The Stability of Complex Dynamics for Two Families of Coquaternionic Quadratic Polynomials

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Information

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Abstract

In this work, we begin by demonstrating that attractors, both periodic and aperiodic, of the one-parameter family of complex quadratic maps $x^2 + c$, where c is a complex number, maintain their stability when we transition from the complex plane $\mathbb{C}$ to the coquaternions $\mathbb{H}_{\text{coq}}$ as the map's phase space. Next, we investigate the same question for a different family of quadratic maps, $x^2 + bx$, and find that this is not the case. In fact, the situation for this family of maps turns out to be quite complicated. Our results show that there are complex attractors that undergo changes in their stability, while others maintain it. However, the most intriguing result is that certain regions of the parameter space, known as bulbs, which correspond to the existence of attracting cycles of some fixed period n, exhibit a mixture of stability behavior when we consider coquaternionic quadratics.

1 Introduction

The iteration of complex quadratic polynomials has a singular feature: it can be demonstrated that a map $a_2x^2 + a_1x + a_0$ is dynamically equivalent to a much simpler quadratic $x^2 + c$. This result offers an apparent advantage: all complex quadratic dynamics can be comprehended by analyzing the dynamics of the family of quadratics $x^2 + c$.

However, there is another side to the story. Despite the wonderful results obtained for the dynamics of the one-parameter family of complex maps $x^2 + c$, see, for example, [5], there is an inevitable sense of lack of diversity. After all, it all boils down to one family of quadratics.

In 2012, we began studying the iteration of quadratic coquaternionic maps with the aim of exploring how their dynamics differ from those of quadratic complex maps. First, in [2], the authors demonstrated that the family of quadratic coquaternionic maps $x^2 + c$ possesses non-isolated sets of coquaternionic fixed points and non-isolated sets of periodic coquaternionic points of period two, something that was only possible, see [1], for a much complicated map.

Subsequently, in [3], the authors established that attractor coexistence is possible for the same family of coquaternionic maps, which is known to be untrue for complex quadratics.

Finally, in [4], the authors computed the coquaternionic fixed points for a distinct family of coquaternionic quadratics, $x^2 + bx$, and determined that this family of coquaternionic quadratics is not dynamically conjugate to $x^2 + c$.

In summary, although dealing with coquaternionic functions presents inherent challenges, we can confidently state that the study of coquaternionic quadratic maps has already revealed a remarkable diversity in admissible dynamics, making it one of the most captivating and intriguing topics in the theory of dynamical systems.

2 Basic results

To ensure completeness, we provide a brief overview of the main concepts and results related to the algebra of coquaternions, which are also referred to as split-quaternions in the literature. This overview is necessary for the remaining sections of the paper.

Let $\{1, \mathsf{i}, \mathsf{j}, \mathsf{k}\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$ with a product given according to the following rules:

$$\begin{cases} \mathsf{i}^2 = -1, \qquad \mathsf{j}^2 = \mathsf{k}^2 = 1, \\ \mathsf{ij} = -\mathsf{ji} = \mathsf{k}. \end{cases} \tag{1}$$

A simple computation allows us to prove that this product generates an associative but non-commutative algebra over $\mathbb{R}$, denoted by $\mathbb{H}_{\rm coq}$, whose elements will be called real coquaternions. It is important to observe that, contrary to what happens in the case of Hamiltonian quaternions, $\mathbb{H}_{\rm coq}$ is not a division algebra. In fact, $\mathbb{H}_{\rm coq}$ contains zero divisors and nilpotent elements: for example, we have $(1+\mathsf{j})(1-\mathsf{j}) = 0$ and $(\mathsf{i}+\mathsf{j})^2 = 0$. In the following, we will identify the space $\mathbb{R}^4$ with $\mathbb{H}_{\rm coq}$ by associating the element $(q_0, q_1, q_2, q_3) \in \mathbb{R}^4$ with the coquaternion $q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$.

Given $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} \in \mathbb{H}_{\rm coq}$, its conjugate $\overline{\mathsf{q}}$ is defined as

$$\overline{\mathsf{q}} = q_0 - q_1\mathsf{i} - q_2\mathsf{j} - q_3\mathsf{k};$$

the number q_0 is called the real part of q and is denoted by $\operatorname{re}\mathsf{q}$ and the vector part of q, denoted by $\operatorname{vec}\mathsf{q}$, is $\operatorname{vec}\mathsf{q} = q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$. In analogy with the complex case, we will identify the set of coquaternions whose vector part is zero with the set $\mathbb{R}$ of real numbers.

