Complemented congruences on Ockham algebras

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Abstract. An Ockham algebra $\mathcal{L} = (L, \wedge, \vee, f, 0, 1)$ that satisfies the identity $f^{2n+m} = f^m$, $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, is called a $\mathrm{K}_{n,m}$-algebra. Generalizing some results obtained in [2], J. Varlet and T. Blyth, in [3, Chapter 8], study congruences on $\mathrm{K}_{1,1}$-algebras. In particular, they describe the complement (when it exists) of a principal congruence and characterize these congruences that are complemented. In this paper we study the same question for $\mathrm{K}_{n,m}$-algebras.

1. Preliminaries

An Ockham algebra $\mathcal{L} = (L, \wedge, \vee, f, 0, 1)$ is an algebra of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and f is a dual endomorphism of this lattice, i.e, $f(0) = 1$, $f(1) = 0$, $f(x \wedge y) = f(x) \vee f(y)$ and $f(x \vee y) = f(x) \wedge f(y)$. The class of Ockham algebras is a variety denoted by $\mathbf{O}$. This variety was introduced in [1]. We write $\mathcal{L} = (L, f)$ for an Ockham algebra $\mathcal{L} = (L, \wedge, \vee, f, 0, 1)$ and we represent both the universe L and the lattice $(L, \wedge, \vee, 0, 1)$ by L. For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, $\mathbf{K}_{n,m}$ is the subvariety of $\mathbf{O}$ characterized by the identity $f^{2n+m} = f^m$. The elements of $\mathbf{K}_{n,m}$ are called $\mathrm{K}_{n,m}$-algebras. For the basic properties of Ockham algebras and $\mathrm{K}_{n,m}$-algebras we refer the reader to [1] and [3].

For each $\mathcal{L} = (L, f) \in \mathbf{O}$, and for all $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, the sets $f^m(L)$ and $L_{n,m} = \{x \in L : f^{2n+m}(x) = f^m(x)\}$ are subuniverses of $\mathcal{L}$. By $f^m(\mathcal{L})$ and $\mathcal{L}_{n,m}$ we represent the subalgebras $(f^m(L), f)$ and $(L_{n,m}, f)$ of $\mathcal{L}$, respectively. Note that $\mathcal{L}_{n,m}$ is the biggest subalgebra of $\mathcal{L}$ that belongs to $\mathbf{K}_{n,m}$. Also it is clear that if $\mathcal{L} \in \mathbf{K}_{n,m}$ then $f^m(\mathcal{L}) \in \mathbf{K}_{n,0}$.

Given $\mathcal{L} = (L, f) \in \mathbf{O}$ we represent the congruence lattice of $\mathcal{L}$ (resp. L) by $\mathrm{Con}\,\mathcal{L}$ (resp. $\mathrm{Con}_{\mathrm{lat}}\,\mathcal{L}$). Given $a, b \in L$, $\theta(a, b)$ (resp. $\theta_{\mathrm{lat}}(a, b)$) stands for the least element of $\mathrm{Con}\,\mathcal{L}$ (resp. $\mathrm{Con}_{\mathrm{lat}}\,\mathcal{L}$) that identifies the elements a and b. On studying principal congruences of $\mathcal{L} \in \mathbf{K}_{n,m}$, it suffices to consider the congruences $\theta(a, b)$ for $a \leq b$ since, for any $\theta \in \mathrm{Con}\,\mathcal{L}$ (resp. $\mathrm{Con}_{\mathrm{lat}}\,\mathcal{L}$) and $x, y \in L$, we have $(x, y) \in \theta$ if and only if $(x \wedge y, x \vee y) \in \theta$. By $\mathbf{0}$ and $\mathbf{1}$ we denote, respectively, the identity and the universal congruence on $\mathcal{L}$. If $\mathcal{L}' = (L', f)$ is a subalgebra of $\mathcal{L}$, we represent by $\theta_{L'}$ a congruence on $\mathcal{L}'$ and $\mathbf{0}_{L'}$ and $\mathbf{1}_{L'}$ represent, respectively, the identity and the universal congruence of $\mathcal{L}'$.

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For all $\mathcal{L} \in \mathbf{O}$, the lattice $\operatorname{Con}\mathcal{L}$ is distributive. Also, for all $\mathcal{L} \in \mathbf{O}$ and any subalgebra $\mathcal{L}'$ of $\mathcal{L}$, each congruence defined on $\mathcal{L}'$ is the restriction of some congruence defined on $\mathcal{L}$. This means that the variety $\mathbf{O}$ satisfies the congruence extension property. We then have the following Lemma:

Lemma 1.1. *If $\mathcal{L} \in \mathbf{O}$, $\mathcal{L}'$ is a subalgebra of $\mathcal{L}$ and $a, b \in L'$, then*

$$\theta(a,b)|_{L'} = \theta_{L'}(a,b). \qquad \square$$

The following result, due to J. Berman, is fundamental in the investigation of congruences defined on $\mathrm{K}_{n,m}$-algebras. It states that any principal congruence on $\mathcal{L} \in \mathbf{K}_{n,m}$ is the join of $2n+m$ principal congruences on the distributive lattice L.

