IMPLICIT OPERATIONS ON CERTAIN CLASSES OF SEMIGROUPS

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ABSTRACT. Varieties of algebras are characterized by identities, where an identity is a formal equality of two terms (i.e., operations defined by means of the underlying operations). Analogously, pseudovarieties of (finite) algebras are defined by pseudo-identities, these being formal equalities of so-called implicit operations (briefly , functions compatible with all homomorphisms). To further explore this analogy to yield results on finite algebras, it is necessary to obtain clear descriptions of implicit operations. This work is a contribution to this project in the area of semigroup theory. All unary implicit operations on semigroups are described, and the implicit operations on certain pseudovarieties of semigroups are given in terms of "generating" operations. The existence of some unusual implicit operations is established based on classical combinatorial theorems about words.

1. INTRODUCTION

This paper is concerned with a new way of looking at pseudovarieties of semigroups : via implicit operations. Roughly put, an implicit operation is a new operation which is preserved by all functions that preserve the old operations (i.e., homomorphisms). Reiterman [8] showed that implicit operations on finite algebras of a finite finitary type form a compact metric space in which the subset of finite composites of old operations is dense. He also showed that pseudovarieties are defined by pseudoidentities, i.e., by formal equalities of implicit operations, thus providing a suitable analog to the classical Birkhoff theorem on varieties.

We start here a systematic study of implicit operations on finite semigroups. Our first positive result is a full constructive description of unary implicit operations. This already allows us to show there is a vast unexplored world of implicit operations compared with what can be found in the literature on pseudovarieties (cf. Eilenberg [4] and Pin [7]).

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We then proceed to show there is apparently more and more complication as the arity n of implicit operations increases. For $n=2,3$, this is based on classical results on avoidable regularities in words on two and three-letter alphabets.

Finally, we restrict our attention to the class $\underline{ZE}$ of all finite semigroups in which idempotents are central. We show that implicit operations on finite groups and explicit operations suffice to obtain all implicit operations on $\underline{ZE}$. This is based on a careful study of sequences of words depending on some simple algebraic and combinatorial lemmas.

For basic notation on semigroups and pseudovarieties, the reader is referred to Lallement [5].

2. BACKGROUND AND NOTATION

A class $\underline{V}$ of finite algebras of a given type is said to be a pseudovariety if it is closed under homomorphic images, subalgebras and finitary direct products. We say $\underline{V}$ is equational if there is a set Σ of identities such that $\underline{V}$ is the class of all finite algebras of the given type which satisfy all the identities in Σ .

EXAMPLE 2.1. The pseudovariety $\underline{N}$ of all finite nilpotent semigroups is not equational since $\underline{N}$ satisfies no nontrivial semigroup identities, i.e., the least equational pseudovariety containing $\underline{N}$ is the class $\underline{S}$ of all finite semigroups. Thus, identities do not suffice to define pseudovarieties.

Let C be any class of algebras of a given type. An n-ary implicit operation π on C associates with each $A \in C$ an n-ary operation $\pi_A : A^n \longrightarrow A$ in such a way that if $A,B \in C$ and $\varphi : A \longrightarrow B$ is a homomorphism then $\pi_B(\varphi(a_1),\dots,\varphi(a_n)) = \varphi(\pi_A(a_1,\dots,a_n))$ for all $a_1,\dots,a_n \in A$. The class of all n-ary implicit operations on C is denoted by $\overline{\Omega}_n C$.

For $A \in C$ and $\pi,\rho \in \overline{\Omega}_n C$, we write $A \models \pi = \rho$ to mean $\pi_A = \rho_A$; we call $\pi = \rho$ a pseudoidentity for C and we then say A satisfies this pseudoidentity. If Σ is a set of pseudoidentities for C, $A \models \Sigma$ means A satisfies all the pseudoidentities in Σ and $[\![\Sigma]\!]$ denotes the class of all $A \in C$ such that $A \models \Sigma$.

