Tree Sign Pattern that Permit Eventual Positivity

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Abstract: Which sign patterns allow a matrix to be eventually positive (EP) is an open question that has received considerable attention. It is known that it is sufficient but not necessary that the positive part be primitive. Considered here are sign patterns, whose undirected graphs are trees (tree sign patterns, TSP's). For a few types of TSP's, it is known that the primitivity of the positive part is necessary and sufficient to alllow EP. Here we show that this is so for all TSP's and conjecture that it is only so for trees. In addition, we consider the least power, at and beyond which all powers, of an EP matrix are positive, the "index". For 2-by-2 matrices, the index is determined in terms of the entries. Unlike the case of primitive matrices, the index can be arbitrarily large.

Keywords: Eventually positive; Index; irreducible, Potentially EP, Positive part, Primitive patterns, Tree sign pattern.

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1 Introduction

An n-by-n matrix A is said to be eventually positive (EP), if there is a least positive integer k_0, called the index of A, such that for all $k \geq k_0$, $A^k > 0$. It is equivalent [3] that A have the strong Perron-Frobenius properties:

1. $\rho(A) \in \sigma(A)$.
2. $\rho(A)$ has algebraic multiplicity 1, and for all $\lambda \in \sigma(A)$ such that $\lambda \neq \rho(A)$, $|\lambda| < \rho(A)$.
3. There are entrywise positive left and right eigenvectors x and y^T associated with $\rho(A)$, such that $Ax = \rho(A)x$ and $y^T A = \rho(A)y^T$.

If A itself is nonnegative ($A \geq 0$), it is well known that A is EP if and only if A is primitive [3], (which depends only on the pattern of the positive entries), and that the index is no more than $(n-1)^2 + 1$. However, A may have some negative entries and still be EP [3], and, in this case, there is no bound on the index in terms of n.

It remains an open problem to describe all square sign patterns $\mathcal{A}$ that allow eventual positivity, as well as to understand the behavior of EP matrices with negative entries. Let $Q(\mathcal{A})$ denote the real matrices with sign pattern indicated by $\mathcal{A}$. We call a sign pattern (square matrix with entries $+, -, 0$) potentially EP (PEP) if there is an EP matrix $A \in Q(\mathcal{A})$. Any real matrix (or sign pattern) may uniquely be decomposed as the sum of a nonnegative matrix (sign pattern) and a nonpositive matrix (sign pattern) that are orthogonal. The former is called the positive part. It is not difficult to show that if the positive part of a sign pattern is primitive, then the sign pattern is PEP [3]. It is natural to ask if the converse is so. However, this is not the case, as demonstrated by the 3-by-3 example.

$$\begin{bmatrix} 1.3 & -0.3 & 0 \\ 1.3 & 0 & -0.3 \\ -0.31 & 0.3 & 1.01 \end{bmatrix}$$

noted in [1]. However, for some sign patterns, the graphs of whose nonzero entries are trees (tree sign patterns, TSP's), the converse has been seen to hold, e.g [4, 5]. These includes paths and stars. Here, we show that the converse holds for every tree. A second purpose is to intiate study of the index of an EP matrix. We are able to calculate the index for 2-by-2 EP matrices.

In section 2, we provide background information and lemmas necessary to prove the main result for TSP's, with the proof given in section 3. In section 4, we calculate the index for 2-by-2 eventually positive matrices in terms of their entries.

2 Background and Preliminary Results

In addition to the necessary Perron-Frobenius background mentioned in the introduction, we give the supporting facts we need for our results. Some are from prior sources.

Lemma 1 *Let* $\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$ *be a sign pattern, with* $\mathcal{A}_{11}$ *and* $\mathcal{A}_{22}$ *square, and suppose that one of* $\mathcal{A}_{12}$ *and* $\mathcal{A}_{21}$ *is nonpositive, while the other is nonnegative. Then* $\mathcal{A}$ *cannot be PEP.*

Roughly, this statement may be found in [1], but it is instructive to mention a proof.

