Abstract. A semigroup $S$ is called generalized $F$-semigroup if there exists a group-congruence on $S$, such that the identity-class contains a greatest element with respect to the natural partial order $\leq_S$ of $S$. Using the concept of anticone all partially ordered groups, which are epimorphic images of a semigroup $(S,.,\leq_S)$, are determined. It is shown that a semigroup $S$ is a generalized $F$-semigroup if and only if $S$ contains an anticone, which is a principal order ideal of $(S,\leq_S)$. Also a characterization by means of the structure of the set of idempotents resp. by the existence of a particular element in $S$ is given. The generalized $F$-semigroups in the following classes are described: monoids, semigroups with zero, trivially ordered semigroups, regular semigroups, bands, inverse semigroups, Clifford-semigroups, inflations of semigroups, and strong semilattices of monoids.

1. Introduction

A semigroup $(S,\cdot)$ is called $F$-inverse if $S$ is inverse and for the least group-congruence $\sigma$ on $S$, every $\sigma$ -class has a greatest element with respect to the natural partial order $\leq_S$ of $S$ (see [16] or [10] for a detailed treatment of this class of semigroups). This concept appeared originally in [19]. A construction of such semigroups was given in [12] by means of groups acting on semilattices with identity obeying certain axioms.

Dropping the condition that the semigroup is inverse we will call a semigroup $S$ an $F$-semigroup if for some group-congruence $\rho$ on $S$ every $\rho$-class of $S$ contains a greatest element with respect to the natural partial order $\leq_S$ of $S$. Recall that for any semigroup $S$, $\leq_S$ is defined by

$$a \leq_S b \text{ if and only if } a = xb = by , xa = a \text{ for some } x, y \in S^1$$

(see [13]). Note that for $e, f \in E(S), e \leq_S f$ iff $e = ef = fe$. In this paper we will more generally study generalized $F$-semigroups, which are semigroups $S$ for which there exists a group-congruence $\rho$ such that the identity-class (only) has a greatest element with respect to the natural partial order $\leq_S$ of $S$ (equivalently, there exists a homomorphism of $S$ onto a group $G$ such that the preimage of the identity element of $G$ has a greatest element with respect to $\leq_S$). Thus we are dealing with semigroups, which are extensions of a subsemigroup $T$ with greatest element by a group (the semigroups of type $T$ were first characterized in [18]). The particular case of $F$-semigroups will be considered in a subsequent paper.

This generalization of $F$-inverse semigroups is motivated by a class of partially ordered semigroups (i.e., semigroups $S$ endowed with a partial order $\leq$ which is compatible with multiplication): $(S,\cdot,\leq)$ is called Dubreil-Jacotin semigroup if there exists an isotone semigroup-homomorphism of $(S,\cdot,\leq)$ onto a partially ordered group $(G,\cdot,\preceq)$ such that the preimage of the negative cone of $G$ is a principal order ideal of $(S,\leq)$. This concept was introduced in [6] (see also [4],

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Specializing the partial order \( \leq \) given on \( S \) to the natural partial order \( \leq_S \) and dropping the compatibility condition for \( \leq_S \) (which is not satisfied, in general) it turns out that in this case the partial order \( \preceq \) given on \( G \) reduces to the equality relation, so that the negative cone of \( G \) consists of the identity element of \( G \) alone. Thus we arrive at the concept of generalized \( F \)-semigroup.

In Section 2 we determine all partially ordered groups, which are isotone semigroup-homomorphic images of an arbitrary semigroup \( S \) - with \( S \) considered as partially ordered by its natural partial order. In the particular case that \( S \) is inverse this question was dealt with in [3], where the greatest such partially ordered group was considered. For this purpose we use the concept of anticone of \( S \) defined in [2] (see also [4]). In Section 3 we specialize the concept of anticone to be principal in the sense that it is also a principal order ideal of \((S, \leq_S)\). In analogy with \( F \)-inverse semigroups we show that for generalized \( F \)-semigroups \( S \) the congruence \( \rho \) appearing in the definition is the least group-congruence on \( S \). Characterizations by the existence of a principal anticone, of a particular element, and by properties of the set of all idempotents are provided. Also, generalized \( F \)-semigroups, which are regular or contain an identity, are considered. The characterization of the latter allows a construction of all generalized \( F \)-inverse monoids. In Section 4 the generalized \( F \)-semigroups in the following classes are described: semigroups with zero, trivially ordered semigroups, bands, inflations of semigroups, and strong semilattices of monoids (in particular, Clifford-semigroups).

### 2. Epimorphic partially ordered groups

Throughout the paper, \( S \) stands for an arbitrary semigroup, unless specified otherwise, and \( \leq_S \) for the natural partial order defined on \( S \). No other partial order on \( S \) will be considered.

Since we are interested in homomorphic images of a semigroup \( S \) onto groups, we first observe that for any group \( G \) and every homomorphism \( \varphi : S \rightarrow G \), \( a \leq_S b \) implies \( a\varphi = b\varphi \), i.e., \( \varphi \) is trivially isotone.

In this Section we give a method for constructing all groups \( G \) and all partial orders on \( G \) such that the partial ordered group \( G \) is a semigroup and order homomorphic image of \( S \). For this purpose we follow the account given in [4, Section 24] using the concept of anticone in a partially ordered semigroup introduced in [2]. Since the natural partial order of \( S \) need not be compatible with multiplication, the theory developed in [4] cannot be applied directly to our case. At several stages other proofs have to be given in order to establish the corresponding results needed in the sequel.

