Global asymptotic stability for neural network models with distributed delays

José J. Oliveira

Departamento de Matemática and CMAT, Escola de Ciências, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal
e-mail: jjoliveira@math.uminho.pt

Abstract

In this paper, we obtain the global asymptotic stability of the zero solution of a general n-dimensional delayed differential system, by imposing a condition of dominance of the nondelayed terms which cancels the delayed effect.

We consider several delayed differential systems in general settings, which allow us to study, as subclasses, the well known neural network models of Hopfield, Cohn-Grossberg, bidirectional associative memory, and static with S-type distributed delays. For these systems, we establish sufficient conditions for the existence of a unique equilibrium and its global asymptotic stability, without using the Lyapunov functional technique. Our results improve and generalize some existing ones.

Key words: Delayed neural network models, distributed delays, time-varying delays, global asymptotic stability, M-matrix.

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1 Introduction

In the last decades, retarded functional differential equations (FDEs) have attracted the attention of an increasing number of scientists due to their potential application in different sciences. Differential equations with delays have served as models in population dynamics, ecology, epidemiology, disease modelling, neural networks.

Neural network models possess good potencial applications in areas such as content-addressable memory, pattern recognition, signal and image processing and optimization (see [1], [2], [3], [4], and references therein). In optimization applications, it is required that the designed neural network converges to a unique and globally asymptotically stable equilibrium. Thus, it is important to achieve sufficient conditions for the systems to possess this dynamic.

In 1983, Cohen and Grossberg [5] proposed and studied the artificial neural network described by a system of ordinary differential equations

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \right], \quad i = 1, \ldots, n \tag{1}$$

and, in 1984, Hopfield [6] studied the particular situation of (1) with $k_i \equiv 1$,

$$\dot{x}_i(t) = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)), \quad i = 1, \ldots, n. \tag{2}$$

The finite switching speed of the amplifiers, communication time, and process of moving images led to the use of time-delays in models (1) and (2). Since then, several sufficient conditions have been obtained to ensure existence and global asymptotic stability of an equilibrium point of different generalizations of models (1) and (2) with delays (see [1], [2], [3], [4], [7], [8], [9], [10], [11] and references therein).

Other neural network models have been studied, such as the static neural network model [12],

$$\dot{x}_i(t) = -x_i(t) + g_i \left(\sum_{j=1}^{n} a_{ij} x_j(t) + I_i \right), \quad i = 1, \ldots, n, \tag{3}$$

also with distributed delays [13], and the bidirectional associative memory neural network [14],

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(y_j(t)) + I_i \\ \\ \dot{y}_i(t) = -y_i(t) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t)) + J_i \end{cases}, \quad i = 1, \ldots, n \tag{4}$$

as well as some other generalizations (see e.g. [15], [16], [17], [18]).

Besides the above cited works, there is an extensive literature dealing with global stability of neural network models with delays. We emphasize however that, in the literature, the usual approach to study the global asymptotic stability of the equilibrium of a system relies on the use of the Lyapunov functional technique. In general, constructing a Lyapunov functional for a concrete n-dimensional FDE is not an easy task. Frequently, a new Lyapunov functional for each model under consideration is required. Contrary to the usual, our techniques (see [19], [20], [21]) do not involve Lyapunov functionals, and our method applies to general systems.

In this paper, we consider general retarded FDEs

$$\dot{x}_i(t) = r_i(t) f_i(x_t), \qquad t \geq 0, \tag{5}$$

with $r_i : [0, +\infty) \to (0, +\infty)$ and $f_i : C_n \to \mathbb{R}$ continuous functions, $i \in I := \{1, \cdots, n\}$. The phase space is the space $C_n := C([-\tau, 0]; \mathbb{R}^n)$, $\tau > 0$, equipped with the sup norm $\|\varphi\| = \sup_{\theta \in [-\tau,0]} |\varphi(\theta)|$, $\varphi \in C_n$, relative to the norm $|x| = \max\{|x_i| : i \in \{1, \cdots, n\}\}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, in $\mathbb{R}^n$. As usual, x_t denotes the function in C_n defined by $x_t(\theta) = x(t + \theta)$, $-\tau \leq \theta \leq 0$.

An equilibrium point $x^* \in \mathbb{R}^n$ of (5) is said to be *globally asymptotically stable* if it is stable and it is a global attractor of all solutions of (5).

We now set some notation. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we say that $x > 0$ if $x_i > 0$ for $i = 1, \ldots, n$ and x^{-1} is the vector given by $x^{-1} := (x_1^{-1}, \ldots, x_n^{-1})$. For $x \in \mathbb{R}^n$, we use x to denote both the real vector and the constant function $\varphi(\theta) = x$ in C_n. For $\varphi = (\varphi_1, \ldots, \varphi_n) \in C_n$ and $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we denote by $a.\varphi$ the function in C_n, $(a_1\varphi_1, \ldots, a_n\varphi_n)$. C_n is supposed to be partially ordered with

$$\varphi \geq \psi \text{ if and only if } \varphi_i(\theta) \geq \psi_i(\theta), \qquad \theta \in [-\tau, 0], \qquad i = 1, \ldots, n.$$

Recall now some concepts from matrix analysis. A real matrix A is said to be *non-negative* if all its entries are non-negative. In this case, we write $A \geq 0$. If all of the entries are positive, A is said to be *positive* and we write $A > 0$. Similarly, if A and B are matrices of equal dimensions, $A \geq B$ or $A > B$ means that $A - B \geq 0$ or $A - B > 0$, respectively.

For a square real matrix $A = [a_{ij}]$ with non-positive off-diagonal entries, i.e., $a_{ij} \leq 0$ for all $i \neq j$, we say that A is an *M-matrix* if all the eigenvalues of A have a non-negative real part, or, equivalently, if all the principal minors of A are non-negative; and A is said to be a *non-singular M-matrix* if all the eigenvalues of A have a positive real part, or, equivalently, if all the principal minors of A are positive. For properties of M-matrices, we refer the reader to [22], Chapter 5.

If $x(t)$ is defined for $t \geq 0$, we say that $x(t)$ is *eventually monotone* if there is $t_0 > 0$ such that $x(t)$ is monotone on $[t_0, +\infty)$.