Lemma 1.2. [1, Corollary of Theorem 1] *If $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $a, b \in L$ are such that $a \leq b$ then*

$$\theta(a,b) = \bigvee_{i=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^i(a), f^i(b)\big). \qquad \square$$

Since many results obtained in this paper use the previous lemma it is useful to remind ourselves about some facts related to distributive lattices.

If L is a distributive lattice and x, y, z, w are elements of L, then:

- R_0) for $z \leq w$, we have $(x,y) \in \theta(z,w)$ if and only if $x \wedge z = y \wedge z$ and $x \vee w = y \vee w$;
- R_1) $\theta(x \wedge y, x) = \theta(y, x \vee y)$;
- R_2) $\theta(x,y) \wedge \theta(z,w) = \theta(x \vee z, x \vee z \vee (y \wedge w)) = \theta(y \wedge w \wedge (x \vee z), y \wedge w)$ (and so $\theta(x,y) \wedge \theta(z,w) = \mathbf{0}$ if and only if $y \wedge w \leq x \vee z$).

Based on Lemma 1.2 and on the fact that the principal congruences on distributive lattices are defined by equations it is easy to prove that a similar situation occurs with $\mathrm{K}_{n,m}$-algebras: if $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$, then $\theta(a,b) \in \operatorname{Con}\mathcal{L}$ is characterized by 2^{2n+m} identities, for any $a, b \in L$.

Theorem 1.3. [5, Theorem 8] *Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $a, b \in L$, with $a \leq b$ and $x, y \in L$. Then $(x,y) \in \theta(a,b)$ if and only if*

$$\begin{aligned} & \Big(x \wedge \bigwedge_{i \in F} f^{2i}(a) \wedge \bigwedge_{j \in G} f^{2j+1}(b)\Big) \vee \bigvee_{k \in T \setminus F} f^{2k}(b) \vee \bigvee_{l \in T' \setminus G} f^{2l+1}(a) \\ = \ & \Big(y \wedge \bigwedge_{i \in F} f^{2i}(a) \wedge \bigwedge_{j \in G} f^{2j+1}(b)\Big) \vee \bigvee_{k \in T \setminus F} f^{2k}(b) \vee \bigvee_{l \in T' \setminus G} f^{2l+1}(a) \end{aligned}$$

for all $F \subseteq T$ and $G \subseteq T'$, where

$T = T' = \{0, 1, 2, ..., n + \frac{m-2}{2}\}$ *if m is even or*
$T = \{0, 1, 2, ..., n + \frac{m-1}{2}\}$, $T' = \{0, 1, 2, ..., n + \frac{m-3}{2}\}$ *if m is odd.* $\square$

In this paper we will also require Theorems 1.4 and 1.7 below, wish are unpublished results of M. Sequeira [6] concerning principal congruences on $\mathrm{K}_{n,m}$-algebras. The proofs are straightforward and are omitted.

The following result is a generalization of [4, Lemma 3.10] and establishes that, given $\mathcal{L} = (L, f) \in \mathbf{O}$, all congruences generated by elements of $L_{1,0}$ are complemented.

Theorem 1.4. *If $\mathcal{L} = (L, f) \in \mathbf{O}$ and $a, b \in L_{1,0}$ with $a \leq b$, then $\theta(a, b)$ is complemented in* $\mathrm{Con}\,\mathcal{L}$, and

$$\begin{aligned} \theta(a,b)' &= \theta\big(f(a)\vee b, 1\big) \vee \theta\big(f(a), f(a)\vee a\big) \vee \theta\big(b, b\vee f(b)\big) \\ &= \theta\big(0, a\wedge f(b)\big) \vee \theta\big(a\wedge f(a), a\big) \vee \theta\big(b\wedge f(b), f(b)\big). \qquad \square \end{aligned}$$

Definition 1.5. By a *p-ladder* in an ordered set E we shall mean a subset of E that consists of two p-chains $a_1 \leq ... \leq a_p$ and $b_1 \leq ...leqb_p$ such that $a_i \leq b_i$ for $i = 1, ..., p$. We shall denote a p-ladder by $(a_i, b_i)_p$.

Example 1.6. Let $T = \{0, 1, ..., n-1\}$ and for $s = 1, ..., n$ let

$$T_s = \{J : J \subseteq T, |J| = s\}.$$

Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $a, b \in L$ be such that $a \leq b$. For $s = 1, ..., n$ let

$$\widetilde{a}_s = \bigwedge_{J\in T_s}\bigvee_{j\in J} f^{2j}(a), \qquad \widetilde{b}_s = \bigwedge_{J\in T_s}\bigvee_{j\in J} f^{2j}(b).$$

