Diagonalizing triangular matrices via orthogonal Pierce decompositions *

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Abstract

A class of sufficient conditions is given to ensure that the sum a + b in a ring R, is equivalent to a sum x + y, which is an orthogonal Pierce decomposition. This is then used to show that a lower triangular matrix, with a regular diagonal is equivalent to its diagonal iff the matrix admits a lower triangular von Neumann inverse.

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1 Introduction

In this paper we shall give sufficient conditions under which the sum m = a + b is equivalent to an element n which admits an orthogonal Pierce Decomposition,

$$n = enf + (1-e)n(1-f) \text{ with } enfR \cap (1-e)n(1-f)R = (0) = Renf \cap R(1-e)n(1-f)$$

for suitable idempotents e and f. We shall then give several special cases and in particular examine the lower triangular matrices with regular diagonal. We begin by giving a few preliminaries.

Given a ring R with identity 1. An element x in R is called (von Neumann) regular if $xx^{-}x = x$ for some inner inverse (von Neumann inverse) x^{-} in R. A reflexive (1-2) inverses of x is any element x^{+}, such that $xx^{+}x = x, x^{+}xx^{+} = x^{+}$. We shall call the elements x and y *orthogonal* (with x regular) if

$$x^{+}y = 0 = yx^{+},$$

for some 1-2 inverse of x. It can be shown [4], that x and y are orthogonal iff

$$xR \cap yR = (0) = Rx \cap Ry. \tag{1}$$

A ring with 1 is called (von Neumann) **finite** if ab = 1 ⇒ ba = 1.
In this paper we shall consider the sum m = a + b, where a is regular and where we assume that the element $u = 1 + a^{+}b$ is a unit in R, for some 1-2 inverse a^{+} of a. We shall also need the idempotents $e = aa^{+}$ and $f = a^{+}a$, as well as the elements $v = 1 + ba^{+}, w = 1 + aa^{+}ba^{+}$ and $z = 1 + a^{+}ba^{+}a$. These elements are related as follows.

Lemma 1.1. (a) If one of the elements u,v,w or z is a unit, then all four are units.
(b) If one of the elements $a^{+}b, ba^{+}, aa^{+}ba^{+}$ or $a^{+}ba^{+}a$ is nilpotent, then all four are nilpotent

Proof. (a) Recall that (1 + pq)x = 1 implies that (1 + qp)(1 - qxp) = 1,
(b) Observe that $(a^{+}n)^{N} = a^{+}(aa^{+}ba^{+})^{N-1}aa^{+}b$ and $(aa^{+}ba)^{M} = a(a^{+}b)^{M}a^{+}$. The rest is clear. We also note that the indices of $a^{+}b$ and ba^{+} differ at most by one as do the indices of $a^{+}n$ and $aa^{+}ba^{+}$. We shall refer to these elements as "being related" to $a^{+}b$.
We next introduce the elements x = $au^{-1}f$ and y = $(1-e)bu^{-1}(1-f)$, which will play a key role in what follows. We first observe that

$$\begin{aligned}
&(i) && a^{+}v = a^{+}w = ua^{+} = za^{+} \text{ and } u^{-1}a^{+} = a^{+}v^{-1} \\
&(ii) && bu = vb \text{ and } bu^{-1} = v^{-1}b \\
&(iii) && v(1-e) = 1-e \text{ and } v^{-1}(1-e) = 1-e \\
&(iv) && m = (1-ew)b + au \\
&(v) && a^{+}m = fu \text{ and } ma^{+} = ve \ \ and \\
&(vi) && fz = uf \ \ and \ \ z(1-f) = 1-f.
\end{aligned} \tag{2}$$

Let us now use these to further characterize x and y. Indeed, x = $au^{-1}f = ev^{-1}a = ev^{-1}mu^{-1}f$ and y = $(1-e)bu^{-1}(1-f) = (1-e)v^{-1}b(1-f) = (1-e)v^{-1}(1-e)bu^{-1}(1-f) = (1-e)v^{-1}[(1-e)b+au]u^{-1}(1-f) = (1-e)v^{-1}mu^{-1}(1-f)$. We also shall need

$$(a) \ \ wx = a = xz \ \ \ (b) \ \ wy = y = yz. \tag{3}$$

The former shows the important fact that x is regular, because a is regular and w is a unit.
We shall denote (unit) equivalence n = pmq, with p and q units, by n ∼ m. We are now ready for our "horizontal" splittings.