If $t(x_1,\dots,x_n)$ is a term of a given type τ in the variables $x_1,\dots,x_n$, then t defines an n-ary implicit operation on any class C of algebras of type τ by letting t_A be the induced n-ary operation on $A \in C$ (where $t_A(a_1,\dots,a_n)$ is obtained by "substituting" $a_i \in A$ for x_i $(i=1,\dots,n)$). An implicit operation of this type is said to be explicit. The set of all n-ary explicit operations on C is represented by $\Omega_n C$.

In $\overline{\Omega}_n C$ it is possible to define a metric distance in the case - which we assume in the following - C is "essentially countable", i.e., up to isomorphism, there are only a countable number of algebras in C. This is the case, for instance, when C consists of finite algebras and the underlying type is finite and finitary. We will not describe here this distance function but rather the convergence of sequences in $\overline{\Omega}_n C$ since this is what will be used in the sequel. A sequence $(\rho_n)_n$ converges in $\overline{\Omega}_m C$ to π if, for every integer k, there is an integer n_k such that $A \models \pi = \rho_n$ for all $n \geq n_k$ and all $A \in C$ with $|A| \leq k$, where $|A|$ denotes the cardinality of A.

Next, we quote two important results of Reiterman [8] .

THEOREM 2.2. $\overline{\Omega}_n C$ is a compact metric space in which $\Omega_n C$ is dense.

THEOREM 2.3. Let $\underline{V}$ be a class of finite algebras of type τ . Then $\underline{V}$ is a pseudovariety if and only if there is a set Σ of pseudoidentities for the class of all finite algebras of type τ such that $\underline{V} = [\![\Sigma]\!]$.

In view of Theorem 2.3, it is natural to study pseudoidentities and implicit operations in order to achieve a better understanding of pseudovarieties.

From here on, we restrict our attention to classes of finite semigroups. For finite semigroups there is one more very natural implicit operation found in the literature besides the explicit operations : the idempotent unary operation x^ω . For an element s of a finite semigroup, the value s^ω of x^ω on s is the idempotent in the subsemigroup generated by s. There is, however, a need for more implicit operations in order to be able to define all pseudovarieties in terms of pseudoidentities.

EXAMPLE 2.4. Let $\underline{Ab}_p$ denote the class of all finite abelian p-groups, and let $\underline{Ab}^p$ denote the class of all finite abelian groups without elements of order p, where p is any prime. We claim $\underline{Ab}_p$ and $\underline{Ab}^p$ cannot be defined by pseudoidentities in which all implicit operations are composites of x^ω and explicit operations.

For, suppose $\underline{Ab}_p = [\![x^\omega = 1, \Sigma]\!]$ where Σ is a set of such pseudoidentities. (In general, $\pi=1$ is an abbreviation of $\pi(x_1,\dots,x_n)y=y=y\pi(x_1,\dots,x_n)$.) Then, every $\pi=\rho$ in Σ can be replaced by an identity of the form $v=1$ or $v=w$. Thus, we may assume Σ is a set of identities.

Let $[\Sigma]$ denote the class of all semigroups satisfying the identities in Σ . Of course, $[\Sigma]$ is a variety and the class $[\Sigma]^F$ of all finite members of $[\Sigma]$ is precisely $[\![\Sigma]\!]$. But $\mathbb{Z} \in [\Sigma]$ since $[\![\Sigma]\!]$ contains cyclic groups of arbitrarily large order and $[\Sigma]$ is a variety. Hence $\mathbb{Z}_q \in [\Sigma]^F \cap [\![x^\omega=1]\!] = \underline{Ab}_p$ for all q, contradicting the definition of $\underline{Ab}_p$. A similar argument works for $\underline{Ab}^p$.

3. UNARY IMPLICIT OPERATIONS

In a cyclic semigroup $\langle a \,;\, a^n = a^{n+k} \rangle$, we call n the index of a and k the period of a ; we also denote by K_a its maximal subgroup.

Let $\underline{S}$ denote the class of all finite semigroups.

