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ASYMPTOTICS FOR GENERATING FUNCTIONS OF THE FUSS–CATALAN NUMBERS

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Abstract. We consider a certain class of polynomials with coefficients in $\mathbb{Z}_M$, all of which admit a unique zero. We prove that the zero of each of those can be given by a (multiple) sum involving the coefficients and a vectorial generalization of the Fuss-Catalan numbers.

We also consider the sequence of the partial sums of the generating function of the d-Fuss-Catalan numbers. Using the holonomy of this sequence, we study its asymptotic behaviour. The main difference from the known case $d = 2$ is, in that one, we have a “closed” expression for the generating function.

1. Introduction

The Catalan numbers were studied by Euler, in the context of enumerating triangulations of regular polygons [5]. Their study by the Mongolian mathematician Antu Ming in the eighteenth century was announced in 1988 by Luo in [10] and further discussed by Larcombe in [9].

These numbers have multiple interpretations and applications, several of which can be found, for example, in [18], which also covers different generalizations of them. Throughout this paper we focus on a couple of these, the d-Fuss-Catalan numbers, for $d \in \mathbb{N} \setminus \{1\}$, whose element of order n, $C_d(n)$, is defined by

$$C_d(n) = \frac{1}{(d-1)n+1}\binom{dn}{n}, \tag{1}$$

and a vectorial generalization of the Catalan numbers, which we will define in (4). $C_d(n)$, introduced by Fuss in [6], counts, for example, the number of partitions of a $n(d-1)+2$-gon into $d+1$-gons and the number of d-ary trees with n internal nodes (see [7]). Recall that the Catalan numbers are the 2-Fuss-Catalan numbers.

The first problem we are interested in is finding the zeros of some polynomials in $\mathbb{Z}_M$, the ring of the integers modulo $M \in \mathbb{N}$. Consider a polynomial $Q = Q(x)$ with coefficients in $\mathbb{Z}_M$ of the form

$$a_d x^d + \cdots + a_1 x + a_0, \quad \text{where } a_i \text{ is nilpotent for } i \geq 2 \text{ and } a_1 \text{ invertible.} \tag{2}$$

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The Chinese remainder theorem and the Hensel lemma guarantee that there exists exactly one zero of Q in $\mathbb{Z}_M$. In this work, we will find a polynomial P in $d+1$ variables such that the zero of any polynomial as in (2) is equal to $P(a_0, a_1^{-1}, a_2, \dots, a_d)$. The coefficients of P are essentially vector generalized Catalan numbers, which are d-Fuss-Catalan numbers if $a_i = 0$ for $1 < i < d$.

The second problem was motivated by sequences presented in OEIS, The On-Line Encyclopedia of Integer Sequences [17]. For $d \in \mathbb{N} \setminus \{1\}$, $r \in \mathbb{R} \setminus \{0\}$, and $n \in \mathbb{N}$, consider the sequence

$$X(d,r,n) = \sum_{k=0}^{n} C_d(k) r^k. \tag{3}$$

In connection with the first problem, we will see that, if p is a prime number and r a multiple of p then, $X(d,r,n)$ is the zero, in $\mathbb{Z}_{p^{n+1}}$ of the polynomial $rx^d - x + 1$.

OEIS, in the sequence A112696 and onwards, presents recurrence formulas for $\big(X(2,r,n)\big)_{n\in\mathbb{N}}$ for some values of r, conjecturing them for some others. In this work, we obtain recurrence formulas for all values of d and r.

We also study the asymptotic behaviour of this sequence, when it diverges. For $d=2$, this was done by Mattarei in [11], using, among other instruments, the generating function of the Catalan numbers $F_2(x) = \frac{1-\sqrt{1-4x}}{2x}$. Elezović, in [3, 4] gives an efficient algorithm for recursive calculations of asymptotic expansions of several sums including $X(2,1,n)$. If $d > 2$ we do not have a nice expression for $F_d(x)$, apart from the equality $F_d(x) = 1 + xF_d(x)^d$.