It is easy to see that the algebra of coquaternions is isomorphic to the algebra of real 2×2 matrices, with the map $\Phi : \mathbb{H}_{\rm coq} \to \mathcal{M}_2(\mathbb{R})$ defined by

$$\Phi(q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}) \;=\; \begin{pmatrix} q_0 + q_3 & q_1 + q_2 \\ q_1 - q_2 & q_0 - q_3 \end{pmatrix}$$

establishing the isomorphism. We call the determinant of q, and denote by $\det\mathsf{q}$, the quantity given by the determinant of the matrix representative of q, i.e.

$$\det\mathsf{q} \;=\; q_0^2 + q_1^2 - q_2^2 - q_3^2.$$

It is a straightforward exercise to demonstrate that the determinant of a coquaternion q can be expressed as $\det\mathsf{q} = \mathsf{q}\,\overline{\mathsf{q}}$.

Finally, it can be shown that a coquaternion q is invertible if and only if its determinant is different from zero. In that case, the expression for the inverse is given by

$$\mathsf{q}^{-1} = \frac{\overline{\mathsf{q}}}{\det\mathsf{q}}.$$

Next, we recall some basic definitions of discrete dynamical systems. Let us consider a coquaternionic map $\mathsf{f} : \mathbb{H}_{\rm coq} \to \mathbb{H}_{\rm coq}$. For $k \in \mathbb{N}$, we shall denote by f^k the k-th iterate of f, inductively defined by

$$\begin{cases} \mathsf{f}^0 = \mathrm{id}_{\mathbb{H}_{\rm coq}} \\ \mathsf{f}^k = \mathsf{f} \circ \mathsf{f}^{k-1}. \end{cases}$$

For a given initial point $\mathsf{q}_0 \in \mathbb{H}_{\mathrm{coq}}$, the orbit of q_0 under the map f is the sequence

$$\mathcal{O}(\mathsf{q}_0) := \left(\mathsf{f}^k(\mathsf{q}_0)\right)_{k\in\mathbb{N}_0}.$$

A point $\mathsf{q} \in \mathbb{H}_{\mathrm{coq}}$ is said to be a periodic point of f, with period $n \in \mathbb{N}$, if we have $\mathsf{f}^n(\mathsf{q}) = \mathsf{q}$, with $\mathsf{f}^k(\mathsf{q}) \neq \mathsf{q}$ for $0 < k < n$; in this case, we say that the set

$$\mathscr{C} = \{\mathsf{q}, \mathsf{f}(\mathsf{q}), \ldots, \mathsf{f}^{n-1}(\mathsf{q})\}$$

is a n-cycle for f, usually written as

$$\mathscr{C} : \mathsf{q}_0 \xrightarrow{\mathsf{f}} \mathsf{q}_1 \xrightarrow{\mathsf{f}} \cdots \xrightarrow{\mathsf{f}} \mathsf{q}_{n-1}$$

with $\mathsf{q}_i = \mathsf{f}^i(\mathsf{q})$. Periodic points of period one are called fixed points.

Finally, there is one last definition relevant for the rest of the paper: we say that two coquaternionic maps $\mathsf{f} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$ and $\mathsf{g} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$ are conjugate if there exists an invertible map $\phi : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$ such that

$$\mathsf{f} \circ \phi = \phi \circ \mathsf{g}.$$

In this case, we say that the corresponding dynamical systems $(\mathbb{H}_{\mathrm{coq}}, \mathsf{f})$ and $(\mathbb{H}_{\mathrm{coq}}, \mathsf{g})$ are dynamically equivalent, since they share the same dynamical characteristics.

3 Coquaternionic Quadratic Maps $\mathsf{q}^2 + c$

We now consider the one-parameter family of coquaternionic quadratic maps

$$\begin{array}{rcl} \mathsf{f}_c : \mathbb{H}_{\mathrm{coq}} & \to & \mathbb{H}_{\mathrm{coq}} \\ \mathsf{q} & \mapsto & \mathsf{q}^2 + c \end{array}$$

with the choice of the parameter c limited to the complex plane $\mathbb{C}$. We will use f_c to denote the restriction of the map f_c to the complex plane i.e. $f_c := \mathsf{f}_c|_{\mathbb{C}}$. Since our goal is to study what happens to the stability of the attractors of f_c when changing the phase space from $\mathbb{C}$ to $\mathbb{H}_{\mathrm{coq}}$, it makes sense to use only complex parameters.

