The overlap gap between left-infinite and right-infinite words

J. C. Costa C. Nogueira M. L. Teixeira

October 10, 2021

Abstract

We study ultimate periodicity properties related to overlaps between the suffixes of a left-infinite word λ and the prefixes of a right-infinite word ρ. The main theorem states that the set of minimum lengths of words x and x' such that $x\lambda_n = \rho_n x'$ or $\lambda_n x = x'\rho_n$ is finite, where n runs over positive integers and λ_n and ρ_n are respectively the suffix of λ and the prefix of ρ of length n, if and only if λ and ρ are ultimately periodic words of the form $\lambda = u^{-\infty}v$ and $\rho = wu^{\infty}$ for some finite words u, v and w.

Keywords. Infinite word; overlap; ultimately periodic.

1 Introduction

The theory of infinite words is an important field in both mathematics and computer science that has recently been subject to major developments largely due to its connections with logic, topology and algebra and to its applications to problems in computer science (see the compendium [7] of Perrin and Pin for an extensive and comprehensive introduction to the theme). In its turn, periodicity is a relevant property of words that has applications in several domains. Phenomena of periodicity in infinite words have been widely studied by researchers and have led to several characterizations of ultimately periodic words [2, 5, 6, 3].

J. C. Costa & M. L. Teixeira: CMAT, Dep. Matemática, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal; e-mail: jcosta@math.uminho.pt, mlurdes@math.uminho.pt

C. Nogueira: CMAT, Escola Superior de Tecnologia e Gestão, Instituto Politécnico de Leiria, Campus 2, Morro do Lena, Alto Vieiro, 2411-901 Leiria, Portugal; e-mail: conceicao.veloso@ipleiria.pt

This paper deals with a combinatorial problem on infinite words that arose in the study of decidability properties of semidirect products of pseudovarieties (varieties of finite semigroups) of the form $\mathbf{V} * \mathbf{D}$, where $\mathbf{D}$ is the pseudovariety of all finite semigroups whose idempotents are right zeros. Pseudovarieties are important in the study of finite semigroups and therefore in formal language theory, while the pseudovarieties $\mathbf{V} * \mathbf{D}$ are of particular relevance in the study of rational language hierarchies. Since the semidirect product $\mathbf{V} * \mathbf{D}$ contains both the pseudovariety $\mathbf{D}$ and its dual $\mathbf{K}$, whose free profinite semigroups are described in terms of finite and left-infinite or right-infinite words (see [1]), it is natural to expect that combinatorial properties involving left-infinite and right-infinite words may arise in the study of semidirect products of that form.

In the investigation made by the authors in [4], we wanted to choose a positive integer n (determining a suffix λ_n of length n for each left-infinite word λ) in such a way that, for two left-infinite words λ and ρ, if λ_n and ρ_n occur in a finite word, then those occurrences would be distant enough. We have found that if one takes a representative for each class of left-infinite words having a common prefix, then the existence of a positive integer n in the above conditions is guaranteed for these representatives. This study is reviewed in Section 4 below.

In the present paper, we study a similar combinatorial problem, motivated by an attempt to extend the above investigation [4]. Instead of two left-infinite words, we take a left-infinite word λ and a right-infinite word ρ. The problem is the same as above, except for the definition of ρ_n which is now the prefix of length n of ρ. This problem is solved in Theorem 3.8 below, the main result of the paper.

2 Preliminaries

In this section, we start by briefly recalling the basic definitions and notations on finite and infinite words. We follow the terminology of Lothaire [6]. Next, we introduce the overlap gap function associated to a pair of words (one left-infinite and the other right-infinite) that motivates this work.

2.1 Finite and infinite words

An *alphabet* is a finite non-empty set. The *free semigroup* and the *free monoid* generated by an alphabet A are denoted, respectively, by A^+ and A^*. The empty word is represented by ε. The length of a word $w \in A^*$ is indicated by $|w|$ and the set of all words over A with length n is denoted by A^n.

