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Extensions of left regular bands by $\mathcal{R}$-unipotent semigroups

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Abstract
In this paper we describe $\mathcal{R}$-unipotent semigroups being regular extensions of a left regular band by an $\mathcal{R}$-unipotent semigroup T as certain subsemigroups of a wreath product of a left regular band by T. We obtain Szendrei's result that each E-unitary $\mathcal{R}$-unipotent semigroup is embeddable into a semidirect product of a left regular band by a group. Further, specialising the first author's notion of λ-semidirect product of a semigroup by a locally $\mathcal{R}$-unipotent semigroup, we provide an answer to an open question raised by the authors in [*Extensions and covers for semigroups whose idempotents form a left regular band*, Semigroup Forum 81 (2010), 51-70].

Keywords Embedding · Extension · Left regular band · R-unipotent semigroup

Mathematics Subject Classification 20M10

1 Introduction

It is well known, due to O'Carroll [7], that each regular extension of a semilattice by a group G, i.e., each E-unitary inverse semigroup with greatest group homomorphic image G, is embeddable into a semidirect product of a semilattice by G. Szendrei [8] extended this result to the class of E-unitary $\mathcal{R}$-unipotent semigroups, i.e., E-unitary regular semigroups with left regular band of idempotents. It is the aim of this paper to establish a similar result for regular extensions of left regular bands by an arbitrary $\mathcal{R}$-unipotent semigroup T, i.e., $\mathcal{R}$-unipotent semigroups S admitting an idempotent pure congruence ρ such that $S/\rho \cong T$.

In Sect. 3 we describe extensions of left regular bands by an $\mathcal{R}$-unipotent semigroup T as certain subsemigroups of a wreath product of a left regular band by T. Some of the ideas

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presented by the author in [1], for the embedding of extensions of regular orthogroups by inverse semigroups, were relevant to our research. For an arbitrary $\mathcal{R}$-unipotent semigroup S, we form an inverse semigroup $C(S)$ of injective partial right translations of S and consider the subsemigroup $C^*(S) = \langle C(S) \cup \{\varphi_s : s \in S\}\rangle$ of the semigroup of all partial transformations of S, where $s \mapsto \varphi_s$, $s \in S$, is the right regular representation of S. We show that $C^*(S)$ is an $\mathcal{R}$-unipotent semigroup and use it to prove that S can be embedded in a semidirect product of a left regular band by S/ρ, for every idempotent pure congruence ρ on S. If the semigroup S is E-unitary and $\mathcal{R}$-unipotent, the least group congruence on S is idempotent pure and so Szendrei's embedding result for E-unitary $\mathcal{R}$-unipotent semigroups follows from ours.

In Sect. 4 we specialize the first author's notion of λ-semidirect product [1,3] and we present an example of an extension of a left regular band by an $\mathcal{R}$-unipotent semigroup T that is not embeddable into a λ-semidirect product of a left regular band by T. This answers an embeddability question raised by the authors in [4].

2 Preliminaries

In this section we recall the notions and summarize the results that are needed in this paper. For standard notation in semigroup theory, defined notations and basic results we refer the reader to the books of Howie [5], Lawson [6] and Petrich [7]. In particular, for a semigroup S, $E(S)$ denotes the subset of idempotents of S and $V(a)$ denotes the set of inverses of an element $a \in S$.

A band B is called left regular if $efe = ef$ for all $e, f \in B$. Any left regular band B is a semilattice of left zero semigroups. In particular, Green's relation $\mathcal{L}$ is the least semilattice congruence on B.

The following characterizations for semigroups whose subset of idempotents forms a left regular band are well known, see [9, Theorem 1].

Proposition 2.1 *The following statements about a regular semigroup S are equivalent:*

(1) *$E(S)$ is a left regular band;*
(2) *each $\mathcal{R}$-class of S contains exactly one idempotent;*
(3) $(\forall e \in E(S))\ (\forall a \in S)\ (\forall a' \in V(a))\ aea'a = ae;$
(4) $(\forall a \in S)\ (\forall a', a'' \in V(a))\ aa' = aa'';$
(5) $(\forall a \in S)\ (\forall a', a'' \in V(a))\ (\forall e \in E(S))\ aea' = aea'';$
(6) $(\forall e, f \in E(S))\ Se \cap Sf = Sef = Sfe.$

As a consequence of the equivalence (1) ⇔ (2) above, regular semigroups whose set of idempotents is a left regular band are called *$\mathcal{R}$-unipotent*. Clearly, a regular subsemigroup of an $\mathcal{R}$-unipotent semigroup S is necessarily $\mathcal{R}$-unipotent, as is the homomorphic image. Further, S is orthodox, whence $V(b)V(a) \subseteq V(ab)$, $V(e) \subseteq E(S)$, for any $a, b \in S$, $e \in E(S)$, and the least inverse semigroup congruence γ on S is given by

$$a\ \gamma\ b \Leftrightarrow V(a) = V(b).$$

Note that γ is idempotent pure, i.e., $s\ \gamma\ e$ implies $s \in E(S)$, for all $s \in S$ and $e \in E(S)$.

A simple but useful observation is the following.

Proposition 2.2 *Let S be an $\mathcal{R}$-unipotent semigroup, and let a, b, ab, a', a'', b', $(ab)' \in S$ be such that $aa'a = a = aa''a$, $bb'b = b$, $ab(ab)'ab = ab$, i.e., a', a'' are pre-inverses of a, b' is a pre-inverse of b, and $(ab)'$ is a pre-inverse of ab. Then the following hold:*

(1) $aa' = aa''$,
(2) $abb'a' = ab(ab)'$.