Let \( X \subseteq S \) and \( a, b \in S \). Define

\[
X : a = \{ x \in S | ax \in X \} \quad \text{and} \quad X : a = \{ x \in S | xa \in X \}.
\]

It is readily seen that

\[
X : ab = (X : a) : b \quad \text{and} \quad X : ab = (X : b) : a.
\]

Say that \( X \neq \emptyset \) is reflexive if \( ab \in X \) implies \( ba \in X \) \((a, b \in S)\). If \( X \) is reflexive then \( X : a = X : a \) for any \( a \in S \), in which case we will use the notation
X : a. Say that X is neat if X is reflexive and X : c \neq \emptyset for all c \in S. If X is a reflexive subsemigroup of S, define

\[ I_X = \{ x \in S | X : x = X \}. \]

Call a subsemigroup H of S an anticone of S if \( I_H \cap H \neq \emptyset \) and both H and \( I_H \) are reflexive and neat. As we will see later, this definition is equivalent to the definition given in [4] in the context of partially ordered semigroups.

A subset \( T \) of a semigroup S is called unitary in S if (i) \( t, ta \in T \) implies that \( a \in T \), and (ii) \( ta, at \in T \) implies that \( a \in T \) (see [5]). If T is reflexive then (i) and (ii) are equivalent.

**Proposition 2.1.** Let H be an anticone of S. Then \( I_H \) is a maximal unitary subsemigroup of S contained in H. In particular, \( I_{I_H} = I_H \) is also an anticone of S, and \( I_H = H \) if and only if H is unitary in S.

**Proof.** Clearly, by the definition of anticone, \( I_H \neq \emptyset \). That \( I_H \) is a unitary subsemigroup follows easily from the fact that \( H : xy = (H : x) : y = (H : y) : x \) for all \( x, y \in S \). If \( x \in I_H \) then \( H : x = H \) and so \( xH \subseteq H \). Let \( k \in I_H \cap H \). Then \( xk \in H \), i.e., \( x \in H : k = H \). Thus \( I_H \subseteq H \).

Next consider any unitary subsemigroup K of S such that \( I_H \subseteq K \subseteq H \). Let \( u \in K \). Since \( I_H \) is neat, choose \( v \in S \) such that \( uv \in I_H \). But K is unitary, so \( v \in K \). If \( z \in H : u \) then \( uz \in H \), so \( vu \in H \), giving \( z \in H : vu \). Since \( I_H \) is reflexive and \( uv \in I_H \), \( vu \in I_H \). Thus \( H : vu = H \) and so \( z \in H \). Since \( H \subseteq H : u \), we get \( H : u = H \), proving \( u \in I_H \). Hence \( K \subseteq I_H \) and so \( I_H \) is a maximal unitary subsemigroup of S contained in H. We now show that \( I_{I_H} = I_H \).

As \( I_H \) is unitary, \( I_{I_H} \subseteq I_H \). If \( x \in I_H \) and \( y \in I_H : x \) then \( xy \in I_H \) and so, since \( I_H \) is unitary, \( y \in I_H \). Since \( I_H \) is a subsemigroup of S, it follows that \( I_H : x = I_H \), that is \( x \in I_{I_H} \). That \( I_H \) is an anticone is now immediate. The assertion follows and the proof is complete.

Let \( H \) be an anticone. Since \( H \) is reflexive, we can define the Dubreil equivalence \( RH \) on S by

\[(a, b) \in RH \iff H : a = H : b.\]

Following the proof in [4, Section 24] we obtain that \( S/R_H \) is a group whose identity is \( I_H \). Also, the binary relation on \( S/R_H \) given by

\[ aR_H \leq bR_H \iff H : b \subseteq H : a \]

is a partial order which is compatible with multiplication. Hence \( G = (S/R_H, \cdot, \leq) \) is a partially ordered group. Moreover, following the arguments given in [4, pages 250-251], \( H \) is the pre-image, under the natural homomorphism, of the set \( \{ xR_H \in S/R_H | xR_H \leq I_H \} \), called the negative cone of \( S/R_H \).
Remark 2.2.  1. Notice that any anticone $H$ of $(S, \leq_S)$ is an order ideal of $(S, \leq_S)$. In fact, if $h \in H$ and $x \in S$, then $hR_H$ belongs to the negative cone of $S/R_H$ and

$$
\begin{align*}
x \leq_S h & \implies x = th = tx \text{ for some } t \in S \\
& \implies xR_H = tR_H.hR_H = tR_H.xR_H \\
& \implies xR_H = hR_H \preceq I_H \\
& \implies x \in H.
\end{align*}
$$

2. From the observation of the beginning of this Section, it follows that the natural homomorphism $\phi : S \to S/R_H$ is isotone.

3. Since $I_H$ is a subsemigroup of $H$ (Proposition ??) and $H$ is an order ideal of $S$, the definition of anticone that we have given is equivalent to the definition given in [4] in the context of partially ordered semigroups.

We summarize the previous results in the following

Theorem 2.3. Let $S$ be a semigroup and $H$ an anticone. Then $S/R_H$, partially ordered by the relation $\preceq$ defined by $aR_H \preceq bR_H \iff H : b \subseteq H : a$, is an (isotone) homomorphic group image of $S$ under the natural homomorphism such that $H$ is the preimage of the negative cone of $(S/R_H, \preceq)$.

The next result shows that every partially ordered group, which is an (isotone) homomorphic image of a semigroup $S$, arises in this way, i.e., is given by an anticone of $S$.

Theorem 2.4. Let $S$ be a semigroup, $G$ a group with compatible partial order $\leq$ and $\varphi : S \to G$ an (isotone) epimorphism. Let $H = \{x \in S : x\varphi \leq 1_G\}$. Then $H$ is an anticone and $\psi : S/R_H \to G$, given by $xR_H \mapsto -x\varphi$, is an isomorphism such that $\psi$ and $\psi^{-1}$ are order preserving.

Proof. To justify that $H$ is an anticone of $S$ we can apply the arguments given in [4, Section 24] since compatibility of the partial order given on $S$ is not used in those arguments. By Theorem ??, $S/R_H$ is a partially ordered group, where $R_H$ denotes the Dubreil equivalence with respect to $H$ and $\preceq$ is the partial order given above. Following the proof of Theorem 24.1 in [4], we obtain that the mapping $\psi : S/R_H \to G$, $(xR_H)\psi = x\varphi$ is an isomorphism such that $\psi$ and $\psi^{-1}$ are order preserving.