A word u is called a *factor* (resp. a *prefix*, resp. a *suffix*) of a word w if there exist words x, y such that $w = xuy$ (resp. $w = uy$, resp. $w = xu$). A word is named *primitive* if it cannot be written in the form u^n with $n > 1$.

Two words u and v are said to be *conjugate* if $u = w_1w_2$ and $v = w_2w_1$ for some words $w_1, w_2 \in A^*$.

A *left-infinite* word on A is a sequence $\lambda = (a_s)_s$ of elements of A indexed by $-\mathbb{N}$, also written $\lambda = \cdots a_{-2}a_{-1}a_0$. The set of all left-infinite words on A will be denoted by $A^{-\mathbb{N}}$. Dually, a *right-infinite* word on A is a sequence $\rho = (a_n)_n$ of letters of A indexed by $\mathbb{N}$, also written $\rho = a_0a_1a_2\cdots$, and $A^{\mathbb{N}}$ denotes the set of all right-infinite words on A. A left-infinite word λ (resp. a right-infinite word ρ) of the form $\lambda = u^{-\infty}w = \cdots uuuw$ (resp. $\rho = wu^{\infty} = wuuu\cdots$), with $u \in A^+$ and $w \in A^*$, is said to be *ultimately periodic*. The word u is called *a root* and $|u|$ *a period* of λ (resp. of ρ). The smallest period p is called the *minimal period* and the roots of length p are called *cyclic roots* of λ (resp. of ρ). Notice that a root is a cyclic root if and only if it is a primitive word. Moreover, cyclic roots of the same infinite word are conjugate.

For any ultimately periodic word $\lambda \in A^{-\mathbb{N}}$ (resp. $\rho \in A^{\mathbb{N}}$) there exist unique words u, w of shortest length such that $\lambda = u^{-\infty}w$ (resp. $\rho = wu^{\infty}$) and this is said to be its *canonical form*. Of course, in this case, u is a primitive word and, if $w \neq \varepsilon$, the first (resp. the last) letters of u and w do not coincide.

2.2 The overlap gap function

For words u and v of equal length, we define:

$$\begin{aligned} lg(u,v) &= \min\{n \in \mathbb{N} : ux = x'v \text{ with } x, x' \in A^n\}, \\ rg(u,v) &= \min\{n \in \mathbb{N} : xu = vx' \text{ with } x, x' \in A^n\}, \\ g(u,v) &= \min\{lg(u,v), rg(u,v)\}. \end{aligned}$$

The non-negative integers $lg(u,v)$, $rg(u,v)$ and $g(u,v)$ are called, respectively, the *left overlap gap*, the *right overlap gap* and the *overlap gap* between u and v. Informally, the overlap gap measures the outside parts of the greatest overlap of the given words. Notice that $lg(u,v) = 0$ if and only if $u = v$ if and only if $rg(u,v) = 0$.

Consider infinite words $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$. For any non-negative integer n, we will denote by λ_n and ρ_n, respectively, the suffix of λ and the prefix of ρ of length n. For each $f \in \{lg, rg, g\}$, we let $f_{\lambda,\rho}$ be the function $f_{\lambda,\rho} : \mathbb{N} \to \mathbb{N}$ defined, for each non-negative integer n, by

$$f_{\lambda,\rho}(n) = f(\lambda_n, \rho_n).$$

The images $lg_{\lambda,\rho}(\mathbb{N})$, $rg_{\lambda,\rho}(\mathbb{N})$ and $g_{\lambda,\rho}(\mathbb{N})$ of the functions $lg_{\lambda,\rho}$, $rg_{\lambda,\rho}$ and $g_{\lambda,\rho}$, will be denoted, respectively, by $LG_{\lambda,\rho}$, $RG_{\lambda,\rho}$ and $G_{\lambda,\rho}$. Usually, when the words λ and ρ are clear from the context they will be omitted in the above notations.