Proof

(1) Since $aa', aa'' \in E(S)$ and $E(S)$ is left regular,

$$aa' = aa''aa' = aa''aa'aa'' = aa''aa'' = aa''.$$

(2) Since $a'abb' \in E(S)$,

$$abb'a'ab = aa'abb'a'abb'b = aa'abb'b = ab,$$

whence the assertion follows from (1) □

In $\mathcal{R}$-unipotent semigroups, the natural partial order has the following characterization:

$$(\forall a, b \in S) \qquad a \leq b \Leftrightarrow \exists e \in E(S) : \ a = be.$$

Proposition 2.3 *On an $\mathcal{R}$-unipotent semigroup S the least inverse semigroup congruence γ is given by*

$$(\forall a, b \in S) \qquad a \ \gamma \ b \Leftrightarrow a = eb, b = fa, \ \text{for some } e, f \in E(S).$$

Proof We have

$$\begin{aligned} a \ \gamma \ b \Leftrightarrow V(a) = V(b) \Rightarrow a &= ab'a = ab'bb'a \\ &= ab'bb'ab'b, \ \text{since } b'a \in E(S) \\ &\quad \text{and } E(S) \text{ is a left regular band} \\ &= ab'b, \ \text{with } ab' \in E(S). \end{aligned}$$

By symmetry of γ it follows $b = ba'a$ with $ba' \in E(S)$.

On the other hand, $ea = b$, $fb = a$ for some $e, f \in E(S)$ trivially implies $a \ \gamma \ b$. □

We recall some facts about injective partial right translations and permissible subsets of an inverse semigroup.

A *one-to-one partial right translation* of a semigroup S is a one-to-one partial transformation ρ which satisfies

– dom ρ is a left ideal of S;
– $\forall x \in S, \forall y \in \text{dom}\,\rho, \ x(y\rho) = (xy)\rho$.

Proposition 2.4 ([7, Lemma V.2.2]) *For any inverse semigroup S, the set of all one-to-one partial right translations is an inverse subsemigroup of $\mathcal{I}(S)$.*

A nonempty subset H of an inverse semigroup S is called *permissible* if it is an order ideal of S, in relation to the natural partial order, and satisfies

$$a, b \in H \Rightarrow ab^{-1}, \ a^{-1}b \in E(S).$$

For any nonempty subsets H and K of an inverse semigroup S, define

$$HK := \{hk : h \in H, k \in K\} \ \text{ and } \ H^{-1} := \{h^{-1} : h \in H\}. \qquad (*)$$

Proposition 2.5 ([7, Lemma V.2.6]) *For any inverse semigroup S, the set of all permissible subsets of S is an inverse semigroup under the multiplication defined by $(*)$, in which H^{-1} is the inverse of H and the natural partial order coincides with inclusion.*

Theorem 2.6 ([7, Theorem V.2.8]) *Let S be an inverse semigroup. The assignment $H \mapsto \rho$, where ρ is defined by*

- dom $\rho = \{a \in S : a^{-1}a = hh^{-1}$, *for some* $h \in H\}$,
- $\forall a \in \operatorname{dom}\rho,\ a\rho = ah$, *where* $h \in H$ *is such that* $a^{-1}a = hh^{-1}$,

is an isomorphism from the inverse semigroup of all permissible subsets of S onto the inverse semigroup of all one-to-one partial right translation of S.

Proposition 2.7 *Let S be an inverse semigroup and H a nonempty subset of S satisfying*

$$h_1, h_2 \in H \Rightarrow h_1h_2^{-1}, h_1^{-1}h_2 \in E(S).$$

Then the set

$$\overline{H} := \{fh : f \in E(S), h \in H\}$$

is a permissible subset of S generated by H.

Proof By definition, $\overline{H}$ is the order ideal of the inverse semigroup S generated by H. Since the natural partial order is compatible with both multiplication and the unary operation $x \mapsto x^{-1}$, given any $f_1h_1, f_2h_2 \in \overline{H}$, we obtain, from $f_1h_1 \leq h_1$ and $f_2h_2 \leq h_2$, that $(f_1h_1)(f_2h_2)^{-1} \leq h_1h_2^{-1}$ and $(f_1h_1)^{-1}(f_2h_2) \leq h_1^{-1}h_2$ which, by hypothesis, imply that $(f_1h_1)(f_2h_2)^{-1}, (f_1h_1)^{-1}(f_2h_2) \in E(S)$. Therefore, $\overline{H}$ is the permissible subset of S generated by H. □

Some of the results presented in this paper generalize work done on E-unitary $\mathcal{R}$-unipotent semigroups. A regular semigroup S is called E*-unitary* if it satisfies, for all $a \in S$,

$$ae \in E(S) \text{ and } e \in E(S) \Rightarrow a \in E(S).$$

An E-unitary regular semigroup is also orthodox. It is well known that in an E-unitary regular semigroup S, for all $a, b \in S$, $ab \in E(S)$ if and only if $ba \in E(S)$. It is also well known that in any E-unitary regular semigroup, the least group congruence σ is idempotent pure.

The notion of extension of a set by a semigroup plays an important role in the study pursued in this paper. A semigroup S is said to be an *extension of $H \subseteq S$ by a semigroup T* if there exists a surjective homomorphism $\phi : S \to T$ such that $E(T)\phi^{-1} = H$. If $E(T)\phi^{-1} = E(S)$, we say that S is an *idempotent pure extension of $H(= E(S))$*. This notion arises naturally in group theory, where a group G is an extension of any of its normal subgroups H by the quotient group G/H. In this paper we are interested in extensions of left regular bands by $\mathcal{R}$-unipotent semigroups.

Each $\mathcal{R}$-unipotent semigroup S is an idempotent pure extension of the left regular band $E(S)$ by the inverse semigroup S/γ via the canonical epimorphism $(\gamma)^\natural$, since γ is idempotent pure.

As referred to in Sect. 1, this paper is concerned with the embeddability of regular semigroups which are extensions of left regular bands by an $\mathcal{R}$-unipotent semigroup T. In Sect. 3 we describe such extensions in terms of the wreath product of a left regular band by T. A wreath product is a particular case of a semidirect product in the sense that it is a semidirect product in which a special action is taken. More precisely, let A and T be semigroups. Consider the semigroup A^T of all mappings from T into A, under pointwise multiplication. The mapping $\phi : T \to \operatorname{End}(A^T)$ defined by $x[f(s\phi)] = (xs)f$, for all $s, x \in T$, is a left action of T on A^T by endomorphisms. The action $s \mapsto s\phi$ is denoted by ${}^sf$ and so $x\,{}^sf = (xs)f$. The semidirect product $A^T * T$ is called the *wreath product of A by T* and is denoted by $A \operatorname{Wr} T$.

3 A wreath product embedding

In this section we prove that any $\mathcal{R}$-unipotent semigroup S can be embedded into a semidirect product of a left regular band by S/ρ, where ρ is an idempotent pure congruence on S.