2 Horizontal Splittings

Consider m = a + b, where a is regular and $u = 1 + a^{+}b$ is a unit. We present the orthogonal Pierce Decomposition of the equivalent elements $\gamma = v^{-1}mu^{-1}$, $\delta = w\gamma$ and $\lambda = \gamma z$. Let us first turn to the splitting of γ,

Theorem 2.1. Let m = a + b, with a regular and u invertible, and let $\gamma = v^{-1}mu^{-1}$. Then

$$\text{(i) } e\gamma(1-f) = 0 = (1-e)\gamma f$$

and

$$\text{(ii) } \gamma = v^{-1}mu^{-1} = \text{x + y}, \quad \text{(Pierce Decomposition)}$$

where $x = e\gamma f = au^{-1}f$ is regular and $y = (1-e)\gamma(1-f) = (1-e)bu^{-1}(1-f)$
(iii) there exists a 1-2 inverse x^+ such that $x^+y = 0 = yx^+$ and hence $xR \cap yR = (0) = Rx \cap Ry$ (orthogonality).

Proof. (i) From (2) we see that $e\gamma(1-f) = aa^+(v^{-1}mu^{-1})(1-f) = au^{-1}(a^+m)u^{-1}(1-f) = au^{-1}(fu)u^{-1}(1-f) = au^{-1}f(1-f) = 0$.
Likewise by symmetry $(1-e)\gamma(1-f) = (1-e)(v^{-1}mu^{-1})f = (1-e)ev^{-1} = 0$. In addition
$e\gamma = aa^+v^{-1}mu^{-1} = au^{-1}a^+mu^{-1} = au^{-1}fuu^{-1} = au^{-1}f = ev^{-1}a = x$ and
$(1-e)v^{-1}mu^{-1}(1-f) = (1-e)v^{-1}[(1-e)b+au]u^{-1}(1-f) = (1-e)v^{-1}(1-e)bu^{-1}(1-f) = (1-e)bu^{-1}(1-f) = (1-e)v^{-1}b(1-f) = \text{y}$.
(ii) This follows from the Pierce decomposition $\gamma = e\gamma f + (1-e)\gamma f + (e)\gamma(1-f) + (1-e)\gamma(1-f)$.
(iii) Since x = $ev^{-1}a$ is regular it is clear that $f(a^+v)e = x^+$ and that for this 1-2 inverse $x^+y = 0 = yx^+$.

We can actually improve this decomposition somewhat by giving the orthogonal Pierce decompositions of δ and λ.

Theorem 2.2. Let m = a + b, with a regular and u invertible. Then
(i) $\delta = (wv^{-1})mu^{-1} = a + y = v^{-1}m(u^{-1}z) = \lambda$, (orthogonal Pierce Decomposition),
where $y = (1-e)\gamma(1-f) = (1-e)bu^{-1}(1-f) = (1-e)v^{-1}b(1-f)$.
(ii) $a^+y = 0 = ya^+$ and hence $aR \cap yR = (0) = Ra \cap Ry$.

Proof. (i) This follows from (3), which shows that $\delta = wv^{-1}mu^{-1} = w(x+y) = a + y = (x+y)z = v^{-1}mu^{-1}z = \lambda$
(ii) It is clear that $a^+y = 0 = ya^+$ and hence that $aR \cap yR = (0) = Ra \cap Ry$.

Needless to say, when the element a^+b and its family members are **nilpotent**, with say $index(a^+b) = N$, then we may rewrite y as

$$\text{y = (1-e)} \sum_{i=0}^{N-1} (-1)^i b(a^+b)^i(1-f).$$

We shall return to this later and use path products to identify this.
The two orthogonal decompositions will be equal under the following conditions.

Corollary 2.1. Let m = a + b, with a regular and u invertible and let x = $au^{-1}f$. Then x = a iff ebf = 0

Proof. $ev^{-1}a = a$ iff $ev^{-1}e = e$. But (1 - e) = v(1 - e) or $ev^{-1}(1-e) = 0$. As such $ev^{-1} = ev^{-1}e = e$, which means that e = ev or $eba^+ = 0$ as desired.

Let us now examine several special cases of the orthogonal decomposition.

Corollary 2.2. Let m = a + b, with a regular and u invertible. Then

m is regular iff y is regular

Proof. Because $wv^{-1}mu^{-1} = a + y$ is an orthogonal Pierce decomposition, we see that m is regular iff a + y is regular iff both a and y are regular. But because a is already regular, we are left with y being regular.