Then $\{\widetilde{a}_s, \widetilde{b}_s : s = 1, ..., n\}$ is an n-ladder consisting of elements that belong to the subalgebra $\mathcal{L}_{1,m}$. Indeed, since $a \leq b$ we have $\widetilde{a}_s \leq \widetilde{b}_s$ for $s = 1, ..., n$. It is also obvious that $\widetilde{a}_1 \leq ... \leq \widetilde{a}_n$ and $\widetilde{b}_1 \leq ... \leq \widetilde{b}_n$. Using the fact that, for all $s \in \{1, ..., n\}$, the map

$$\begin{aligned} \varphi_s : T_s &\rightarrow T_s \\ J &\mapsto \begin{cases} \{j+1 \,|\, j \in J\} & \text{if } n-1 \notin J \\ \{j+1 \,|\, j \in J \setminus \{n-1\}\} \cup \{0\} & \text{if } n-1 \in J \end{cases} \end{aligned}$$

is surjective and that $f^{2n+m}(a) = f^m(a)$ and $f^{2n+m}(b) = f^m(b)$, it is easy to see that $\widetilde{a}_s, \widetilde{b}_s \in L_{1,m}$. In fact,

- if m is even then

$$\begin{aligned} f^{2+m}(\widetilde{a}_s) &= \bigwedge_{J\in T_s}\bigvee_{j\in J} f^{2j+2+m}(a) = \bigwedge_{J\in T_s}\bigvee_{k\in\varphi_s(J)} f^{2k+m}(a) \\ &= \bigwedge_{K\in T_s}\bigvee_{k\in K} f^{2k+m}(a) = f^m(\widetilde{a}_s); \end{aligned}$$

• if m is odd then

$$f^{2+m}(\widetilde{a}_s) = \bigvee_{J\in T_s} \bigwedge_{j\in J} f^{2j+2+m}(a) = \bigvee_{J\in T_s} \bigwedge_{k\in\varphi_s(J)} f^{2k+m}(a)$$
$$= \bigvee_{K\in T_s} \bigwedge_{k\in K} f^{2k+m}(a) = f^m(\widetilde{a}_s).$$

In both cases we conclude that $\widetilde{a}_s \in L_{1,m}$.

Using the n-ladder $(\widetilde{a}_s, \widetilde{b}_s)_n$ defined on the previous example, M. Sequeira establishes that each principal congruence $\theta(a,b)$ defined on an algebra $\mathcal{L} \in \mathbf{K}_{n,m}$ is the join of a finite number of principal congruences generated by elements of $L_{1,m}$.

Theorem 1.7. *Let $\mathcal{L} = (L,f) \in \mathbf{K}_{n,m}$ and $a,b \in L$ such that $a \leq b$. Then*

$$\theta(a,b) = \bigvee_{s=1}^{n} \theta(\widetilde{a}_s, \widetilde{b}_s). \qquad \square$$

Corollary 1.8. *Let $\mathcal{L} = (L,f) \in \mathbf{K}_{n,m}$ and $a,b \in L$ such that $a \leq b$. Then*

$$\theta(a,b) = \bigvee_{s=1}^{n} \bigvee_{j=0}^{m+1} \theta_{\mathrm{lat}}\big(f^j(\widetilde{a}_s), f^j(\widetilde{b}_s)\big). \qquad \square$$

Since any congruence, defined on an algebra $\mathcal{A}$, is the join of principal congruences, it follows from Theorem 1.7 that each congruence θ, defined on an algebra $\mathcal{L}$ of $\mathbf{K}_{n,m}$ is the join of principal congruences generated by elements of $L_{1,m}$.

The purpose of this paper is to characterize the principal congruences $\theta(a,b)$ on $\mathcal{L} \in \mathbf{K}_{n,m}$ that are complemented. This will be achieved by studying congruences θ on $\mathcal{L} \in \mathbf{K}_{n,m}$ that can be represented in the form $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s)$, for some p-ladder $\theta(c_s, d_s)_p$ of elements of $L_{1,m}$.

2. The congruences

Let $\mathcal{L} \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$ and $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s)$ for some p-ladder $\theta(c_s, d_s)_p$ of elements of $L_{1,m}$. If each $\theta(c_s, d_s)$ is complemented, it is obvious that θ is also complemented, with $\theta' = \bigwedge_{s=1}^{p} \theta(c_s, d_s)'$. The condition of θ being complemented is not sufficient for each $\theta(c_s, d_s)$ to be complemented (Example 2.11). Furthermore, if θ is complemented, we can obtain the description of θ' without knowing whether each $\theta(c_s, d_s)$ is complemented or not.

In order to determine the complement of θ (if θ is complemented) we need to establish some further results.

Lemma 2.1. *If $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$, and $a, b \in L$ are such that $a \leq b$, then*

$$(\forall k \geq m)(\forall q \in \mathbb{N}) \quad \theta(a,b)|_{f^k(L)} = \theta\big(f^{q2n}(a), f^{q2n}(b)\big)|_{f^k(L)}.$$

Proof. If $\big(f^k(x), f^k(y)\big) \in \theta(a,b)$, then both $f^k(x)$ and $f^k(y)$ satisfy the 2^{2n+m} equations of Theorem 1.3. Applying f^{q2n} to each equation, since $k \geq m$, we get $\big(f^k(x),\, f^k(y)\big) \in \theta\big(f^{q2n}(a), f^{q2n}(b)\big)$. The converse follows from the fact that $\theta\big(f^{q2n}(a), f^{q2n}(b)\big) \leq \theta(a,b)$. □

Lemma 2.2. *Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$, $i \in \mathbb{N}_0$, $k \in \mathbb{N}$ with $k \geq m$ and $a, b \in L$ with $a \leq b$.*
Then, for any $x, y \in L$,

$$(x,y) \in \theta_{\text{lat}}\big(f^i(a), f^i(b)\big) \Rightarrow \big(f^k(x), f^k(y)\big) \in \theta_{\text{lat}}\big(f^t(a), f^t(b)\big),$$

for some $t \in \{m, ..., 2n+m-1\}$.