We use some well-known results for holonomic sequences such as the Poincaré-Perron Theorem in [13, 12], and Corollary 1.6 of [8] to prove that

$$X(d,r,n) \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{d}}{(d-1)^{\frac{3}{2}}} \frac{A(d)r}{A(d)r-1} \left(A(d)\, r\right)^n n^{-\frac{3}{2}},$$

where $A(d) = \frac{d^d}{(d-1)^{d-1}}$, and $A(d)|r| > 1$.

2. Preliminaries

The Catalan numbers have a lot of generalizations. In this work we are interested in the d-Fuss-Catalan numbers, defined in (1), and the natural vectorial generalization, $C_{\vec{v}}(\vec{n})$, seen, for example, in [2] and a more general case in [14]. $C_{\vec{v}}(\vec{n})$ is defined by

$$C_{\vec{v}}(\vec{n}) = \frac{1}{(\vec{v}-\vec{1})\cdot\vec{n}+1} \binom{\vec{v}\cdot\vec{n}}{\vec{n}} = \frac{1}{\vec{v}\cdot\vec{n}+1} \binom{\vec{v}\cdot\vec{n}+1}{\vec{n}} \tag{4}$$

where, given $s \in \mathbb{N}$, $\vec{n} \in \mathbb{N}_0^s$ and $\vec{v} \in \mathbb{N}^s$, $\vec{v}\cdot\vec{n}$ denotes the inner product of $\vec{n}$ and $\vec{v}$ and $\binom{\vec{v}\cdot\vec{n}}{\vec{n}}$ is the multinomial coefficient $\frac{(\vec{v}\cdot\vec{n})!}{n_1!\cdots n_s!(\vec{v}\cdot\vec{n}-(n_1+\cdots+n_s))!}$.

$C_{\vec{v}}(\vec{n})$ is, for example, the number of ways that $\vec{v}\cdot\vec{n}$ people can be seated at a (round) table in such a way that, for all $i = 1, \dots, s$, there exist n_i groups of v_i people giving a v_i-hand shake with no crossings between different groups [2]. Of course, this

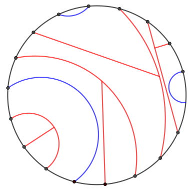

Figure 1: This is one of the $92\,810$ possible configuration for 18 people to be seated around a table, as referred to in the text for $\vec{n}=(3,4)$ and $\vec{v}=(2,3)$.

is the same as the number of subdivisions of $\vec{v}\cdot\vec{n}$ points on a circumference in n_i sets of v_i point groups without crossing.

$C_{\vec{v}}(\vec{n})$ is also is the number of polygonal dissections of an $(\vec{v}-\vec{1})\cdot\vec{n}+2$-gon into $n_1+\cdots+n_s$ polygons with n_i of them having v_i+1 edges, for $i=1,\ldots,s$. This can be found, for example, in [15].

Analogously with what happens with the Catalan numbers [16] and Fuss-Catalan numbers [6], these generalized Catalan numbers satisfy a recurrence relation that is an easy consequence of a result of Rhoades in [14] stating, in particular, that, if $\vec{r}\in\mathbb{N}_0^s$, $\vec{v}\in\mathbb{N}^s$, $m\in\mathbb{N}$ then

$$\sum_{\vec{r}_1+\cdots+\vec{r}_m=\vec{r}} C_{\vec{v}}(\vec{r}_1)\cdots C_{\vec{v}}(\vec{r}_m)=\frac{m}{m+\vec{v}\cdot\vec{r}}\binom{m+\vec{v}\cdot\vec{r}}{\vec{r}}. \tag{5}$$

LEMMA 1. *For $s\in\mathbb{N}$, $\vec{n}\in\mathbb{N}_0^s$ and $\vec{v}\in\mathbb{N}^s$ we have*

$$\forall\vec{n}\in\mathbb{N}_0^s\setminus\{\vec{0}\}\quad C_{\vec{v}}(\vec{n})=\sum_{i=1}^{s}\left(\sum_{\vec{r}_1+\cdots+\vec{r}_{v_i}=\vec{n}-\vec{e}_i} C_{\vec{v}}(\vec{r}_1)\cdots C_{\vec{v}}(\vec{r}_{v_i})\right)^{1} \tag{6}$$

where $\vec{e}_i$ is the unit-vector with 1 in its i^{th} coordinate.