LEMMA 3.1. Let $\pi \in \overline{\Omega}_1\underline{S}$ be such that $\pi_A(a) \notin K_a$ for some $A \in \underline{S}$ and $a \in A$. Then $\pi \in \Omega_1\underline{S}$.

PROOF. By Theorem 2.3, there is a sequence $(x^{\alpha_n})_n$ of words such that $\lim_{n\to\infty} x^{\alpha_n} = \pi$ in $\overline{\Omega}_1\underline{S}$. Now, if $\alpha_n \geq |A|$, then $\pi_A(a) \in K_a$, so that the set of exponents $\{\alpha_n : n = 1,2,\ldots\}$ must be bounded. Hence, there is a a constant sequence $(x^\alpha)_n$ converging to π in $\overline{\Omega}_1\underline{S}$, that is $\underline{S} \models \pi = x^\alpha$, whence $\pi \in \Omega_1\underline{S}$.

LEMMA 3.2. Let $\pi \in \overline{\Omega}_1\underline{S}$. Then, for $A \in \underline{S}$ and $a \in A$,
a) $\pi_A(a^s) = (\pi_A(a))^s$ for all positive integers s, and
b) $\pi_A(a^\omega) = a^\omega$.

PROOF. (a) Just note that $b \longmapsto b^s$ defines an endomorphism of any cyclic semigroup.
(b) This follows from (a) noting that there is an integer s such that $b^\omega = b^s$ for all $b \in A$.

Let $\underline{G}$ denote the class of all finite groups.

PROPOSITION 3.3 Let $\pi \in \overline{\Omega}_1\underline{S}$. Then, either $\pi \in \Omega_1\underline{S}$, or $\pi(x) = \pi(x^\omega x)$ so that π is completely determined by its restriction to $\underline{G}$.

PROOF. Suppose $\pi \notin \Omega_1\underline{S}$. Then, by Lemma 3.1, $\pi_A(a) \in K_a$ for all $A \in \underline{S}$ and $a \in A$. Thus, $\pi_A(a) = \pi_A(a)a^\omega$ since a^ω is the neutral element of K_a, whence $\pi_A(a) = \pi_A(a)\pi_A(a^\omega) = \pi_A(a^\omega a)$ applying Lemma 3.2 twice. Hence $\pi(x) = \pi(x^\omega x)$ as claimed.

We now study the unary implicit operations on $\underline{G}$. Let $\pi \in \overline{\Omega}_1\underline{G}$. Then, for $A \in \underline{G}$ and $a \in A$, $\pi_A(a) = a^{\alpha(n)}$ for some function $\alpha : \mathbb{N} \longrightarrow \mathbb{N}_0$ where n=ord a is the order of a (since $\pi_A(a) = \pi_{<a>}(a)$). The following Lemma gives the arithmetic conditions on such a function α which insure that the formula $\pi_A(a) = a^{\alpha(n)}$ defines an implicit operation on $\underline{G}$. We write $m|n$ in case m divides n. We denote by $\mathbb{N}$ the set of all positive integers and we let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

LEMMA 3.4. The following are equivalent for a function $\alpha : \mathbb{N} \longrightarrow \mathbb{N}_0$.
i) α defines an implicit operation on $\underline{G}$.
ii) $d|n$ implies $d|\alpha(n) - \alpha(d)$.

PROOF. (i) means, for every $A,B \in \underline{G}$, every homomorphism $h : A \to B$, and every $a \in A$, $h(\pi_A(a)) = \pi_B(h(a))$, i.e.,

$h(a^{\alpha(n)}) = (h(a))^{\alpha(d)}$ where n=ord a and d = ord h(a). Since h is a

homomorphism, $d|n$, whence $(h(a))^{\alpha(n)} = (h(a))^{\alpha(d)}$ if and only if $d|\ \alpha(n)-\alpha(d)$. This proves (ii) $\Rightarrow$ (i). For the converse, just use the above cyclic groups A and B of orders n and d and generators a and b, respectively, and the homomorphism $h : A \to B$ sending a to b.

Let $\mathbb{P}$ denote the set of all primes.