Proof. Let $x, y \in L$. If $(x,y) \in \theta_{\text{lat}}\big(f^i(a), f^i(b)\big)$, with $i \in \mathbb{N}_0$, then $\big(f^k(x), f^k(y)\big) \in \theta_{\text{lat}}\big(f^k(f^i(a)), f^k(f^i(b))\big)$. Since $k \geq m$ we have that $f^{k+i}(a) = f^t(a)$ and $f^{k+i}(b) = f^t(b)$, for some $t \in \{m, ..., 2n+m-1\}$. □

For each $x \in \mathbb{Q}_0$, we will denote by $\lceil x \rceil$ the smallest element of $\mathbb{N}$ that is greater than or equal to x.

Lemma 2.3. *If $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $a, b \in L$ are such that $a \leq b$, then*

$$\theta(a,b)|_{f^m(L)} = \bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)|_{f^m(L)}.$$

Proof. By Lemma 1.2 we have $\theta(a,b) = \bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)$ and it is obvious that

$$\bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)|_{f^m(L)} \leq \Big(\bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)\Big)|_{f^m(L)}.$$

Let $x, y \in L$ and suppose that $(x,y) \in \Big(\bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)\Big)|_{f^m(L)}$. Then $x, y \in f^m(L)$ and $(x,y) \in \bigvee_{k=0}^{2n+m-1} \theta_{\text{lat}}\big(f^k(a), f^k(b)\big)$.

Consequently there exist $s \in \mathbb{N}$ and $x_0 = x,\, x_1,\, ...,\, x_s = y \in L$ such that, for all $v \in \{0, ..., s-1\}$, $(x_v, x_{v+1}) \in \theta_{\text{lat}}\big(f^{k_v}(a), f^{k_v}(b)\big)$, for some $k_v \in \{0, ..., 2n+m-1\}$. Let $q = \lceil m/2n \rceil$. By Lemma 2.2 it follows that $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\text{lat}}\big(f^{t_v}(a), f^{t_v}(b)\big)$, with $t_v \in \{m, ..., 2n+m-1\}$. Since $q2n \geq m$, then $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\text{lat}}\big(f^{t_v}(a), f^{t_v}(b)\big)|_{f^m(L)}$.

Thus we have that

$$\big(f^{q2n}(x), f^{q2n}(y)\big) \in \bigvee_{k=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^k(a), f^k(b)\big)|_{f^m(L)},$$

with $f^{q2n}(x) = x$ and $f^{q2n}(y) = y$ since $x, y \in f^m(L)$. Therefore

$$\Big(\bigvee_{k=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^k(a), f^k(b)\big)\Big)\Big|_{f^m(L)} \leq \bigvee_{k=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^k(a), f^k(b)\big)|_{f^m(L)}. \qquad \square$$

Using this result we prove that:

Lemma 2.4. *If $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$ and $a_i, b_i \in L$ are such that $a_i \leq b_i$ for $i \in \{1, ..., p\}$, then*

$$\Big(\bigvee_{i=1}^{p} \theta(a_i, b_i)\Big)\Big|_{f^m(L)} = \bigvee_{i=1}^{p} \theta(a_i, b_i)|_{f^m(L)}.$$

Proof. Let $x, y \in L$ and suppose that $(x, y) \in \big(\bigvee_{i=1}^{p} \theta(a_i, b_i)\big)|_{f^m(L)}$. Then $x, y \in f^m(L)$ and $(x, y) \in \bigvee_{i=1}^{p} \theta(a_i, b_i)$. By Lemma 1.2, it follows that

$$(x, y) \in \bigvee_{i=1}^{p} \bigvee_{k=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^k(a_i), f^k(b_i)\big).$$

This means that there exist $s \in \mathbb{N}$ and $x_0 = x, x_1, ..., x_s = y \in L$ such that, for each $v \in \{0, ..., s-1\}$, $(x_v, x_{v+1}) \in \theta_{\mathrm{lat}}\big(f^{k_v}(a_{i_v}), f^{k_v}(b_{i_v})\big)$, for some $i_v \in \{1, ..., p\}$ and $k_v \in \{0, ..., 2n+m-1\}$. Let $q = \lceil m/2n \rceil$. By Lemma 2.2, we know that $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\mathrm{lat}}\big(f^{t_v}(a_{i_v}), f^{t_v}(b_{i_v})\big)$, with $t_v \in \{m, ..., 2n+m-1\}$. Then $\big(f^{q2n}(x_v), f^{q2n}(x_{v+1})\big) \in \theta_{\mathrm{lat}}\big(f^{t_v}(a_{i_v}), f^{t_v}(b_{i_v})\big)|_{f^m(L)}$. Since $f^{q2n}(x) = x$ and $f^{q2n}(y) = y$ it follows that

$$(x, y) \in \bigvee_{i=1}^{p} \bigvee_{k=0}^{2n+m-1} \theta_{\mathrm{lat}}\big(f^k(a_i), f^k(b_i)\big)|_{f^m(L)}.$$