Corollary 2.5. Let $\varphi : S \to G$ be an isotone epimorphism where $G$ is a group with compatible partial order $\leq$. Then $\leq$ is trivial if and only if the anticone $H = \{x \in S : x\varphi \leq 1_G\}$ is unitary in $S$.

Proof. By Theorem 2.4, since $\psi$ is an isomorphism, $I_H = 1_G\varphi^{-1}$. If $\leq$ is trivial then clearly $H = I_H$, by definition of $H$. Conversely, if $H = I_H$ and $a\varphi \leq b\varphi$ ($a, b \in S$) then, by Theorem ??, $aR_H \preceq bR_H$, i.e., $H : b \subseteq H : a$. Hence for any $x \in S$ such that $bx \in I_H$, $ax \in I_H$. So $(bx)\varphi = 1_G = (ax)\varphi$ giving $b\varphi = a\varphi$. Thus, $\leq$ is trivial if and only if $H = I_H$, and this is equivalent to $H$ be unitary, by Proposition ??.
Example 2.6. Let \( B \) be a band, \((G, \leq)\) a partially ordered group and let 
\( S = B \times G \) be their direct product. Then the natural partial order on \( S \) is given by 
\[
(e, a) \leq_S (f, b) \iff e \leq_B f \text{ and } a = b.
\]

Notice that \( \leq_S \) is not compatible with multiplication, in general. The projection \( \varphi : S \to G \), defined by \((e, a) \varphi = a\), is an isotone epimorphism. By Theorem ??, the set \( H = \{(e, a) \in S : a \leq 1_G\} \) is an anticone of \( S \) and the mapping \( \psi : S/R_H \to G \) defined by \( xR_H \mapsto x\varphi \) is an isomorphism such that \( \psi \) and \( \psi^{-1} \) are order preserving. By Corollary ??, the anticone \( H \) is not unitary if the partial order \( \leq \) on \( G \) is not trivial.

Example 2.7. Let \( S \) be an inverse semigroup. Then the natural partial order of \( S \) has the form:
\[
a \leq_S b \iff a = eb \text{ for some } e \in E_S \text{ (see [16]).}
\]

It was shown in [17] that \( H = \{h \in S : e \leq h \text{ for some } e \in E_S\} \) is the least anticone of \( S \) yielding the greatest isotone homomorphic group image of \( S \). The latter is given by the congruence \( \sigma \) on \( S \) defined by:
\[
a \sigma b \iff ea = eb \text{ for some } e \in E_S;
\]

in fact, \( R_H = \sigma \) by [3]. We show that \( H \) is unitary in \( S \). Let \( h, ha \in H \). Then 
\[
e \leq_S h, f \leq_S ha \text{ for some } e, f \in E_S, \text{ whence } e = jh, f = iha \text{ for some } i, j \in E_S.
\]
Since the idempotents of \( S \) commute, we get \( jf = ijha = ie a \), where \( ie \in E_S \). Thus \( jf \leq_S a \) with \( jf \in E_S \); hence \( a \in H \) and so \( H \) is unitary.

We next introduce a class of semigroups, which contain (unitary) anticones: the class of \( E \)-inversive, \( E \)-unitary semigroups.

(i) A semigroup \( S \) is called \( E \)-inversive if for every \( a \in S \) there exists \( x \in S \) such that \( ax \in E_S \) (see [5], Ex. 3.2 (8)). In this case there also exists \( y \in S \) such that \( ay, ya \in E_S \). Examples are provided by periodic (in particular, finite) or regular semigroups (see [14]).

(ii) \( S \) is called \( E \)-unitary if \( E_S \) is unitary in \( S \), that is, if \( e, ea \in E_S \) implies that \( a \in E_S \), and if \( e, ae \in E_S \) implies that \( a \in E_S \). In fact, these two conditions on \( S \) are equivalent (see the beginning of Section 3, in [14]).

Let \( S \) be an \( E \)-unitary semigroup and \( a, b \in S \) such that \( ab \in E_S \). Then
\[
(ba)^3 = babab = b(ab)^2a = b(ab)a = (ba)^2
\]
and
\[
(ba)^4 = (ba)^2.
\]
Hence \((ba)^2 \in E_S \) and \((ba)(ba)^2 = (ba)^3 = (ba)^2 \in E_S \). It follows that \( ba \in E_S \). So \( E_S \) is reflexive.
If $S$ is also $E$-inversive, easy calculations show that $E_S$ is a neat subsemigroup of $S$ and $I_{E_S} = E_S$. Hence $E_S$ is an anticone of $S$. Also, if $H$ is an anticone of $S$, then by Theorems ?? and ??, $H = \{x \in S : x \varphi \leq 1_G\}$, $\varphi$ being the natural homomorphism $\varphi : S \to S/R_H = G$. Since, for every idempotent $e \in S$, $e \varphi = 1_G$, it follows that $E_S \subseteq H$. Thus we have the following

**Proposition 2.8.** Every $E$-inversive, $E$-unitary semigroup $S$ has a (least) anticone, namely $H = E_S$.

Notice that since by Theorem ?? every anticone of a semigroup $S$ gives rise to a group $G$, which is an isotone homomorphic image of $S$, the result of Proposition ?? is implicitly contained in [1] Theorem 3.1.

3. **Generalized $F$-semigroups**

We will now specialize our study to the case of semigroups $S$ containing an anticone $H$ with a greatest element, i.e., an anticone which (by Remark ??) is a principal order ideal of $(S, \leq_S)$. Such an anticone will be called a **principal anticone**. This additional condition leads to the class of generalized $F$-semigroup. We call a semigroup **generalized $F$-semigroup** if there exists a group-congruence $\rho$ on $S$ such that the identity $\rho$-class $1_G \in G = S/\rho$ has a greatest element $\xi$. The element $\xi$ will be called a **pivot** of $S$.