Proof. Suppose $\mathcal{A}$ is PEP. Then $\mathcal{A}$ cannot be reducible, so $\mathcal{A}_{12} \neq 0$ and $\mathcal{A}_{21} \neq 0$. Suppose wlog that $\mathcal{A}_{12}$ is nonpositive.

Let $A \in Q(\mathcal{A})$ be eventually positive with spectral radius ρ, and let $x > 0$ and $y^T > 0$ be right and left eigenvectors for ρ, respectively. Then $(A - \rho I)x =$

0, and $y^T(A - \rho I) = 0$. Let $x^T = [x_1^T \ x_2^T]$ and $y^T = [y_1^T \ y_2^T]$, partitioned conformally with the partition of A. Then,

$$(A_{11} - \rho I)x_1 + A_{12}x_2 = 0,$$

$$y_1^T(A_{11} - \rho I) + y_2^T A_{21} = 0.$$

The first equation implies $(A_{11} - \rho I)x_1 \geq 0$, while the second implies $y_1^T(A_{11} - \rho I) \leq 0$. But then, it follows both that $y_1^T(A_{11} - \rho I)x_1 > 0$ and $y_1^T(A_{11} - \rho I)x_1 < 0$, a contradiction. □

Lemma 2 *Let $A \in M_n(\mathbb{R})$ be EP and $\alpha > 0$, then $A + \alpha I$ is EP.*

Proof. Since $\rho(A) \in \sigma(A)$, $\rho(A) + \alpha \in \rho(A + \alpha I)$ with the same positive left and right eigenvectors. The other eigenvalues are also translated by α, so that none can catch up with $\rho(A) + \alpha$ in absolute value. Then $A + \alpha I$ has the required spectral properties to be EP. □

A useful corollary now follows.

Corollary 1 *If $\mathcal{A}$ is a PEP sign pattern and $\mathcal{A}$' is the result of replacing all diagonal entries with +'s, then A' is also PEP.*

Proof. Apply Lemma 2 to an EP realization of A, with $\alpha > 0$ large enough. □

Lemma 3 *Let $A \in M_n(\mathbb{R})$ be EP, then no nontrival signature similarity of A is EP.*

Proof. Suppose that $A^k > 0$ and SAS is EP with $S = I \oplus (-I)$. Then $(SAS)^p = SA^pS$ for all positive integers p. But, then $(SAS)^k = \begin{bmatrix} + & - \\ - & + \end{bmatrix}$ in blocks and no power of $(SAS)^k$ can be positive, meaning that SAS cannot be EP. □

Corollary 2 *If $\mathcal{A}$ is a PEP sign pattern, then no nontrivial signature similarity of A can be PEP.*

3 Main Result for Tree Sign Pattern

Theorem 1 *Suppose that $\mathcal{A}$ is a TSP. Then $\mathcal{A}$ is PEP if and only if the positive part of $\mathcal{A}$ ($\mathcal{A}^+$) is primitive.*

Proof. The sufficiency of the primitivity of $\mathcal{A}^+$ is independent of TSP. So, we focus upon the necessity. If $\mathcal{A}$ were reducible, then every power of every matrix with sign pattern $\mathcal{A}$ would be as well, so that $\mathcal{A}$ could not be PEP. Thus, $\mathcal{A}$ being PEP implies that $\mathcal{A}$ is irreducible. Then, every off-diagonal entry of $\mathcal{A}$ allowed to be nonzero is nonzero, as $\mathcal{A}$ is a TSP. This means that for every edge in the tree, the symmetrically placed off-diagonal signs of $\mathcal{A}$ must be $+/+, +/-$, or $-/-$.

We first rule out the case $+/-$ using lemma 1. Since each edge of a tree is a cut-edge, after permutation similarity (which does not change PEP), $\mathcal{A}$ appears in the form of lemma 1, with each of $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$ having one nonzero entry which are of opposite sign. But, then $\mathcal{A}$ is not PEP, so that by lemma 1 the case $+/-$ cannot occur.