Example 2.1 *Consider the left-infinite word $\lambda = (baa)^{-\infty}$ and the right-infinite word $\rho = (aab)^{\infty}$ over the alphabet $A = \{a, b\}$. Then, for every $k \in \mathbb{N}$,*

$$\begin{array}{rclclclrclrcl} lg(0) &=& lg(1) &=& lg(2+3k) &=& 0, & lg(3+3k) &=& 1, & lg(4+3k) &=& 2, \\ rg(0) &=& rg(1) &=& rg(2+3k) &=& 0, & rg(3+3k) &=& 2, & rg(4+3k) &=& 1, \\ g(0) &=& g(1) &=& g(2+3k) &=& 0, & g(3+3k) &=& 1, & g(4+3k) &=& 1. \end{array}$$

Therefore, $LG = RG = \{0, 1, 2\}$ and $G = \{0, 1\}$.

The following observation is useful.

Lemma 2.2 *Let $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$. For every $n \in \mathbb{N}$,*

$$lg(n+1) \in \{0, lg(n)+1\}, \quad rg(n+1) \geq rg(n) - 1.$$

Proof. Suppose $\lambda = (a_s)_{s\in-\mathbb{N}}$ and $\rho = (b_t)_{t\in\mathbb{N}}$. For a fixed $n \in \mathbb{N}$, let $p = lg(n)$ and notice that $\lambda_{n+1} = a_{-n}\lambda_n$ and $\rho_{n+1} = \rho_n b_n$. Then $\lambda_n x = x'\rho_n$ for some $x, x' \in A^p$, whence $\lambda_{n+1} x b_n = a_{-n} x' \rho_{n+1}$. Thus, $lg(n+1) \leq p+1$. Suppose that $lg(n+1) \notin \{0, p+1\}$. Hence $\lambda_{n+1} z = z'\rho_{n+1}$ for some words z and z' with $1 \leq |z| = |z'| \leq p$. Then $z = wb_n$ and $z' = a_{-n}w'$ for some words w and w', whence $\lambda_n w = w'\rho_n$. From this equality one deduces that $lg(n) < p$ which contradicts the assumption that $p = lg(n)$. Therefore $lg(n+1) \in \{0, p+1\} = \{0, lg(n)+1\}$.

Now, suppose that $z\lambda_{n+1} = \rho_{n+1} z'$ for some words z and z', so that $za_{-n}\lambda_n = \rho_n b_n z'$. Hence $rg(n+1) \geq rg(n) - 1$. ■

Example 2.3 *Consider the left-infinite word $\lambda = (b^2a^2)^{-\infty}cab$ and the right-infinite word $\rho = (ab^2a)^{\infty}$ over the alphabet $A = \{a, b, c\}$. The values of the functions lg, rg and g for these words are shown in the following table*

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
$lg(n)$	0	1	0	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$rg(n)$	0	1	0	3	3	3	5	5	4	3	6	5	4	3	6	$\cdots$
$g(n)$	0	1	0	1	2	3	4	5	4	3	6	5	4	3	6	$\cdots$

So, $LG = \mathbb{N}$, $RG = \{0, 1, 3, 4, 5, 6\}$ and $G = \{0, 1, 2, 3, 4, 5, 6\}$.

3 Characterization of overlap functions with finite image

The purpose of this paper is to characterize the pairs of words in $A^{-\mathbb{N}} \times A^{\mathbb{N}}$ for which the overlap gap function has finite image.

3.1 Overlap gap versus right overlap gap

We begin by showing that $g_{\lambda,\rho}(\mathbb{N})$ being finite or not depends exclusively on the function $rg_{\lambda,\rho}$.