For this family of quadratics f_c, we know that

$$\mathsf{q}_1 = \frac{1}{2}(1 - \sqrt{1-4c}) \qquad \mathsf{q}_2 = \frac{1}{2}(1 + \sqrt{1-4c})$$

are the complex fixed points and

$$\mathsf{p}_1 = \frac{1}{2}(-1 - \sqrt{-3-4c}) \qquad \mathsf{p}_2 = \frac{1}{2}(-1 + \sqrt{-3-4c})$$

are the complex periodic points of period two. Moreover, from [2], we have that the fixed point q_1 and the 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ are attractors for the complex map f_c for parameter values inside the cardioid $|1 - \sqrt{1-4c}| = 1$, and inside the circle $|c+1| = 1/4$, respectively.

Since there is no appropriate concept of derivative for coquaternionic maps, the most suitable method to analyze the stability of a given periodic point of period n is to treat f_c^n as a function from $\mathbb{R}^4$ to $\mathbb{R}^4$ and evaluate the magnitude of the eigenvalues of its corresponding Jacobian matrix. As it is widely recognized, if all eigenvalues of this matrix have a modulus less than one, then the periodic point is considered to be attractive.

From the multiplication rules (1), it follows that

$$\mathsf{f}_c(\mathsf{q}) = (c_0 + q_0^2 - q_1^2 + q_2^2 + q_3^2,\ c_1 + 2q_0q_1,\ 2q_0q_2,\ 2q_0q_3)$$

for $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$ and $c = c_0 + c_1\mathsf{i}$. Hence, the Jacobian matrix of the map f_c, computed at a given point $\mathsf{q} = q_0 + q_1\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k}$, is given by

$$\mathbf{J}_c(\mathsf{q}) = \begin{pmatrix} 2q_0 & -2q_1 & 2q_2 & 2q_3 \\ 2q_1 & 2q_0 & 0 & 0 \\ 2q_2 & 0 & 2q_0 & 0 \\ 2q_3 & 0 & 0 & 2q_0 \end{pmatrix},$$

and its four eigenvalues are

$$\begin{aligned}\lambda_1(\mathsf{q}) = \lambda_2(\mathsf{q}) &= 2q_0\\ \lambda_{3,4}(\mathsf{q}) &= 2q_0 \pm 2\sqrt{-q_1^2+q_2^2+q_3^2}.\end{aligned}$$

Now, we are ready to present our first result.

Theorem 1. *The complex fixed point* $\mathsf{q}_1 = \frac{1}{2}(1-\sqrt{1-4c})$ *is an attractor for* f_c*, for parameter values inside the cardioid* $|1-\sqrt{1-4c}| = 1$.

Proof. First, let us remember that the square root of a complex number $a+b\,\mathsf{i}$ can be written as

$$\sqrt{a+b\,\mathsf{i}} = \frac{1}{2}\sqrt{\sqrt{a^2+b^2}+a} \pm \frac{1}{2}\sqrt{\sqrt{a^2+b^2}-a)}\,\mathsf{i} \tag{2}$$

Thus, the real part of the complex fixed point q_1 is given by

$$\operatorname{re}\mathsf{q}_1 = \frac{1}{2} - \frac{1}{4}\sqrt{\sqrt{(1-4c_0)^2+16c_1^2}+1-4c_0}$$

while its vector part, in this case equal to its imaginary part, is given by

$$\operatorname{vec}\mathsf{q}_1 = \pm\frac{1}{4}\sqrt{\sqrt{(1-4c_0)^2+16c_1^2}-1+4c_0}$$

Then, the first two eigenvalues of the Jacobian matrix evaluated at the fixed point q_1 are given by

$$\lambda_1(\mathsf{q}_1) = \lambda_2(\mathsf{q}_1) = 1 - \frac{1}{2}\sqrt{\sqrt{(1-4c_0)^2+16c_1^2}+1-4c_0}$$

After a lengthy computation, we can conclude that for parameter values lying inside the parabola $c_0 = -3/4 + 1/4c_1^2$, except for the points on the horizontal half-line from $(1/4, 0)$ to the right, as shown in Fig. 1, the modulus of the first two eigenvalues of the Jacobian matrix evaluated at q_1 is less than one (for the points on the aforementioned half-line, the eigenvalues have modulus equal to one).