The remainder of this paper is organized as follows: In Section 2, a criterion for the global asymptotic stability of the equilibrium point $x = 0$ of the general system (5) is presented. Then we obtain a sufficient condition for the existence of a unique equilibrium point, and for its global asymptotic stability, of the system

$$\dot{x}_i(t) = -r_i(t)k_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \quad i = 1, \dots, n, \quad t \geq 0, \tag{6}$$

which will be applied to the study of the stability of several neural network models. In Section 3, we consider two types of neural network models with distributed delays, the Cohen-Grossberg and the static. For both, we give sufficient conditions for the existence and global asymptotic stability of an equilibrium point improving known results in the literature. Finally, in Section 4, we give a similar result for a general neural network system with discrete time-varying delays, which has, as subclass, several Cohen-Grossberg and bidirectional associative memory neural network models.

2 Main Results

Let $C_n := C([-\tau, 0]; \mathbb{R}^n)$ be equipped with the supremum norm $\|\cdot\|$ relative to the norm $|\cdot|$ in $\mathbb{R}^n$. In the phase space C_n, consider a nonautonomous system of delayed differential equations of the form

$$\dot{x}_i(t) = r_i(t)f_i(x_t), \quad i = 1, \dots, n, \quad t \geq 0, \tag{7}$$

where $r_i : [0, +\infty) \to (0, +\infty)$ and $f_i : C_n \to \mathbb{R}$ are continuous functions, $i \in I := \{1, \dots, n\}$.

For (7) the following hypotheses will be considered:

(H1) $r_i(t)$ is uniformly bounded on $[0, +\infty)$ and $\int_0^\infty r_i(t)dt = \infty$, $i \in I$;
(H2) **(i)** f_i is bounded on bounded sets of C_n, $i \in I$;
(ii) for all $\varphi \in C_n$ such that $\|\varphi\| = |\varphi(0)| > 0$, then $\varphi_i(0)f_i(\varphi) < 0$ for all $i \in I$ such that $|\varphi_i(0)| = \|\varphi\|$.

Note that **(H2)(ii)** implies that $x = 0$ is the unique equilibrium of (7).

The following result was proven in [21]:

Lemma 2.1 *[21] Consider the equation*

$$\dot{y}(t) = f(t, y_t), \quad t \geq t_0, \tag{8}$$

where $t_0 \in \mathbb{R}$ and $f : [t_0, +\infty) \times C_n \to \mathbb{R}^n$ is a continuous function, $f = (f_1, \ldots, f_n)$ satisfying:

(H2*) *for all $t \geq t_0$ and $\varphi \in C_n$ such that $|\varphi(\theta)| < |\varphi(0)|$, for $\theta \in [-\tau, 0)$, then $\varphi_i(0) f_i(t, \varphi) < 0$ for some $i \in I$ such that $|\varphi(0)| = |\varphi_i(0)|$.*

Then, the solutions of (8) are defined and bounded for $t \geq t_0$. Moreover, if $y(t)$ is a solution of (8) and $|y(t)| \leq K$ for $t \in [t_0 - \tau, t_0]$, then $|y(t)| \leq K$ for $t \geq t_0$.

Now, we state our main result on the global asymptotic stability of the equilibrium $x = 0$ of (7). We remark that the arguments used in the proof can be found in [19] and [21].

Theorem 2.2 *Assume* **(H1)**-**(H2)**. *Then the equilibrium $x = 0$ of (7) is globally asymptotically stable.*

Proof. From Lemma 2.1, we deduce that all solutions are defined and bounded on $[0, +\infty)$, and that $x = 0$ is uniformly stable. It remains to prove that zero is globally attractive.

Let $x(t) = (x_i(t))_{i=1}^n$ be a solution to (7). Set

$$-v_i = \liminf_{t \to +\infty} x_i(t), \quad u_i = \limsup_{t \to +\infty} x_i(t), \quad i \in I,$$

and

$$v = \max_{i \in I}\{v_i\}, \quad u = \max_{i \in I}\{u_i\}.$$

Note that $u, v \in \mathbb{R}$ and $-v \leq u$.

It is sufficient to prove that $\max(u, v) = 0$. Assume e.g. that $|v| \leq u$, so that $\max(u, v) = u$. (The situation is analogous for $|u| \leq v$.)

Let $i \in I$ such that $u_i = u$ and fix $\epsilon > 0$. There is $T = T(\epsilon) > 0$ such that $\|x_t\| < u_\epsilon := u + \epsilon$ for $t \geq T$.

As in [21], first we prove that there is a sequence $(t_k)_{k \in \mathbb{N}}$ with

$$t_k \nearrow +\infty, \quad x_i(t_k) \to u, \quad \text{and} \quad f_i(x_{t_k}) \to 0, \text{ as } k \to +\infty. \tag{9}$$

Case 1. Assume that $x_i(t)$ is eventually monotone. In this case, $\lim_{t \to +\infty} x_i(t) = u$ and for t large, either $\dot{x}_i(t) \leq 0$ or $\dot{x}_i(t) \geq 0$. Assume e.g. that $\dot{x}_i(t) \leq 0$ for t large (the situation $\dot{x}_i(t) \geq 0$ is analogous). Then $f_i(x_t) \leq 0$ for t large, hence

$$\limsup_{t \to +\infty} f_i(x_t) = c \leq 0.$$

If $c < 0$, then there is $t_0 > 0$ such that $f_i(x_t) < c/2$ for $t \geq t_0$, implying that

$$x_i(t) \leq x_i(t_0) + \frac{c}{2} \int_{t_0}^{t} r_i(s) ds.$$

From **(H1)** and the above inequality, we obtain $x_i(t) \to -\infty$ as $t \to +\infty$, which is not possible. Thus $c = 0$, which proves (9).

Case 2. Assume that $x_i(t)$ is not eventually monotone. In this case there is a sequence $(t_k)_{k \in \mathbb{N}}$ such that $t_k \nearrow +\infty$, $\dot{x}_i(t_k) = 0$ and $x_i(t_k) \to u$, as $k \to \infty$. Then $f_i(x_{t_k}) = 0$ for all $k \in \mathbb{N}$, and (9) holds.

Now we have to show that $u = 0$, hence $v = 0$ as well.