Throughout this section S is an $\mathcal{R}$-unipotent semigroup, for any $x \in S$, x' denotes an inverse of x and γ denotes the least inverse semigroup congruence on S. We generalize the definition of permissible subset for subsets of $\mathcal{R}$-unipotent semigroups. A nonempty subset H of S is said to be *permissible* if $H = \{h \in S : h\gamma \in H_0\}$ for some permissible subset H_0 of S/γ.

We have the following characterization.

Proposition 3.1 *A nonempty subset H of S is permissible if and only if*

(1) $\forall h_1, h_2 \in H,\ \ h_1h_2',\ \ h_1'h_2 \in E(S)$ *for some (whence all)* $h_1' \in V(h_1), h_2' \in V(h_2)$;
(2) $\forall h \in H,\ \forall f \in E(S)),\ \ fh \in H$.

Proof Let H be a nonempty subset of S. Suppose that H is permissible. Then $H = \{h \in S : h\gamma \in H_0\}$ for some permissible subset H_0 of S/γ. Now, for $h_1, h_2 \in H$, we have $(h_1h_2')\gamma = (h_1\gamma)\,(h_2\gamma)^{-1} \in E(S/\gamma)$, since $h_1\gamma, h_2\gamma \in H_0$ and H_0 is a permissible subset of S/γ. It follows, by Lallement's Lemma and since γ is idempotent pure that $h_1h_2' \in E(S)$. Similarly, $h_1'h_2 \in E(S)$. Moreover, for $h \in H$ and $f \in E(S)$, we have $(fh)\gamma = (f\gamma)(h\gamma)$, where $h\gamma \in H_0$ and $f\gamma \in E(S/\gamma)$, and so $(fh)\gamma \leq h\gamma$. Since H_0 is an order ideal (H_0 is permissible) we obtain $(fh)\gamma \in H_0$, giving $(fh \in H$.

Conversely, assume that (1) and (2) hold. We show that H is permissible. Let $H_0 := \{h\gamma : h \in H\}$. Clearly $H = \{h \in S : h\gamma \in H_0\}$. We show that the subset H_0 of S/γ is permissible. For $h\gamma \in H_0$ and $a\gamma \in S/\gamma$, we have

$$\begin{aligned} a\gamma \leq h\gamma &\Rightarrow a\gamma = (e\gamma)(h\gamma),\ \ e \in E(S) \\ &\Rightarrow a\gamma = (eh)\gamma \\ &\Rightarrow a = f(eh),\ \ f \in E(S) \qquad \text{[Proposition 2.3]} \\ &\Rightarrow a = (fe)h,\ \ fe \in E(S) \\ &\Rightarrow a \in H,\ \ \text{by (2)}, \end{aligned}$$

giving $a\gamma \in H\gamma$. Further, let $h_1\gamma, h_2\gamma \in H_0$. We show that

$$(h_1\gamma)^{-1}(h_2\gamma), (h_1\gamma)(h_2\gamma)^{-1} \in E(S/\gamma).$$

We have

$$(h_1\gamma)^{-1}(h_2\gamma) = (h_1'\gamma)(h_2\gamma) = (h_1'h_2)\gamma$$

and so, since $h_1, h_2 \in H$, it follows by (1) that $h_1'h_2 \in E(S)$. Thus, $(h_1\gamma)^{-1}(h_2\gamma) \in E(S/\gamma)$. Similarly, $(h_1\gamma)(h_2\gamma)^{-1} \in E(S/\gamma)$. Hence, H_0 is permissible. □

Proposition 3.2 *The set of all permissible subsets of S forms an inverse semigroup with respect to set product where the inverse of any permissible subset H is $H^{-1} = \{h' \in V(h) : h \in H\}$. Moreover, the assignment $H \mapsto H\gamma := \{h\gamma \in S/\rho : h \in H\}$ defines an isomorphism from the inverse semigroup of all permissible subsets of S onto the inverse semigroup of all permissible subsets of S/γ.*

Proof Let H and K be permissible subsets of S. By definition, $H = \{h \in S : h\gamma \in H_0\}$ and $K = \{k \in S : k\gamma \in K_0\}$, for some permissible subsets H_0 and K_0 of S/γ. By Proposition 2.5, H_0K_0 is a permissible subset of the inverse semigroup S/γ. Thus, by definition, the set $\{l \in S : l \in H_0K_0\}$ is a permissible subset of S. We show that $HK = \{l \in S : l\gamma \in H_0K_0\}$. It is clear that $HK \subseteq \{l \in S : l\gamma \in H_0K_0\}$. Conversely, take $l \in S$ such that $l\gamma = (h\gamma)(k\gamma) = (hk)\gamma)$ for some $h \in H$ and $k \in K$. From $l\gamma = (hk)\gamma$ we obtain $l = e(hk) = (eh)k$, for some $e \in E(S)$. Since H is permissible, it follows that $eh \in H$ and so that $l \in HK$. This shows that the set of all permissible subsets of S forms a semigroup. It is clear by definition that the assignment defined in the proposition is a mapping; denote it by Γ. Let $H = \{h \in S : h\gamma \in H_0\}$ be a permissible subset of S where H_0 is a permissible subset of S/γ. The γ-classes of S are nonempty, therefore $H\Gamma = H_0$. This implies that Γ is surjective, and it is also injective since H_0 uniquely determines H. Moreover, the argument in the previous paragraph shows that Γ is a homomorphism, Altogether, Γ is, indeed, an isomorphism whence we obtain that the set of all permissible subsets of S forms an inverse semigroup. This implies that, for every permissible subset H as above, we have $H^{-1} = (H\gamma)^{-1} = H_0^{-1}$ whence

$$\begin{aligned} H^{-1} &= \{h' \in S : h'\gamma \in H_0^{-1} = \{h' \in S : (h'\gamma)^{-1} \in H_0\} \\ &= \{h' \in S : h \in V(h'), h\gamma \in H_0\} = \{h' \in V(h) : h\gamma \in H\}. \end{aligned}$$

□

Proposition 3.3 *For a nonempty subset H of S such that*

$$h_1, h_2 \in H \Rightarrow h_1h_2', h_1'h_2 \in E(S), \qquad (\dagger)$$

the set $\overline{H} := \{fh : f \in E(S), h \in H\}$ is the permissible subset of S generated by H.