THEOREM 3.5. Let $\lambda_i : \mathbb{P} \to \mathbb{N}$ $(i=0,1,\ldots)$ be given functions and define $\alpha(p^k) = \sum_{i=0}^{k-1} \lambda_i(p)p^i$. Then, there is one and only one extension of α to $\mathbb{N}$ (with $\alpha(n)$ defined up to congruence modulo n) such that α defines an implicit operation on $\underline{G}$. Moreover, every $\pi \in \overline{\Omega}_1\underline{G}$ can be obtained in this way.

PROOF. Let $\pi \in \overline{\Omega}_1\underline{G}$ and let $\alpha : \mathbb{N} \to \mathbb{N}_0$ be any function defining π. Then, by Lemma 3.4, for each $p \in \mathbb{P}$, $p \mid \alpha(p^2) - \alpha(p)$ and so we may assume $\alpha(p^2) = \alpha(p) + \lambda_1(p)p$ for some $\lambda_1(p) \in \mathbb{N}_0$ (since adding any multiple of p^2 to $\alpha(p^2)$ does not change π). An easy induction gives that we may assume $\alpha(p^n) = \alpha(p) + \lambda_1(p)p + \ldots + \lambda_{n-1}(p)p^{n-1}$ with $\lambda_i(p) \in \mathbb{N}_0$ $(i=1,\ldots,n-1)$ independent of n. Finally, take $\lambda_0(p) = \alpha(p)$.

Next we show there is a unique extension $\beta : \mathbb{N} \to \mathbb{N}_0$ of any α defined by a sequence $(\lambda_i)_i$ of functions on prime powers as in the statement of the Theorem so that β defines an implicit operation on $\underline{G}$ (where uniqueness of $\beta(n)$ is again up to congruence modulo n). Indeed, suppose β is such an extension. Let $n \in \mathbb{N}$ and let

$n = p_1^{k_1} \ldots p_k^{k_r}$ be a factorization of n into powers of distinct primes.

Then, by Lemma 3.4, $\beta(n)$ is a solution to the system of congruences

$x \equiv \alpha(p_i^{k_i}) \pmod{p_i^{k_i}}$ $(i=1,\ldots,r)$. By the Chinese Remainder

Theorem, this solution always exists and is unique modulo n. This establishes uniqueness and gives a way to define β to establish existence. Thus, it remains to show that β, defined in this way, defines in turn an implicit operation on $\underline{G}$. Now, if $d \mid n$, then

$d = p_1^{\ell_1} \ldots p_r^{\ell_r}$ for some $\ell_i \le k_i$ $(i=1,\ldots,r)$. Hence

$\beta(n) \equiv \alpha(p_i^{k_i})$ and $\beta(d) \equiv \alpha(p_i^{\ell_i})$ both mod $p_i^{\ell_i}$. Since $\alpha(p_i^{k_i}) \equiv \alpha(p_i^{\ell_i}) \pmod{p_i^{\ell_i}}$ by the definition of $\alpha(p_i^{k_i})$, it follows that $p_i^{\ell_i} \mid \beta(n) - \beta(d)$ for $i=1,\ldots,r$ and so $d \mid \beta(n) - \beta(d)$.

By Lemma 3.4, β defines an implicit operation on $\underline{G}$.

COROLLARY 3.6. Every pseudovariety of abelian groups is of the form $[\![\pi=1, xy=yx]\!]$ for some $\pi \in \overline{\Omega}_1\underline{S}$.

PROOF. Let $\underline{V}$ be any pseudovariety of abelian groups. Define $\lambda_i : \mathbb{P} \to \mathbb{N}_0$ by $\lambda_i(p) = 0$ if $\mathbb{Z}_{p^{i+1}} \in \underline{V}$ and $\lambda_i(p) = 1$ otherwise. Let α be as in Theorem 3.5 and $\pi \in \overline{\Omega}_1\underline{S}$ be the implicit operation determined by α. We claim $\underline{V} = [\![\pi=1, xy=yx]\!]$.