For $t \geq T$, we have $\|x_t\| < u_\epsilon$ and from **(H1)** and **(H2)** we conclude that there is $K > 0$ such that $|\dot{x}_j(t)| = |r_j(t) f_j(x_t)| < K$, $t \geq T, j \in I$. It follows that $x(t)$ and $\dot{x}(t)$ are uniformly bounded on $[0, +\infty)$, thus $\{x_{t_k} : k \in \mathbb{N}\} \subseteq C_n$ is bounded and equicontinuous. By Ascoli-Arzelà Theorem, for a subsequence, still denoted by (x_{t_k}), we have $x_{t_k} \to \varphi$ for some $\varphi \in C_n$. Since $\|x_{t_k}\| \leq u_\epsilon$ and $\epsilon > 0$ is arbitrary, then $\|\varphi\| \leq u$. From (9), we get $\varphi_i(0) = u$ and $f_i(\varphi) = 0$. Clearly $\|\varphi\| = |\varphi_i(0)| = u$ and from hypothesis **(H2)(ii)** we conclude that $u = 0$, and the theorem is proven. ■

In applications, neural networks models often take the form

$$\dot{x}_i(t) = -r_i(t) k_i(x_i(t))[b_i(x_i(t)) + f_i(x_t)], \quad i = 1, \ldots, n, \quad t \geq 0, \tag{10}$$

where $r_i : [0, +\infty) \to (0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_i : C_n \to \mathbb{R}$ are continuous functions, $i \in I$.

In the sequel, for (10) the following hypotheses will be considered:

(A1) for each $i \in I$, there is $\beta_i > 0$ such that

$$(b_i(u) - b_i(v))/(u - v) \geq \beta_i, \quad \forall u, v \in \mathbb{R}, u \neq v;$$

(A2) $f_i : C_n \to \mathbb{R}$ are Lipschitz functions with constants l_i, $i \in I$.

Here, we give sufficient conditions for the existence, uniqueness and global asymptotic stability of the equilibrium point for system (10). To prove the existence and uniqueness of the equilibrium, we make use of arguments in recent literature [2], [11], [16], and [23]. First, we state the following lemma.

Lemma 2.3 *[24] If $H : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous and injective function such* *that*

$$\lim_{|x|\to\infty} |H(x)| = \infty,$$

then H is a homeomorphism of $\mathbb{R}^n$.

Lemma 2.4 *Assume* **(A1)**, **(A2)** *and $\beta_i > l_i$ for $i \in I$. Then system (10) has a unique equilibrium point $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$.*

Proof. Define the continuous map

$$\begin{aligned} H : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto (b_1(x_1) + f_1(x), \ldots, b_n(x_n) + f_n(x)), \quad x = (x_1, \cdots, x_n). \end{aligned}$$

First, we prove that H is injective. By way of contradiction, assume that there exist $x, y \in \mathbb{R}^n$, with $x \neq y$, such that $H(x) = H(y)$. It follows that $b_i(x_i) + f_i(x) = b_i(y_i) + f_i(y)$ for all $i \in I$, hence

$$|b_i(x_i) - b_i(y_i)| = |f_i(x) - f_i(y)|, \quad i \in I,$$

and from the hypotheses we have

$$\beta_i |x_i - y_i| \leq l_i |x - y| < \beta_i |x - y|, \quad i \in I,$$

which is a contradiction.

Now we prove that $\lim_{|x|\to\infty} |H(x)| = \infty$. Let $\gamma := \min_{i\in I}(\beta_i - l_i) > 0$. For $x \in \mathbb{R}^n$ and $i_0 \in I$ such that $|x_{i_0}| = |x|$, we have

$$\begin{aligned} |H(x)| &\geq |b_{i_0}(x_{i_0}) + f_{i_0}(x)| \\ &= |(b_{i_0}(x_{i_0}) - b_{i_0}(0)) + (f_{i_0}(x) - f_{i_0}(0)) + (b_{i_0}(0) + f_{i_0}(0))| \\ &\geq (\beta_{i_0} - l_{i_0})|x_{i_0}| - |b_{i_0}(0) + f_{i_0}(0)| \\ &\geq \gamma |x| - |b(0) + f(0)|, \end{aligned}$$

then $|H(x)| \to +\infty$, as $|x| \to \infty$.

From the above lemma we conclude that H is a homeomorphism, hence there is a unique $x^* \in \mathbb{R}^n$ such that $H(x^*) = 0$, i.e., x^* is the unique equilibrium point of (10). ■

Lemma 2.5 *Assume* **(A1)**, **(A2)** *and* $\beta_i > l_i$ *for all* $i \in I$. *Suppose that* $x^* = 0$ *is the equilibrium of (10). Then the function* $g = (g_1, \ldots, g_n) : C_n \to \mathbb{R}^n$ *defined by* $g_i(\varphi) = -k_i(\varphi_i(0))[b_i(\varphi_i(0)) + f_i(\varphi)]$, *satisfies* **(H2)**.

Proof. Clearly g satisfies **(H2)(i)**.

Let $\varphi \in C_n$ be such that $\|\varphi\| = |\varphi(0)| > 0$ and consider $i \in I$ such that $|\varphi_i(0)| = \|\varphi\|$.

Since $x^* = 0$ is the equilibrium, then $b_j(0)+f_j(0) = 0$ for all $j \in I$. If $\varphi_i(0) > 0$, then $\|\varphi\| = \varphi_i(0)$ and from the hypotheses we conclude that

$$k_i(\varphi_i(0))(b_i(\varphi_i(0)) + f_i(\varphi)) = k_i(\varphi_i(0))[(b_i(\varphi_i(0)) - b_i(0)) + (f_i(\varphi) - f_i(0))]$$

$$\geq k_i(\varphi_i(0))(\beta_i - l_i)\|\varphi\| > 0.$$

Analogously for the situation $\varphi_i(0) < 0$. ∎

Assume that $x^* = (x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ is the equilibrium point of (10). By translating it to the origin by the change $\bar{x}(t) = x(t) - x^*$, (10) becomes

$$\dot{\bar{x}}_i(t) = -r_i(t)\bar{k}_i(\bar{x}_i(t))[\bar{b}_i(\bar{x}_i(t)) + \bar{f}_i(\bar{x}_t)], \quad i \in I, \quad t \geq 0, \tag{11}$$

with $\bar{k}_i(u) = k_i(u + x_i^*)$, $\bar{b}_i(u) = b_i(u + x_i^*) - b_i(x_i^*)$ and $\bar{f}_i(\varphi) = f_i(x^* + \varphi) - f_i(x^*)$. Clearly b_i and f_i satisfy **(A1)** and **(A2)** if and only if $\bar{b}_i$ and $\bar{f}_i$ satisfy **(A1)**, **(A2)**. From Lemmas 2.4 and 2.5, and Theorem 2.2, we have the following result:

Theorem 2.6 *Assume* **(H1)**, **(A1)** *and* **(A2)**. *If* $\beta_i > l_i$ *for all* $i \in I$, *then system (10) has a unique equilibrium point which is globally asymptotically stable.*

3 Global stability for neural network models with distributed delays

In this section, we shall apply the study in last section to two different types of neural network models with distributed delays, improving recent stability results in the literature (see examples below).