Proposition 3.1 *Let $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$ and suppose that $G_{\lambda,\rho}$ has a maximum element M. Then $2M$ is an upper bound for $RG_{\lambda,\rho}$.*

Proof. Assume that $rg(n) > 2M$ for some $n \in \mathbb{N}$. Hence, Lemma 2.2 implies that, for every $i \in \{0, 1, \ldots, M\}$,

$$rg(n+i) \geq rg(n) - i > M. \tag{1}$$

So, by definition of g and M, it follows that $lg(n+i) \leq M$ for all $i \in \{0, 1, \ldots, M\}$. If $lg(n+i) \neq 0$ for all such i, then we deduce from Lemma 2.2 that $lg(n+M) = lg(n) + M > M$. As this does not happen by hypothesis, one deduces that $lg(n+j) = 0$ for some $j \in \{0, 1, \ldots, M\}$. Since, for every $k \in \mathbb{N}$, $lg(k) = 0$ if and only if $rg(k) = 0$, it follows that $rg(n+j) = 0$. This is in contradiction with (1) and, so, we conclude that $rg(n) \leq 2M$, thus proving the proposition. ■

The next result follows immediately from the previous proposition.

Corollary 3.2 *Let $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$. The set $G_{\lambda,\rho}$ is finite if and only if the set $RG_{\lambda,\rho}$ is finite.*

As a consequence, we will now focus our attention on the function $rg_{\lambda,\rho}$.

3.2 The rg function for ultimately periodic words

In Examples 2.1 and 2.3, we have considered ultimately periodic words $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$ with the same cyclic roots. In both examples $\big(rg_{\lambda,\rho}(n)\big)_{n\in\mathbb{N}}$ is an ultimately periodic sequence. This is generally true for words like that, as we prove below.

In this section, we fix two ultimately periodic words $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$ with the same cyclic roots and let

$$\lambda = u^{-\infty} w_1 \quad \text{and} \quad \rho = w_2 v^{\infty}$$

be their canonical forms, so that u and v are conjugate words and λ and ρ have the same minimal period $p \geq 1$. Let

$$m' = \min\{|w_1|, |w_2|\}, \quad m = \max\{|w_1|, |w_2|\} \quad \text{and} \quad M = m + p - 1.$$

The other case being symmetric, we consider only the case in which $m = |w_1|$ and $m' = |w_2|$. We begin by showing that M is an upper bound for $rg_{\lambda,\rho}$.

Lemma 3.3 *For every $n \in \mathbb{N}$, $rg(n) \leq M$.*

Proof. That the condition $rg(n) \leq M$ holds for every $n \leq M$ is trivial. Consider now an $n > M$ so that λ_n and ρ_n can be written in the forms

$$\lambda_n = u'u^{\ell_1}w_1 \quad \text{and} \quad \rho_n = w_2 v^{\ell_2} v',$$

where ℓ_1 and ℓ_2 are positive integers and u' and v' are, respectively, a proper suffix of u and a proper prefix of v.

The suffix of length p of ρ_n is a conjugate of v and so of u. Hence, there exists a word $z \in A^*$ such that $|z| \leq p-1$, u is a suffix of $vv'z$ and $v'z$ is a prefix of v^2. Therefore, λ_n is a suffix of $\rho_n z w_1$ and

$$rg(n) \leq |zw_1| = m + |z| \leq M, \tag{2}$$

which concludes the proof. ∎

Notice that, in (2), the equality $rg(n) = |zw_1|$ holds if and only if $rg(n) \geq m$. So, in case $n > M$ and $rg(n) \geq m$, the value of $rg(n)$ depends on z only. Moreover, we show in Lemma 3.4 below that the inequality $rg(n) \geq m$ holds for every integer n greater than or equal to the threshold $n_0 = m+m'+p-1$. This value of n_0 cannot be decreased, as shown in the following example. Indeed, for an arbitrary integer $i \geq 1$, if $\lambda = a^{-\infty}ba^i$ and $\rho = a^iba^\infty$, then $n_0 = 2i+2$ and $rg(n_0 - 1) = 0 < i+1 = m$.