Let $A \in \underline{V}$ and $a \in A$ be an element of order n. Let $n = p_1^{k_1} \cdots p_r^{k_r}$ be a factorization of n into products of distinct primes. Then $\alpha(n) \equiv \alpha(p_i^{k_i}) \pmod{p_i^{k_i}}$ $(i=1,\ldots,r)$ and so $\alpha(n) \equiv 0 \pmod n$ since $\alpha(p_i^{k_i}) = \sum_{j=0}^{k_i-1} \lambda_j(p)p^j = 0$ as $\mathbb{Z}_{p_i^{k_i}} \in \underline{V}$. Hence,

$\pi_A(a) = a^{\alpha(n)} = 1$. Whence $\underline{V} \models \pi = 1$.

Conversely, let $A \in [\![\pi=1, xy=yx]\!]$. Then A is an abelian group since, if $a \in A$, then $\pi_A(a) = a^s$ por some $s \geq 1$ and a^s is an identity element in A because $A \models \pi = 1$. Thus, A is isomorphic to a direct product $\mathbb{Z}_{p_1^{k_1}} \times \ldots \times \mathbb{Z}_{p_s^{k_s}}$ of cyclic groups (where the p_i are not necessarily distinct primes). If $A \notin \underline{V}$, then some $\mathbb{Z}_{p_i^{k_i}} \notin \underline{V}$ and so, if $\ell = \min\{m : \mathbb{Z}_{p_i^m} \notin \underline{V}\}$ and $a \in A$ is an element of order p_i^{ℓ}, then $\pi_A(a) = a^{\alpha(p_i^{\ell})} = a^{p_i^{\ell-1}} \neq 1_A$, whence $A \not\models \pi = 1$.

Hence $A \in \underline{V}$.

Using Corollary 3.6, Almeida [2] has observed that there are finite sets Σ of pseudoidentities such that there is no algorithm to decide whether a given finite semigroup S lies in the pseudovariety $[\![\Sigma]\!]$.

COROLLARY 3.7. $|\overline{\Omega}_1\underline{S}| = 2^{\aleph_0}$.

PROOF. This follows easily from Corollary 3.6 since there are that many pseudovarieties of abelian groups.

4. SOME UNUSUAL BINARY AND TERNARY OPERATIONS

In this section we use some classical results of the combinatorial theory of words to produce, for the values 2 and 3 of n, n-ary implicit operations which are not composites of (n-1)-ary implicit operations and explicit operations.

THEOREM 4.1. Let $\underline{LJ}_1$ denote the class of all finite semigroups S all of whose submonoids eSe $(e^2=e \in S)$ are semilattices. Then, for every pseudovariety $\underline{V}$ containing $\underline{LJ}_1$, there exist binary implicit operations on $\underline{V}$ which are not finite composites of explicit and unary implicit operations.

PROOF. Let $A = \{ a,b \}$ be a two-letter alphabet and consider the endomorphism μ of the free semigroup A^+ on A defined by $\mu(a) = ab$, $\mu(b) = ba$. Consider the sequence $(w_n)_n$ of the words $w_n = \mu^n(a)$ obtained by iteration of μ on a. This sequence of words was first studied by Thuë and Morse and it is known to consist of cube-free words, i.e., no w_n has a factor of the form u^3 with $u \in A^+$ (see Lothaire [6, Chapter 2] for details).

Since $\overline{\Omega}_2\underline{V}$ is compact by Theorem 2.2, $(w_n)_n$ admits a convergent subsequence. We show no subsequence $(w_{\varphi(n)})_n$ can converge in $\overline{\Omega}_2\underline{S}$ to an implicit operation which is a finite composite of explicit and unary implicit operations. Suppose, on the contrary, that $\lim_{n\to\infty} w_{\varphi(n)} = \pi$ where $\pi(a,b)$ is such an implicit operation. Notice that π cannot be an explicit operation.