Corollary 2.3. Let m = a + b, with a regular, u invertible and R finite. Then the following are equivalent
(i) m $\sim$ a
(ii) y = 0.
(iii) ar - sa = bu^{-1} has a solution pair in R.
In which case m is regular in R, and $u^{-1}a^+$ is a 1-2 inverse of m.

Proof. (i) ⇒(ii). If a is equivalent to m, then m is regular. In addition, because m is equivalent to a + y, we know that a + y is regular and p(a + y)q = a for some units p and q in R. Hence aR ≅ (a + y)R (isomorphic). But a and y are orthogonal and thus (a + y)R = aR $\dot{+}$ yR (internal direct sum). As such $aR \subseteq aR \dot{+} yR = (a+y)R$ and $aR \cong (a+y)R$. Using the finiteness of R, this ensures that $aR = (a+y)R = aR \dot{+} yR$ [5], which in turn forces y = 0. (ii) The remaining parts are clear.

Remarks

(i) The equivalence is false when R is not finite as seen from the example [3], where a = $\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and b = $\begin{bmatrix} 0 & 1-pq \\ 0 & 0 \end{bmatrix}$ with qp = 1 ≠ pq.

(ii) Even if R is finite, we cannot when m regular in R, deduce that y = 0, as seen from the real matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which is regular in $\mathbb{R}_{2\times 2}$, but for which y = m.

Let us next induce invertibility via the Jacobson radical.

Corollary 2.4. Let m = a + b, with a regular and b ∈ J, the Jacobson radical of R.
Then the following are equivalent
(i) m is regular in R.
(ii) y = 0
(iii) m is equivalent to a over R.
(iv) ar - sa = bu^{-1} has a solution pair r,s in R. (Roth Condition in R)
In which case $u^{-1}a^{+} = a^{+}v^{-1}$ is a reflexive inverse of m.

Proof. (i)⇒(ii) For any 1-2 inverse a^{+}, it is clear that $a^{+}b \in J$ and hence that u = 1 + $a^{+}b$ is a unit in R. Likewise v, w and z as well as their inverses are all units in R. From theorem (2.2) it follows that $\delta = wv^{-1}mu^{-1} = a + y$, where y = $(1-e)bu^{-1}(1-f) \in J$. Consequently, if m is regular in R, then so is δ and hence, since the decomposition is **orthogonal**, we may conclude that y is regular in R and lies in J. This forces y = 0.
(i)⇒(ii) This is clear from theorem (2.2).
(ii)⇒(iii) Clear from the Pierce decomposition
(iii)⇒(i) If m is equivalent to a and a is regular, then m is also regular.
(ii)⇔(i) This is clear.
Lastly from $m = vw^{-1}au$ we see that $m^{+} = u^{-1}(a^{+}w)v^{-1} = u^{-1}(a^{+}v)v^{-1} = u^{-1}a^{+}$.
We note that this proof is independent of the 1-2 inverse of a that we selected.

Corollary 2.5. Let m = a + b, with a,b ∈ S a subring of R. Further we assume that
(i) a regular in S, and (ii) b ∈ J_S, the Jacobson radical of S.
Then the following are equivalent
(i) m is regular in S.
(ii) y = 0
(iii) m ∼a over S.
(iv) ar - sa = bu^{-1} has a solution pair r,s in S.

If in addition $b \in J_R$ then these are also equivalent to
(v) $m \sim a$ over R
(vi) m is regular in R

Proof. The equivalence of the first four conditions follows from corollary (2.4). It is further clear that (iii) ⇒ (v) and (i) ⇒ (vi).
If in addition $b \in J_R$ and m is regular over R, then y is also regular over R and lies in J_R forcing it to vanish.

Corollary 2.6. Let m = a + b, with a,b ∈ S a subring of R. Further we assume that
(i) a regular in S, and (ii) b ∈ J_S, the Jacobson radical of S, and (iii) R is finite.
Then the following are equivalent
(i) m is regular in S.
(ii) y = 0
(iii) m ∼ a in S
(iv) the Roth condition holds in S

(v) m $\sim$ a in R.
In which case m is regular in R

Proof. Combine corollaries (2.5) and (2.3).

Let us next consider the ring of lower triangular matrices over a ring R'.