Proof. For $i=1,\ldots,s$ such that $n_i>0$, using (5) for $m=v_i$ and $\vec{r}=\vec{n}-\vec{e}_i$, we obtain

$$\begin{aligned}\sum_{\vec{r}_1+\cdots+\vec{r}_{v_i}=\vec{n}-\vec{e}_i} C_{\vec{v}}(\vec{r}_1)\cdots C_{\vec{v}}(\vec{r}_{v_i}) &= \frac{v_i}{v_i+\vec{v}\cdot(\vec{n}-\vec{e}_i)}\binom{v_i+\vec{v}\cdot(\vec{n}-\vec{e}_i)}{\vec{n}-\vec{e}_i}\\ &=\frac{v_i}{\vec{v}\cdot\vec{n}}\binom{\vec{v}\cdot\vec{n}}{\vec{n}-\vec{e}_i}\\ &=\frac{(\vec{v}\cdot\vec{n})!}{(\vec{v}\cdot\vec{n})n_1!\cdots n_s!\left((\vec{v}-\vec{1})\cdot\vec{n}+1\right)!}v_i n_i\end{aligned}$$

and then

$$\sum_{i=1}^{s}\sum_{\vec{r}_1+\cdots+\vec{r}_{v_i}=\vec{n}-\vec{e}_i} C_{\vec{v}}(\vec{r}_1)\cdots C_{\vec{v}}(\vec{r}_{v_i})=\frac{(\vec{v}\cdot\vec{n})!}{n_1!\cdots n_s!\left((\vec{v}-\vec{1})\cdot\vec{n}+1\right)!},$$

[1] As $\vec{n}\neq\vec{0}$ the sum is never empty, although the second summation is, if $n_i=0$

completing the proof. $\square$

Recall that a sequence $(a_n)_{n\in\mathbb{N}}$ is holonomic of order s ($s\in\mathbb{N}$) and degree t ($t\in\mathbb{N}_0$) if there exist $p_0, p_1, \dots, p_s$ polynomials in n such that p_0 never vanishes (to simplify), the maximum of their degrees is t and

$$\forall n\in\mathbb{N} \quad \left[n>s \;\Rightarrow\; p_0(n)a_n = \sum_{i=1}^{s} p_i(n)a_{n-s} \right].$$

It is well known (the proof can be made, for example, using the Stirling approximation) that

$$C_d(n) \sim \frac{1}{\sqrt{2\pi}} \frac{\sqrt{d}}{(d-1)^{\frac{3}{2}}} \left(\frac{d^d}{(d-1)^{d-1}} \right)^n n^{-\frac{3}{2}}. \tag{7}$$

In the article [1] one can find good approximations of binomials of the form $\binom{dn}{n}$.

3. The zero of polynomials of particular kind

As it was said in the Introduction, any polynomial of the form (2) has a unique zero in $\mathbb{Z}_M$. This is a consequence of the following result, which is just a version of Hensel's Lemma applied to this kind of polynomials, and of the Chinese Remainder Theorem.

LEMMA 2. *Let p be a prime number and $Q=Q(x)$ a polynomial of the form $a_dx^d+\cdots+a_1x+a_0$, where p divides a_i for $i\geq 2$ and p do not divide a_1. Then, for all $k\in\mathbb{N}$, the congruence $Q(x)\equiv 0 \mod p^k$ has a unique solution.*

Proof. If $k=1$ then the result is trivial as $Q(x)\equiv 0 \mod p$ is equivalent to $a_1x+a_0\equiv 0 \mod p$ and a_1 is invertible modulo p. For $m\geq 1$, if x_m is the unique solution of $Q(x)\equiv 0 \mod p^m$, then all solutions of $Q(x)\equiv 0 \mod p^{m+1}$ are of the form $x=x_m+sp^m$, with $s\in\mathbb{Z}$. As p divides a_i for $i\geq 2$ and p^m divides $Q(x_m)$,

$$\begin{aligned} Q(x)\equiv 0 \mod p^{m+1} &\Leftrightarrow Q(x_m)+a_1p^ms\equiv 0 \mod p^{m+1} \\ &\Leftrightarrow \frac{Q(x_m)}{p^m}+a_1s\equiv 0 \mod p \end{aligned}$$

and the conclusion follows as this last congruence has only one solution modulo p. $\square$