Taking into account Lemma 2.3, we have $(x, y) \in \bigvee_{i=1}^{p} \theta(a_i, b_i)|_{f^m(L)}$ and so $\big(\bigvee_{i=1}^{p} \theta(a_i, b_i)\big)|_{f^m(L)} \leq \bigvee_{i=1}^{p} \theta(a_i, b_i)|_{f^m(L)}$. Since the converse inequality is obvious, we conclude that $\big(\bigvee_{i=1}^{p} \theta(a_i, b_i)\big)|_{f^m(L)} = \bigvee_{i=1}^{p} \theta(a_i, b_i)|_{f^m(L)}$. $\square$

For each $\mathcal{L} = (L, f) \in \mathbf{O}$, let $\mathrm{Con}'\,\mathcal{L}$ be the lattice of complemented congruences on $\mathcal{L}$.

Lemma 2.5. *Let $\mathcal{L} = (L, f) \in \mathbf{O}$ and $k \in \mathbb{N}$. If $\theta \in \mathrm{Con}'\,\mathcal{L}$, then $\theta|_{f^k(L)} \in \mathrm{Con}'\, f^k(\mathcal{L})$. In fact, if θ' is the complement of θ in $\mathrm{Con}\,\mathcal{L}$, then $\theta'|_{f^k(L)}$ is the complement of $\theta|_{f^k(L)}$ in $\mathrm{Con}\, f^k(\mathcal{L})$.*

Proof. Let θ' be the complement of θ in $\operatorname{Con}\mathcal{L}$. From $\theta \wedge \theta' = \mathbf{0}$ and $\theta|_{f^k(L)} \wedge \theta'|_{f^k(L)} \leq \theta \wedge \theta'$, it follows that $\theta|_{f^k(L)} \wedge \theta'|_{f^k(L)} = \mathbf{0}_{f^k(L)}$. Since $(0,1) \in \theta \vee \theta'$, there exist x_0, x_1, ..., $x_n \in L$ such that

$$0 = x_0 \overset{\theta}{\equiv} x_1 \overset{\theta'}{\equiv} x_2 \overset{\theta}{\equiv} ... \overset{\theta'}{\equiv} x_{n-1} \overset{\theta}{\equiv} x_n = 1.$$

Applying f^k to each element we then obtain

$$f^k(0) = f^k(x_0) \overset{\theta}{\equiv} f^k(x_1) \overset{\theta'}{\equiv} f^k(x_2) \overset{\theta}{\equiv} ... \overset{\theta'}{\equiv} f^k(x_{n-1}) \overset{\theta}{\equiv} f^k(x_n) = f^k(1)$$

and so $\big(f^k(x_0), f^k(x_n)\big) \in \theta|_{f^k(L)} \vee \theta'|_{f^k(L)}$. In both cases, k odd or k even, it is obvious that $(0,1) \in \theta|_{f^k(L)} \vee \theta'|_{f^k(L)}$, whence we have $\theta|_{f^k(L)} \vee \theta'|_{f^k(L)} = \mathbf{1}_{f^k(L)}$. Therefore $\theta'|_{f^k(L)}$ is the complement of $\theta|_{f^k(L)}$ in $\operatorname{Con} f^k(\mathcal{L})$. □

Definition 2.6. By a *m-pair*, $m \in \mathbb{N}$, we shall mean the ordered pair (k,l) such that

$$(k,l) = \begin{cases} (m, m+1) & \text{if } m \text{ is even;} \\ (m+1, m) & \text{if } m \text{ is odd.} \end{cases}$$

It is useful to notice that, if (k,l) is a m-pair then k is always even, and l is always odd.

In what follows, we consider $\mathcal{L} \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$ and $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s)$, for some p-ladder $(c_s, d_s)_p$ of elements of $L_{1,m}$. Moreover, (k,l) denotes an m-pair.

Suppose that θ is complemented. As we will see, the description of the complement of θ is related to the description of the complement of principal congruences generated by elements of $L_{1,0}$ (Theorem 1.4).