Proof Let H be a nonempty subset of S as in the hypothesis. Then $H\gamma = \{h\gamma : h \in H\}$ is a nonempty subset of the inverse semigroup S/γ that satisfies

$$(h_1\gamma)(h_2\gamma)^{-1},\ (h_1\gamma)^{-1}(h_2\gamma) \in E(S/\gamma),$$

for all $h_1\gamma, h_2\gamma \in H\gamma$. Thus, by Proposition 6, the set

$$\overline{H\gamma} := \{(f\gamma)(h\gamma) :\ f\gamma \in E(S/\gamma),\ h\gamma \in H\gamma\}$$

is the permissible subset of S/γ generated by $H\gamma$. By Proposition 8 we now obtain that $(\overline{H\gamma})\Gamma^{-1}$ is a permissible subset of S. Hence $\overline{H} = (\overline{H\gamma})\Gamma^{-1}$ is a permissible subset of S generated by H. □

For $h \in S$, consider the mapping

$$\begin{aligned} \omega^h : Sh' &\to Sh, \\ xh' &\mapsto xh'h \end{aligned}$$

where h' is any inverse of h. The mapping ω^h is independent of the choice of h' in $V(h)$ since, by Proposition 2.1 (4), $Sh' = Shh' = Shh^* = Sh^*$, for all $h', h^* \in V(h)$. It is also worth calling the attention here to the fact that, in particular, if $e \in E(S)$ then $\omega^e = 1_{Se}$, the identity mapping on the set Se.

Proposition 3.4 *For each $h \in S$, the mapping $\omega^h$is a bijection, and so ω^h belongs to the symmetric inverse semigroup $\mathcal{I}(S)$. Furthermore, the equalities $(\omega^h)^{-1} = \omega^{h'}$ and $\omega^{h_1}\omega^{h_2} = \omega^{h_1h_2}$ hold in $\mathcal{I}(S)$ for every $h, h_1, h_2 \in S$ and $h' \in V(h)$.*

Proof For all $x \in S$, $(xh')\omega^h \in Sh$, and

$$(xh')(\omega^h\omega^{h'}) = ((xh')h)h' = (xh')(hh') = (xh')\omega^{hh'}.$$

Since $hh' \in E(S)$, and so $\omega^{hh'}$ is the identity mapping on $Shh' = Sh'$, we obtain that $\omega^h \omega^{h'} = 1_{\mathrm{dom}\,\omega^h}$ for all mutually inverse elements h, h'. This implies that $\omega^h \in \mathcal{I}(S)$ and $(\omega^h)^{-1} = \omega^{h'}$ for all $h, h' \in S$ with $h' \in V(h)$.

Now let $h_1, h_2 \in S$. By Proposition 2.1 (6), we see that

$$\mathrm{ran}\,(\omega^{h_1}\omega^{h_2}) = (Sh_1 \cap Sh_2')\omega^{h_2} = S(h_1'h_1h_2h_2'h_2) = Sh_1h_2 = \mathrm{ran}\,\omega^{h_1h_2}.$$

Moreover, we have

$$\mathrm{dom}\,(\omega^{h_1}\omega^{h_2}) = \mathrm{ran}\,(\omega^{h_2'}\omega^{h_1'}) = \mathrm{ran}\,\omega^{h_2'h_1'} = \mathrm{dom}\,\omega^{h_1h_2}.$$

Since $h_2'h_1' \in V(h_1h_2)$, we obtain by definition that

$$(xh_2'h_1')(\omega^{h_1}\omega^{h_2}) = xh_2'h_1'h_1h_2 = (xh_2'h_1')\omega^{h_1h_2},$$

completing the proof of the equality $\omega^{h_1}\omega^{h_2} = \omega^{h_1h_2}$. □

For any nonempty subset H of S with property (†), define

$$\varphi_H := \bigcup_{h\in H} \omega^h.$$

By Proposition 3.4, this property implies that

$$(\omega^{h_1})^{-1}\omega^{h_2} = \omega^{h_1'h_2} = 1_{Sh_1'h_2},$$

and similarly, $\omega^{h_1}(\omega^{h_2})^{-1} = 1_{Sh_1h_2'}$. Hence we immediately obtain the first two statements of the following proposition by applying [6, Proposition 1.2.1].

Proposition 3.5 *Let H, K be nonempty subsets of S having property (†). Then*

(1) $\varphi_H \in \mathcal{I}(S)$ *such that* $\mathrm{dom}\,\varphi_H = SH^{-1}(= \bigcup_{h'\in H^{-1}} Sh')$ *and* $\mathrm{ran}\,\varphi_H = SH(= \bigcup_{h\in H} Sh)$;
(2) $\varphi_H\varphi_K = \varphi_{HK}$ *and* $\varphi_H^{-1} = \varphi_{H^{-1}}$;
(3) $\varphi_H^2 = \varphi_H$, *or equivalently,* $\varphi_H = 1_{SH^{-1}}$ *if and only if* $H \subseteq E(S)$;
(4) $\varphi_H = \varphi_{\overline{H}}$ *if H satisfies condition* (†);
(5) *if S is an inverse semigroup and H is a permissible subset of S, then φ_H is just the one-to-one partial right translation ρ of S assigned to H in Theorem 2.6.*

Proof (3) Obviously, $\varphi_H = 1_{SH^{-1}}$ if and only if $\omega^h = 1_{Sh^{-1}}$ for every $h \in H$, and this is the case precisely if $h \in E(S)$ for every $h \in H$.
(4) Since $\omega^{fh} \subseteq \omega^h$ for $f \in E(S)$ and $h \in H$, we have $\bigcup_{h\in H, f\in E(S)} \omega^{fh} = \bigcup_{h\in H} \omega^h$, whence $\varphi_H = \varphi_{\overline{H}}$.
(5) can be easily checked by the definition of ρ. □

Define the subset $C(S) = \{\varphi_H : H \text{ is a permissible subset of } S\}$ of $\mathcal{I}(S)$. In particular, if S is an inverse semigroup then statement (5) of the previous proposition implies that $C(S)$ is the inverse semigroup of all one-to-one partial right translations of S. More generally, the following holds.