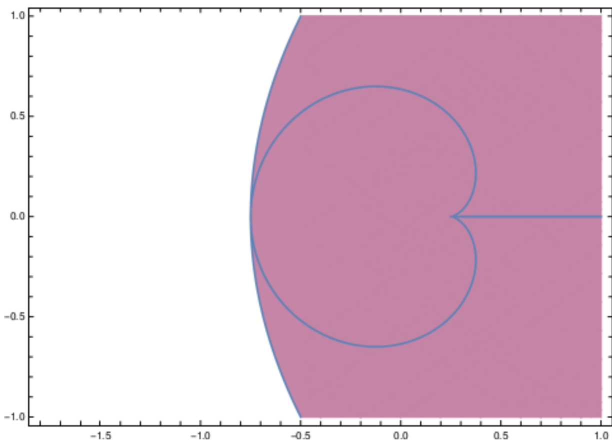

Figure 1: The regions for which the modulus of the eigenvalues of the Jacobian matrix evaluated at the fixed point q_1 are less than one: the points inside the parabola $c_0 = -3/4 + 1/4c_1^2$, except the horizontal half-line from the point $(1/4, 0)$ to the right, for the first two, and the points inside the cardioid $|1-\sqrt{1-4c}| = 1$, for the others.

On the other hand, the remaining two eigenvalues of the Jacobian matrix evaluated at the fixed point q_1 are given by

$$\begin{aligned}\lambda_{3,4}(\mathsf{q}_1) = 1 - \frac{\sqrt{2}}{2}&\sqrt{\sqrt{(1-4c_0)^2+16c_1^2}+1-4c_0} \pm\\ &\pm \frac{\sqrt{2}}{2}\sqrt{\sqrt{(1-4c_0)^2+16c_1^2}-1+4c_0}\,\mathsf{i}\end{aligned}$$

for which we can say that their modulus are less than one for parameter values inside the cardioid $|1-\sqrt{1-4c}| = 1$, see Fig. 1. Therefore, we conclude that all four eigenvalues have modulus less than one for parameter values inside the cardioid $|1-\sqrt{1-4c}| = 1$. $\square$

Based on these results, we can conclude that the complex fixed point q_1 is an attractor for f_c precisely for the same parameter values that it does for the complex map f_c, indicating that its stability remains constant regardless of whether we consider the coquaternion phase space $\mathbb{H}_{\mathrm{coq}}$.

A similar result can be stated for the complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ of the coquaternionic maps f_c.

Theorem 2. *The complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$, with $\mathsf{p}_1 = \frac{1}{2}(-1-\sqrt{-3-4c})$ and $\mathsf{p}_2 = \frac{1}{2}(-1+\sqrt{-3-4c})$, is an attractor for f_c, for parameter values inside the circle $|c+1| = 1/4$.*

Proof. In order to evaluate for which parameter values the complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ is an attractor of f_c, we are going to compute the products $\lambda_i(\mathsf{p}_1)\,\lambda_i(\mathsf{p}_2)$, for $i = 1, \ldots, 4$. From (2), we have

$$\lambda_1(\mathsf{p}_1)\,\lambda_1(\mathsf{p}_2) = \lambda_2(\mathsf{p}_1)\,\lambda_2(\mathsf{p}_2) = 1 + \frac{1}{2}\big(3 + 4c_0 - \sqrt{(3+4c_0)^2 + 16c_1^2}\big)$$

$$\lambda_{3,4}(\mathsf{p}_1)\,\lambda_{3,4}(\mathsf{p}_2) = 1 - \sqrt{(3+4c_0)^2 + 16c_1^2} \pm$$
$$\pm\sqrt{2}\sqrt{3 + 4c_0 + \sqrt{(3+4c_0)^2 + 16c_1^2}}\,\mathsf{i}.$$

These expressions allow us to say that all four products $\lambda_i(\mathsf{p}_1)\,\lambda_i(\mathsf{p}_2)$ have modulus less than one for parameter values inside the circle $|c+1| = 1/4$, i.e. we conclude that the complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ is an attractor for f_c, for c inside the circle $|c+1| = 1/4$. $\square$

From this last theorem, we are able to say that the complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ is an attractor of f_c for exactly the same parameter values for which it is an attractor for f_c, i.e. $\{\mathsf{p}_1, \mathsf{p}_2\}$ does not change its stability when the phase space goes from the complex plane $\mathbb{C}$ to the coquaternions $\mathbb{H}_{\mathrm{coq}}$.

Since the analytic study of the stability of cycles with period longer than two is not feasible, we decided to investigate computationally whether the results obtained above for the fixed point and the 2-cycle would hold for other attractors of the complex maps f_c. The results of this investigation are now presented in the form of a conjecture.