By Lemmas 2.4 and 2.1

$$\theta|_{f^m(L)} = \bigvee_{s=1}^{p} \theta\big(f^{q2n}(c_s), f^{q2n}(d_s)\big)|_{f^m(L)},$$

for all $q \in \mathbb{N}$. If we take $q = \lceil m/2n \rceil$, then $f^{2qn}(c_s)$, $f^{2qn}(d_s) \in f^m(L)$ and consequently, by Lemma 1.1

$$\theta|_{f^m(L)} = \bigvee_{s=1}^{p} \theta_{f^m(L)}\big(f^{q2n}(c_s), f^{q2n}(d_s)\big).$$

Since $c_s, d_s \in L_{1,m}$ and $q2n \geq m$, we have that $f^{q2n}(c_s), f^{q2n}(d_s) \in L_{1,0}$ and $q2n = m + r$, for some $r \in \mathbb{N}_0$. For each $x \in L_{1,m}$, is easy to see that

- if m is even, $f^{q2n}(x) = f^m(x)$ and $f^{q2n+1}(x) = f^{m+1}(x)$,
- if m is odd, $f^{q2n}(x) = f^{m+1}(x)$ and $f^{q2n+1}(x) = f^m(x)$.

By Theorem 1.4 and Lemmas 1.1 and 2.4, we know that each congruence $\theta_{f^m(L)}\big(f^{q2n}(c_s), f^{q2n}(d_s)\big)$ is complemented in $\operatorname{Con} f^m(\mathcal{L})$ and that

$$\theta_{f^m(L)}\big(f^{q2n}(c_s), f^{q2n}(d_s)\big)'$$

$$\begin{aligned}
&= \theta_{f^m(L)}\big(f^{q2n}(d_s) \vee f^{q2n+1}(c_s), 1\big) \vee \theta_{f^m(L)}\big(f^{q2n}(d_s), f^{q2n}(d_s) \vee f^{q2n+1}(d_s)\big)\\
&\quad \vee \theta_{f^m(L)}\big(f^{q2n+1}(c_s), f^{q2n+1}(c_s) \vee f^{q2n}(c_s)\big)\\
&= \theta_{f^m(L)}\big(f^k(d_s) \vee f^l(c_s), 1\big) \vee \theta_{f^m(L)}\big(f^k(d_s), f^k(d_s) \vee f^l(d_s)\big)\\
&\quad \vee \theta_{f^m(L)}\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big)\\
&= \theta\big(f^k(d_s) \vee f^l(c_s), 1\big)|_{f^m(L)} \vee \theta\big(f^k(d_s), f^k(d_s) \vee f^l(d_s)\big)|_{f^m(L)}\\
&\quad \vee \theta\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big)|_{f^m(L)}\\
&= \big[\theta\big(f^k(d_s) \vee f^l(c_s), 1\big) \vee \theta\big(f^k(d_s), f^k(d_s) \vee f^l(d_s)\big)\\
&\quad \vee \theta\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big)\big]|_{f^m(L)}.
\end{aligned}$$

Let $\varphi(c_s, d_s)$ stand for

$$\theta\big(f^k(d_s) \vee f^l(c_s), 1\big) \vee \theta\big(f^k(d_s), f^k(d_s) \vee f^l(d_s)\big) \vee \theta\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big).$$

Since

$$\theta|_{f^m(L)} = \bigvee_{s=1}^{p} \theta_{f^m(L)}\big(f^{q2n}(c_s), f^{q2n}(d_s)\big)$$

and since each congruence $\theta_{f^m(L)}\big(f^{q2n}(c_s), f^{q2n}(d_s)\big)$ is complemented, it follows that $\theta|_{f^m(L)}$ is complemented with:

$$\begin{aligned}
(\theta|_{f^m(L)})' &= \bigwedge_{s=1}^{p} \theta_{f^m(L)}\big(f^{2nq}(c_s), f^{2nq}(d_s)\big)' = \bigwedge_{s=1}^{p} \big(\varphi(c_s, d_s)|_{f^m(L)}\big)\\
&= \Big(\bigwedge_{s=1}^{p} \varphi(c_s, d_s)\Big)\Big|_{f^m(L)}.
\end{aligned}$$

From Lemma 2.5, we know that $(\theta|_{f^m(L)})' = \theta'|_{f^m(L)}$. Consequently we have $\theta'|_{f^m(L)} = \big(\bigwedge_{s=1}^{p} \varphi(c_s, d_s)\big)|_{f^m(L)}$.

By φ we represent $\bigwedge_{s=1}^{p} \varphi(c_s, d_s)$.

Using the fact that $f^{k+1}(x) = f^l(x)$ and $f^{l+1}(x) = f^k(x)$, for all $x \in L_{1,m}$, and defining $d_0 = 0$ and $c_{p+1} = 1$, it can be shown that φ can be expressed in the form

$$\varphi = \bigvee_{i=1}^{p+1} \bigvee_{j=i-1}^{p} \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big).$$