Next, suppose that there are $-/-$ pairs and that $\mathcal{A}$ is PEP. Then, as is well-known, there is a (nontrivial) signature similarity of $\mathcal{A}$ that changes the $-/-$ pairs into $+/+$ pairs, without changing any $+/+$ pairs or any diagonal entries. Call this pattern $\mathcal{A}$'. Then, after changing all diagonal entries of $\mathcal{A}$ to $+$, the resulting pattern $\mathcal{A}$" is PEP, by corollary 1. But $\mathcal{A}$" is a signature similarity of the result of changing all diagonal entries of $\mathcal{A}$ to $+$. Since $\mathcal{A}$ is PEP, this means that the signature similarity was trivial by corollary 2 and there were no $-/-$ pairs. We conclude that evey pair associated with an edge of our tree is $+/+$.

Now, if at least one diagonal entry of $\mathcal{A}$ is $+$ (a one cycle), $\mathcal{A}^+$ would be primitive, as it is irreducible and the gcd of the cycle lengths would be 1 [2]. A + diagonal entry of $\mathcal{A}$ is necessary, as well. Assume for the moment that all diagonal entries are $-$ or 0.

Then consider $\mathcal{A}^{2k+1}$. Each diagonal entry is a sum of path products from i to i. But each of these must include an odd number of diagonal entries (as there are no other odd length cycles) and thus be negative if they are nonzero. We conclude that $\mathcal{A}^{2k+1}$ cannot be positive for any k and that $\mathcal{A}$ is not PEP.

So in order to be PEP, at least one diagonal entry must be + and $\mathcal{A}^+$ be primitive, which completes the proof. □

We remark that if we have any directed graph in which there are no odd-length cycles, then a sign pattern with this graph and off-diagonal nonzero entries + can be PEP only if there are + diagonal entries. This follws part of the above proof.

To make a conjecture that there is such a theorem only for trees, we define the notion of a "pattern" of a sign pattern. This is simply the arrangement of zero and nonzero off-diagonal entries. The diagonal is free. Often, such a pattern may be described in terms of a graph. Such as for TSP's.

Conjecture: The only patterns for which all PEP sign patterns require that the positive part be primitive are the TSP's.

4 The Index of a 2-by-2 EP Matrix

To be EP, a 2-by-2 matrix must have both off-diagonal entries positive and at least one diagonal entry positive by theorem 1. Then, after positive diagonal similarity, scaling and possible permutation similarity, which do not change the index (the exponent k_0 for which A^{k_0} and all subsequent powers are positive), A may be then taken to have the form

$$A = \begin{bmatrix} a & 1 \\ 1 & -b \end{bmatrix}$$

in which $a > b \geq 0$. (If $-b$ were positive, the exponent would be 1 and no analysis would be necessary. If b were $\geq a$, no power would be positive.) In this event, all even powers are positive, but odd powers may not be, so that this case already exhibits the possibility that the sequence of powers enters and exits the positive matrices. In fact, it is easily seen that a power is positive if and only if its $2,2$ entry is positive. Now, the eigenvalues of A are

$$r = \frac{1}{2}(a - \ b) + \frac{1}{2}\sqrt{(a+b)^2+4} \ and \ s = \frac{1}{2}(a-b) - \frac{1}{2}\sqrt{(a+b)^2+4}$$

with corresponding orthogonal eigenvectors

$$p = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\sqrt{(a+b)^2+4} \\ 1 \end{bmatrix} \ and \ q = \begin{bmatrix} \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}\sqrt{(a+b)^2+4} \\ 1 \end{bmatrix}$$

of the same length.

Let

$$P = \left[\begin{array}{c|c} \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\sqrt{(a+b)^2+4} & \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}\sqrt{(a+b)^2+4} \\ \hline 1 & 1 \end{array}\right],$$

then the inverse of P is

$$\frac{1}{\sqrt{(a+b)^2+4}} \left[\begin{array}{c|c} 1 & \frac{1}{2}\sqrt{(a+b)^2+4} - \frac{1}{2}a - \frac{1}{2}b \\ \hline -1 & \frac{1}{2}\sqrt{(a+b)^2+4} + \frac{1}{2}a + \frac{1}{2}b \end{array}\right].$$

And $A^k = P \begin{bmatrix} r^k & 0 \\ 0 & s^k \end{bmatrix} P^{-1}$.