Corollary 2.7. Let R' be a ring with 1, R = $M_n(R')$ and let S be the subring of lower triangular matrices over R'. Suppose M = A + B with A diagonal and regular, and B strictly lower triangular.
The following are equivalent.
(i) M is regular in S
(ii) Y = 0
(iii) M $\sim$ A over S
(iv) AX - YA = BU^{-1} has a solution in S
If in addition R = $M_n(R')$ is finite then these are also equivalent to
(v) M $\sim$ A over R.
In this case M is regular in R.

Proof. The result follows from corollary (2.6).
The fact that regularity in R is not sufficient for the above, can be seen from the the same real matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which is regular in $\mathbb{R}_{2\times 2}$, but not in the subring of 2×2 lower triangular matrices.

3 Triangular Matrix Reduction

Let us now perform the actual direct reduction of M, to the orthogonal Pierce decomposition. The final canonical form will be identified with A + Y, and the entries in Y will as such be computed with aid of path products. The identification of the two methods will be completed by showing that the unital triangular matrices that are used precisely match the matrices V^{-1} and $U^{-1}Z$, thereby completing the circle. Consider

$$M = \begin{bmatrix} a_1 & & & 0 \\ \mathbf{b}_2^T & a_2 & & \\ \vdots & & \ddots & \\ & \mathbf{b}_n^T & & a_n \end{bmatrix} = A + B \in S,$$

where A = $diag(a_1, .., a_n)$, is a diagonal matrix with all entries regular, $\mathbf{b}_k^T$ is a row of size $1 \times (k-1)$, and B is strictly lower triangular. As such the latter lies in J, the Jacobson radical of S. We denote the leading $k \times k$ principal submatrix of a matrix Y by Y_k and partition $M_k = \begin{bmatrix} M_{k-1} & 0 \\ \mathbf{b}_k^T & a_k \end{bmatrix}$ for k = 1,2,3,..,n. In particular $M_0 = \emptyset$ and $M_1 = a_1$.
We now have

Theorem 3.1. If M is as above then there exist $n \times n$ matrices α and $\beta \in S_1$ such that $\alpha M \beta$ = N = A + C, where
(a) $C = \begin{bmatrix} 0 & & & 0 \\ \mathbf{c}_2^T & 0 & & \\ \vdots & & \ddots & \\ & \mathbf{c}_n^T & & 0 \end{bmatrix}$,
(b) $\mathbf{c}_k^T = (1 - e_k)\mathbf{b}_k^T \pi_{k-1}(I - A_{k-1}^+ A_{k-1})$, in which
(c) $\pi_k = \begin{bmatrix} Q_2 & 0 \\ 0 & I_{k-2} \end{bmatrix} \begin{bmatrix} Q_3 & 0 \\ 0 & I_{k-3} \end{bmatrix} ... \begin{bmatrix} Q_k & 0 \\ 0 & I_{k-k} \end{bmatrix}$ is $k \times k$,
(d) $Q_k = \begin{bmatrix} I_{k-1} & 0 \\ -a_k^+ \mathbf{b}_k^T \pi_{k-1}(I - A_{k-1}^+ A_{k-1}) & 1 \end{bmatrix}$ is $k \times k$.
In particular, $\pi_1 = 1 = Q_1$ and $Q_2 = \begin{bmatrix} 1 & 0 \\ -a_2^+ a_{21} 1.(1 - a_1 a_1) & 1 \end{bmatrix}$.

Proof. The second part is a recurrence for the $\mathbf{c}_k^T$, because N_{k-1} depends on the previous $\mathbf{c}_{k-1}^T, .., \mathbf{c}_2^T$ and the last part is a recurrence for the Q_k because π_{k-1} depends on $Q_2, ..., Q_{k-1}$.

We shall use induction on k, and begin by introducing the k by k matrices $P_k = \begin{bmatrix} I_{k-1} & 0 \\ -\mathbf{b}_k^T \pi_{k-1} A_{k-1}^+ & 1 \end{bmatrix}$ with $P_1 = 1 = Q_1$, and the $k \times k$ products $\Delta_k = \begin{bmatrix} P_k & 0 \\ 0 & I_{k-k} \end{bmatrix} \begin{bmatrix} P_3 & 0 \\ 0 & I_{k-3} \end{bmatrix} ... \begin{bmatrix} P_2 & 0 \\ 0 & I_{k-2} \end{bmatrix}$.