If a semigroup $S$ has a principal anticone $H$ with greatest element $\xi$, i.e. $H = (\xi) = \{x \in S : x \leq_S \xi\}$, then $R_H$ is a group congruence. Using the natural homomorphism of $S$ onto the group $S/R_H$ whose identity is $I_H$, we have

$$t, ta \in H \implies t, ta \leq_S \xi \implies tR_H.aR_H = \xi R_H = tR_H$$
$$\implies aR_H = 1_{S/R_H} = I_H$$
$$\implies a \in I_H \subseteq H. \quad \text{[by Proposition ??]}$$

Hence $H$ is unitary and so, by Proposition ??, $H = I_H$. It follows that the identity $R_H$-class $I_H$ has a greatest element. So $S$ is a generalized $F$-semigroup with pivot $\xi$.

Conversely, let $S$ be a generalized $F$-semigroup, $\rho$ a corresponding group congruence on $S$ and $\varphi : S \to G = S/\rho$ the natural epimorphism. Considering on $G$ the identity relation for $\leq$ we have by Theorem ??, that $H = \{x \in S : x \varphi = 1_G\}$ is an anticone of $S$. By hypothesis, the identity $\rho$-class $1_G \in S/\rho$, that is, $H = 1_G\varphi^{-1}$ has a greatest element $\xi$, say. Therefore $H$ is a principal (hence unitary) anticone and $H = I_H = (\xi)$.

We have proved the following characterization:

**Theorem 3.1.** Let $S$ be a semigroup. Then $S$ is a generalized $F$-semigroup if and only if $S$ has a principal (unitary) anticone $H$. In this case $H = I_H = (\xi)$, where $\xi$ is a pivot of $S$.

**Remark 3.2.** 1. An unitary anticone is not necessarily principal. Indeed, consider any $E$-unitary inverse semigroup $S$. By Proposition ??, $E_S$ is an unitary anticone and by [10] Proposition 7.1.3, $E_S$ contains a greatest element if and only if $S$ has an identity.
2. Since for any anticone $H$ of a semigroup $S$, $I_H$ is unitary (by Proposition ??), the natural partial order on $I_H$ is just the restriction of $\leq_S$ to $I_H$.

3. If $S$ is a generalized $F$-semigroup then any group $G$ appearing in the definition admits as a compatible partial order only the identity relation (by Theorem ?? and Corollary ??). Hence the negative cone of $G$ consists of the identity alone.

Our next aim is to show that the group in the definition of generalized $F$-semigroup is unique. We show even more:

**Theorem 3.3.** Let $S$ be a generalized $F$-semigroup and $\rho$ a corresponding group congruence. Then $\rho$ is the least group congruence on $S$. In particular, both the congruence and the pivot of $S$ are uniquely determined.

**Proof.** Let $\tau$ be any group congruence on $S$ and $a, b \in S$ be such that $a\rho b$. If $c \in (a\rho)^{-1} = (b\rho)^{-1}$ then $c\rho = (a\rho)^{-1}$ so that $(c\rho)(a\rho) = I_H$, the identity of $S/R_H$ ($H$ being the principal (unitary) anticone of $S$ corresponding to $\rho$ in Theorem ??). Therefore, $ca \in I_H = H = [\xi]$, by Theorem ??, that is, $ca \leq_S \xi$. Similarly, $cb \leq_S \xi$. If $\psi$ denotes the natural homomorphism corresponding to $\tau$, then it follows that $(c\psi)(a\psi) = \xi\psi = (c\psi)(b\psi)$ (see the beginning of Section ??). Therefore, $a\psi = b\psi$ (by cancellation), that is, $a\tau b$.

Due to the definition, the knowledge of semigroups $T$ containing a greatest element is relevant to the study of generalized $F$-semigroups. A characterization of such semigroups $T$ was given in [18]. Here we provide an independent proof. For this purpose, we show first

**Lemma 3.4.** Let $S$ be a semigroup with greatest element $\xi$. Then $\xi^3 = \xi^2$ and $\xi^2 \in E_S$.

**Proof.** By hypothesis $\xi^2 \leq_S \xi$. If $\xi^2 = \xi$ then $\xi \in E_S$. If $\xi^2 <_S \xi$ then $\xi^2 = x\xi = \xi y = x\xi^2$ for some $x, y \in S$. Thus $\xi^3 = x\xi^2 = \xi^2$ and so $\xi^2 \in E_S$. ■

**Theorem 3.5.** ([18]) A semigroup $S$ admits a greatest element if and only if $S$ is one of the following types:

(i) $S$ is a band with identity;

(ii) $S = T \cup \{\xi\}$, where $T$ is a band with identity $e$ such that $\xi^2 = e$ and $a\xi = \xi a = a$ for every $a \in T$.

**Proof.** If $S$ is a semigroup of type (i) then the identity $e \in S$ is the greatest element of $S$. Also, if $S$ is of type (ii) then $a\xi = \xi a = a$ for every $a \in T$ implies that $a \leq \xi$ (since $a \in E_S$). Thus $\xi$ is the greatest element of $S$.

Conversely, let $S$ be a semigroup with greatest element $\xi$. Then, for every $a \in S$, $a \leq \xi$. If $\xi \in E_S$, it follows by [15], Lemma 2.1, that $a \in E_S$. Hence $S$ is a band with identity $\xi$, i.e., $S$ is of type (i). If $\xi \notin E_S$ then we have
1. \( T = S \setminus \{ \xi \} \) is a subsemigroup of \( S \):

Let \( a, b \in T \); then \( a \leq_S \xi \) and so \( a = x\xi = \xi y = xa \) for some \( x, y \in S \). Assume that \( ab \notin T \). Then \( ab = \xi \) and

\[
a = x\xi = xa.b = ab = \xi ,
\]
a contradiction. Thus \( ab \in T \).

2. \( a\xi = a\xi^2 \), \( \xi a = \xi^2 a \) for every \( a \in S \):

If \( a = \xi \) then by Lemma ??

\[
a\xi = \xi^2 = \xi^3 = \xi.\xi^2 = a\xi^2
\]
and similarly \( \xi a = \xi^2 a \).