Proposition 3.6 *The set $C(S)$ forms an inverse subsemigroup of $\mathcal{I}(S)$, and consequently, of the semigroup $\mathcal{PT}(S)$ of all partial transformations on S. Moreover, the mapping $C(S) \to C(S/\gamma)$, $\varphi_H \mapsto \varphi_{H\gamma}$ is an isomorphism, and so $C(S)$ is isomorphic to the inverse semigroup $C(S/\gamma)$ of all one-to-one partial right translations of S/γ.*

Proof Proposition 3.5 (1) and (2) show that $C(S)$ is an inverse subsemigroup of $\mathcal{I}(S)$. Denote the mapping $C(S) \to C(S/\gamma)$, $\varphi_H \mapsto \varphi_{H\gamma}$ by Ξ. Consider the mappings from the inverse semigroup of all permissible subsets of S to $C(S)$ and from the inverse semigroup of all permissible subsets of S/γ to $C(S/\gamma)$ given by the assignments $H \mapsto \varphi_H$ and $H_0 \mapsto \varphi_{H_0}$, respectively, and denote them Φ and Φ_0. By Proposition 3.5 (2) we also obtain that Φ is a surjective homomorphism. Theorem 2.6 and Proposition 3.5 (5) imply that Φ_0 is an isomorphism, and we have seen in the proof of Proposition 3.2 that the mapping Γ from the inverse semigroup of all permissible subsets of S into that of S/γ defined by the rule $H \mapsto H\gamma$ is also an isomorphism. It is clear by definition that $\Gamma\Phi_0 = \Phi\Xi$. Since $\Gamma\Phi_0$ is an isomorphism, this equality implies that the surjective homomorphism Φ is also injective. Therefore Φ is also an isomorphism whence the same follows for Ξ. □

For each $s \in S$, let $\varphi_s : S \to Ss$ be defined by $x \mapsto xs$. Let

$$C^*(S) = \langle C(S) \cup \{\varphi_s : s \in S\}\rangle$$

be the subsemigroup of $\mathcal{PT}(S)$ generated by $C(S) \cup \{\varphi_s : s \in S\}$.

Proposition 3.7 *For* $\lambda = [\varphi_{s_1}]\varphi_{H_1} \cdots \varphi_{s_n}[\varphi_{H_n}] \in C^*(S)$, *where* $n \in \mathbb{N}$ *and* $[\cdots]$ *may occur or not, let* $\overline{\lambda} = [\omega^{s_1}]\varphi_{H_1} \cdots \omega^{s_n}[\varphi_{H_n}]$. *Then*

(1) $\operatorname{ran}\lambda = \operatorname{ran}\overline{\lambda}$;
(2) $\operatorname{dom}\overline{\lambda} \subseteq \operatorname{dom}\lambda$;
(3) $\lambda\,|_{\operatorname{dom}\overline{\lambda}} = \overline{\lambda}$.

Proof We prove the result by induction on n. We first show (1), (2) and (3) for $\lambda \in \{\varphi_s, \varphi_s\varphi_H, \varphi_H\varphi_s\}$, where the proof for φ_s and $\varphi_s\varphi_H$ gives the assertion for $n = 1$, and the proof for $\varphi_H\varphi_s$ is needed in the induction step. For $\lambda = \varphi_s$, $\overline{\lambda} = \omega^s$ and so $\operatorname{ran}\lambda = Ss = \operatorname{ran}\overline{\lambda}$, $\operatorname{dom}\overline{\lambda} = Ss' \subseteq S = \operatorname{dom}\lambda$ and, for $xs' \in Ss'$, $(xs')\lambda = (xs')\varphi_s = xs's = (xs')\omega^s$. For $\lambda = \varphi_s\varphi_H$, $\overline{\lambda} = \omega^s\varphi_H$ and so $\operatorname{ran}\lambda = \bigcup_{h\in H} Sh = \operatorname{ran}\overline{\lambda}$, since $xsh = (xs)s'sh$,

$$\operatorname{dom}\overline{\lambda} = \{xs' \in S : (\exists y \in S)\, xs's = yh'\} \subseteq \{x \in S : (\exists y \in S)\, xs = yh'\} = \operatorname{dom}\lambda$$

and, for $xs' \in \operatorname{dom}\overline{\lambda}$,

$$(xs')\lambda = (xs')\varphi_s\varphi_H = (xs's)\varphi_H = (yh')\varphi_H = yh'h = xs'sh = (xs')\overline{\lambda}.$$

For $\lambda = \varphi_H\varphi_s$, $\overline{\lambda} = \varphi_H\omega^s$ and so

$$\operatorname{ran}\lambda = \{xh'hs : x \in S, h \in H \text{ and } xh'h = ys' \text{ for some } y \in S\} = \operatorname{ran}\overline{\lambda},$$
$$\operatorname{dom}\overline{\lambda} = \{xh' : x \in S \text{ and } xh'h = ys' \text{ for some } y \in S\} \subseteq \{xh' : x \in S\} = \operatorname{dom}\lambda$$

and, for $xh' \in \operatorname{dom}\overline{\lambda}$,

$$(xh')\lambda = (xh')\varphi_H\varphi_s = (xh'h)\varphi_s = (ys')\varphi_s = ys's = xh'hs = (xh')\overline{\lambda}.$$

Now let $n \in \mathbb{N}$ and suppose that (1), (2) and (3) are satisfied for $\lambda = [\varphi_{s_1}]\varphi_{H_1} \cdots \varphi_{s_n}[\varphi_{H_n}]$. Let $\theta = [\varphi_{s_1}]\varphi_{H_1} \cdots \varphi_{s_{n+1}}[\varphi_{H_{n+1}}]$. We have

