Incorporating a Four-Dimensional Filter Line Search Method into an Interior Point Framework

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Abstract

Here we incorporate a four-dimensional filter line search method into an infeasible primal-dual interior point framework for nonlinear programming. Each entry in the filter has four components measuring dual feasibility, complementarity, primal feasibility and optimality. Three measures arise directly from the first order optimality conditions of the problem and the fourth is the objective function, so that convergence to a stationary point that is a minimizer is guaranteed. The primary assessment of the method has been done with a well-known collection of small problems.

Key words: Nonlinear optimization, interior point, filter method

MSC 2000: 90C51, 90C30

1 Introduction

In this paper we consider a nonlinear constrained optimization problem in the following form:

\[
\min_{x \in \mathbb{R}^n} \ F(x)
\]

\[
\text{s.t. } h(x) \geq 0
\]

where \(h_i : \mathbb{R}^n \rightarrow \mathbb{R}\) for \(i = 1, \ldots, m\) and \(F : \mathbb{R}^n \rightarrow \mathbb{R}\) are nonlinear and twice continuously differentiable functions. Interior point methods based on a logarithmic barrier function have been widely used for nonlinear programming [9, 11, 12]. To allow convergence from poor starting points, barrier and augmented Lagrangian merit functions may be used [7]. Some line search frameworks use penalty merit functions to enforce progress toward the solution. As an alternative to merit functions, Fletcher and Leyffer [4] proposed a filter method as a tool to guarantee global convergence in
algorithms for nonlinear optimization. This technique incorporates the concept of non-dominance to build a filter that is able to accept trial points if they improve either the objective function or the constraints violation, instead of a combination of those two measures defined by a merit function. The filter replaces the use of merit functions, so avoiding the update of penalty parameters that are associated with the penalization of the constraints in a merit function. The filter technique has already been adapted to interior point methods. In [13, 14, 15], a filter line search strategy incorporated in a barrier type method is used. The two components of each entry in the filter are the barrier objective function and the constraints violation. In [10], a two-dimensional filter is used in a primal-dual interior point method context. The two entries, measuring quasi-centrality and optimality, combine the three criteria of the first order optimality conditions. A three-dimensional filter based line search strategy has already been tested in [2, 3]. The three components of the filter measure feasibility, centrality and optimality and are present in the first order KKT conditions of the barrier problem associated with the problem (1). The optimality measure relies on the norm of the gradient of the Lagrangian function. Convergence to stationary points may be proved, although convergence to a local minimizer is not guaranteed.

In this paper we propose a four-dimensional filter line search method to incorporate into a primal-dual interior point framework. The three criteria of the first order optimality conditions are used separately to define three measures, and the objective function, $F(x)$, is the other so that convergence to a stationary point that is a minimizer is guaranteed.

The paper is organized as follows. Section 2 presents the interior point paradigm and Section 3 introduces the novel filter line search method that relies on four components and presents the acceptance conditions used to accept a point in the filter. The experimental results and the conclusions make Section 4.

## 2 The primal-dual interior point paradigm

In this interior point paradigm, problem (1) is reformulated as an equality constrained problem by using nonnegative slack variables $w$, as follows:

$$
\begin{align*}
\min_{x \in \mathbb{R}^n, w \in \mathbb{R}^m} & \quad F(x) \\
\text{s.t.} & \quad h(x) - w = 0 \\
& \quad w \geq 0,
\end{align*}
$$

(2)

and the first order or Karush-Kuhn-Tucker (KKT) optimality conditions for a minimum of (2) are written as

$$
\begin{align*}
\nabla_x \mathcal{L}(x, w, y, v) &= 0 \\
y - v &= 0 \\
W y &= 0 \\
h(x) - w &= 0 \\
w \geq 0, \quad v \geq 0
\end{align*}
$$

(3)

where $y, v \in \mathbb{R}^m$ are the vector of Lagrange multipliers, $W = diag(w_i)$ is a diagonal matrix, and $\nabla_x \mathcal{L}$ is the gradient with respect to $x$ of the Lagrangian function defined
by
\[ L(x, w, y, v) = F(x) - y^T(h(x) - w) - v^T w. \]