Then $\pi = \lim_{n\to\infty} v_n$ where $v_n \in A^+$ is given, for all n, by the same finite expression in a,b using the operations $(x,y) \mapsto xy$ and operations of the form $x \mapsto x^{\theta(n)}$, where θ is a given strictly increasing function from $\mathbb{N}$ into itself (non-strictly increasing θ leads to explicit $x \mapsto x^{\theta(n)}$). In particular, if $(v_n)_n$ is not constant (which is the case since $\pi \notin \Omega_2\underline{S}$), v_n has a factor u^3 for all $n \geq 3$ where $u \in A^+$ is independent of n.

Here, we recall that there exists a finite semigroup $S_k \in \underline{LJ}_1 \subseteq \underline{V}$ such that an identity $v=w$ holds in S_k if and only if the words v and w have the same initial and terminal segments of length k-1 and the same factors of length k (cf. Almeida [1]) . Let $k=3|u|$ and let $\ell=|S_k|$. Then, by definition of convergence of sequences in $\overline{\Omega}_2\underline{V}$, there exists $n_0 \geq 3$ such that $S \models w_{\varphi(n)} = \pi = v_n$ for all $n \geq n_0$ and all $S \in \underline{V}$ with $|S| \leq \ell$. In particular, $S_k \models w_{\varphi(n)} = v_n$ so that u^3 is a factor of $w_{\varphi(n)}$, which is impossible since $w_{\varphi(n)}$ is cube-free.

Hence, no accumulation point of the sequence $(w_n)_n$ in $\overline{\Omega}_2\underline{V}$ can be a finite composite of explicit and unary implicit operations.

THEOREM 4.2. For each pseudovariety $\underline{V}$ containing $\underline{LJ}_1$ there exist ternary implicit operations on $\underline{V}$ which are not finite composites of explicit and binary implicit operations.

PROOF. Let $A = \{a,b,c\}$ be a three-letter alphabet. The proof proceeds along the same lines as the proof of Theorem 4.1 working with any infinite sequence of square-free words in A^+ (see also Lothaire [6, Chapter 2] for the existence of such sequences). The only other ingredient is the observation that there are no square-free words of length 4 on a two-letter alphabet. We leave the details to the reader.

In view of Theorems 4.1 and 4.2, it is natural to ask whether in general there are n-ary implicit operations on $\underline{V} \supseteq \underline{LJ}_1$ which are not composites of explicit and (n-1)-ary implicit operations. If the same type of argument as the one used above is going to be applied, one is apparently still lacking theorems on avoidable regularities in words on n-letter alphabets.

5. FINITE SEMIGROUPS IN WHICH IDEMPOTENTS ARE CENTRAL

In spite of the apparent chaos of implicit operations discovered in the previons section, we proceed to clarify their structure. Here, we restrict our attention to the pseudovariety $\underline{ZE} = [\![x^\omega y = yx^\omega]\!]$ of all finite semigroups in which all idempotents are central.

Let $\pi \in \overline{\Omega}_n\underline{ZE}$. We say that π has the kernel property if, for every $S \in \underline{ZE}$ and $s_1,\dots s_n \in S$, $\pi_S(s_1,\dots,s_n)$ belongs to the minimal ideal of the subsemigroup of S generated by $s_1,\dots,s_n$.

LEMMA 5.1. $\underline{ZE} \models (xy)^\omega = x^\omega y^\omega$.

PROOF. Let $S \in \underline{ZE}$ and let n be such that $S \models x^\omega = x^n$. Then

$$
\begin{aligned}
S \models (xy)^n &= xy(xy)^{n-1}(xy)^n(xy)^n \\
&= x(xy)^n y(xy)^{x-1}(xy)^n && \text{since idempotents are central} \\
&= x^2u_2(xy)^n && \text{where } u_2 = y(xy)^{n-1}y(xy)^{n-1} \\
&= x^k u_k(xy)^n && \text{for some } u_k \text{, by induction on } k \\
&= x^n x^n u_n(xy)^n && \text{since } S \models x^\omega = x^n \\
&= x^n(xy)^n && \text{by the above.}
\end{aligned}
$$