Conjecture 1. Every complex attractor, either periodic or aperiodic, for f_c is still an attractor for the coquaternionic map f_c.

This assertion resulted from selecting parameter values from 2,000,000 randomly chosen inside the circle $|c| < 2$ that corresponded to maps f_c with a periodic or aperiodic attractor. Then, for each parameter value, we computed the iterate $\mathsf{f}_c^n(\mathsf{q})$, for 100 randomly chosen points q within a small coquaternionic neighborhood of a point belonging to the complex attractor of f_c, for a large value of n. In all instances, we observed that $\mathsf{f}_c^n(\mathsf{q})$ approached the complex plane and converged to the complex attractor of f_c. $\square$

The first part of the process described above is easily recognizable as the identification of which of the randomly chosen parameter values belong to the Mandelbrot set $\mathcal{M}(f_c)$. If we generalize the definition of Mandelbrot set for the coquaternionic maps f_c as the complex parameter values for which the map possesses an attractor, either complex or coquaternionic, we have that the conjecture above is equivalent to saying that the Mandelbrot set $\mathcal{M}(f_c)$ is contained in the Mandelbrot set, $\mathcal{M}(\mathsf{f}_c)$, for the coquaternionic family f_c.

Let us conclude this section with a comment regarding the Mandelbrot sets associated with these families of maps: in [2], the authors showed that

$$\mathcal{P}_8 = \left\{-\frac{1}{2} + \frac{c_1}{2}\mathsf{i} + q_2\mathsf{j} + q_3\mathsf{k} :\, q_2^2 + q_3^2 = \frac{c_1^2 - 4c_0 - 3}{4}\right\},$$

corresponding to the choice of a complex parameter $c = c_0 + c_1\mathsf{i}$, with $c_1^2 > 4c_0 + 3$, is a set of attractive coquaternionic points of period 2, for parameter values inside the ellipse $16(c_0+1)^2 + 2c_1^2 = 1$. This means that, for parameter values inside the circle $|c+1| = 1/4$, the map f_c has a complex 2-cycle attractor but also a coquaternionic attracting 2-cycle. Moreover, one can easily observe that there are parameter values inside the ellipse given above for which f_c has coquaternionic attracting 2-cycles but f_c has no attractor. Therefore, we conclude that the Mandelbrot set for the coquaternionic family f_c contains the Mandelbrot set for the corresponding complex family f_c, but does not coincide with it.

4 Coquaternionic Quadratic Maps $\mathsf{q}^2 + b\,\mathsf{q}$

The results presented in the previous section, which allowed us to claim that any attractor of x^2+c, periodic or aperiodic, does not alter its stability by changing the phase space from complex numbers $\mathbb{C}$ to the coquaternions $\mathbb{H}_{\text{coq}}$, may actually seem trivial, but let us see what happens when we pose the exact same question for a different family of coquaternionic quadratics maps.

Consider the one-parameter family of coquaternionic quadratic maps

$$\begin{array}{rcl}\mathsf{f}_b : \mathbb{H}_{\text{coq}} & \to & \mathbb{H}_{\text{coq}} \\ \mathsf{q} & \mapsto & \mathsf{q}^2 + b\,\mathsf{q}\end{array}$$

with $b \in \mathbb{C}$, such that $\operatorname{re} b \geq 1$. Again, it will be useful to introduce the complex map obtained by restricting f_b to the complex plane $f_b := \mathsf{f}_b|_{\mathbb{C}}$. A straightforward computation, see [4], allows us to say that f_b has two complex fixed points,

$$\mathsf{q}_1 = 0 \qquad \mathsf{q}_2 = 1 - b$$

and two complex periodic points of period two

$$\mathsf{p}_1 = \tfrac{1}{2}(-1 - b + \sqrt{-3 - 2b + b^2})$$

$$\mathsf{p}_2 = \tfrac{1}{2}(-1 - b - \sqrt{-3 - 2b + b^2}).$$

Since this family f_b of complex quadratic maps is conjugated to the simpler complex family $z^2 + c$, its Mandelbrot set is, to the best of our knowledge, not often depicted in the literature. Therefore, we found appropriate to show it here.