The system (3) is equivalent to the system
\[
\begin{align*}
\nabla F(x) - A(x)^T y &= 0 \\
W y &= 0 \\
h(x) - w &= 0 \\
w &\geq 0, \quad y &\geq 0
\end{align*}
\] (4)

where \( A(x) \) is the Jacobian matrix of the constraint functions \( h(x) \). If the second equation of conditions (4) is perturbed, we get the KKT perturbed system of equations
\[
\begin{align*}
\nabla F(x) - A(x)^T y &= 0 \\
W y - \mu e &= 0 \\
h(x) - w &= 0 \\
w &\geq 0, \quad y &\geq 0
\end{align*}
\] (5)

where \( e \) is a vector of unit \( m \) elements and \( \mu \) is a positive parameter called barrier parameter [9, 12]. This perturbed system is equivalent to the KKT conditions of the barrier problem associated with problem (2), in the sense that they have the same solution,
\[
\min_{x \in \mathbb{R}^n, w \in \mathbb{R}^m} \varphi_{\mu}(x, w)
\quad \text{s.t.} \quad h(x) - w = 0
\] (6)

where \( \varphi_{\mu}(x, w) \equiv F(x) - \mu \sum_{i=1}^{m} \log(w_i) \) is the logarithmic barrier function. Applying the Newton’s method to solve (5), the following system, after symmetrization, arises
\[
\begin{bmatrix}
-H & 0 & A(x)^T \\
0 & -W^{-1}Y & -I \\
A(x) & -I & 0
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta w \\
\Delta y
\end{bmatrix}
= \begin{bmatrix}
\nabla F(x) - A(x)^T y \\
-\mu W^{-1} e + y \\
w - h(x)
\end{bmatrix}
\] (7)

where \( Y = \text{diag}(y_i) \) is a diagonal matrix,
\[
H = \nabla^2 F(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x)
\]
is the Hessian matrix of the Lagrangian function.

Since the second equation in (7) can be used to eliminate \( \Delta w \) without producing any off-diagonal fill-in in the remaining system, one obtains
\[
\Delta w = W Y^{-1} (\mu W^{-1} e - y - \Delta y),
\] (8)

and the resulting reduced KKT system
\[
\begin{bmatrix}
-H & A(x)^T \\
A(x) & W Y^{-1}
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
= \begin{bmatrix}
\nabla F(x) - A(c)^T y \\
w - h(x) + W Y^{-1} (\mu W^{-1} e - y)
\end{bmatrix}
\] (9)
to compute the search directions $\Delta x$, $\Delta w$, $\Delta y$. This interior point based method implements a line search procedure combined with a backtracking strategy to compute a step size $\alpha_k$, at each iteration $k$, and define a new approximation by

$$
\begin{align*}
x_{k+1} &= x_k + \alpha_k \Delta x_k \\
w_{k+1} &= w_k + \alpha_k \Delta w_k \\
y_{k+1} &= y_k + \alpha_k \Delta y_k
\end{align*}
$$

where equal step sizes are used with primal and dual directions. The choice of the step size $\alpha_k$ is a very important issue in nonconvex optimization and in the interior point context aims:

1. to ensure the nonnegativity of the slack and dual variables;
2. to enforce progress towards feasibility, complementarity and optimality.

Here we propose a four-dimensional filter method combined with a backtracking strategy to define new approximations to the primal, slack and dual variables that give a sufficient reduction in one of the filter measures. The backtracking strategy defines a decreasing sequence of step sizes

$$
\alpha_{k,l} \in (0, \alpha_{\text{max}}^k], l = 0, 1, \ldots,
$$

with $\lim_{l} \alpha_{k,l} = 0$, until a set of acceptance conditions are satisfied. Here, the index $l$ denotes the iteration counter for the inner loop. The parameter $\alpha_{\text{max}}^k$ represents the longest step size that can be taken along the direction before violating the nonnegativity conditions $w_k \geq 0$, $y_k \geq 0$. If the initial approximations for the slack and dual variables satisfy $w_0 > 0$, $y_0 > 0$, the maximal step size $\alpha_{\text{max}}^k \in (0, 1]$ is defined by

$$
\alpha_{\text{max}}^k = \min \{1, \varepsilon \min \{-w_i^k(\Delta w_i^k)^{-1}, -y_i^k(\Delta y_i^k)^{-1}\} \}
$$

for all $i$ such that $\Delta w_i^k < 0$ and $\Delta y_i^k < 0$, and $\varepsilon \in (0, 1)$ is a fixed parameter.