Consider the following generalization of the Cohen-Grossberg model (1),

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) + \sum_{j=1}^{n} f_{ij}(x_{j,t})\right], \quad i \in 1,\ldots,n, \tag{12}$$

where $k_i : \mathbb{R} \to (0,+\infty)$, $b_i : \mathbb{R} \to \mathbb{R}$ and $f_{ij} : C_1 \to \mathbb{R}$ are continuous functions, $i,j = 1,\ldots,n$.

Remark 3.1 Model (12) generalizes several neural network models, which have been studied in [3], [7], [10], [15], [17], [25], [26].

For system (12), we assume **(A1)** and

(A3) $f_{ij} : C_1 \to \mathbb{R}$ are Lipschitz functions with constants l_{ij}, $i,j \in I$.

Define the square real matrices,

$$B = diag(\beta_1,\ldots,\beta_n), \quad A = [l_{ij}] \quad \text{and} \quad N = B - A, \tag{13}$$

where $\beta_1,\ldots,\beta_n$ are as in **(A1)**.

Theorem 3.1 *Assume* **(A1)** *and* **(A3)**. *If N is a non-singular M-matrix, then there is a unique equilibrium point of (12), which is globally asymptotically stable.*

Proof. If N is a non-singular M-matrix, then (see [22]) there is $d = (d_1,\ldots,d_n) > 0$ such that $Nd > 0$, i.e.,

$$\beta_i d_i > \sum_{j=1}^{n} l_{ij} d_j, \quad i \in I. \tag{14}$$

The change $y_i(t) = d_i^{-1} x_i(t)$ transforms (12) into

$$\dot{y}_i(t) = -k_i(d_i y_i(t)) d_i^{-1}\left[b_i(d_i y_i(t)) + \sum_{j=1}^{n} f_{ij}(d_j y_{j,t})\right], \quad i \in I. \tag{15}$$

Defining, for each $i \in I$,

$$\bar{f}_i(\varphi) = d_i^{-1} \sum_{j=1}^{n} f_{ij}(d_j \varphi_j), \quad \varphi = (\varphi_1,\ldots,\varphi_n) \in C_n,$$

$$\bar{b}_i(u) = d_i^{-1} b_i(d_i u), \quad \bar{k}_i = k_i(d_i u), \quad u \in \mathbb{R},$$

system (15) has the form

$$\dot{y}_i(t) = -\bar{k}_i(y_i(t))[\bar{b}_i(y_i(t)) + \bar{f}_i(y_t)], \quad i \in I, \quad t \geq 0. \tag{16}$$

For $\varphi, \psi \in C_n$ and $i \in I$, we have

$$|\bar{f}_i(\varphi) - \bar{f}_i(\psi)| = d_i^{-1} \left| \sum_{j=1}^{n} f_{ij}(d_j \varphi_j) - \sum_{j=1}^{n} f_{ij}(d_j \psi_j) \right| \leq \left(d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j \right) \|\varphi - \psi\|,$$

thus $\bar{f}_i$ is a Lipschitz function with constant $l_i := d_i^{-1} \sum_{j=1}^{n} l_{ij} d_j$, $i \in I$. Moreover, $\bar{b}_i$ satisfies **(A1)** with $\bar{\beta}_i = \beta_i$, and from (14) we have $\beta_i > l_i$, $i \in I$. The conclusion follows now from Theorem 2.6. ∎

Example 3.1 Consider the Cohen-Grossberg neural network model with discrete delays

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum_{j=1}^{n} \sum_{p=1}^{P} a_{ij}^{(p)} f_j(x_j(t - \tau_{ij}^{(p)})) + J_i \right], i \in I, \tag{17}$$

where $P \in \mathbb{N}$, $J_i, a_{ij}^{(p)} \in \mathbb{R}$, $\tau_{ij}^{(p)} \geq 0$, and $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $i, j = 1, \ldots, n$, $p = 1, \ldots, P$, recently studied in [2] and [27]. Let $\tau = \max\{\tau_{ij}^{(p)} : i, j \in I, p = 1, \ldots, P\}$.

System (17) has the form (12) for $f_{ij}(\varphi) = -\sum_{p=1}^{P} a_{ij}^{(p)} f_j(\varphi(-\tau_{ij}^{(p)}))$, $\varphi \in C_1 = C([-\tau, 0], \mathbb{R})$. Since $f_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants l_i, f_{ij} is also a Lipschitz function, with Lipschitz constant $l_{ij} = \sum_{p=1}^{P} |a_{ij}^{(p)}| l_j$, for all $i, j \in I$. Theorem 3.1 applied to system (17) gives the following result:

Corollary 3.2 *Assume* **(A1)** *and that $f_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant l_i, for all $i \in I$. If $N := B - A$, where $B = diag(\beta_1, \ldots, \beta_n)$ and $A = [l_{ij}]$ with $l_{ij} = \sum_{p=1}^{P} |a_{ij}^{(p)}| l_j$, is a non-singular M-matrix, then there is a unique equilibrium point of (17), which is globally asymptotically stable.*

Remark 3.2 For system (17), the existence and uniqueness of an equilibrium point was already obtained by Y. Chen [2], but he assumed the following additional hypotheses:

(i) For each $i \in I$, there exist $\underline{k}_i, \overline{k}_i > 0$ such that

$$0 < \underline{k}_i \leq k_i(u) \leq \overline{k}_i, \quad \forall u \in \mathbb{R};$$

(ii) $\underline{N} := B\underline{K} - A\overline{K}$ is a non-singular M-matrix, where $\underline{K} = diag(\underline{k}_1, \ldots, \underline{k}_n)$ and $\overline{K} = diag(\overline{k}_1, \ldots, \overline{k}_n)$.

Note that, if **(i)** holds, then $\underline{N}$ is a non-singular M-matrix which implies that N is a non-singular M-matrix. But the reverse is not truth. The above Corollary 3.2 improves strongly the criterion in [2].