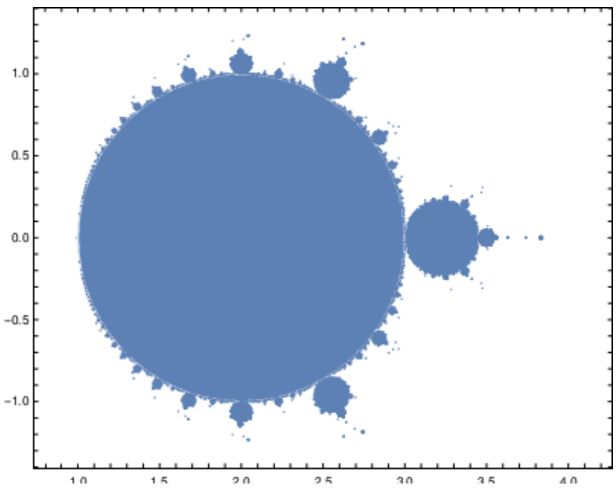

Figure 2: The Mandelbrot set $\mathcal{M}(f_b)$, for the complex quadratic maps $f_b(z) = z^2 + b\,z$, with $b \in \mathbb{C}$, such that $\operatorname{re} b \geq 1$.

In the graphical representation of the Mandelbrot set $\mathcal{M}(f_b)$ given in Fig. 2, we can easily identify both discs $|b-2| \leq 1$ and $|b - (2 + \sqrt{3/2})| \leq \sqrt{3/2} - 1$ corresponding to parameters values for which the complex fixed point q_2 and the complex 2-cycle $\{\mathsf{p}_1, \mathsf{p}_2\}$ are attractors, respectively.

In [4], the authors proved the following result.

Theorem 3. *The complex fixed point* $\mathsf{q}_2 = 1 - b$ *is an attractor for* f_b*, for parameter values inside the circle* $|b - 2| = 1$.

In the same paper, the authors claimed to have computational evidence for the following statement.

Conjecture 2. The complex 2-cycle $\{\mathrm{p}_1, \mathrm{p}_2\}$, with $\mathrm{p}_1 = \frac{1}{2}(-1-b+\sqrt{-3-2b+b^2})$ and $\mathrm{p}_2 = \frac{1}{2}(-1-b-\sqrt{-3-2b+b^2})$, is an attractor for f_b, for parameter values inside the circle $|b-(2+\sqrt{3/2})| = \sqrt{3/2}-1$.

Both these results mean that, in agreement with what was stated before for the simpler family of quadratic maps, there is no change in the stability of the complex fixed point and the 2-cycle attractors, when we consider the family of coquaternionic quadratic maps f_b. The point now is whether this is also true for other complex attractors of f_b, e.g. the complex 3-cycle which is an attractor for parameter values in the bulb tangent to the circle $|b-2|=1$.

In Fig. 3, we show a zoom of the Mandelbrot set $\mathcal{M}(f_b)$ where we highlighted three bulbs, $\mathcal{B}_4$, $\mathcal{B}_3$, and $\mathcal{B}_5$, tangent to the main circle $|c-2|=1$, corresponding to parameter values for which f_b has a 4-cycle, a 3-cycle, and a 5-cycle attractor, respectively.

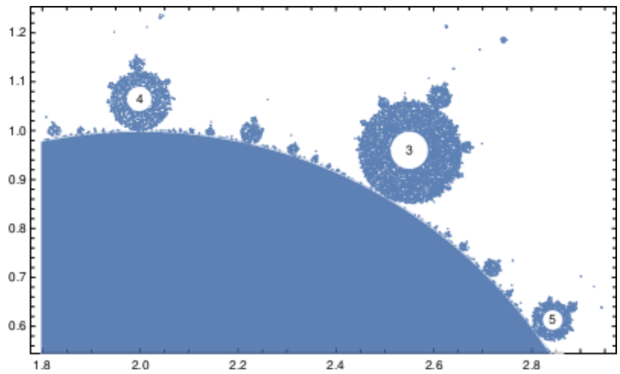

Figure 3: A zoom of the Mandelbrot set $\mathcal{M}(f_b)$ for the complex quadratic maps $f_b(z) = z^2 + b\,z$, with $b \in \mathbb{C}$, where the bulbs tangent to the main circle corresponding to the existence of a 4-cycle, a 3-cycle, and a 5-cycle complex attractors are marked.

For the first two bulbs, $\mathcal{B}_4$ and $\mathcal{B}_3$, we have computational evidence to claim the following statements.

Conjecture 3. For every parameter value $b \in \mathcal{B}_4$, the complex 4-cycle, which was an attractor for f_b, is not an attractor for f_b.

Conjecture 4. For every parameter value $b \in \mathcal{B}_3$, the complex 3-cycle, which was an attractor for f_b, is not an attractor for f_b.