### 3 Four-dimensional filter line search method

In order to define the components of each entry in the filter and the corresponding acceptance conditions, the following notation is used:

$$
\begin{align*}
u &= (x, w, y), & \Delta &= (\Delta x, \Delta w, \Delta y), \\
u^1 &= (x, w), & \Delta^1 &= (\Delta x, \Delta w), \\
u^2 &= (w, y), & \Delta^2 &= (\Delta w, \Delta y), \\
u^3 &= (x, y), & \Delta^3 &= (\Delta x, \Delta y).
\end{align*}
$$

The optimality conditions (4) define a set of natural measures to assess the algorithm progress. Some combinations of these measures may be used to define the components of each entry in the filter, see for example [10]. We use the three conditions separately. Further, to be able to guarantee convergence to stationary points that are minimizers, we introduce $F$ as the fourth measure in the filter [5]. Table 1 lists the four components for the herein proposed filter.
Table 1: Components of the four-dimensional filter

<table>
<thead>
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<th>measure</th>
<th>component</th>
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<tbody>
<tr>
<td>primal feasibility</td>
<td>$\theta pf(u^1) \equiv |h(x) - w|_2$</td>
</tr>
<tr>
<td>complementarity</td>
<td>$\theta c(u^2) \equiv |Wy|_2$</td>
</tr>
<tr>
<td>dual feasibility</td>
<td>$\theta df(w^3) \equiv |\nabla F(x) - A(x)^T y|_2$</td>
</tr>
<tr>
<td>optimality</td>
<td>$F(x)$</td>
</tr>
</tbody>
</table>

3.1 The acceptance conditions

In this algorithm, the trial point $u_k(\alpha_{k,l}) = u_k + \alpha_{k,l}\Delta_k$ is acceptable by the filter, if it leads to sufficient progress in one of the four measures compared to the current iterate,

$$
\theta pf(u^1_k(\alpha_{k,l})) \leq (1 - \gamma_1)\theta pf(u^1_k) \quad \text{or} \quad \theta c(u^2_k(\alpha_{k,l})) \leq (1 - \gamma_2)\theta c(u^2_k)
$$

or

$$
\theta df(u^3_k(\alpha_{k,l})) \leq (1 - \gamma_3)\theta df(u^3_k) \quad \text{or} \quad F(x_k(\alpha_{k,l})) \leq F(x_k) - \gamma_4\theta pf(u^1_k)
$$

where $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in (0, 1)$ are fixed constants.

However, to prevent convergence to a point that is nonoptimal, and whenever for the trial step size $\alpha_{k,l}$, the following switching conditions

$$
m_k(\alpha_{k,l}) < 0 \quad \text{and} \quad [-m_k(\alpha_{k,l})]^{s_0} [\alpha_{k,l}]^{1-s_0} > \delta [\theta pf(u^1_k)]^{s_1}
$$

and

$$
[-m_k(\alpha_{k,l})]^{s_0} [\alpha_{k,l}]^{1-s_0} > \delta [\theta df(u^3_k)]^{s_3}
$$

hold, with fixed constants $\delta > 0$, $s_1, s_2, s_3 > 1$, $s_0 \geq 1$, where

$$
m_k(\alpha) = \alpha \nabla F(x_k)^T \Delta x_k,
$$

then the trial point must satisfy the Armijo condition with respect to the optimality measure

$$
F(x_k(\alpha_{k,l})) \leq F(x_k) + \eta_1 m_k(\alpha_{k,l}),
$$

instead of (11) to be acceptable. Here, $\eta_1 \in (0, 0.5)$ is a constant.

According to previous publications on filter methods (for example [13]), a trial step size $\alpha_{k,l}$ is called a $F$-step if (13) holds. Similarly, if a $F$-step is accepted as the final step size $\alpha_k$ in iteration $k$, then $k$ is referred to as a $F$-type iteration.

3.2 The four-dimensional filter

The filter is a set that contains combinations of the four measures $\theta pf$, $\theta c$, $\theta df$, and $F$ that are prohibited for a successful trial point and is initialized to

$$
\mathcal{F}_0 \subseteq \left\{(\theta pf, \theta c, \theta df, F) \in \mathbb{R}^4 : \theta pf \geq \theta pf^{\max}, \theta c \geq \theta c^{\max}, \theta df \geq \theta df^{\max}, F \geq F^{\max}\right\},
$$

for the nonnegative constants $\theta pf^{\max}$, $\theta c^{\max}$, $\theta df^{\max}$ and $F^{\max}$. The filter is updated according to

$$
\mathcal{F}_{k+1} = \mathcal{F}_k \cup \left\{(\theta pf, \theta c, \theta df, F) \in \mathbb{R}^4 : \theta pf \geq (1 - \gamma_1)\theta pf(u^1_k) \quad \text{and} \quad \theta c \geq (1 - \gamma_2)\theta c(u^2_k)
$$

and

$$
\theta df \geq (1 - \gamma_3)\theta df(u^3_k) \quad \text{and} \quad F \geq F(x_k) - \gamma_4\theta pf(u^1_k)\right\},
$$

(15)
whenever the accepted step size satisfies (11). However, when for the accepted step size the conditions (12) and (13) hold, the filter remains unchanged.