We now present an expression for the zero of polynomials of the form (2), for $M\in\mathbb{N}$. All the operations in this section are made in the ring $\mathbb{Z}_M$ and it is clear that all the "infinite" sums referred to here only have a finite number of non-zero terms.

Let $d\geq 2$ and $\vec{v}=(v_2,\dots,v_d)\in\mathbb{N}^{d-1}$. Consider, for $\vec{x}=(x_2,\dots,x_d)$ whose coordinates are all nilpotent in $\mathbb{Z}_M$, the (finite) sum in $\mathbb{Z}_M$

$$y_{\vec{v}}(\vec{x}) = \sum_{\vec{n}\in\mathbb{N}_0^{d-1}} C_{\vec{v}}(\vec{n})x_2^{n_2}\cdots x_d^{n_d}, \quad \text{where } \vec{n}=(n_2,\dots,n_d). \tag{8}$$

Notice that $y_{\vec{v}}(\vec{x})$ is always invertible as it is a sum of 1 with a nilpotent element.

LEMMA 3. *With the above notation,*

$$y_{\vec{v}}(\vec{x}) = 1 + x_2 y_{\vec{v}}(\vec{x})^{v_2} + \cdots x_d y_{\vec{v}}(\vec{x})^{v_d}. \tag{9}$$

Proof. It is easy to see by comparing the terms of the sums that, for $i = 2, \dots, m$,

$$\begin{aligned} x_i \cdot y_{\vec{v}}(\vec{x})^{v_i} &= \sum_{\vec{n} \in \mathbb{N}_0^{d-1}} \left(\sum_{\vec{r}_1 + \cdots + \vec{r}_{v_i} = \vec{n}} C_{\vec{v}}(\vec{r}_1) \cdots C_{\vec{v}}(\vec{r}_{v_i}) \right) x_2^{n_2} \cdots x_d^{n_d} \cdot x_i \\ &= \sum_{\vec{n} \in \mathbb{N}_0^{d-1}, n_i \geq 1} \left(\sum_{\vec{r}_1 + \cdots + \vec{r}_{v_i} = \vec{n} - \vec{e}_i} C_{\vec{v}}(\vec{r}_1) \cdots C_{\vec{v}}(\vec{r}_{v_i}) \right) x_2^{n_2} \cdots x_d^{n_d} \end{aligned}$$

and then, denoting by z the right side of (9),

$$\begin{aligned} z &= 1 + \sum_{i=2}^{d} \left(\sum_{\vec{n} \in \mathbb{N}_0^{d-1}, n_i \geq 1} \left(\sum_{\vec{r}_1 + \cdots + \vec{r}_{v_i} = \vec{n} - \vec{e}_i} C_{\vec{v}}(\vec{r}_1) \cdots C_{\vec{v}}(\vec{r}_{v_i}) \right) x_2^{n_2} \cdots x_d^{n_d} \right) \\ &= 1 + \sum_{\vec{n} \in \mathbb{N}_0^{d-1} \setminus \{\vec{0}\}} \left(\sum_{i=2}^{d} \sum_{\vec{r}_1 + \cdots + \vec{r}_{v_i} = \vec{n} - \vec{e}_i} C_{\vec{v}}(\vec{r}_1) \cdots C_{\vec{v}}(\vec{r}_{v_i}) \right) x_2^{n_2} \cdots x_d^{n_d} \end{aligned}$$

and the conclusion follows using (6) and the fact that $C_{\vec{v}}(\vec{0}) = 1$. □

We are now in the conditions to show an (algebraic) expression for the zero of a polynomial as in (2), whose existence and uniqueness are guaranteed by Lemma 2 and the Chinese Remainder Theorem.