$$\begin{aligned}
\operatorname{ran}\theta &= \big(\operatorname{ran}([\varphi_{s_1}]\varphi_{H_1} \cdots \varphi_{s_n}\varphi_{H_n}) \cap \operatorname{dom}(\varphi_{s_{n+1}}\varphi_{H_{n+1}})\big)\,(\varphi_{s_{n+1}}[\varphi_{H_{n+1}}])\\
&= \big(\operatorname{ran}([\omega^{s_1}]\varphi_{H_1} \cdots \omega^{s_n}\varphi_{H_n}) \cap \operatorname{dom}(\varphi_{s_{n+1}}\varphi_{H_{n+1}})\big)\,(\varphi_{s_{n+1}}[\varphi_{H_{n+1}}])\\
&= \operatorname{ran}([\omega^{s_1}]\varphi_{H_1} \cdots \omega^{s_n}\varphi_{H_n}\varphi_{s_{n+1}}[\varphi_{H_{n+1}}])\\
&= \operatorname{ran}(\varphi_K\varphi_{s_{n+1}}[\varphi_{H_{n+1}}]), \text{ where } K = [\{s_1\}]H_1 \cdots \{s_n\}H_n\\
&= \big(\operatorname{ran}(\varphi_K\varphi_{s_{n+1}}) \cap \operatorname{dom}([\varphi_{H_{n+1}}])\big)\,[\varphi_{H_{n+1}}]\\
&= \big(\operatorname{ran}(\varphi_K\omega^{s_{n+1}}) \cap \operatorname{dom}([\varphi_{H_{n+1}}])\big)\,[\varphi_{H_{n+1}}]\\
&= \operatorname{ran}(\varphi_K\omega^{s_{n+1}}[\varphi_{H_{n+1}}])\\
&= \operatorname{ran}\overline{\theta}
\end{aligned}$$

and

$$\begin{aligned}\operatorname{dom}\theta &= \big(\operatorname{ran}([\varphi_{s_1}]\varphi_{H_1}) \cap \operatorname{dom}(\varphi_{s_2}\varphi_{H_2}\cdots\varphi_{s_{n+1}}[\varphi_{H_{n+1}}])\big)([\varphi_{s_1}]\varphi_{H_1})^{-1}\\ &\supseteq \big(\operatorname{ran}([\varphi_{s_1}]\varphi_{H_1}) \cap \operatorname{dom}(\omega^{s_2}\varphi_{H_2}\cdots\omega^{s_{n+1}}[\varphi_{H_{n+1}}])\big)([\varphi_{s_1}]\varphi_{H_1})^{-1}\\ &= \operatorname{dom}([\varphi_{s_1}]\varphi_{H_1}\omega^{s_2}\varphi_{H_2}\cdots\omega^{s_{n+1}}[\varphi_{H_{n+1}}])\\ &= \operatorname{dom}([\varphi_{s_1}]\varphi_K),\ \text{where}\ K = H_1\cdots\{s_n\}H_n\{s_{n+1}\}[H_{n+1}]\\ &\supseteq \operatorname{dom}([\omega^{s_1}]\varphi_K)\\ &= \operatorname{dom}\overline{\theta}.\end{aligned}$$

Finally, we show that $\theta\ |_{\operatorname{dom}\overline{\theta}} = \overline{\theta}$. Let $x \in \operatorname{dom}([\omega^{s_1}]\varphi_{H_1}\cdots\omega^{s_{n+1}}[\varphi_{H_{n+1}}])$. On the one hand, we have that $x \in \operatorname{dom}([\omega^{s_1}]\varphi_{H_1}\cdots\omega^{s_n}\varphi_{H_n})$ and so, by hypothesis,

$$x([\varphi_{s_1}]\varphi_{H_1}\cdots\varphi_{s_n}\varphi_{H_n}) = x([\omega^{s_1}\varphi_{H_1}\cdots\omega^{s_n}\varphi_{H_n}).$$

On the other hand, we have that $x([\omega^{s_1}]\varphi_{H_1}\cdots\omega^{s_n}\varphi_{H_n}) \in \operatorname{dom}(\omega^{s_{n+1}}[\varphi_{H_{n+1}}])$ and so

$$\big(x([\omega^{s_1}]\varphi_{H_1}\cdots\omega^{s_n}\varphi_{H_n})\big)(\varphi_{s_{n+1}}[\varphi_{H_{n+1}}]) = \big(x([\omega^{s_1}]\varphi_{H_1}\cdots\omega^{s_n}\varphi_{H_n})\big)(\omega^{s_{n+1}}[\varphi_{H_{n+1}}]).$$

Hence

$$x\theta = x([\varphi_{s_1}]\varphi_{H_1}\varphi_{s_2}\varphi_{H_2}\cdots\varphi_{s_{n+1}}[\varphi_{H_{n+1}}]) = x([\omega^{s_1}]\varphi_{H_1}\omega^{s_2}\varphi_{H_2}\cdots\omega^{s_{n+1}}[\varphi_{H_{n+1}}]) = x\overline{\theta}.$$

□

Lemma 3.8 *Let $\lambda, \mu \in C^*(S)$ where $\lambda = [\varphi_{s_1}]\varphi_{H_1}\cdots\varphi_{s_n}[\varphi_{H_n}]$, and suppose that K is a permissible subset of S such that $[\{s_1\}]H_1\cdots\{s_n\}[H_n] \subseteq K$. Then we have*

(1) *$\lambda \in E(C^*(S))$ if and only if $[\{s_1\}]H_1\cdots\{s_n\}[H_n] \subseteq E(S)$;*
(2) *if $\lambda, \mu \in E(C^*(S))$ then $\overline{\lambda\mu} = \overline{\lambda}\overline{\mu}$;*
(3) *$\overline{\lambda}^{-1} \in V(\lambda)$;*
(4) *$\lambda\varphi_K^{-1} = \lambda\overline{\lambda}^{-1}$ and $\lambda\varphi_K^{-1}\lambda = \lambda = \lambda\varphi_K^{-1}\varphi_K$;*
(5) *if $[\{s_1\}]H_1\cdots\{s_n\}[H_n] \subseteq E(S)$ and $H_i = t_i\rho$ for any idempotent pure congruence ρ on S and for some $t_i \in S$ $(i = 1, \ldots, n)$ then $\lambda\varphi_{(s_1t_1\cdots s_nt_n)\rho} = \lambda$.*

Proof (1)–(2) These statements follow immediately from Proposition 3.7 since λ is idempotent if and only if $\overline{\lambda}$ is the identity mapping on $\operatorname{dom}\overline{\lambda} = \operatorname{ran}\overline{\lambda}$.

(3) By definition, it suffices to notice that Proposition 3.7 implies $\overline{\lambda}^{-1}\lambda$ to be the identity mapping on $\operatorname{dom}\overline{\lambda}^{-1} = \operatorname{ran}\overline{\lambda} = \operatorname{ran}\lambda$.