If \( a \neq \xi \) then \( a <_S \xi \) and so \( a = x\xi = \xi y = xa \), for some \( x, y \in S \). It follows by Lemma ?? that

\[
a\xi = x\xi.\xi = x\xi^2 = x\xi^3 = x\xi.\xi^2 = a\xi^2
\]
and similarly \( \xi a = \xi^2 a \).

3. \( \xi^2 \in T \) is the identity of \( T \):

Since \( \xi \notin E_S \), \( \xi^2 \in S \setminus \{ \xi \} = T \). Let \( a \in T \). Then \( a <_S \xi \) and so \( a = x\xi = \xi y = xa \) for some \( x, y \in S \). Therefore, by 2.,

\[
a\xi^2 = a\xi = x\xi.\xi = x\xi^2 = x\xi = a.
\]
Similarly, \( \xi^2 a = a \).

4. \( T = S \setminus \{ \xi \} \) is a band:

By 2. and Lemma ??, \( a <_S \xi \) for every \( a \in T \) implies that \( a \leq_S \xi^2 \). Since by Lemma ??, \( \xi^2 \in E_S \) it follows by [15], Lemma 2.1, that \( a \in E_S \). Hence by 1., \( T \) is a band.

We have shown that \( S = T \cup \{ \xi \} \), where \( T \) is a band with identity \( \xi^2 \) such that \( a\xi = a\xi^2 = a \) and \( \xi a = \xi^2 a = a \) for every \( a \in T \). Therefore, \( S \) is of type \((ii)\).

\[\square\]

**Corollary 3.6.** If \( S \) is a generalized \( F \)-semigroup with pivot \( \xi \) then either \( (\xi) = E_S \) or \( (\xi) = E_S \cup \{ \xi \} \) with \( \xi^2 \in E_S \) and \( e\xi = \xi e = e \) for all \( e \in E_S \).
Proof. By Theorem ?, $H = (\xi]$ is a principal anticone of $S$, hence a subsemigroup of $S$ with greatest element $\xi$ (note that by Remark ??. the natural partial order on $H$ is the restriction of $\leq_S$ to $H$). Therefore by Lemma ??, $\xi^2 \in E_S$. Since $e\varphi = 1_G$ for any $e \in E_S$, where $\varphi$ is the corresponding natural homomorphism, we have $E_S \subseteq (\xi]$. The assertion now follows from Theorem ??.

This description of the identity class yields the following properties of a generalized $F$-semigroup.

**Proposition 3.7.** Every generalized $F$-semigroup $S$ with pivot $\xi$ is $E$-inversive. Furthermore, $E_S$ is a subsemigroup of $S$ with greatest element $\xi^2$.

**Proof.** By Corollary ??, either $(\xi] = E_S$ or $(\xi] = E_S \cup \{\xi\}$ where $\xi^2$ is the identity of $E_S$. By the proof of Theorem ??, $T = E_S$ is a subsemigroup of $S$. It follows that $E_S$ contains a greatest element: $\xi^2$. We show now that $S$ is $E$-inversive. Let $a \in S$ and $\varphi : S \to G = S/\rho$ the surjective homomorphism satisfying $1_G\varphi^{-1} = (\xi]$. Then we have

$$a\varphi \in G \implies (a\varphi)^{-1} = b\varphi \text{ for some } b \in S$$

$$\implies ab \in 1_G\varphi^{-1} = (\xi]$$

$$\implies ab \in E_S \text{ or } ab = \xi$$

$$\implies ab \in E_S \text{ or } abab = \xi^2 \in E_S.$$ 

Hence $S$ is $E$-inversive.

The two properties given in Proposition ?? are not sufficient for a semigroup to be a generalized $F$-semigroup. For example, consider the multiplicative monoid $S$ of natural numbers together with 0; then $S$ is $E$-inversive and $E_S = \{0, 1\}$ is a subsemigroup with greatest element 1. If $S$ was a generalized $F$-semigroup with pivot $\xi$ then by Proposition ??, $\xi^2 = 1$ and so $\xi = 1$. Hence $(\xi] = \{0, 1\}$, which is not unitary, a contradiction (see Theorem ??).

The next theorem establishes a characterization of a generalized $F$-semigroup in terms of the idempotents of $S$. This result has several applications (see Section ??).

**Theorem 3.8.** Let $S$ be a semigroup. Then $S$ is a generalized $F$-semigroup with pivot $\xi$ if and only if $S$ is $E$-inversive, $\xi$ is an upper bound of $E_S$ and $E_S \cup \{\xi\}$ is unitary.

**Proof.** Necessity follows by Proposition ??, Corollary ?? and Theorem ??.

Conversely, let $S$ be $E$-inversive, $\xi$ be an upper bound of $E_S$ and $E_S \cup \{\xi\}$ be unitary. Suppose first that $\xi \in E_S$. Then $S$ is an $E$-inversive and $E$-unitary semigroup. It follows by Proposition ??, that $H = E_S$ is a (unitary) anticone with greatest element $\xi$. Thus by Theorem ??, $S$ is a generalized $F$-semigroup with pivot $\xi$. Suppose now that $\xi \notin E_S$. We show that $H = E_S \cup \{\xi\}$ is a principal anticone of $S$.

1. $H$ is a subsemigroup of $S$:
Let \( h,k \in H \). Since \( S \) is \( E \)-inversive, there exists \( x \in S \) such that \( hkx \in ES \subseteq H \). Since \( H \) is unitary, we then have, successively \( kx \in H \), \( x \in H \) and finally \( hk \in H \).