Lemma 3.4 *Let $n \geq n_0$. Then $rg(n) \geq m$.*

Proof. The result is clear for $m = 0$. So, assume that $m > 0$ and let a be the first letter of w_1. We have $\lambda_n = u'u^{\ell_1}w_1$ and $\rho_n = w_2v^{\ell_2}v'$, with $\ell_1, \ell_2 \geq 0$. Moreover,

$$x\lambda_n = \rho_n x' = w_2 t y w_1$$

with $rg(n) = |x| = |x'|$ and $|y| = p-1$. Suppose, by way of contradiction, that $rg(n) < m$. Hence ya is a factor of $v^{\ell_2}v'$, so that it is a conjugate of both v and u. Since $u = by$ for some letter b, it follows that ya and by have the same number of occurrences of a. So $a = b$, a contradiction with λ being in canonical form. Therefore $rg(n) \geq m$, thus proving the lemma. ∎

We end this section with the description of the ultimately periodic nature of the function $rg_{\lambda,\rho}$.

Proposition 3.5 *Let $n \geq n_0$ and write $\lambda_n = u'u^{\ell_1}w_1$ and $\rho_n = w_2v^{\ell_2}v'$ with $|u'|, |v'| < p$. Then*

i) $rg(n+p) = rg(n)$;

ii) $rg(n) = m$ *if and only if* $u = v''v'$ *where* v'' *is such that* $v = v'v''$*;*

iii) $rg(n+1) = \begin{cases} M & \text{if } rg(n) = m \\ rg(n) - 1 & \text{if } rg(n) > m \end{cases}$.

Proof. Notice that $\lambda_{n+p} = u'u^{\ell_1+1}w_1$ and $\rho_{n+p} = w_2 v^{\ell_2+1} v'$. So, for arbitrary words y and y', $y\lambda_n = \rho_n y'$ if and only if $y\lambda_{n+p} = \rho_{n+p} y'$. This proves i).

As a consequence, we may assume that $n > M$. Hence $\ell_1, \ell_2 \geq 1$ and, as observed after the proof of Lemma 3.3, $rg(n) = |zw_1|$, where z is a proper suffix of u. Let $x\lambda_n = \rho_n x'$, with $rg(n) = |x| = |x'|$. Hence $x' = zw_1$ and u is a suffix of $v^{\ell_2}v'z$.

Suppose that $rg(n) = m$, so $z = \varepsilon$ and $x' = w_1$. Moreover v' is a suffix of u. So, $u = v''v'$ where v'' is such that $v = v'v''$. Conversely, assume that $u = v''v'$ with $v = v'v''$. Since $|w_2| \leq |w_1|$, this implies that λ_n is a suffix of $\rho_n w_1 = w_2 v' u^{\ell_2} w_1$, meaning that $rg(n) = m$. This shows ii).

For iii), let us consider again $rg(n) = m$. As observed above, $x\lambda_{n+p} = \rho_{n+p}x'$. Let $a \in A$ and $u'' \in A^{p-1}$ be such that $u = au''$. Then $\rho_{n+p} = \rho_{n+1}u''$, so that

$$xv'''\lambda_{n+1} = \rho_{n+1}u''x'$$

where v''' is the prefix of λ_{n+p} of length $p-1$. Hence $rg(n+1) \leq |u''x'| = M$. Suppose, by way of contradiction, that $rg(n+1) < M$. Then $y\lambda_{n+1} = \rho_{n+1}y'$ with $|y| < M$. Hence $y\lambda_{n+1} = ybu'u^{\ell_1}w_1$ and $\rho_{n+1}y' = \rho_n ay' = w_2 v' u^{\ell_2} ay'$, where $b \in A$. Since $|w_1| = m$ and $m < |ay'| < m + p$, we deduce that u is a proper conjugate of itself. This is a contradiction with the fact that u is a primitive word. Hence $rg(n+1) = M$.