We further need the n by n matrices

$$\alpha_k = \begin{bmatrix} P_k & 0 \\ 0 & I_{n-k} \end{bmatrix} ... \begin{bmatrix} P_2 & 0 \\ 0 & I_{n-2} \end{bmatrix} \text{ and } \beta_k = \begin{bmatrix} Q_2 & 0 \\ 0 & I_{n-2} \end{bmatrix} \begin{bmatrix} Q_3 & 0 \\ 0 & I_{n-3} \end{bmatrix} ... \begin{bmatrix} Q_k & 0 \\ 0 & I_{n-k} \end{bmatrix}$$

and set $\alpha_n = \alpha$ and $\beta_n = \beta$. It is clear that $\alpha_k = diag(\Delta_k \oplus I_{n-k})$ and $\beta_k = diag(\pi_k \oplus I_{n-k})$.

We next partition $M = \begin{bmatrix} M_k & 0 \\ ? & ? \end{bmatrix}$ where $M_k = \begin{bmatrix} M_{k-1} & 0 \\ \mathbf{b}_k^T & a_k \end{bmatrix}$ is a k by k matrix, for k = 1,2,3,..,n. In particular $M_0 = \emptyset$ and $M_1 = a_1$.

Recalling that $N_k = A_k + C_k = \begin{bmatrix} a_1 & & 0 \\ \mathbf{c}_2^T & a_2 & \\ \vdots & & \\ & \mathbf{c}_k^T & a_k \end{bmatrix}$, we claim by induction that

$$\alpha_{k-1} M \beta_{k-1} = \begin{bmatrix} N_{k-1} & 0 & & & 0 \\ \mathbf{b}_k^T \pi_{k-1} & a_k & & & \\ \mathbf{b}_{k+1}(\pi_{k-1} \oplus 1) & & a_{k+1} & & \\ \vdots & & & & \\ \mathbf{b}_r^T(\pi_{k-1} \oplus I_{r-k}) & & & a_{r+1} & \\ \vdots & & & & \\ \mathbf{b}_n^T(\pi_{k-1} \oplus I_{n-k}) & & & & a_n \end{bmatrix}.$$

To do this we multiply by one more row sweep and one more column sweep, so that we reduce $\mathbf{b}_k^T \pi_{k-1}$ into the next $\mathbf{c}_k^T$. Indeed, we arrive at $\begin{bmatrix} P_k & 0 \\ 0 & I_{n-k} \end{bmatrix} \alpha_{k-1} M \beta_{k-1} \begin{bmatrix} Q_k & 0 \\ 0 & I_{n-k} \end{bmatrix} = \alpha_k M \beta_k$.

We first compute the effect on the k by k block:

$$P_k \begin{bmatrix} N_{k-1} & 0 \\ \mathbf{b}_k^T \pi_{k-1} & a_k \end{bmatrix} Q_k = \begin{bmatrix} I_{k-1} & 0 \\ -\mathbf{b}_k^T \pi_{k-1} A_{k-1}^+ & 1 \end{bmatrix} \begin{bmatrix} N_{k-1} & 0 \\ \mathbf{b}_k^T \pi_{k-1} & a_k \end{bmatrix} \begin{bmatrix} I_{k-1} & 0 \\ -a_k^+ \mathbf{b}_k^T \pi_{k-1}(I - A_{k-1}^+ A_{k-1}) & 1 \end{bmatrix} = \begin{bmatrix} N_{k-1} & 0 \\ X & a_k \end{bmatrix},$$

where X = $(1 - a_k a_k^+) \mathbf{b}_k^T \pi_{k-1}(I - A_{k-1}^+ A_{k-1})$.

Let us now examine the rest of the matrix. The post multiplication by Q_k only affects the first k columns of $\alpha_{k-1}^T M \beta_{k-1}$, however since we do not want to cut up the vectors $\mathbf{b}_r^T$, we shall multiply successive rows by $Q_k, diag(Q_k \oplus 1), ..., diag(Q_k \oplus I_{r-k-1})$,... r = k+1,k+2 ,...This gives for r = k+1, ..,n, that

$$\mathbf{b}_r^T \begin{bmatrix} Q_2 & 0 \\ 0 & I_{r-3} \end{bmatrix} \begin{bmatrix} Q_3 & 0 \\ 0 & I_{r-4} \end{bmatrix} ... \begin{bmatrix} Q_{k-1} & 0 \\ 0 & I_{r-k} \end{bmatrix} \begin{bmatrix} Q_k & 0 \\ 0 & I_{r-k-1} \end{bmatrix} = \mathbf{b}_r^T diag(\pi_k \oplus I_{r-k-1}), \ .. \ .$$

This shows that we have obtained the next stage with k replaced by k+1, i.e.