2. \( H \) is reflexive:

Let \( a,b \in S \) be such that \( ab \in H \). Consider first the case \( ab \in ES \). Then,

\[
(ba)^3 = b(ab)^2a = (ba)^2 \implies (ba)^2 \in ES \subseteq H.
\]

Since \( (ba)(ba)^2 = (ba)^2 \in H \) and since \( H \) is unitary, we have that \( ba \in H \). Consider next the case \( ab = \xi \). By 1., \( H \) is a subsemigroup (with greatest element \( \xi \)). Thus by Lemma ??, \( \xi^3 = \xi^2 \),

\[
(ba)^4 = b(ab)^3a = b\xi^3a = b\xi^2a = (ba)^3
\]

and so \( (ba)^3 \in ES \subseteq H \). Thus \( (ba)^3(ba) = (ba)^3 \in H \); since \( H \) is unitary, it follows that \( ba \in H \).

3. \( H \) is neat:

This follows from 2. and the fact that \( S \) is \( E \)-inversive and \( ES \subseteq H \).

4. \( I_H = H \):

Since by 1., \( H \) is a subsemigroup of \( S \), \( H \subseteq H : x \) for any \( x \in H \). Also, because \( H \) is unitary, \( H : x \subseteq H \). Thus \( H = H : x \) for any \( x \in H \). Thus \( H \subseteq I_H \). Conversely, let \( a \in I_H \); then \( H : a = H \) and \( h \in H = H : a \implies ah \in H \implies a \in H \) (since \( H \) is unitary).

We have shown that \( H \) is an anticone. Since, by hypothesis, \( \xi \in H \) is an upper bound of \( ES \subseteq ES \cup \{\xi\} = H \), \( \xi \) is the greatest element of \( H \). Sufficiency now follows by Theorem ??.

Notice that in Theorem ?? the attribute ”with pivot \( \xi \)” is essential. In fact, consider the following example.

**Example 3.9.** Let \( T = \{0,1\} \) be the two element semilattice and let \( S = \{0,1,a\} \) with \( a0 = 0a = 0 \), \( a1 = 1a = 1 \), \( a^2 = 1 \) (see Theorem ??). Then \( a \in S \) is the greatest element of \( S \) and \( S \) satisfies the conditions of Theorem ?? with \( \xi = a \). Hence \( S \) is generalized \( F \)-semigroup with pivot \( \xi = a \). Now, 1 is also an upper bound of \( ES \), but \( ES \cup \{1\} = ES \) is not unitary in \( S \) since \( a.1 = 1 \in ES \), \( a \notin ES \). This means that \( S \) is not a generalized \( F \)-semigroup with pivot \( \xi = 1 \).

As an immediate consequence of Theorem ??, we give a characterization of those elements of a semigroup \( S \) which may serve as pivot of \( S \). Notice that by Theorem ?? there is at most one such element.
Corollary 3.10. Let $S$ be a semigroup. Then $S$ is a generalized $F$-semigroup with pivot $\xi$ if and only if (i) $\xi^2$ is the greatest idempotent of $S$ and $\xi^2 \leq_S \xi$, (ii) for any $a \in S$ there exists $a' \in S$ that $aa' \leq S \xi^2$, (iii) $E_S \cup \{\xi\}$ is unitary in $S$.

Note that the conditions of Corollary 3.10 also characterize those order ideals of a semigroup $(S, \leq_S)$ which are (principal) anticones of $S$.

As a special case of Theorem 3.9, consider a semigroup $S$ such that $E_S$ has a greatest element. Then we obtain the following

Corollary 3.11. Let $S$ be a semigroup containing a greatest idempotent $e$. Then $S$ is a generalized $F$-semigroup with pivot $e$ if and only if $S$ is $E$-inversive and $E$-unitary.

The condition imposed on $S$ in Corollary 3.11 is certainly satisfied if $S$ has an identity. In this case it is easy to show that the identity, being a maximal element of $(S, \leq_S)$, is the pivot of $S$. Thus, we obtain a characterization of generalized $F$-monoids:

Corollary 3.12. Let $S$ be a monoid. Then $S$ is a generalized $F$-semigroup if and only if $S$ is $E$-inversive and $E$-unitary.

Next we study generalized $F$-semigroups which are regular. We begin with the more general situation where only the pivot of $S$ is regular. First we show

Proposition 3.13. For a generalized $F$-semigroup with pivot $\xi$ the following are equivalent:

(i) $\xi$ is regular; (ii) $\xi$ is (the greatest) idempotent; (iii) $S$ is $E$-unitary.

Proof. By hypothesis, there exists a group $G$ and a surjective homomorphism $\varphi : S \to G$ such that $1_G \varphi^{-1} = \{\xi\}$.

(i) $\implies$ (ii). Let $\xi' \in S$ be such that $\xi = \xi \xi' \xi$. Since $\xi \xi' \in E_S$, we have that $(\xi \xi') \varphi = 1_G$ so that $\xi \xi' \in \{\xi\}$. Hence $\xi \xi' \leq_S \xi$ and so,

$$\xi \xi' = x \xi = \xi y = x \xi \xi'$$

for some $x, y \in S^1$. Thus $\xi = x \xi = \xi \xi' \in E_S$. (It follows by Theorem 3.9 that $\xi$ is the greatest idempotent.)

(ii) $\implies$ (iii). This follows from Corollary 3.11.

(iii) $\implies$ (i). Since by Theorem 3.9, $\{\xi\}$ is a semigroup with greatest element $\xi$, $\xi^3 = \xi^2 \in E_S$ by Lemma 3.10. Thus, by hypothesis, $\xi^2 \xi \in E_S$ implies that $\xi \in E_S$. Hence $\xi$ is regular.

As a consequence of Proposition 3.13, the conditions of Corollary 3.10 characterize the generalized $F$-semigroups with regular pivot. Also they yield a characterization of the regular generalized $F$-semigroups:

Theorem 3.14. Let $S$ be a regular semigroup. Then $S$ is a generalized $F$-semigroup if and only if $S$ is an $E$-unitary monoid.
Proof. Let $S$ be a regular semigroup. Then $S$ is $E$-inversive. If $S$ is an $E$-unitary monoid it follows from Corollary ?? that $S$ is a generalized $F$-semigroup.