3.2 Static neural network models with S-type distributed delays

Consider the following generalization of the static model (3),

$$\dot{x}_i(t) = -k_i(x_i(t))\left[b_i(x_i(t)) + f_i\left(\sum_{j=1}^{n}\omega_{ij}\int_{-\tau}^{0} x_j(t+\theta)d\eta_{ij}(\theta) + J_i\right)\right], \quad (18)$$

$i \in I$, where $J_i, \omega_{ij} \in \mathbb{R}$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions and $\eta_{ij} : [-\tau, 0] \to \mathbb{R}$ are normalized bounded variation functions, i.e., $\eta_{ij} \in BV([-\tau, 0]; \mathbb{R})$ with $Var_{[-\tau,0]}\eta_{ij} = 1$, $i, j = 1, \dots, n$. Assume the hypothesis:

(A4) $f_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with constant l_i, for $i \in I$.

For each $i \in I$, the function defined by

$$\bar{f}_i(\varphi) = f_i\left(\sum_{j=1}^{n}\omega_{ij}\int_{-\tau}^{0} \varphi_j(\theta)d\eta_{ij}(\theta) + J_i\right), \qquad \varphi = (\varphi_1, \cdots, \varphi_n) \in C_n$$

is a Lipschitz function with constant $l_i \sum_{j=1}^{n} |\omega_{ij}|$. Define the following square real matrices:

$$B = diag(\beta_1, \dots, \beta_n) \quad \text{and} \quad M = B - [l_i|\omega_{ij}|]. \quad (19)$$

We have the following result:

Theorem 3.3 *Assume* **(A1)** *and* **(A4)***. If M is a non-singular M-matrix, then there is a unique equilibrium point of (18), which is globally asymptotically stable.*

Proof. The proof is analogous to the proof of Theorem 3.1, so it is omitted. ∎

Example 3.2 Consider the static neural network model with S-type distributed delay studied in [13]

$$\dot{x}_i(t) = -b_i(\lambda)x_i(t) + f_i\left(\sum_{j=1}^{n}\omega_{ij}(\lambda)\int_{-\tau(\lambda)}^{0} x_j(t+\theta)d\eta_{ij}(\lambda,\theta) + J_i(\lambda)\right), \quad (20)$$

$i \in I$, where $\lambda \in \Lambda \subseteq \mathbb{R}$ is a real parameter, $f_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $b_i, \tau : \Lambda \to [0, +\infty)$ and $J_i, \omega_{ij} : \Lambda \to \mathbb{R}$ are real functions with $0 \leq \tau(\lambda) \leq \tau$ for some $\tau > 0$, and, for each $\lambda \in \Lambda$, $\theta \mapsto \eta_{ij}(\lambda, \theta)$ are normalized bounded variation functions on $[-\tau(\lambda), 0]$, $i, j \in I$.

Suppose that, for each $i, j \in I$, there exist $\underline{b}_i, \overline{\omega}_{ij} > 0$ such that,

$$0 < \underline{b}_i \leq b_i(\lambda), \quad \text{and} \quad |\omega_{ij}(\lambda)| \leq \overline{\omega}_{ij}, \quad \text{for all } \lambda \in \Lambda.$$

Assume that the functions f_i satisfy **(A4)**, and define the following square real matrices:

$$B(\lambda) = diag(b_1(\lambda), \dots, b_n(\lambda)), \quad M(\lambda) = B(\lambda) - [l_i|\omega_{ij}(\lambda)|], \quad \lambda \in \Lambda,$$

$$\underline{B} = diag(\underline{b}_1, \dots, \underline{b}_n) \quad \text{and} \quad \underline{M} = \underline{B} - [l_i\overline{\omega}_{ij}].$$

Definition 3.1 *System (20) is said to be globally asymptotically robust stable on Λ if, for each $\lambda \in \Lambda$, there is an equilibrium point of (20) which is globally asymptotically stable.*

The next result is an immediate consequence of Theorem 3.3.

Corollary 3.4 *Assume* **(A4)**. *If $\underline{M}$ is a non-singular M-matrix, then system (20) is globally asymptotically robust stable on Λ.*

Proof. Let $\lambda_0 \in \Lambda$. Since $\underline{M} \leq M(\lambda_0)$ and $\underline{M}$ is a non-singular M-matrix, then $M(\lambda_0)$ is a non-singular M-matrix as well (see [22]), thus we have the result from Theorem 3.3. ∎

Remark 3.3 Besides the assumptions in Corollary 3.4, Wang and Wang [13] assumed that the maps $\lambda \mapsto b_i(\lambda)$ were bounded and that, for each $\lambda \in \Lambda$, $\theta \mapsto \eta_{ij}(\lambda, \theta)$ were nondecreasing normalized bounded variation function on $[-\tau(\lambda), 0]$. Thus the last result improves the main result in [13].

Remark 3.4 The results in this section also hold for non-autonomous models of the form (10), if the functions $r_i(t)$ satisfy **(H1)**.

4 Global stability of neural network models with discrete time-varying delays

Consider the following neural network model:

$$\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + \sum_{j=1}^{n} \sum_{p=1}^{P} h_{ij}^{(p)}(x_j(t - \tau_{ij}^{(p)}(t))) \right], \quad i \in I, \quad (21)$$

where $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, h_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ and $\tau_{ij}^{(p)} : [0, +\infty) \to [0, +\infty)$ are continuous functions such that, $h_{ij}^{(p)}$ are Lipschitz functions with constants $l_{ij}^{(p)}$, $\tau_{ij}^{(p)}$ are bounded and **(A1)** holds for b_i, $i, j = 1, \ldots, n$, $p = 1, \ldots, P$.

System (21) is a generalization of several neural network models with discrete time-varying delays [8], [9], [16]. It is important to note that the general setting of (21) allows us to consider the bidirectional associative memory neural network models in [16] and [18] as subclasses.

Let $\tau \geq 0$ be such that $0 \leq \tau_{ij}^{(p)}(t) \leq \tau$ for all $t \geq 0$, $i, j \in I$ and $p \in \{1, \ldots, P\}$, and define the square real matrices

$$B = diag(\beta_1, \ldots, \beta_n) \qquad \text{and} \qquad N := B - [l_{ij}],$$

where $\beta_1, \ldots, \beta_n$ are as in **(A1)** and $l_{ij} = \sum_{p=1}^{P} l_{ij}^{(p)}$.