We finally assume that $rg(n) > m$. Whence $|z| \neq 0$ and $\rho_{n+1} = w_2 v^{\ell_2} v' a$ where a is the first letter of z. Analogously, $\lambda_{n+1} = bu'u^{\ell_1}w_1$ where b is the last letter of x. Then $t\lambda_{n+1} = \rho_{n+1}t'$ where t and t' are defined by $x = tb$ and $x' = at'$. Hence, $rg(n+1) = rg(n) - 1$ by Lemma 2.2. This concludes the proof of iii) and of the proposition. ∎

3.3 The main result

In the last section, we have shown that ultimately periodic words with the same cyclic roots determine right overlap gap functions with finite image. We next prove that this happens only for that kind of infinite words. For that, we recall first the following property of infinite words.

Lemma 3.6 *Let* α *be a right-infinite word over an alphabet* S*. There is a letter* $s \in S$ *and a positive integer* k *such that* α *has an infinite number of occurrences of factors of the form* srs *with* $r \in S^{k-1}$*.*

Proof. Each word of length $|S| + 1$ has at least two occurrences of some letter. So there exist $s \in S$ and an infinite number of occurrences in α of factors of the form srs with $|r| < |S|$. Further, there are infinitely many such words r of length $k-1$ for at least one $1 \le k \le |S|$. ■

We now show the above announced result.

Theorem 3.7 *Consider two infinite words $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$. The set $RG_{\lambda,\rho}$ is finite if and only if $\lambda = u^{-\infty}w_1$ and $\rho = w_2u^{\infty}$ for some words $u, w_1, w_2 \in A^*$.*

Proof. If λ and ρ are of the forms $\lambda = u^{-\infty}w_1$ and $\rho = w_2u^{\infty}$, then RG is finite by Lemma 3.3. Conversely, assume that RG is a finite set and notice that the sequence $(rg(n))_{n\in\mathbb{N}}$ is a right-infinite word over the alphabet RG. By Lemma 3.6 there are positive integers k and s and a sequence $(\ell_i)_{i\in\mathbb{N}}$ such that $rg(\ell_i) = rg(\ell_i + k) = s$ for every i. Moreover, since A^k is a finite set, there are words u and v in A^k, such that for infinitely many $i \in \mathbb{N}$ we have

$$\begin{array}{rclrcl} \lambda_{\ell_i} & = & t_i\lambda_s, & \rho_{\ell_i} & = & \rho_s t_i, \\ \lambda_{\ell_i+k} & = & vt_i\lambda_s, & \rho_{\ell_i+k} & = & \rho_s t_i u, \end{array}$$

where $t_iu = vt_i$. From the equality $t_iu = vt_i$ it follows that $t_i = u'u^q = v^qv'$ with $q \in \mathbb{N}$, u' a proper suffix of u and v' a proper prefix of v. Hence $\lambda = u^{-\infty}\lambda_s$ and $\rho = \rho_s v^{\infty}$ with conjugate u and v, which concludes the proof of the theorem. ■

Combining Corollary 3.2 with Theorem 3.7, we deduce the main result of the paper.

Theorem 3.8 *For two given words $\lambda \in A^{-\mathbb{N}}$ and $\rho \in A^{\mathbb{N}}$, the set $G_{\lambda,\rho}$ is finite if and only if $\lambda = u^{-\infty}w_1$ and $\rho = w_2u^{\infty}$ for some words $u, w_1, w_2 \in A^*$.*

4 The one-sided version

Theorem 3.8 above characterizes the pairs of words in $A^{-\mathbb{N}} \times A^{\mathbb{N}}$ for which the overlap gap function has finite image. We now state and prove a one-sided version of this result, involving two left-infinite words λ and ρ. It should be noted that, in this new situation, the definitions in Section 2.2 need to be adapted. To do this, simply change the definition of ρ_n, which now denotes the suffix of length n of ρ. This is a simpler problem that was already treated in [4] but with a slightly different definition of the overlap gap function and not announced with full generality.