Conversely, if $S$ is a regular generalized $F$-semigroup with pivot $\xi$ then by Proposition ??, $\xi$ is the greatest idempotent of $S$ and $S$ is $E$-unitary. Following the proof of Proposition 7.1.3 in [10], we show that $\xi$ is the identity of $S$. Let $a \in S$ and $a' \in S$ be such that $a = aa'a$. Since $aa', a'a \in E_S$ we have by Corollary ??, that $aa', a'a \leq_S \xi$ and so $a'a\xi = a'a$ and $\xi aa' = aa'$. Hence, $a\xi = \xi a = a$ and so $\xi$ is the identity of $S$.

Example 3.15. Let $B$ be a band with identity $1_B$, let $G$ be a group with identity $1_G$ and let $S = B \times G$ be their direct product. Then $S$ is a regular monoid with identity $(1_B, 1_G)$ and $E_S = \{(e, 1_G) \in S : e \in B\}$. Simple calculations show that $S$ is $E$-unitary. Thus $S$ is a generalized $F$-semigroup. The corresponding group is the given group $G$ and $(1_B, 1_G)$ is the greatest element of its identity class since $\varphi : S \to G$, $(e, a) \varphi = a$, is a surjective homomorphism.

A construction of all regular generalized $F$-semigroups is given in [8].

4. Examples

In this section we characterize in several classes of semigroups those members which are generalized $F$-semigroups. Moreover, two types of constructions are investigated with the aim to produce generalized $F$-semigroups: inflations of semigroups and strong semilattices of monoids. The proofs concerning the last two ones are not given because they consist of extensive calculations.

1. Every group $G$ is a (generalized) $F$-semigroup (the identity relation on $G$ is the desired group congruence).

2. Every semigroup $S$ with greatest element $\xi$ is a generalized $F$-semigroup (the universal relation on $S$ is the corresponding group congruence).

3. A band $B$ is a generalized $F$-semigroup if and only if $B$ has an identity (this is a consequence of 2. and of Theorem ??).

In the class of all monoids the generalized $F$-semigroups were characterized by Corollary ??). For a much bigger class of semigroups, we have

4. Let $S$ be a semigroup containing a maximal element $m$, which is idempotent. Then $S$ is a generalized $F$-semigroup if and only if $S$ is $E$-inversive, $E$-unitary and has a greatest idempotent (this follows from Theorem ?? and Corollary ??).

5. Let $S$ be a trivially ordered semigroup (i.e., the natural partial order of $S$ is the identity relation). Then $S$ is a generalized $F$-semigroup if and only if $S$ is a group. (Necessity: Since by Theorem ??, $S$ is $E$-inversive and $E_S = \{\xi\}$, $S$ is regular by [14], Proposition 3; hence $S$ is a group by [16], Lemma II.2.10.) Examples of trivially ordered semigroups $S$ (without zero) are provided by weakly cancellative semigroups, right-(left-) simple semigroups, right-(left-) stratified semigroups, in particular, completely simple semigroups (see [7]).
6. Let \( S \) be a semigroup with zero. Then \( S \) is a generalized \( F \)-semigroup if and only if \( S \) has a greatest element (that is, \( S \) is of type \((i)\) or \((ii)\)) in Theorem 7.2. Note that 0\( \varphi \) is the zero of \( G = S/\rho \), whence \( |G| = 1 \).

In the class of all regular semigroups, the generalized \( F \)-semigroups were characterized by Theorem 7.2 as the \( E \)-unitary monoids. The inverse case deserves to be mentioned separately. Note that every \( E \)-unitary inverse semigroup is isomorphic to a McAlister \( P \)-semigroup \( P \), and that \( P \) has an identity if and only if \( Y \) has a greatest element (see [10] Theorem 7.1.1). Thus we obtain

7. Let \( S \) be an inverse semigroup. Then \( S \) is a generalized \( F \)-semigroup if and only if \( S \) is isomorphic to a \( P \)-semigroup \( P(Y,G;X) \) such that \( Y \) has a greatest element with respect to \( \leq X \).

Remark 4.1. This result provides a method for the construction of all generalized \( F \)-inverse semigroups. Take a lower directed partially ordered set \( X \) (see [16], Lemma VII.1.3), a principal order ideal \( Y \) of \( X \), which is also a subsemilattice, and a group \( G \) acting on the left by order-automorphisms on \( X \) such that \( G.Y = X \); then \( S = P(Y,G;X) \) is a generalized \( F \)-inverse semigroup. Conversely, every such semigroup can be constructed in this way. It is worth noting the difference of this construction with that of all \( F \)-inverse semigroups: by [11], Theorem 2.8, a semigroup \( S \) is \( F \)-inverse if and only if \( S \) is isomorphic to \( P(Y,G;X) \) constructed as above with \( X \) a semilattice instead of a lower directed partially ordered set (see also [16], Proposition VII.5.11).

In the following, for two constructions necessary and sufficient conditions on the ingredients are given, which allow to produce further examples of generalized \( F \)-semigroups.

8. Inflations of semigroups

Let \( T \) be a semigroup; for every \( \alpha \in T \) let \( T_\alpha \) be a set such that \( T_\alpha \cap T_\beta = \emptyset \) for all \( \alpha \neq \beta \) in \( T \) and \( T_\alpha \cap T = \{\alpha\} \) for any \( \alpha \in T \). On \( S = \cup_{\alpha \in T} T_\alpha \) define a multiplication by

\[
a.b = \alpha \beta \quad \text{if} \quad a \in T_\alpha, b \in T_\beta.
\]

Then \( S \) is a semigroup called an inflation of \( T \). If \( T \) satisfies the condition that for every \( \alpha \in T \) there exist \( \beta, \gamma \in T \) such that \( \alpha = \beta \alpha = \alpha \gamma \) (for example, if \( T \) has an identity or if \( T \) is regular), the natural partial order on \( S \) was characterized in [7]:

\[
a \leq_S b \ (a \in T_\alpha, b \in T_\beta) \quad \text{if and only if} \quad a = b \text{ or } a = \alpha \leq_T \beta.
\]

In particular, if \( a, b \in T_\alpha \) then \( a \leq_S b \) if and only if \( a = \alpha \).