This is proved by induction on p. The anchor point is $p = 1$ and for this value of p the result is immediate. In fact, if we define $d_0 = 0$ and $c_2 = 1$, we have $f^k(d_0) = 0, f^l(d_0) = 1, f^k(c_2) = 1$ and $f^l(c_2) = 0$, so

$$\begin{aligned}
\varphi &= \bigwedge_{s=1}^{1} \varphi(c_s, d_s) \\
&= \theta\big(f^l(c_1), f^l(c_1) \vee f^k(c_1)\big) \\
&\quad \vee \theta\big(f^l(c_1) \vee f^k(d_1), 1\big) \\
&\quad \vee \theta\big(f^k(d_1), f^k(d_1) \vee f^l(d_1)\big) \\
&= \theta\big(f^l(c_1) \vee f^k(d_0), f^l(c_1) \vee f^k(d_0) \vee [f^l(d_0) \wedge f^k(c_1)]\big) \\
&\quad \vee \theta\big(f^l(c_1) \vee f^k(d_1), f^l(c_1) \vee f^k(d_1) \vee [f^l(d_0) \wedge f^k(c_2)]\big) \\
&\quad \vee \theta\big(f^l(c_2) \vee f^k(d_1), f^l(c_2) \vee f^k(d_1) \vee [f^l(d_1) \wedge f^k(c_2)]\big) \\
&= \bigvee_{i=1}^{2} \bigvee_{j=i-1}^{1} \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big)
\end{aligned}$$

We omit the proof of the inductive step since, although routine, it is very long.

Finally, we can obtain the description of the complement of θ:

Theorem 2.7. *Let $\mathcal{L} \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$, $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s)$ for some p-ladder $(c_s, d_s)_p$ of elements of $L_{1,m}$ and let (k, l) be an m-pair.*
Then

(a) $\theta \vee \varphi = \mathbf{1}$,

(b) if θ is complemented then $\theta' = \varphi$.

Proof. (a) For $s \in \{1, ..., p\}$, $\theta(c_s, d_s) \vee \varphi(c_s, d_s) = \mathbf{1}$. In fact,

- $\big(0, f^l(d_s) \wedge f^k(c_s)\big) \in \theta\big(0, f^l(d_s) \wedge f^k(c_s)\big) = \theta\big(f^k(d_s) \vee f^l(c_s), 1\big)$;
- $\big(f^l(d_s) \wedge f^k(c_s), f^l(d_s) \wedge f^k(d_s)\big) \in \theta(c_s, d_s)$;
- $\big(f^l(d_s) \wedge f^k(d_s), f^l(d_s)\big) \in \theta\big(f^k(d_s), f^k(d_s) \vee f^l(d_s)\big)$;
- $\big(f^l(d_s), f^l(c_s)\big) \in \theta(c_s, d_s)$;
- $\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big) \in \theta\big(f^l(c_s), f^l(c_s) \vee f^k(c_s)\big)$;
- $\big(f^l(c_s) \vee f^k(c_s), f^l(c_s) \vee f^k(d_s)\big) \in \theta(c_s, d_s)$;
- $\big(f^l(c_s) \vee f^k(d_s), 1\big) \in \theta\big(f^l(c_s) \vee f^k(d_s), 1\big)$.

Consequently, $\theta \vee \varphi = \mathbf{1}$.

(b) Suppose now that θ is complemented. From (a) it follows that $\theta' \leq \varphi$. It remains to prove that $\varphi \leq \theta'$.

As we have already seen, $\theta'|_{f^m(L)} = \varphi|_{f^m(L)}$.

Let $d_0 = 0$ and $c_{p+1} = 1$. Since

$$\varphi = \bigvee_{i=1}^{p+1} \bigvee_{j=i-1}^{p} \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big)$$

we have by Lemma 2.4

$$\varphi|_{f^m(L)} = \bigvee_{i=1}^{p+1} \bigvee_{j=i-1}^{p} \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big)|_{f^m(L)}$$

From $\theta'|_{f^m(L)} = \varphi|_{f^m(L)}$, we conclude that θ' identifies each pair

$$\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big)$$

that occurs in $\varphi|_{f^m(L)}$. Since φ is the least congruence that identifies each of these pairs, we have $\varphi \leq \theta'$. Thus, $\theta' = \varphi$. $\square$

Using the description of the complement of θ, we establish a necessary and sufficient condition for θ to be complemented.

Theorem 2.8. *Let $\mathcal{L} \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$, $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s)$ for some p-ladder $(c_s, d_s)_p$ of elements of $L_{1,m}$, and let (k, l) be an m-pair. The congruence θ is complemented if and only if, for all $s \in \{1, ..., p\}$ and $i \in \{1, ..., p+1\}$*

$$d_s \wedge f^l(d_{i-1}) \wedge f^k(c_{j+1}) \leq c_s \vee f^l(c_i) \vee f^k(d_j), \text{ for } j \in \{i-1, ..., p\}$$

$$d_s \wedge f^l(d_j) \wedge f^k(c_i) \leq c_s \vee f^l(c_{j+1}) \vee f^k(d_{i-1}), \text{ for } j \in \{i, ..., p\}.$$

Proof. By Theorem 2.7, θ is complemented if and only if $\theta \wedge \varphi = \mathbf{0}$. We also know that

$$\varphi = \bigvee_{i=1}^{p+1} \bigvee_{j=i-1}^{p} \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big).$$

So, θ is complemented if and only if for all $s \in \{1, ..., p\}$, all $i \in \{1, ..., p+1\}$ and all $j \in \{i-1, ..., p\}$,

$$\theta(c_s, d_s) \wedge \theta\big(f^l(c_i) \vee f^k(d_j), f^l(c_i) \vee f^k(d_j) \vee [f^l(d_{i-1}) \wedge f^k(c_{j+1})]\big) = \mathbf{0}.$$