$$\alpha_k M \beta_k = \begin{bmatrix} N_k & 0 & & & & 0 \\ \mathbf{b}_{k+1}^T \pi_k & a_{k+1} & & & & \\ \mathbf{b}_{k+2}(\pi_k \oplus 1) & & a_{k+2} & & & \\ \vdots & & & & & \\ & \mathbf{b}_r^T(\pi_k \oplus I_{r-k-1}) & & & a_r & \\ \vdots & & & & & \\ & & & \mathbf{b}_n^T(\pi_k \oplus I_{n-k-1}) & & a_n \end{bmatrix}.$$

When k = n, we arrive at the final "vertical"reduction $\alpha M \beta = A + C$.

We thus obtain the consistency conditions for M to be regular in S, in the form $\mathbf{c}_k^T = 0$ for k = 2,..,n.

4 Identification of The Entries

Let us conclude by identifying the horizontal and vertical reductions. We have $\alpha M\beta = A + C$ on the one hand and $V^{-1}MU^{-1}Z = A + Y$ on the other. We claim that $\alpha = V^{-1}, U^{-1}Z = \beta$ and Y = C. To see this we denote the leading $k \times k$ submatrix of $V = I + BA^+$ by V_k and observe that $V_{k+1} = \begin{bmatrix} V_k & 0 \\ \mathbf{b}_{k+1}^T A_k^+ & 1 \end{bmatrix}$ and hence that $V_{k+1}^{-1} = \begin{bmatrix} V_k^{-1} & 0 \\ -\mathbf{b}_{k+1}^T A_k^+ V_k^{-1} & 1 \end{bmatrix}$.

On the other hand, from the product form of Δ_k we see that $\Delta_{k+1} = \begin{bmatrix} \Delta_k & 0 \\ -\mathbf{b}_{k+1}^T A_k^+ \Delta_k & 1 \end{bmatrix}$. Lastly, because $\Delta_2 = V_2^{-1}$, we may conclude that $\alpha_k = V_k^{-1}$, for all k and thus $\alpha = V^{-1}$. From the product form of β_k we deduce that

$$\beta = \begin{bmatrix} 1 & & & 0 \\ -\mathbf{q}_2^T & 1 & & \\ \vdots & & \ddots & \\ & -\mathbf{q}_n^T & & 1 \end{bmatrix} = \begin{bmatrix} \pi_k & 0 \\ ? & ? \end{bmatrix},$$

and can as such express the consistency matrix in term of the original matrices as

$$\beta = I - A^+B\beta(I - F)$$

Since $\beta F = F$, this can be rewritten as $(I + A^+B)\beta = I + A^+BA^+A = Z$. This implies that $U\beta = Z$, $\beta = U^{-1}Z$ and lastly, that C = Y.

5 Path Products

The consistency conditions are given by $Y = (I - E)BU^{-1}(I - F) = 0$, where $BU^{-1} = \sum_{k=0}^{N-1} (-1)^k B(A^+B)^k$. Now because $B(A^+B)^k$ has k+1 zero diagonals. It follows that the entry $B(A^+B)^k$ will vanish when $p \leq k + 1$ or when $q > n - k - 1$. This means that we only get nonzero entries when $k \leq p - 2$ and $k \leq n - q - 1$. Now set r = min{p-2,n-q-1}. Then

$$(BU^{-1})_{pq} = b_{pq} + \sum_{k=1}^{r} (-1)^k \sum_{p>i_1>i_2>..>i_k>q} b_{p,i_1} a_{i_1}^+ b_{i_1,i_2} a_{i_2}^+ ... b_{i_{k-1},i_k} a_{i_k}^+ b_{i_k q}$$

These products can now be expressed using weighted path products. Indeed, we take n nodes S_i and draw a weighted arc from S_i to S_j of weight b_{ij}, when the entry b_{ij} does not vanish, and add a loop at each node of weight a_i^+. The sum is taken over all k step paths from S_i to S_j with $i > j$, in the weighted di-graph. Each product is exactly the product of the weights along one particular k step path.

We close with the remark that if $u = 1 + a^+b$ is a unit for some inner inverse a^+ of a, then it may not be a unit for another 1-2 inverses $\tilde{a}$. For example take a = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, b = $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, with $a^+ = a$ and a second 1-2 inverse $\tilde{a} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{Z}_2$.

It would be of interest to see if one can weaken the above conditions to "quasi-similar matrices? Indeed, for which rings do similarity and quasi-similarity coincide ?

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