THEOREM 1. *Let $M \in \mathbb{N}$ and $P(x) = a_d x^d + \cdots + a_1 x + a_0$ be a polynomial in $\mathbb{Z}_M$ as in (2). Then the unique zero x of the polynomial is equal to the (finite) sum*

$$x_0 = -a_1^{-1} a_0 \sum_{\vec{n} = (n_2, \dots, n_d) \in \mathbb{N}_0^{d-1}} (-1)^{\vec{v} \cdot \vec{n}} C_{\vec{v}}(\vec{n}) a_0^{(\vec{v} - \vec{1}) \cdot \vec{n}} a_1^{-\vec{v} \cdot \vec{n}} a_2^{n_2} \cdots a_d^{n_d}, \tag{10}$$

where $\vec{v} = (2, 3, \dots, d)$ and $\vec{1} = (1, \dots, 1)$.

Moreover x_0 is invertible if and only if a_0 is invertible.

Proof. We find a solution x_0 of the form $x_1 y$, where $y = y_{\vec{v}}(\vec{x})$ is defined in (8) for $x_2, \dots, x_d$ nilpotents. Using equality (9),

$$\begin{aligned} P(x_1 y) = 0 &\iff \sum_{i=2}^{d} a_i x_1^i y^i + a_1 x_1 y + a_0 = 0 \\ &\iff \sum_{i=2}^{d} a_i x_1^i y^i + a_1 x_1 (1 + x_2 y^2 + \cdots x_d y^d) + a_0 = 0 \\ &\iff \sum_{i=2}^{d} (a_i x_1^i + a_1 x_1 x_i) y^i + a_1 x_1 + a_0 = 0. \end{aligned}$$

So, if we choose

$$\begin{cases} x_1 = -a_0 a_1^{-1} \\ x_i = -a_i a_1^{-1} x_1^{i-1} = (-1)^i a_0^{i-1} a_1^{-i} a_i, \quad i \geq 2, \end{cases}$$

we obtain the referred solution.

The last observation is an immediate consequence of the fact that y is invertible, as mentioned before. □

For example, the zero of the polynomial $a_d x^d + a_1 x + a_0$ is, with the previous notation, equal to the sum

$$x_0 = -\sum_{k \in \mathbb{N}_0} (-1)^{dk} C_d(k) a_0^{(d-1)k+1} a_1^{-dk-1} a_d^k.$$

In particular, if p is a prime number and r a multiple of p then, for $n \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} C_d(k) r^k$$

is a solution of the congruence $rx^d - x + 1 \mod p^{n+1}$.

The rate of growth, in n, of this sum, for all $r \neq 0$, follows from Theorem 3.

REMARK 1. Suppose we have a polynomial $Q(x) = \sum_{i=0}^{d} a_i x^i$ in $\mathbb{Z}_M$ such that a_i are nilpotent for $i \leq d-2$, and a_{d-1} and a_d are invertible, which can be seen as a kind of reverse form of (2).

Q may have more than one solution, as we can see, for example, if $Q(x) = x^3 + x^2 + 3x + 9$ and $M = 27$, but only one is invertible. To prove this, consider the polynomial $Q^*(y) = \sum_{i=0}^{d} a_i y^{d-i}$, of the form (2), noticing that $y^d Q(y^{-1}) = Q^*(y)$, for invertible y.

4. Holonomic sequences related to Fuss-Catalan numbers

For $d \in \mathbb{N} \setminus \{1\}$, $r \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, consider $X(d,r,n)$ defined in (3). We intend to obtain a recurrence relation for the sequence $\big(X(d,r,n)\big)_{n \in \mathbb{N}}$, generalizing some cases referred to in OEIS, as mentioned in the Introduction.

For $n,k \in \mathbb{N}$, we let $(n)_k$ denote the *falling factorial* $\prod_{i=0}^{k-1}(n-i)$ $(= \frac{n!}{(n-k)!})$. Notice that $(n)_k$ is a polynomial in n of degree k.