Theorem 4.1 *Assume* **(A1)**, $0 \leq \tau_{ij}^{(p)}(t) \leq \tau$ *and* $h_{ij}^{(p)}$ *are Lipschitz functions with constants* $l_{ij}^{(p)}$, $i, j \in I$, $p \in \{1, \ldots, P\}$.

If N is a non-singular M-matrix, then there is a unique equilibrium point of (21), which is globally asymptotically stable.

Proof. Since N is a non-singular M-matrix, then (see [22]) there is $d = (d_1, \ldots, d_n) > 0$ such that $Nd > 0$, i.e.,

$$\beta_i > d_i^{-1} \left(\sum_{j=1}^{n} l_{ij} d_j \right), \quad i \in I. \quad (22)$$

The change $z_i(t) = d_i^{-1} x_i(t)$ transforms (21) into

$$\dot{z}_i(t) = -\bar{k}_i(z_i(t))[\bar{b}_i(z_i(t)) + h_i(t, z_t)], \quad i \in I, \quad t \geq 0, \quad (23)$$

where

$$h_i(t,\varphi) = d_i^{-1}\left[\sum_{j=1}^n \sum_{p=1}^P h_{ij}^{(p)}(d_j\varphi_j(-\tau_{ij}^{(p)}(t)))\right], \quad t \geq 0, \varphi \in C_n, i \in I,$$

$$\bar{k}_i(u) = k_i(d_i u), \quad \bar{b}_i(u) = d_i^{-1} b_i(d_i u), \quad u \in \mathbb{R}, i \in I.$$

Note that $(\bar{b}_i(u) - \bar{b}_i(v))/(u - v) \geq \beta_i$ for $u, v \in \mathbb{R}$, $u \neq v$, i.e., condition **(A1)** is satisfied by the functions $\bar{b}_i(u)$, $i \in I$. For $\varphi, \psi \in C_n$ and $t \geq 0$ we have

$$|h_i(t,\varphi) - h_i(t,\psi)| \leq \left(d_i^{-1}\sum_{j=1}^n l_{ij} d_j\right) \|\varphi - \psi\|, \quad i \in I,$$

that is, $h_i(t,\cdot)$ is a uniform Lipschitz function on C_n for all $t \geq 0$, with Lipschitz constant $l_i := d_i^{-1}\sum_{j=1}^n l_{ij} d_j < \beta_i$.

Observe that system (23) has an equilibrium point $y^* = (y_1^*, \cdots, y_n^*) \in \mathbb{R}^n$ if and only if $H(y^*) = 0$, where

$$H(y) = \left(\bar{b}_i(y_i) + d_i^{-1}\sum_{j=1}^n \sum_{p=1}^P h_{ij}^{(p)}(d_j y_j)\right)_{i=1}^n, \quad y = (y_1, \cdots, y_n) \in \mathbb{R}^n.$$

Arguing as in the proof of Lemma 2.4, we conclude that there is a unique point $y^* = (y_1^*, \cdots, y_n^*)$ such that $H(y^*) = 0$.

By translating the equilibrium to the origin by the change $y_i(t) = z_i(t) - y_i^*$, (23) becomes

$$\dot{y}_i(t) = g_i(t, y_t), \quad t \geq 0, i \in I, \tag{24}$$

where $g = (g_1, \ldots, g_n) : [0, +\infty) \times C_n \to \mathbb{R}^n$ is defined by

$$g_i(t,\varphi) = -\bar{k}_i(\varphi_i(0) + y_i^*)[\bar{b}_i(\varphi_i(0) + y_i^*) + h_i(t, \varphi + y^*)], \quad \varphi \in C_n, t \geq 0, i \in I.$$

Arguing as in the proof of Lemma 2.5, we conclude that g satisfies **(H2*)**, thus from Lemma 2.1 all solutions of (24) are defined and bounded on $[0, +\infty)$.

Let $y(t) = (y_i(t))_{i=1}^n$ be a solution of (24). Set

$$-v_i = \liminf_{t\to+\infty} y_i(t), \quad u_i = \limsup_{t\to+\infty} y_i(t), \quad i \in I,$$

and

$$v = \max_{i\in I}\{v_i\}, \quad u = \max_{i\in I}\{u_i\}.$$

Note that $u, v \in \mathbb{R}$ and $-v \leq u$.

It is sufficient to prove that $\max(u,v)=0$. Assume e.g. that $|v|\le u$, so that $\max(u,v)=u$. (The situation $|u|\le v$ is analogous).

Fix $\epsilon>0$ and let $T=T(\epsilon)>0$ be such that $\|y_t\|<u_\epsilon:=u+\epsilon$ for $t\ge T$. Let $i\in I$ such that $u_i=u$.

Arguing as in the proof of Theorem 2.2, we conclude that there is a positive real sequence $(t_k)_{k\in\mathbb{N}}$ such that

$$t_k\nearrow+\infty,\quad y_i(t_k)\to u\quad\text{and}\quad g_i(t_k,y_{t_k})\to 0,\quad\text{as}\quad k\to+\infty. \tag{25}$$

From our hypotheses, clearly we have g bounded on $[0,+\infty)\times K$ for all bounded sets $K\subseteq C_n$. Since $\|y_t\|<u_\epsilon$ for $t\ge T$, we have $(\dot{y}_j(t))_{j=1}^n$ bounded. Hence $y(t)$ and $\dot{y}(t)$ are uniformly bounded on $[0,+\infty)$, thus $\{y_{t_k}:k\in\mathbb{N}\}\subseteq C_n$ is bounded and equicontinuous. By Ascoli-Arzelà Theorem, for a subsequence, still denoted by (y_{t_k}), we have $y_{t_k}\to\varphi$ for some $\varphi\in C_n$. Since $\|y_{t_k}\|\le u_\epsilon$ and $\epsilon>0$ is arbitrary, then $\|\varphi\|\le u$. Moreover, from (25) we get $\varphi_i(0)=u$.

Since the sequence $\left(\left(\tau_{ij}^{(p)}(t_k)\right)\right)_{k\in\mathbb{N}}$ in $\mathbb{R}^{Pn^2}$ is bounded, there is a subsequence of (t_k), still denoted by (t_k), which converges to a point $(\tau_{ij}^{(p)*})\in[0,\tau]^{Pn^2}$. Thus

$$g_i(t_k,y_{t_k})\to c_i\quad\text{as } k\to+\infty,$$

with

$$c_i:=-\bar{k}_i(\varphi_i(0)+y_i^*)[\bar{b}_i(\varphi_i(0)+y_i^*)+\bar{h}_i(\varphi)],$$

where

$$\bar{h}_i(\varphi):=d_i^{-1}\left[\sum_{j=1}^n\sum_{p=1}^P h_{ij}^{(p)}(d_j(\varphi_j(-\tau_{ij}^{(p)*})+y_j^*))\right].$$

Since y^* is the equilibrium point of (23), we have $\bar{b}_j(y_j^*)+\bar{h}_j(0)=0$ for all $j\in I$.