(4) By the assumption on K, we have $\operatorname{dom}\overline{\lambda} \subseteq \operatorname{dom}\varphi_K$, $\operatorname{ran}\overline{\lambda} \subseteq \operatorname{ran}\varphi_K$, and $\overline{\lambda}$ is a restriction of φ_K. This implies the first equality of statement (4), and the rest follows by applying (3), and that $\varphi_K^{-1}\varphi_K$ is the identity mapping on $\operatorname{ran}\varphi_K$.

(5) Let $K = \overline{(s_1t_1\cdots s_nt_n)\rho}$. Since $[\{s_1\}](t_1\rho)\cdots\{s_n\}[(t_n\rho)] \subseteq E(S)$ by assumption, and ρ is an idempotent pure congruence, we see that $K \subseteq E(S)$. Hence $\varphi_K \in E(C(S))$ and $\varphi_K = \varphi_K^{-1}\varphi_K$ follow, and the last equality of (4) implies the equality to be proved. □

Theorem 3.9 *$C^*(S)$ is an $\mathcal{R}$-unipotent semigroup.*

Proof Lemma 3.8 (1) and (3) imply that $E(C^*(S))$ is a band and $C^*(S)$ is a regular semigroup. We show that $E(C^*(S))$ is left regular. Let $\lambda, \mu \in E(C^*(S))$. By Proposition 3.7 and Lemma 3.8 (2), $\operatorname{ran}(\lambda\mu) = \operatorname{ran}\overline{\lambda\mu} = \operatorname{ran}(\overline{\lambda}\overline{\mu})$ and so

$$\operatorname{ran}(\lambda\mu) = \operatorname{ran}\overline{\lambda} \cap \operatorname{ran}\overline{\mu} \subseteq \operatorname{ran}\overline{\lambda} \subseteq \operatorname{dom}\lambda,$$

whence $\mathrm{dom}\,(\lambda\mu) = \mathrm{dom}\,(\lambda\mu\lambda)$. Next, for $x \in \mathrm{dom}\,(\lambda\mu)$, there is $y \in S$ such that $x(\lambda\mu) = y\overline{\lambda\mu}$. We compute as follows:

$$x(\lambda\mu) = y\overline{\lambda\mu} = y(\overline{\lambda\mu}\,\overline{\lambda}) = (y\overline{\lambda\mu})\overline{\lambda} = (x\lambda\mu)\lambda = x(\lambda\mu\lambda).$$

Summarizing, we obtain $\lambda\mu = \lambda\mu\lambda$, whence $E(C^*(S))$ is left regular. □

Let S be an $\mathcal{R}$-unipotent semigroup with identity element 1_S and let ρ be an idempotent pure congruence on S. Then each ρ-class $u\rho$ satisfies condition (†). By Proposition 3.5 (4) we have $\varphi_{u\rho} = \varphi_{\overline{u\rho}}$, whence by Proposition 3.5 (2) $\varphi_{u\rho}^{-1} = \varphi_{\overline{u\rho}}^{-1} = \varphi_{\overline{u\rho}^{-1}}$ follows. From Lemma 3.8 (4) we directly obtain $\varphi_s\varphi_{s\rho}^{-1}\varphi_s = \varphi_s$ and $\varphi_{u\rho}\varphi_s\varphi_{(us)\rho}^{-1}\varphi_{u\rho}\varphi_s = \varphi_{u\rho}\varphi_s$. Consider the wreath product $E(C^*(S))\,\mathrm{Wr}\,S/\rho$ and let Ψ be the correspondence defined by $s\Psi = (f_s, s\rho)$, for all $s \in S$, where $f_s \in C^*(S)^{S/\rho}$ is defined by $(u\rho)f_s = \varphi_{u\rho}\varphi_s\varphi_{(us)\rho}^{-1}$. By Proposition 2.2 and the remark above we get $(u\rho)f_s = \varphi_{u\rho}\varphi_s\varphi_{s\rho}^{-1}\varphi_{u\rho}^{-1} = \varphi_{u\rho}\varphi_s\varphi_{s'}\varphi_{u\rho}^{-1} = \varphi_{u\rho}\varphi_{ss'}\varphi_{u\rho}^{-1}$ with $s' \in V(s)$, whence $(u\rho)f_s \in E(C^*(S))$ by Lemma 3.8 (1). Consequently Ψ is a mapping from S into $E(C^*(S))\,\mathrm{Wr}\,S/\rho$. Note in particular that from the above we also get $f_s = f_{ss'}$, since $ss' = ss'(ss')'$.

We prove that Ψ is a morphism. Let $u\rho \in S/\rho$. We compute

$$\begin{aligned}(u\rho)f_s{}^{s\rho}f_t &= (u\rho)f_s(us)\rho f_t\\ &= \varphi_{u\rho}\varphi_s\varphi_{(us)\rho}^{-1}\varphi_{(us)\rho}\varphi_t\varphi_{(ust)\rho}^{-1}\\ &= \varphi_{u\rho}\varphi_s\varphi_t\varphi_{(ust)\rho}^{-1}, \text{ by Lemma 3.8(4), since } \overline{u\rho}\{s\} \subseteq \overline{(us)\rho}\\ &= \varphi_{u\rho}\varphi_{st}\varphi_{(ust)\rho}^{-1}\\ &= (u\rho)f_{st}.\end{aligned}$$

We prove that Ψ is injective. Note first that $\overline{(1_S)\rho} = E(S)$ whence $\varphi_{(1_S)\rho} = \varphi_{E(S)}$, the identity mapping on S. Let now $(f_s, s\rho) = (f_t, t\rho)$. It follows $f_s = f_t$ and $s\rho = t\rho$. Also,

$$\begin{aligned}f_s = f_t &\Rightarrow ((1_S)\rho)f_s = ((1_S)\rho)f_t \Rightarrow \varphi_{ss'} = \varphi_{tt'}\\ &\Rightarrow (1_S)\varphi_{ss'} = (1_S)\varphi_{tt'} \Rightarrow ss' = tt'\\ &\Rightarrow (s,t) \in \mathcal{R}.\end{aligned}$$

Since $s\rho = t\rho$, we now obtain $(s,t) \in \mathcal{R} \cap \rho$. Thus, $t's = t't$ (since $t's \in E(S)$, $t's\ \mathcal{R}\ t't$ and S is $\mathcal{R}$-unipotent) and so,

$$s = ss's = tt's = tt't = t.$$

We have the following theorem.