Since $c_s, d_s \in L_{1,m}$, Lemma 1.2, and results R_1) and R_2), give easily that θ is complemented if and only if for all $s \in \{1, ..., p\}$, all $i \in \{1, ..., p+1\}$ and all $j \in \{i-1, ..., p\}$ we have that

$$d_s \wedge f^l(d_{i-1}) \wedge f^k(c_{j+1}) \leq c_s \vee f^l(c_i) \vee f^k(d_j)$$

and

$$d_s \wedge f^l(d_j) \wedge f^k(c_i) \leq c_s \vee f^l(c_{j+1}) \vee f^k(d_{i-1}).$$

Notice that these two conditions coincide when $j = i - 1$. $\square$

An immediate consequence of this theorem is the following Corollary, which is a generalization of Theorem 8.10 of [3]:

Corollary 2.9. *Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $c, d \in L_{1,m}$ with $c \leq d$. Let (k, l) be an m-pair. Then, $\theta(c, d)$ is complemented if and only if*

a) $d \leq c \vee f^l(c) \vee f^k(d)$;
b) $d \wedge f^k(c) \wedge f^l(d) \leq c$;
c) $d \wedge f^k(c) \leq c \vee f^l(c)$;
d) $d \wedge f^l(d) \leq c \vee f^k(d)$. □

Another consequence of Theorem 2.8, which we state as a theorem and is the main result of this paper, is the characterization of the complemented principal congruences on $\mathcal{L} \in \mathbf{K}_{n,m}$.

Theorem 2.10. *Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$ and $a, b \in L$ such that $a \leq b$. Let $\widetilde{b}_0 = 0$ and $\widetilde{a}_{n+1} = 1$. Let (k, l) be an m-pair.*
Then $\theta(a, b)$ is complemented if and only if, for all $s \in \{1, ..., n\}$ and $i \in \{1, ..., n+1\}$

$$\widetilde{b}_s \wedge f^l(\widetilde{b}_{i-1}) \wedge f^k(\widetilde{a}_{j+1}) \leq \widetilde{a}_s \vee f^l(\widetilde{a}_i) \vee f^k(\widetilde{b}_j), \text{ for } j \in \{i-1, ..., n\} \text{ and}$$

$$\widetilde{b}_s \wedge f^l(\widetilde{b}_j) \wedge f^k(\widetilde{a}_i) \leq \widetilde{a}_s \vee f^l(\widetilde{a}_{j+1}) \vee f^k(\widetilde{b}_{i-1}), \text{ for } j \in \{i, ..., n\}.$$

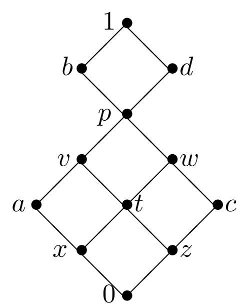
In this case

$$\theta(a,b)' = \bigwedge_{s=1}^{n} \big[\theta\big(f^k(\widetilde{b}_s) \vee f^l(\widetilde{a}_s), 1\big) \vee \theta\big(f^k(\widetilde{b}_s), f^k(\widetilde{b}_s) \vee f^l(\widetilde{b}_s)\big) \vee \theta\big(f^l(\widetilde{a}_s), f^l(\widetilde{a}_s) \vee f^k(\widetilde{a}_s)\big)\big].$$

Proof. By Theorem 1.7, we know that $\theta(a, b) = \bigvee_{s=1}^{n} \theta(\widetilde{a}_s, \widetilde{b}_s)$, where $(\widetilde{a}_s, \widetilde{b}_s)_n$ is an n-ladder of elements of $L_{1,m}$. So, the result follows immediately from Theorems 2.7 and 2.8. □

Let $\mathcal{L} = (L, f) \in \mathbf{K}_{n,m}$, $p \in \mathbb{N}$ and $\theta = \bigvee_{s=1}^{p} \theta(c_s, d_s) \in \operatorname{Con} \mathcal{L}$ for some p--ladder $(c_s, d_s)_p$ of elements of $L_{1,m}$. The following example shows that if θ is complemented, each $\theta(c_s, d_s)$ is not necessarily complemented.

Example 2.11. Let L be the lattice described below



made into a $K_{2,1}$-algebra by defining f as follows

y	x	z	a	t	c	v	w	p	b	d	0	1
$f(y)$	b	d	b	p	d	p	p	p	c	a	1	0

By Theorem 1.7,

$$\theta(x,t) = \theta(\widetilde{x}_1, \widetilde{t}_1) \vee \theta(\widetilde{x}_2, \widetilde{t}_2) = \theta(0,t) \vee \theta(w,p).$$

Since $(f(0), f(t)) = (1,p) \in \theta(0,t)$ and, since p is a fixed point, $\theta(0,t)$ is the universal congruence. Consequently $\theta(x,t)$ is also the universal congruence, which is obviously complemented. However, $\theta(w,p)$ does not satisfy condition b) of the Corollary 2.9 and so $\theta(w,p)$ is not complemented.

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