If $\varphi_i(0)=u>0$, then

$$\bar{b}_i(\varphi_i(0)+y_i^*)+\bar{h}_i(\varphi)=\bar{b}_i(\varphi_i(0)+y_i^*)-\bar{b}_i(y_i^*)+\bar{h}_i(\varphi)-\bar{h}_i(0)$$

$$\ge\beta_i\varphi_i(0)-d_i^{-1}\sum_{j=1}^n l_{ij}d_j\|\varphi\|=\left(\beta_i-d_i^{-1}\sum_{j=1}^n l_{ij}d_j\right)u>0.$$

Since $\bar{k}_i(u+y_i^*)>0$, we have $c_i\ne 0$, which contradicts (25). Hence $u=0$ and then all solutions $y(t)$ of (24) verify $y(t)\to 0$ as $t\to+\infty$, that is, the equilibrium point of (21) is globally asymptotically stable. ■

Example 4.1 Consider the Cohen-Grossberg neural network model studied in [9]

$$\dot{x}_i(t) = -k_i(x_i(t)) \Bigg[b_i(x_i(t)) - \sum_{j=1}^{n} c_{ij} g_j(x_j(t)) \\ - \sum_{j=1}^{n} a_{ij} f_j(x_j(t - \tau_{ij}(t))) + J_i \Bigg], \quad i \in I, \tag{26}$$

where $a_{ij}, c_{ij}, J_i \in \mathbb{R}$ and $\tau_{ij} : [0, +\infty) \to [0, +\infty)$, $k_i : \mathbb{R} \to (0, +\infty)$, $b_i, f_i, g_i : \mathbb{R} \to \mathbb{R}$ are continuous functions, $i, j \in I$, with τ_{ij} bounded.

Assume that the functions b_i satisfy **(A1)** and $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i and γ_i, for $i \in I$. Define the square real matrices

$$B = diag(\beta_1, \dots, \beta_n) \quad \text{and} \quad N = B - [|c_{ij}|\gamma_j] - [|a_{ij}|\theta_j],$$

where $\beta_1, \dots, \beta_n$ are as in **(A1)**.

Clearly, (26) is a particular situation of (21). From Theorem 4.1 we have the following result:

Corollary 4.2 *Assume* **(A1)** *and that $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i and γ_i, $i = 1, \dots, n$.*

If N is a non-singular M-matrix, then there is an equilibrium point of (26), which is globally asymptotically stable.

Remark 4.1 Model (26) was studied in [9] and [11]. Chen and Rong [9] proved that all solutions of (26) converge exponentially to the equilibrium point with the additional hypotheses:

(i) $\tau_{ij}(t)$ are continuously differentiable functions with $\tau'_{ij}(t) \le 1$ for all $t \ge 0$ and $i, j \in I$;
(ii) There are $\underline{k}_i, \overline{k}_i > 0$ such that

$$0 < \underline{k}_i \le k_i(u) \le \overline{k}_i, \quad u \in \mathbb{R}, i \in I.$$

Without condition **(i)** and assuming that there is $\underline{k}_i > 0$ such that $\underline{k}_i \le k_i(u)$ for all $u \in \mathbb{R}$, $i \in I$, instead of **(ii)**, Song and Cao [11] proved the exponential stability of (26). In a forthcoming paper, the exponential stability of the equilibrium of general models (12) and (21) will be addressed.

Example 4.2 Consider the Hopfield neural network model

$$\dot{x}_i(t) = -d_i(\lambda)x_i(t) + \sum_{j=1}^{n} c_{ij}(\lambda)g_j(x_j(t)) + \sum_{j=1}^{n} a_{ij}(\lambda)f_j(x_j(t-\tau_{ij}(t)))+ \\ +J_i(\lambda), \quad i \in I, \tag{27}$$

where $\lambda \in \Lambda \subseteq \mathbb{R}$ is a real parameter, $\tau_{ij} : [0, +\infty) \to [0, +\infty)$ are bounded continuous functions, $a_{ij}, c_{ij}, d_i, J_i : \Lambda \to \mathbb{R}$ are real functions and $f_i, g_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions with constants θ_i, γ_i for $i, j \in I$.

Note that, for each $\lambda \in \Lambda$, (27) looks like (26) when $k_i(u) \equiv 1$ and $b_i(u) = d_i(\lambda)u$ for $u \in \mathbb{R}$, $i \in I$.

Assume that there are square real matrices $\bar{A} = [\bar{a}_{ij}] \geq 0$, $\bar{C} = [\bar{c}_{ij}] \geq 0$ and $\underline{D} = diag(\underline{d}_1, \dots, \underline{d}_n)$, with $\underline{d}_i > 0$ for all $i \in I$, such that, for each $\lambda \in \Lambda$,

$$|a_{ij}(\lambda)| \leq \bar{a}_{ij}, \quad |c_{ij}(\lambda)| \leq \bar{c}_{ij}, \quad \text{and} \quad 0 < \underline{d}_i \leq d_i(\lambda), \quad \forall i, j \in I.$$

For each $\lambda \in \Lambda$, define

$$D(\lambda) = diag(d_1(\lambda), \dots, d_n(\lambda)), \quad M(\lambda) = D(\lambda) - [|a_{ij}(\lambda)|\theta_j] - [|c_{ij}(\lambda)|\gamma_j] \quad \text{and}$$

$$\underline{M} = \underline{D} - [\bar{a}_{ij}\theta_j] - [\bar{c}_{ij}\gamma_j].$$

From Theorem 4.1 we have the following result:

Corollary 4.3 *If $\underline{M}$ is a non-singular M-matrix, then system (27) is globally asymptotically robust stable on Λ.*

Proof. Let $\lambda_0 \in \Lambda$. Since $\underline{M} \leq M(\lambda_0)$ and $\underline{M}$ is a non-singular M-matrix, then (see [22]) $M(\lambda_0)$ is also a non-singular M-matrix and the result follows from Theorem 4.1. ∎

Remark 4.2 In [10], the global asymptotic robust stability of the Hopfield model (27) with discrete independent delays $\tau_{ij}(t) \equiv \tau_{ij}$ was proved. Hence, our Corollary 4.3 is a generalization of the main result in [10].