Analogously, $S \models (xy)^n = (xy)^n y^n$. Furthermore,

$$
\begin{aligned}
S \models x^n y^n &= xx^{n-1}\, y^n\, x^n\, y^n \\
&= xyy^{n-1}\, x^{n-1}\, x^n\, y^n \\
&= xyxx^{n-1}\, y^{n-1}\, x^{n-1}\, x^n\, y^n \\
&= (xy)^2(y^{n-1}x^{n-1})^2\, x^n\, y^n \\
&= (xy)^k(y^{n-1}x^{n-1})^k\, x^n\, y^n && \text{by induction on } k \\
&= (xy)^n(xy)^n(y^{n-1}x^{n-1})^n\, x^n\, y^n && \text{since } S \models x^\omega = x^n \\
&= (xy)^n\, x^n\, y^n && \text{by the above.}
\end{aligned}
$$

Hence $S \models x^n\, y^n = (xy)^n\, x^n\, y^n = (xy)^n\, y^n = (xy)^n$.

PROPOSITION 5.2. Let $\pi \in \overline{\Omega}_n \underline{ZE}$. Then π has the kernel property if and only if $\pi(x_1,\dots x_n) = \rho(ex_1,\dots,ex_n)$ where $e=(x_1\dots x_n)^\omega$ for some $\rho \in \overline{\Omega}_n \underline{G}$.

PROOF: Suppose π has the kernel property and let ρ be the restriction of π to $\underline{G}$. Then $\pi(x_1,\dots,x_n) = \pi(x_1,\dots,x_n)e$ since, for each $S\in\underline{ZE}$ and each $s_1,\dots,s_n \in S$, $f = (s_1\dots s_n)^\omega$ is the neutral element of the kernel of the subsemigroup generated by $s_1,\dots,s_n$ by Lemma 5.1. On the other hand, since idempotents are central, the mapping $s \mapsto sf$ is an endomorphism of S. Since π is an implicit operation on $\underline{ZE}$, it follows that $\pi(x_1,\dots,x_n)e = \pi(ex_1,\dots,ex_n) = \rho(ex_1,\dots,ex_n)$. Hence, π is of the required form. The converse is obvious.

LEMMA 5.3. Let $S \in \underline{ZE}$ and let $n=|S|$. Then, for each word w with a number $|w|_x$ of occurrences of a variable x at least n, $S \models w = x^\omega w$.

PROOF. Let $s \in S$ and $t_0,t_1,\dots,t_n \in S^1$. Let $a_k = t_0st_1s\dots t_ks$ $(k=0,\dots,n-1)$. If the a_k $(k=0,\dots,n-1)$ are all distinct, then $a_k = s^\omega$ for some k and so $a_nt_n = s^\omega a_nt_n$. Otherwise, let $k < \ell$ with $a_k = a_\ell$. Then

$$\begin{aligned} a_k = a_\ell &= a_k(t_{k+1}s\dots t_\ell s) \\ &= a_k(t_{k+1}s\dots t_\ell s)^\omega \quad \text{by the preceding line} \\ &= a_k(t_{k+1}\, s\dots t_\ell s)^\omega s^\omega \quad \text{by Lemma 5.1} \\ &= s^\omega\, a_k \quad \text{by the above .} \end{aligned}$$

Hence $a_nt_n = s^\omega a_nt_n$.

We are now ready for the main result of this section .

THEOREM 5.4. Every implicit operation on $\underline{ZE}$ is of the form

$$w_0\ \rho_1(ey_1,\dots,ey_n)\ w_1\dots\ \rho_k(ey_1,\dots,ey_n)\ w_k$$

where each w_i is a word not involving the variables $y_1,\dots,y_n$ (with w_0,w_k possibly empty), $e=(y_1\dots y_n)^\omega$, and each $\rho_i \in \overline{\Omega}_n \underline{G}$.