As it can be expected, the structure of \( S \) depends heavily on that of \( T \), in particular, the property to be a generalized \( F \)-semigroup.

Theorem 4.2. Let \( S = \cup_{\alpha \in T} T_\alpha \) be an inflation of the semigroup \( T \) such that for every \( \alpha \in T \) there exist \( \beta, \gamma \in T \) with \( \alpha = \beta \alpha = \alpha \gamma \). Then \( S \) is a generalized \( F \)-semigroup if and only if

\((i)\) \( T \) is a generalized \( F \)-semigroup with pivot \( \xi \), say;
(ii) \(|T_\alpha| = 1\) for every \(\alpha \in T\) with \(\alpha <_T \xi\);
(iii) \(|T_\xi| \leq 2\).

A particular case of inflations should be mentioned.

**Corollary 4.3.** Let \(G\) be a group and let \(S = \cup_{g \in G} T_g\) be an inflation of \(G\). Then \(S\) is a generalized \(F\)-semigroup if and only if \(|T_1| \leq 2\).

9. Strong semilattices of monoids

Let \(Y\) be a semilattice and for every \(\alpha \in Y\) let \(S_\alpha\) be a monoid (whose identity is \(1_\alpha\)) such that \(S_\alpha \cap S_\beta = \emptyset\) for all \(\alpha \neq \beta\) in \(Y\). For any \(\alpha, \beta \in Y\) with \(\beta \leq Y \alpha\), let \(\varphi_{\alpha, \beta} : S_\alpha \to S_\beta\) be a homomorphism such that \(\varphi_{\alpha, \alpha} = \text{id}_{S_\alpha}\) for every \(\alpha \in Y\) and \(\varphi_{\alpha, \beta} \circ \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma}\) for \(\gamma \leq Y \beta \leq Y \alpha\) in \(Y\). On \(S = \cup_{\alpha \in Y} S_\alpha\) define a multiplication by

\[a \cdot b = (a \varphi_{\alpha, \alpha})(b \varphi_{\beta, \alpha\beta})\text{ if } a \in S_\alpha, b \in S_\beta,\]

where \(\alpha\beta = \inf\{\alpha, \beta\}\) in \(Y\). The semigroup \(S\) is called strong semilattice of monoids and is denoted by \(S = [Y; S_\alpha, \varphi_{\alpha, \beta}]\). By [15], the natural partial order on \(S\) is characterized by

\[a \leq_S b \text{ (} a \in S_\alpha, b \in S_\beta \text{) if and only if } \alpha \leq Y \beta, a \leq_\alpha b \varphi_{\beta, \alpha},\]

where \(\leq_\alpha\) denotes the natural partial order on \(S_\alpha\) (\(\alpha \in Y\)).

**Proposition 4.4.** Let \(S\) be a strong semilattice of monoids. Then \(S\) is a generalized \(F\)-semigroup if and only if \(S\) is an \(E\)-inversive, \(E\)-unitary monoid.

**Theorem 4.5.** Let \(S = [Y; S_\alpha, \varphi_{\alpha, \beta}]\) be a strong semilattice of monoids. Then \(S\) is a generalized \(F\)-semigroup if and only if

(i) \(Y\) has a greatest element \(\omega\) and for every \(\alpha \in Y\), \(\varphi_{\omega, \alpha}\) is a monoid-homomorphism;
(ii) \(S_\alpha\) is \(E\)-unitary for any \(\alpha \in Y\) and \(\varphi_{\alpha, \beta}\) is idempotent pure, for all \(\beta \leq Y \alpha\) in \(Y\);
(iii) For every \(\alpha \in Y\) and \(a \in S_\alpha\) there exist \(\beta \leq Y \alpha\) in \(Y\) and \(x \in S_\beta\) such that \((a \varphi_{\alpha, \beta})x \in E_{S_\beta}\).

**Remark 4.6.** Concerning condition (iii) notice that it is possible that no component \(S_\alpha\) of \(S\) is \(E\)-inversive but that \(S\) is so. For example, let \(Y\) be a chain, unbounded from below, \(S_\alpha = (N, .)\) \((0 \notin N)\), \(\varphi_{\alpha, \alpha} = \text{id}_{S_\alpha}\) for every \(\alpha \in Y\), and for all \(\beta < Y \alpha, a \in S_\alpha, a \varphi_{\alpha, \beta} = 1_\beta\) (the identity of \(S_\beta\)). Then for any \(a \in S, a \in S_\alpha\) say, \(a 1_\beta = 1_\beta \in E_S\) whenever \(\beta < Y \alpha\).

Two particular cases of this construction should be mentioned.
Corollary 4.7. Let $S = [Y; S_\alpha, \varphi_{\alpha, \beta}]$ be a strong semilattice of unipotent monoids (i.e., $E_{S_\alpha} = \{1_\alpha\}$ for every $\alpha \in Y$). Then $S$ is a generalized $F$-semigroup if and only if

(i) $Y$ has a greatest element;
(ii) $\varphi_{\alpha, \beta}$ is idempotent pure for all $\beta \leq Y \alpha$ in $Y$;
(iii) for every $\alpha \in Y$ and $a \in S_\alpha$ there exists $\beta \leq Y \alpha$ in $Y$ such that $(a \varphi_{\alpha, \beta})x \in E_{S_\beta}$.

The second particular case is a specialization of Corollary 4.7, supposing that every $S_\alpha (\alpha \in Y)$ is a group, that is, $S$ is a Clifford-semigroup.

Corollary 4.8. Let $S = [Y; G_\alpha, \varphi_{\alpha, \beta}]$ be a strong semilattice of groups. Then $S$ is a generalized $F$-semigroup if and only if $Y$ has a greatest element and $\varphi_{\alpha, \beta}$ is injective for all $\beta \leq Y \alpha$ in $Y$.

References


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