THEOREM 2. *Let $d \in \mathbb{N} \setminus \{1\}$, and $r \in \mathbb{R} \setminus \{0\}$. Then $\big(X(d,r,n)\big)_{n \in \mathbb{N}}$ is a holonomic sequence of order 2 and degree $d-1$. More precisely, for $p_0(n) = \big((d-1)n+1\big)_{d-1}$, $p_2(n) = d\big(dn-1\big)_{d-1}$ and $p_1 = p_0 + rp_2$, we have*

$$\forall n \in \mathbb{N} \setminus \{1\} \quad p_0(n) X(d,r,n) = p_1(n) X(d,r,n-1) - rp_2(n) X(d,r,n-2).$$

Proof. As

$$\begin{aligned}
&p_1(n)X(d,r,n-1) - rp_2(n)X(d,r,n-2)\\
&\quad = p_0(n)\sum_{k=0}^{n-1} C_d(k)r^k + p_2(n)\sum_{k=0}^{n-1} C_d(k)r^{k+1} - p_2(n)\sum_{k=0}^{n-2} C_d(k)r^{k+1}\\
&\quad = p_0(n)\sum_{k=0}^{n-1} C_d(k)r^k + p_2(n)C_d(n-1)r^n\\
&\quad = p_0(n)X(d,r,n) - p_0(n)C_d(n)r^n + p_2(n)C_d(n-1)r^n,
\end{aligned}$$

we only need to prove that $p_0(n)C_d(n) = p_2(n)C_d(n-1)$. In fact,

$$\begin{aligned}
\frac{C_d(n)}{C_d(n-1)} &= \frac{((d-1)(n-1)+1)\binom{dn}{n}}{((d-1)n+1)\binom{d(n-1)}{n-1}}\\
&= \frac{((d-1)(n-1)+1)}{((d-1)n+1)} \frac{(n-1)!\big((d-1)(n-1)\big)!\,\big(dn\big)!}{n!\big((d-1)n\big)!\,\big(d(n-1)\big)!}\\
&= \frac{\big((d-1)(n-1)+1\big)!\,\big(dn\big)!}{n\big((d-1)n+1\big)!\,\big(d(n-1)\big)!}\\
&= \frac{\big(dn\big)_d}{n\big((d-1)n+1\big)_{d-1}}\\
&= \frac{d\big(dn-1\big)_{d-1}}{\big((d-1)n+1\big)_{d-1}},
\end{aligned}$$

which concludes the proof. □

The following observation will be useful in the next section.

REMARK 2. Notice that a constant sequence satisfies the recurrence referred to in the previous theorem. As a consequence, if $(Z_n)_n$ is a non-constant solution of the recurrence, then $\langle (Z_n)_n, (1)_n\rangle$ is a basis of the space of solutions of the recurrence.

Notice also that the characteristic polynomial of the recurrence, $p_0(n)x^2 - p_1(n)x - rp_2(n)$, has the zeros 1 and $\frac{rp_2(n)}{p_0(n)}$ and that

$$\lim_{n\to\infty} \frac{rp_2(n)}{p_0(n)} = \frac{rd^d}{(d-1)^{d-1}}.$$

5. Asymptotics for Generating Functions of the Fuss-Catalan Numbers

We are now in conditions to establish the asymptotic behaviour of the sequence $(X(d,r,n))_n$, when $\frac{|r|d^d}{(d-1)^{d-1}} > 1$ which, using (7), is when it diverges.

We use the following asymptotic behaviour: if $a,b \in \mathbb{Z}$, with $a \neq 0$, then

$$\prod_{j=2}^{n+1}(aj+b) = a^n \prod_{j=2}^{n+1}(j+\tfrac{b}{a}) = a^n \frac{\Gamma(n+2+\frac{b}{a})}{\Gamma(2+\frac{b}{a})} \sim \frac{\Gamma(n)}{\Gamma(2+\frac{b}{a})}\, a^n n^{2+\frac{b}{a}} \tag{11}$$

as $\Gamma(x+\alpha)\sim\Gamma(x)x^{\alpha}$ when $x\to+\infty$.

