since there exist only a finite number of ω's in the factorization of x, the iteration of this process leads after a finite number of steps to a factorization of x as in the statement. Notice that, moreover, this factorization is effectively computable.

Suppose now that

$$x = u_0 x_1^{\omega} u_1 x_2^{\omega} \cdots x_k^{\omega} u_k \quad \text{and} \quad y = v_0 y_1^{\omega} v_1 y_2^{\omega} \cdots y_m^{\omega} v_m$$

are factorizations as in the statement. Then, as one can easily verify, we have, with the notations of Theorem 3.6,

$$x = [u_0 x_1^{+\infty}, I_x, x_k^{-\infty} u_k] \quad \text{and} \quad y = [v_0 y_1^{+\infty}, I_y, y_m^{-\infty} v_m].$$

Therefore, the application of Theorem 3.6, permits to deduce that $x = y$ if and only if $u_0 x_1^{+\infty} = v_0 y_1^{+\infty}$, $I_x = I_y$ and $x_k^{-\infty} u_k = y_m^{-\infty} v_m$.

It remains to show that the equality $x = y$ is decidable. This is equivalent to the decidability of the equalities $u_0 x_1^{+\infty} = v_0 y_1^{+\infty}$, $x_k^{-\infty} u_k = y_m^{-\infty} v_m$ and $I_x = I_y$. Now, each word $w \in A^{\mathbb{N}} \cup A^{-\mathbb{N}} \cup A^{\tilde{\mathbb{Z}}}$ appearing in one such equality is an ultimately periodical word and so it admits a canonical form. Moreover this canonical form is effectively computable since w is a word already given in the form

- $w = uv^{+\infty}$ ($u \in A^*$, $v \in A^+$) if w is a right-infinite word;
- $w = v^{-\infty}u$ ($u \in A^*$, $v \in A^+$) if w is a left-infinite word;
- $w = v^{\infty}$ or $w = v^{-\infty}ut^{+\infty}$ ($u \in A^*$, $v, t \in A^+$) if w is a bi-infinite word.

We can therefore conclude that the equality $x = y$ is effectively decidable, which concludes the proof. $\square$

Example. Let $x \in \hat{F}_{\{a,b\}}(\mathcal{L}\mathbf{Sl})$ be the ω-word given by

$$x = ba^2ba^2(baba^2baba^2)^{\omega} b(((ab)^{\omega}b)^{\omega}(ba)^{\omega}b)^{\omega}.$$

Then, we may also write x in the following forms

$$\begin{aligned} x &= ba^2ba^2(baba^2baba^2)^{\omega} b((ab)^{\omega}b)^{\omega}(ba)^{\omega} b((ab)^{\omega}b)^{\omega}(ba)^{\omega} b \\ &= ba^2ba^2(baba^2baba^2)^{\omega} b(ab)^{\omega} b(ab)^{\omega} b(ba)^{\omega} b(ab)^{\omega} b(ab)^{\omega} b(ba)^{\omega} b. \end{aligned}$$

If I_x is the set of bi-infinite words described in Theorem 7.1, we have, with the words already given in their canonical form,

$$I_x = \{(a^2bab)^\infty, (ab)^\infty, (ababa)^{-\infty}b(ab)^{+\infty}, (ab)^{-\infty}b(ab)^{+\infty}, (ab)^{-\infty}b(ba)^{+\infty}\}.$$

Notice that $ba^2ba^2(baba^2baba^2)^{+\infty} = ba(aba^2b)^{+\infty}$ and that $(ba)^{-\infty}b = (ab)^{-\infty}$. Hence, as one can easily verify, x can be written, for instance,

$$x = ba(aba^2b)^\omega(ab)^\omega(ba)^\omega b^3(ab)^\omega. \blacksquare$$

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