Finally, when the backtracking line search cannot find a trial step size \( \alpha_{k,l} \) that satisfies the above criteria, we define a minimum desired step size \( \alpha_{k}^{\text{min}} \), using linear models of the involved functions,

\[
\alpha_{k}^{\text{min}} = \xi \begin{cases} 
\min \{ \gamma_1, \pi_1, \pi_2, \pi_3, \pi_4 \} , & \text{if } m_k(\alpha_{k,l}) < 0 \\
\min \{ \gamma_1, \pi_1 \} , & \text{if } m_k(\alpha_{k,l}) = 0 \\
\min \{ \gamma_1, \pi_1 \} , & \text{if } m_k(\alpha_{k,l}) > 0 \\
\pi_1 , & \text{otherwise}
\end{cases}
\]

(16)

where

\[
\pi_1 = \frac{\gamma_4 \theta_{pf}(u_k^1)}{-m_k(\alpha_{k,l})} , \quad \pi_2 = \frac{\delta [\theta_{pf}(u_k^1)]^{s_1}}{-m_k(\alpha_{k,l})} , \quad \pi_3 = \frac{\delta [\theta_c(u_k^2)]^{s_2}}{-m_k(\alpha_{k,l})} , \quad \pi_4 = \frac{\delta [\theta_{df}(u_k^2)]^{s_3}}{-m_k(\alpha_{k,l})}
\]

for positive constants \( \theta_{pf}^{\text{min}}, \theta_c^{\text{min}}, \theta_{df}^{\text{min}} \) and a safety factor \( \xi \in (0,1] \).

Like in [15] and whenever the backtracking line search finds a trial step size \( \alpha_{k,l} < \alpha_k^{\text{min}} \), the algorithm reverts to a restoration phase. Here, the algorithm tries to find a new iterate \( u_{k+1} \) that is acceptable to the current filter, i.e., (11) holds, by reducing either the primal feasibility measure or the complementarity within an iterative process.

### 3.3 Restoration phase

The task of the restoration phase is to compute a new iterate acceptable to the filter by decreasing either the primal feasibility or the complementarity, whenever the backtracking line search procedure cannot make sufficient progress and the step size becomes too small. Thus, the restoration algorithm works with the new functions

\[
\theta_{pf}^2(u^1) = \frac{1}{2} \| h(x) - w \|^2_2 \quad \text{and} \quad \theta_c^2(u^2) = \frac{1}{2} \| Wy \|^2_2
\]

and the steps \( \Delta^1 \) and \( \Delta^2 \) that are descent directions for \( \theta_{pf}^2(u^1) \) and \( \theta_c^2(u^2) \), respectively.

Using a backtracking strategy, the algorithm selects, at each iteration \( k \), a step size \( \alpha_k \in (0, \alpha_k^{\text{max}}] \) to define a new trial point \( u_k(\alpha_k) = u_k + \alpha_k \Delta_k \) that satisfies either

\[
\theta_{pf}^2(u_k^1(\alpha_k)) \leq \theta_{pf}^2(u_k^1) + \alpha_k \eta_2 \nabla \theta_{pf}^2(u_k^1)^T \Delta_k^1
\]

or

\[
\theta_c^2(u_k^2(\alpha_k)) \leq \theta_c^2(u_k^2) + \alpha_k \eta_3 \nabla \theta_c^2(u_k^2)^T \Delta_k^2
\]

for constants \( \eta_2 \) and \( \eta_3 \) in the set \((0,0.5] \).
3.4 Setting the barrier parameter

To guarantee a positive decreasing sequence of μ values, the barrier parameter is updated by a formula that couples the theoretical requirement defined on the first order KKT conditions (5) with a simple heuristic. Thus, μ is updated by

$$\mu_{k+1} = \max \left\{ \epsilon, \min \left\{ \kappa_\mu \mu_k m, \delta_\mu \frac{w^T_{k+1} y_{k+1}}{m} \right\} \right\}$$

(17)

where the constants \( \kappa_\mu, \delta_\mu \in (0, 1) \) and the tolerance \( \epsilon \) is used to prevent μ from becoming too small so avoiding numerical difficulties at the end of the iterative process.

3.5 Termination criteria

The termination criteria consider dual and primal feasibility and complementarity measures

$$\max \left\{ \| \nabla F(x) - A(x) y \|_\infty, \| h(x) - w \|_\infty, \| W y \|_\infty \right\} \leq \epsilon_{tol},$$

(18)

where

$$s = \max \left\{ 1, 0.01 \frac{\| y \|_1}{m} \right\}$$

and \( \epsilon_{tol} > 0 \) is the error tolerance.