Theorem 4.1 *Consider two left-infinite words $\lambda, \rho \in A^{-\mathbb{N}}$. The set $G_{\lambda,\rho}$ is finite if and only if $\rho = \lambda w$ or $\lambda = \rho w$ for some word $w \in A^*$.*

Proof. Suppose first that $\rho = \lambda w$ or $\lambda = \rho w$ for some word $w \in A^*$. Let $k = |w|$ and let n be an arbitrary non-negative integer. In case $\rho = \lambda w$, one has

$$\lambda_n w = \rho_{n+k} = w' \rho_n$$

where w' is the prefix of ρ_{n+k} of length k. This shows that $lg_{\lambda,\rho}(n) \leq k$ and so, since n is arbitrary, that $LG_{\lambda,\rho}$ is a finite set. In case $\lambda = \rho w$, one deduces symmetrically that $rg_{\lambda,\rho}(n) \leq k$ and that $RG_{\lambda,\rho}$ is finite. It follows, in both cases, that $G_{\lambda,\rho}$ is a finite set.

Suppose now that $G_{\lambda,\rho}$ is a finite set. Then, there exists a constant $k \in \mathbb{N}$ such that $g_{\lambda,\rho}(n) = k$ for an infinite number of non-negative integers n. By definition, $g_{\lambda,\rho}(n) = \min\{lg_{\lambda,\rho}(n), rg_{\lambda,\rho}(n)\}$ for every $n \in \mathbb{N}$. Hence, one of the functions $lg_{\lambda,\rho}$ and $rg_{\lambda,\rho}$ take the same value k for an infinite sequence $n_0 < n_1 < \cdots < n_i < \cdots$ of natural numbers n_i. In the first case, that means that, for each $i \in \mathbb{N}$,

$$\lambda_{n_i} x_i = x'_i \rho_{n_i}$$

for some $x_i, x'_i \in A^k$. Therefore, $x_i = \rho_k$ for every i such that $n_i \geq k$. Moreover, denoting ρ_k by w, $\lambda w = \rho$. In the second case one deduces symmetrically that $\lambda = \rho w$, with $w = \lambda_k$. This concludes the proof of the theorem. ∎

A symmetric result is valid for right-infinite words.

Acknowledgments

The authors would like to thank the anonymous referees for their helpful comments and suggestions to improve the readability of the paper.

This work was partially financed by Portuguese Funds through FCT (Fundação para a Ciência e a Tecnologia) within the Projects UIDB/00013/2020, UIDP/00013/2020 and PEst-C/MAT/UI0013/2014.

References

[1] J. Almeida, *Finite Semigroups and Universal Algebra*, (World Scientific, Singapore, 1995). English translation.

[2] R. Assous, M. Pouzet, Une caractérisation des mots périodiques, *Discrete Mathematics* **25** (1979) 1–5.

[3] J. C. Costa, Biinfinite words with maximal recurrent unbordered factors, *Theor. Comput. Sci.* **290** (2003) 2053–2061.

[4] J. C. Costa, C. Nogueira, M. L. Teixeira, On κ-reducibility of pseudovarieties of the form $\mathbf{V} * \mathbf{D}$, *Int. J. Algebra Comput.* **27** (2017) 23–40.

[5] A. Ehrenfeucht, D. M. Silberger, Periodicity and unbordered segments of words, *Discrete Mathematics* **26** (1979) 101–109.

[6] M. Lothaire, *Algebraic Combinatorics on Words*, (Cambridge University Press, 2002).

[7] D. Perrin, J.-É. Pin, *Infinite Words. Automata, Semigroups, Logic and Games*, (Elsevier, Amsterdam, 2004).