PROOF. Let $\pi \in \overline{\Omega}_m \underline{ZE}$. By Theorem 2.2, there is a sequence $(v_k)_k$ of words on an m-letter alphabet $\{x_1,\dots,x_m\}$ such that $\lim_{k\to\infty} v_k = \pi$.

Let $J = \{j : 1 \le j \le m, \{|v_k|_{x_j} : k = 1,2,\dots\}$ is unbounded $\}$.

Then, by considering a subsequence of $(v_k)_k$ if necessary, we may assume that the word u obtained from v_k by removing all x_j with $j \in J$ is the same for all k and $|v_k|_{x_j} \geq k$ for all $j \in J$. Write

$v_k = w_0 v_{k1} w_1 \dots v_{kr} w_r$ where $w_0 w_1 \dots w_r = u$.

Given $\ell \in \mathbb{N}$, there exists k_ℓ such that $S \models \pi = v_k$ for all $k \geq k_\ell$ and all $S \in \underline{ZE}$ with $|S| \leq \ell$. In particular, for $k \geq \max\{\ell, k_\ell\}$, $S \models \pi = w_k = x_j^{\omega} w_k$ $(j \in J)$ by Lemma 5.3.

Let $\sigma_{ki} = v_{ki} \prod_{j \in J} x_j^{\omega}$ $(i=1,\dots,r)$ and let $\sigma_k = w_0 \sigma_{k1} w_1 \dots \sigma_{kr} w_r$. By the above, $\lim_{k\to\infty} \sigma_k = \pi$. Since $\overline{\Omega}_m \underline{ZE}$ is compact, we may assume each sequence $(\sigma_{ki})_k$ converges, say to π_i. Clearly each σ_{ki} has the kernel property and, therefore, so does each π_i. To complete the proof, it suffices to quote Proposition 5.2.

COROLLARY 5.5 (Almeida and Reilly [3]) Every pseudovariety $\underline{V} \subseteq \underline{N}$ is of the form $[[\, x^{\omega} = 0, \Sigma \,]]$ for some set Σ of identities.

PROOF. Since $\underline{V} \models x^{\omega} = 0$, we see that every implicit operation on $\underline{V}$ with the kernel property is constant with value 0 on each $S \in \underline{V}$. Thus, if $\underline{V} = [[\, x^{\omega} = 0, \Sigma \,]]$ where Σ is a set of pseudoidentities, then each element of Σ may be replaced by an identity of one of the forms $v=0$ or $v=w$ in view of Theorem 5.4. Hence, we may assume Σ is a set of identities, as claimed.

For the pseudovariety $\underline{Com}$ of all finite commutative semigroups, we can give a complete description of the implicit operations on $\underline{Com}$.

THEOREM 5.6. Every n-ary implicit operation on $\underline{Com}$ is a product of unary implicit operations (these being viewed as n-ary operations depending on only one variable).

PROOF. Let $\pi \in \overline{\Omega}_n \underline{Com}$. Then, by Theorem 2.2, there is a sequence $(w_k)_k$ of words in $\{x_1,\dots,x_n\}^+$ such that $\lim_{k\to\infty} w_k = \pi$. Since we are working with commutative semigroups, we may take $w_k = x_1^{\alpha_{k1}} \dots x_n^{\alpha_{kn}}$ for $k=1,2,\dots$. Since $\overline{\Omega}_n \underline{Com}$ is compact, we may assume each sequence $(x_i^{\alpha_{ki}})_k$ converges, say $\pi_i = \lim_{k\to\infty} x_i^{\alpha_{ki}}$. Then $\pi_i \in \overline{\Omega}_1 \underline{Com}$ $(i=1,\dots,n)$ and $\pi = \pi_1 \dots \pi_n$.

Theorems 3.5 and 5.6 are, so far, rare complete characterizations of certain sets of implicit operations. In general, it should be more feasible to obtain results like Theorem 5.4 in which implicit operations on a pseudovariety are described modulo the knowledge of implicit operations of some special kind. In this direction, we propose the following conjecture : every implicit operation on $\underline{S}$ is a finite composite of explicit operations and implicit operations which assume only regular values.

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