Theorem 3.10 *Let ρ be an idempotent pure congruence on an $\mathcal{R}$-unipotent semigroup S with identity element. Then S is embeddable into a semidirect product of a left regular band by S/ρ.*

Let now S be an $\mathcal{R}$-unipotent semigroup having no identity element, and let ρ be an idempotent pure congruence on S. Then ρ can be extended to an idempotent pure congruence ρ^1 on S^1 by defining $\rho^1 = \rho \cup \{(1,1)\}$, and S^1/ρ^1 is nothing but S/ρ with the identity element $\{1\}$ adjoined. By Theorem 3.10 there is an embedding $\Psi : S^1 \to E(C^*(S^1))\,\mathrm{Wr}\,(S/\rho)^{\{1\}}$ mapping S into the subsemigroup of $E(C^*(S^1))\,\mathrm{Wr}\,(S/\rho)^{\{1\}}$ which consists of all elements whose second component belongs to S/ρ. Thus we obtain the main result of the paper.

Corollary 3.11 *Let S be an $\mathcal{R}$-unipotent semigroup and let ρ be an idempotent pure congruence on S. Then S is embeddable into a semidirect product of a left regular band by S/ρ.*

If S is an E-unitary $\mathcal{R}$-unipotent semigroup, then the least group congruence σ on S is idempotent pure and so, by Corollary 3.11, we obtain Szendrei's embedding result for E-unitary $\mathcal{R}$-unipotent semigroups.

Corollary 3.12 ([8, Proposition 2.1]) *Every E-unitary $\mathcal{R}$-unipotent semigroup can be embedded into a semidirect product of a left regular band by a group.*

4 Embedding in a λ-semidirect product

In connection with an embedding of a certain type of $\mathcal{R}$-unipotent semigroups into a λ-semidirect product of a left regular band by an $\mathcal{R}$-unipotent semigroup, the authors raise, in [4], the question of "whether any idempotent pure extension of an $\mathcal{R}$-unipotent semigroup T is embeddable into a λ-semidirect product of a left regular band by T". In this section, we provide a negative answer to this question.

The notion of λ-semidirect product of semigroups was introduced by the first author in [2] for inverse semigroups, see also [6], and later generalized for locally $\mathcal{R}$-unipotent semigroups in [3]. We recall this concept for the special case of $\mathcal{R}$-unipotent semigroups. Let T be an $\mathcal{R}$-unipotent semigroup acting on a semigroup S by endomorphisms on the left. On the set

$$S *_\lambda T := \left\{(a, x) \in S \times T : {}^{xx'}a = a, \text{ for } x' \in V(x)\right\}$$

the equality $(a, x)(b, y) = \left({}^{xy(xy)'}a\, {}^{x}b, xy\right)$ defines a binary operation. $S *_\lambda T$ with the above operation is called a λ-*semidirect product* of S by T.

Proposition 4.1 ([3, Theorem 3]) *Let $S *_\lambda T$ be a λ-semidirect product of a semigroup S by an $\mathcal{R}$-unipotent semigroup T. Then*

(1) *$S *_\lambda T$ is a semigroup;*
(2) *$E_{S*_\lambda T} = \{(a, x) \in S *_\lambda T \mid a \in E(S), x \in E(T)\}$;*
(3) *for $(a, x) \in S *_\lambda T$, $V(a, x) = \{({}^{x'}a', x') \mid a' \in V(a), x' \in V(x)\}$;*
(4) *if S is regular, then so is $S *_\lambda T$;*
(5) *if S is $\mathcal{R}$-unipotent, then so is $S *_\lambda T$.*

We prove a necessary condition for an $\mathcal{R}$-unipotent semigroup to be embeddable in a λ-semidirect product of a left regular band by an inverse semigroup.

Proposition 4.2 *Let S be an orthodox semigroup which is embeddable in a λ-semidirect product of a left regular band B by an inverse semigroup T. Then S satisfies the following condition:*

$$\text{(E)} \qquad se \in E(S) \Rightarrow se = s^2e, \text{ for all } e \in E(S),\ s \in S.$$

Proof Let $s \in S$ and $e \in E(S)$ be such that $se \in E(S)$. Because of the embedding of S in $B*_\lambda T$, B being a left regular band and T an inverse semigroup, we can write $s = (b, t)$ and $e = (f, r)$, for some $b, f \in B$ and $t, r \in T$, where ${}^{tt'}b = b$ and ${}^{rr'}f = f$. Since e is an idempotent of S, we have that $r \in E(T)$ and $f = {}^{r}f$. Also $se = (b, t)(f, r) = ({}^{(tr)(tr)'}b\,{}^{t}f, tr)$ is, by hypothesis, an idempotent, we have that $tr \in E(T)$ and ${}^{tr}({}^{tr}b\,{}^{t}f) = {}^{tr}b\,{}^{t}f = {}^{tr}b\,{}^{tr}f = {}^{tr}(bf)$.

Also, $t^2r = t^2r^2 = t \cdot tr \cdot r = tr \cdot tr = tr$, since idempotents commute in T. We then obtain

$$\begin{aligned} s^2e = (b,t)({}^{tr}b^t f, tr) &= ({}^{t^2r}b^t({}^{tr}b^t f), t^2r) \\ &= ({}^{tr}b^{tr}b^{t^2} f, tr) \\ &= ({}^{tr}b^{t^2}({}^{r} f), tr) \\ &= ({}^{tr}(bf), tr) \\ &= se. \end{aligned}$$

□

The following example provides a negative answer to the open question raised in [4].

Example 4.3 Let $S = \{1, s, e, f\}$ and consider the operation defined in S by the following Cayley table:

	1	s	e	f
1	1	s	e	f
s	s	1	f	e
e	e	e	e	e
f	f	f	f	f

Clearly, S is an $\mathcal{R}$-unipotent semigroup. The least inverse semigroup congruence γ on S is idempotent pure and so S is an idempotent pure extension of $E(S)$ by S/γ. Moreover, $se = f \in E(S)$ and $se = f \neq e = 1e = s^2e$, that is, S does not satisfy condition (E). Thus, by Proposition 4.2, S is not embeddable in a λ-semidirect product of a left regular band by S/γ.

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