4 Experimental results and conclusions

To test this interior point framework with the herein proposed four-dimensional filter line search technique we selected 109 constrained problems from the Hock and Schittkowski (HS) collection [8]. This preliminary selection aims to consider small and simple to code problems. The tests were done in double precision arithmetic with a Pentium 4. The algorithm is coded in the C programming language and includes an interface to AMPL to read the problems that are coded in the AMPL modeling language [6].

Our algorithm is a quasi-Newton based method in the sense that a symmetric positive definite quasi-Newton BFGS approximation, \( B_k \), is used to approximate the Hessian of the Lagrangian \( H \), at each iteration \( k \). In the first iteration, we may set \( B_0 = I \) or \( B_0 = \) positive definite modification of \( \nabla^2 F(x_0) \), depending on the characteristics of the problem to be solved.

4.1 Initial approximations

The algorithm implements two alternatives to initialize the primal and the dual variables. One uses the usual published initial values, \( x_0 \), as mentioned in [8], and sets all the dual variables to one. The other uses the published \( x_0 \) to define the initial dual variables, \( y_0 \), and new primal variables, \( \tilde{x}_0 \), by solving the simplified reduced system:

$$\begin{bmatrix} -(B_0 + I) & A^T(x_0) \\ A(x_0) & I \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \nabla F(x_0) \\ 0 \end{bmatrix}.$$
Further, if $\|y_0\|_\infty > 10^3$ then $y_0$ is component by component set to one. However, if $\|\hat{x}_0\|_\infty > 10^9\|x_0\|_\infty$ then $\hat{x}_0 = x_0$.

The nonnegativity of the initial slack variables are ensured by computing $w_0 = \max\{|h(x_0)|, \epsilon_w\}$, for the previously defined $x_0$, and a fixed positive constant $\epsilon_w$.

### 4.2 Setting user defined parameters

The chosen values for some of the constants are similar to the ones proposed in [15]:

- $\theta_{pf}^{max} = 10^4 \max\{1, \theta_{pf}(u_0^{1})\}$,
- $\theta_{pf}^{min} = 10^{-4} \max\{1, \theta_{pf}(u_0^{2})\}$,
- $\theta_c^{max} = 10^4 \max\{1, \theta_c(u_0^{3})\}$,
- $\theta_c^{min} = 10^{-4} \max\{1, \theta_c(u_0^{3})\}$,
- $F^{max} = 10^4 \max\{1, F(x_0)\}$,
- $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 10^{-5}$,
- $\delta = 1$, $s_1 = s_2 = s_3 = 1.1$,
- $s_o = 2.3$, $\eta_1 = \eta_2 = \eta_3 = 10^{-4}$,
- $\xi = 0.05$.

The other parameters are set as follows: $\epsilon = 0.95$, $\delta = \kappa = 0.1$, $\epsilon = 10^{-9}$, $\epsilon_w = 0.01$ and $\epsilon_{tol} = 10^{-6}$.

### 4.3 Comparative results

Table 2 summarizes the results obtained with the herein proposed four-dimensional filter line search interior point method. The table reports the number of iterations required to obtain a solution according to the termination criteria in (18), $N_{it}$, and the objective function value, $F(x^*)$. Except in 11 problems, the number of function evaluations was $N_{it} + 1$. Results inside parentheses were obtained with the parameter $\epsilon_{tol}$ set to $= 10^{-4}$. In all problems, our algorithm converges to the solution within a reasonable number of iterations.

For a comparative purpose we compare our results with the IPOPT, a filter line search barrier based method [13, 14, 15]. The results obtained by IPOPT are reported in the file "Ipopt-table.pdf" under


We noticed differences, some are rather small, in the objective function value in 32 problems. They are listed in Table 3. For the remaining problems used in this study, the herein proposed filter line search interior point method converges to the solutions reported in "Ipopt-table.pdf". The table reveals that we were able to get better solutions in eight problems. They are emphasized in the table. We may then conclude that the four-dimensional filter line search interior point based method is effective in reaching the solution of small nonlinear constrained optimization problems. In the future, different combinations of the criteria involved in the first order optimality conditions (4) will be analyzed, tested and compared with the present proposal.

### References

Table 2: Number of iterations and function values of our study

<table>
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<tr>
<th>Prob</th>
<th>Nit</th>
<th>$F(x^*)$</th>
<th>Prob</th>
<th>Nit</th>
<th>$F(x^*)$</th>
<th>Prob</th>
<th>Nit</th>
<th>$F(x^*)$</th>
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Table 3: Function values obtained by IPOPT

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