REMARK 3. In order to apply Corollary 1.6 of [8] in the next theorem we draw the attention to the fact that, if p and q are two polynomials of the same degree s and q is never zero in $\mathbb{N}$, then

$$\sum_{n=1}^{\infty}\left|\frac{p(n+1)}{q(n+1)}-\frac{p(n)}{q(n)}\right|<\infty,$$

as the degree of the polynomial, in n, $p(n+1)q(n)-p(n)q(n+1)$ is at most $2s-2$.

THEOREM 3. *With the above notation, if* $A(d)=\frac{d^d}{(d-1)^{d-1}}$ *and* $A(d)|r|>1$,

$$X(d,r,n)\sim\frac{1}{\sqrt{2\pi}}\frac{\sqrt{d}}{(d-1)^{\frac{3}{2}}}\frac{A(d)r}{A(d)r-1}\,(A(d)\,r)^n\,n^{-\frac{3}{2}}.$$

Proof. By Remark 2, the zeros of the characteristic polynomial of the recurrence equation converge, when n tends to infinity, to different numbers, namely $A(d)r$ and 1. Therefore, and using Remark 3 for $p=p_i$, $i=1,2$ and $q=p_0$, we are in the conditions to apply Corollary 1.6 of [8]. In particular, there exists a solution $(Y_n)_n$ of the recurrence equation such that $Y_n\sim\prod_{j=2}^{n+1}\frac{rp_2(j)}{p_0(j)}$. Notice that, using (11), we have

$$\begin{aligned}
\prod_{j=2}^{n+1}\frac{rp_2(j)}{p_0(j)}&=\prod_{j=2}^{n+1}\frac{rd(dj-1)_{d-1}}{((d-1)j+1)_{d-1}}=(rd)^n\prod_{i=1}^{d-1}\prod_{j=2}^{n+1}\frac{dj-i}{(d-1)j+2-i}\\
&\sim r^nd^n\prod_{i=1}^{d-1}\frac{\Gamma(2+\frac{2-i}{d-1})}{\Gamma(2-\frac{i}{d})}\left(\frac{d}{d-1}\right)^n n^{-\frac{i}{d}-\frac{2-i}{d-1}}\\
&=k_d\,r^nd^n\left(\frac{d}{d-1}\right)^{(d-1)n}n^{-\frac{3}{2}},\quad\text{where } k_d=\left(\prod_{i=1}^{d-1}\frac{\Gamma(2+\frac{2-i}{d-1})}{\Gamma(2-\frac{i}{d})}\right)\\
&=k_d\left(\frac{d^d}{(d-1)^{d-1}}\,r\right)^n n^{-\frac{3}{2}}.
\end{aligned}$$

As $\langle(Y_n)_n,(1)_n\rangle$ is a basis of the space of solutions of the recurrence, there exist $a,b\in\mathbb{R}$ such that, letting X_n denote $X(d,r,n)$, $(X_n)_n=a(Y_n)_n+b(1)_n$ and then

$$X_n\sim aY_n\sim a\,k_d(A(d)r)^n n^{-\frac{3}{2}}.\tag{12}$$

To calculate $a\,k_d$, using (7), we have

$$\frac{X_n-X_{n-1}}{Y_n}=\frac{C_d(n)\,r^n}{Y_n}\underset{n}{\longrightarrow}\frac{1}{k_d\sqrt{2\pi}}\frac{\sqrt{d}}{(d-1)^{\frac{3}{2}}}$$

and, on the other hand, using (12),

$$\frac{X_n-X_{n-1}}{Y_n}=\frac{Y_n-Y_{n-1}}{Y_n}\underset{n}{\longrightarrow}a\left(1-\frac{1}{A(d)r}\right),$$

from where we obtain

$$ak_d = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{d}}{(d-1)^{\frac{3}{2}}} \frac{A(d)r}{A(d)r-1},$$

concluding the proof. $\square$

REMARK 4. Although it is not relevant, we would like to point out that k_d referred to in the above proof is equal to $\frac{1}{\sqrt{2\pi}}\left(\frac{d}{d-1}\right)^{d+\frac{1}{2}}$.

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