Then the 2,2 entry of A^k is

$\sqrt{(a+b)^2+4}(r^k\left(\frac{1}{2}\sqrt{(a+b)^2+4} - \frac{1}{2}a - \frac{1}{2}b\right) + s^k\left(\frac{1}{2}\sqrt{(a+b)^2+4} + \frac{1}{2}a + \frac{1}{2}b\right))$.

When k is odd, the positivity of the 2,2 entry is equivalent to

$$r^k\left(\tfrac{1}{2}\sqrt{(a+b)^2+4}-\tfrac{1}{2}a-\tfrac{1}{2}b\right)-|s|^k\left(\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2}\sqrt{(a+b)^2+4}\right)>0$$

which gives

$$\frac{r^k}{|s|^k}>\frac{\left(\sqrt{(a+b)^2+4}+a+b\right)}{\left(\sqrt{(a+b)^2+4}-a-b\right)}.$$

Since $a>b$, then $|s|=(b-a)+\sqrt{(a+b)^2+4}$, which gives

$$\left(\frac{(a-\ b)+\sqrt{(a+b)^2+4}}{(b-a)+\sqrt{(a+b)^2+4)}}\right)^k>\frac{\left(\sqrt{(a+b)^2+4}+a+b\right)}{\left(\sqrt{(a+b)^2+4}-a-b\right).}$$

So, in case $n=2$, as soon as an odd power is positive the index of an EP matrix has been attained. The first such odd power could be any number.

Theorem 2 *If $A\in M_2(\mathbb{R})$ is EP, the possible values for index of A are $1,2$ or any odd number greater than 3, and any of these may occur. If the first odd power that is positive is more than 3, then that odd power is the index. If the first is 3, then the index is 2. If A is positive, its index is 1. In general, all even powers are positive.*

Since all even powers are positive and positive odd powers play a crucial role in determining the index when $n=2$, we wonder if sufficiently many consecutive positive powers insure that the index has been attained for other specific values of n.

It is natural to ask when the positive part of an EP matrix is primitive, whether there is a relation between the index of an EP matrix and its primitive part. In case $n=2$, the index is at least that of the primitive part. However, this is not so in general.

Example 1 *Let* $A=\begin{bmatrix}20&20&-1\\-1&20&20\\20&-1&-10\end{bmatrix}$. *Then index$(A)=2$, while index$(A^+)=$ 3.*

In seems likely, however, that for TSP's the index of the primitive part of an EP matrix is no more than that of the EP matrix itself, as in the 2-by-2 case.

We conclude with three natural questions.

1. If sign pattern $\mathcal{A}$ is PEP, then there are some EP matrices in $Q(\mathcal{A})$. If $\mathcal{A}$ is nonnegative, there are only EP matrices in $Q(\mathcal{A})$. But, if $\mathcal{A}$ has some negative entries, we suspect that there are also non-EP matrices in $Q(\mathcal{A})$. This is clear by the strong Perron-Frobenius conditions if there is a "-" on the diagonal. But how may this be nicely proven in general?

2. If $\mathcal{A}$ is PEP, the index of EP matrices in $Q(\mathcal{A})$ may very (again if some entries are "-"). what may be said about this variation? It seems likely, for example, that it increases if the negative entries get larger relative to the positive ones (in absolute value) and eventually the matrices then pass out of EP.

3. Among the entries of a PEP sign pattern, some may be "critical" (even in the nonnegative case) in the sense that replacing them with 0 (individually) may result in the pattern, no longer being EP. How may such critical entries be identified? If there are critical entries, the sign pattern may be viewed as "minimal", and if not, the pattern properly contains minimal PEP patterns. Understanding the distinction seems important.

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