Both these claims are the result of selecting which parameter values, from 1,000,000 randomly chosen inside circles containing each bulb, corresponded to the existence of attracting cycles of f_b. Then, for all these values, we computed the iterate $\mathrm{f}_b^n(\mathrm{q})$, for a randomly chosen point q in a small coquaternionic neighborhood of a point of the attractor, again for a large choice of n, and confirmed that it did not converged to the complex plane. □

These results are quite different from everything we had before: for parameter values inside $\mathcal{B}_4$ and $\mathcal{B}_3$, the complex cycle is no longer an attractor for the coquaternionic map f_b, i.e. the cycles change their stability with the phase space going from $\mathbb{C}$ to $\mathbb{H}_{\mathrm{coq}}$. Next, we asked the same question for parameter values inside the bulb $\mathcal{B}_5$ and the answer we obtained was quite unexpected.

Conjecture 5. For parameter values $b \in \mathcal{B}_5$, the complex attracting 5-cycles for f_b exhibit a mixture of stability behavior when we consider the coquaternionic quadratics f_b.

For parameter values belonging to $\mathcal{B}_5$, our computational results are summarized in Fig. 4: in blue we have values such that the attracting complex 5-cycle for f_b is still an attractor for the coquaternionic map f_b, while in orange we have values such that the complex 5-cycle attractor for f_b is not an attractor for f_b. □

After obtaining these results, the subsequent inquiry was to determine the specific points on the Mandelbrot set $\mathcal{M}(f_b)$ that correspond to the existence of complex attractors, periodic or aperiodic, of the coquaternionic

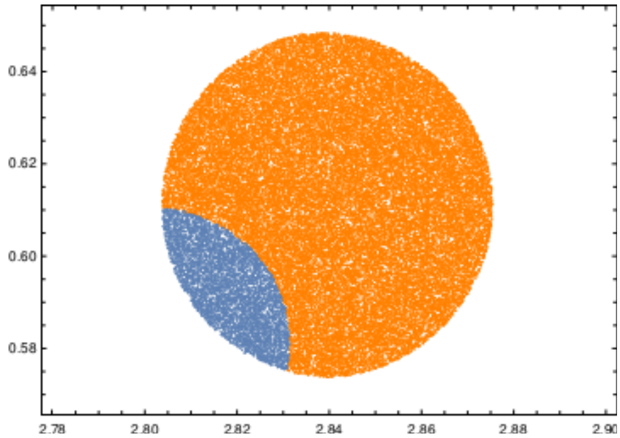

Figure 4: The bulb from the Mandelbrot set $\mathcal{M}(f_b)$ tangent to the main circle corresponding to the existence of attracting complex 5-cycles: in blue we represent parameter values such that the 5-cycle is an attractor for the coquaternionic map f_b, while in orange we represent parameter values such that the 5-cycle is not an attractor for f_b.

quadratic map f_b. It is worth noting that this task is highly demanding in terms of computation, and therefore, the results presented should be interpreted with caution, as they represent a work in progress.

Our computational results are summarized in Fig. 5, where the values $b \in \mathbb{C}$ such that the coquaternionic map f_b has a complex attractor are represented in blue, while in red are represented parameter values such that f_b has no complex attractor.

Upon examining Fig. 5, one might be inclined to assume that only the bulbs tangent the main circle, corresponding to parameter values $b = b_0 + b_1\,\mathrm{i}$ with b_1's modulus exceeding a certain threshold, correspond to coquaternionic maps lacking a complex attractor. However, upon closer inspection of the zoomed-in Fig. 4, we can conclude that this is not the case.

To conclude this section, we will take a closer look at the parameter space region where there is a secondary bulb $\mathcal{B}_{13}$, tangent to the $\mathcal{B}_6$ bulb, corresponding to coquaternionic maps that exhibit a mixture of stability behavior. Our computational results for this region are summarized in Fig. 6.

Through this example, we aim to emphasize that the criteria for determining which coquaternionic maps f_b have complex attractors are not expected to be simple.

5 Conclusions

In this work, we begin by showing that attractors, both periodic and aperiodic, of the one-parameter family of complex quadratic maps x^2+c, where c is a complex number, maintain their stability when we change the map's phase space from the complex plane $\mathbb{C}$ to the coquaternions $\mathbb{H}_{\mathrm{coq}}$. Next, we investigate the same question for the one-parameter family of quadratic maps, $x^2 + bx$, and find that this is not the case. In fact, the situation for this family of maps turns out to be quite complicated, since we show that there are complex attractors that undergo changes in their stability, while others maintain it. However, the most intriguing result is that certain regions of the parameter space, known as bulbs, which correspond to the existence of attracting cycles of some fixed period n, exhibit a mixture of stability behavior when we consider coquaternionic quadratics. Finally, we present the result of our investigation regarding the stability of all complex attracting cycles of f_b when we consider the coquaternionic quadratic maps f_b.