It is important to note that the general setting of (21) allows us to consider the bidirectional associative memory neural network model with delays as a subclass.

Example 4.3 Consider the following model:

$$
\begin{cases}
\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) + \sum\limits_{p=1}^{P} g_i^{(p)}(x_i(t-\omega_i^{(p)}(t))) \right. \\
\qquad\qquad\qquad\qquad \left. - \sum\limits_{j=1}^{m} \sum\limits_{p=1}^{P} f_{ij}^{(p)}(y_j(t-\tau_{ij}^{(p)}(t))) \right], \; i = 1, \dots, n, \\
\\
\dot{y}_j(t) = -h_j(y_j(t)) \left[a_j(y_j(t)) + \sum\limits_{p=1}^{P} f_j^{(p)}(y_j(t-\rho_j^{(p)}(t))) \right. \\
\qquad\qquad\qquad\qquad \left. - \sum\limits_{i=1}^{n} \sum\limits_{p=1}^{P} g_{ji}^{(p)}(x_i(t-\sigma_{ji}^{(p)}(t))) \right], \; j = 1, \dots, m,
\end{cases}
\tag{28}
$$

for $t \geq 0$ and $n, m, P \in \mathbb{N}$, where $k_i, h_j : \mathbb{R} \to (0, +\infty)$, $b_i, a_j, g_i^{(p)}, f_j^{(p)}, g_{ji}^{(p)}, f_{ij}^{(p)} : \mathbb{R} \to \mathbb{R}$ are continuous functions and $\omega_i^{(p)}, \rho_j^{(p)}, \tau_{ij}^{(p)}, \sigma_{ji}^{(p)} : [0, +\infty) \to [0, +\infty)$ are bounded continuous functions, $i = 1, \dots, n$, $j = 1, \dots, m$ e $p = 1, \dots, P$.

Arik [15] and Wang and Zou [17] studied the bidirectional associative memory neural network model with discrete delays described by

$$
\begin{cases}
\dot{x}_i(t) = -x_i(t) + \sum\limits_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\
\\
\dot{y}_i(t) = -y_i(t) + \sum\limits_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_{ij})) + J_i
\end{cases}
, \qquad i = 1, \dots, n. \tag{29}
$$

Wang and Zou [18] incorporated inhibitory self-connections terms into model (29), and considered the following system

$$
\begin{cases}
\dot{x}_i(t) = -x_i(t) + c_{ii} g_i(x_i(t-d_{ii})) + \sum\limits_{j=1}^{n} a_{ij} f_j(y_j(t-\tau_{ij})) + I_i \\
\\
\dot{y}_i(t) = -y_i(t) + l_{ii} f_i(y_i(t-m_{ii})) + \sum\limits_{j=1}^{n} b_{ij} g_j(x_j(t-\sigma_{ij})) + J_i
\end{cases}
\quad i \in I. \tag{30}
$$

Recently, the following bidirectional associative memory neural network model with time-varying delays was considered in [16]:

$$
\begin{cases}
\dot{x}_i(t) = -k_i(x_i(t)) \left[b_i(x_i(t)) - \sum\limits_{j=1}^{m} c_{ij} f_j(\lambda_j y_j(t - \tau_{ij}(t))) + I_i \right], \\
\hfill i = 1, \dots, n, \\
\\
\dot{y}_j(t) = -h_j(y_j(t)) \left[a_j(y_j(t)) - \sum\limits_{i=1}^{n} d_{ji} g_i(\mu_i x_i(t - \sigma_{ji}(t))) + J_j \right], \\
\hfill j = 1, \dots, m,
\end{cases} \quad . \tag{31}
$$

Model (28), here considered for the first time, arises as a generalization of all these models. Since (28) is a particular situation of (21), from Theorem 4.1 we have the following result:

Corollary 4.4 *Suppose that a_j and b_i satisfy* **(A1)** *with constants α_j and β_i, respectively, that $k_i(u) > 0$ and $h_j(u) > 0$ for all $u \in \mathbb{R}$; $f_j^{(p)}, g_i^{(p)}, f_{ij}^{(p)}, g_{ji}^{(p)}$ are Lipschitz functions with Lipschitz constants $\theta_j^{(p)}, \gamma_i^{(p)}, \theta_{ij}^{(p)}, \gamma_{ji}^{(p)}$ respectively, and that $\omega_i^{(p)}, \rho_j^{(p)}, \tau_{ij}^{(p)}, \sigma_{ji}^{(p)}$ are bounded continuous functions, for $i = 1, \dots, n$, $j = 1, \dots, m$ and $p = 1, \dots, P$.*

Define

$$
N := \begin{bmatrix} B - G_d & -F \\ \\ -G & A - F_d \end{bmatrix}_{(n+m)\times(n+m)},
$$

where

$$
B = diag(\beta_1, \dots, \beta_n), \qquad A = diag(\alpha_1, \dots, \alpha_m)
$$

$$
G_d = diag\left(\sum_{p=1}^{P} \gamma_1^{(p)}, \dots, \sum_{p=1}^{P} \gamma_n^{(p)} \right), \qquad F_d = diag\left(\sum_{p=1}^{P} \theta_1^{(p)}, \dots, \sum_{p=1}^{P} \theta_m^{(p)} \right),
$$

$$
G = \left[\sum_{p=1}^{P} \gamma_{ji}^{(p)} \right]_{m\times n}, \qquad F = \left[\sum_{p=1}^{P} \theta_{ij}^{(p)} \right]_{n\times m}.
$$

If N is a non-singular M-matrix, then there is a unique equilibrium point of (28), which is globally asymptotically stable.

Remark 4.3 As remarked, (28) is a generalization of models (29), (30) and (31). With the same hypotheses of Corollary 4.4, the exponential stability of (29) and (30) was obtained in [15] and [18]. In [16], the same stability was obtained for system (31) with the additional hypotheses $k_i(u) \geq k_i > 0$ and $h_i(u) \geq h_i > 0$, $u \in \mathbb{R}$, $i \in I$. As mentioned in Remark 4.1, the question of the exponential asymptotic stability for delayed neural networks will be addressed in the general framework of systems of the form (12) and (21).

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