To finish, we would like to emphasize that this study represents the initial step towards investigating the Mandelbrot set for coquaternionic families of quadratics, including $x^2 + c$ and $x^2 + bx$. Future work will build upon these findings.

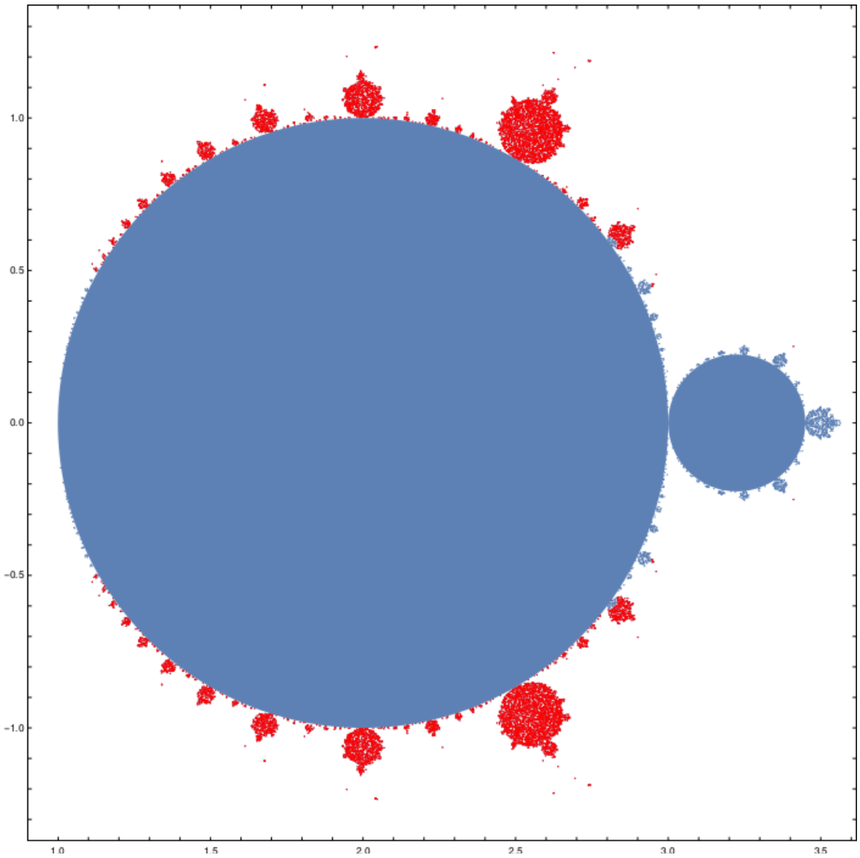

Figure 5: The parameter space $b \in \mathbb{C}$, with $\mathrm{re}\, b > 1$, where blue points represent values such that the coquaternionic map f_b has a complex attractor, and red points represent values such that f_b has no complex attractor.

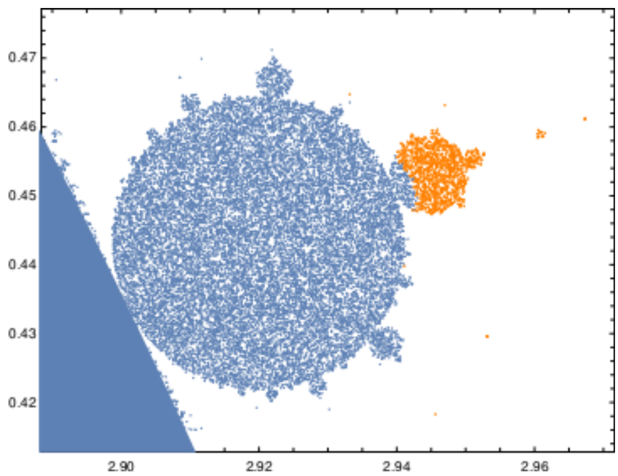

Figure 6: A detail of the parameter space $b \in \mathbb{C}$, where it is shown a secondary bulb $\mathcal{B}_{13}$, corresponding to values such that f_b has an attracting 13-cycle, for which the coquaternionic map f_b exhibit a mixture of stability behavior: blue points represent values such that f_b has a complex attractor, and orange points represent values such that f_b has no complex attractor.

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