Алгебра и анализ
Том 32 (2020), №3

Dedicated to Nina Nikolaevna Ural′tseva on the occasion of her 85th birthday

LAGRANGE MULTIPLIERS FOR EVOLUTION PROBLEMS WITH CONSTRAINTS ON THE DERIVATIVES

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We prove the existence of generalized Lagrange multipliers for a class of evolution problems for linear differential operators of different types subject to constraints on the derivatives. Those Lagrange multipliers and the respective solutions are stable for the vanishing of the coercive parameter and are naturally associated with evolution variational inequalities with time-dependent convex sets of gradient type. We apply these results to the sandpile problem, to superconductivity problems, to flows of thick fluids, to problems with the biharmonic operator, and to first order vector fields of subelliptic type.

§1. Introduction

Variational inequalities with constant gradient constraints appeared in 1967 to solve the equilibrium elastic-plastic torsion with arbitrary cross section (see references in the recent survey [14]). It was later shown that the Lagrange multiplier associated with the yield criteria of von Mises is uniquely determined by a bounded positive function under general assumptions (see [7] and its references).

The first evolution model with a gradient constraint was proposed in 1986 to model a poured pile shape (see [11]) and was treated a decade later as a variational inequality and as an "infinitely fast/slow" diffusion limit, after the

Ключевые слова: variational inequalities, sandpile problem, superconductivity problems, flows of thick fluids, problems with the biharmonic operator, first order vector fields of subelliptic type.

The research of A. Azevedo and L. Santos was partially supported by the Research Centre of Mathematics of the University of Minho with the Portuguese Funds from the "Fundação para a Ciência e a Tecnologia," through the Project UID/MAT/00013/2013, and the one by J. F. Rodrigues was done partially in the framework of the Project PTDC/MAT-PUR/28686/2017.

earlier mathematics study [16], which was extended to the case of parabolic variational inequalities with nonconstant gradient constraint and also to evolution quasivariational inequalities (see [14] for references).

For instance, in [17], for a smooth and strictly positive threshold $g = g(x,t)$, with $x \in \Omega$, a smooth bounded domain of $\mathbb{R}^d$, and $t \in [0,T]$, the unique solution $u = u(x,t)$ in the convex set

$$\mathbb{K}_{g(t)} = \{v \in H^1_0(\Omega)\colon |\nabla v| \leqslant g(t) \ \text{ a.e. in } \Omega\} \tag{1.1}$$

to the variational inequality was studied for $\alpha > 0$ and a suitable given function f:

$$\int_\Omega \partial_t u(t)\,(v - u(t)) + \alpha \int_\Omega \nabla u(t) \cdot \nabla(v - u(t)) \geqslant \int_\Omega f(t)\,(v - u(t)). \tag{1.2}$$

Here the inequality is assumed to be fulfilled for $v \in \mathbb{K}_{g(t)}$ and a.e. $t \in (0,T)$, and u is subject to the condition $u(0) = h \in \mathbb{K}_{g(0)}$. In particular, under the special assumptions $\partial_t g^2 \geqslant 0$ and $\Delta g^2 \leqslant 0$ (including the case of $g = \text{const} > 0$) and with $f \in L^\infty(0,T)$ spatially homogeneous, it was shown in [17] that the Lagrange multiplier problem, i.e., to find a pair (μ, u) solving the equation

$$\int_\Omega \partial_t u(t) w + \int_\Omega \mu \nabla u(t) \cdot \nabla w = \int_\Omega f(t)\,w \tag{1.3}$$

for a.e. $t \in (0,T)$ and for all $w \in H^1_0(\Omega)$ under the conditions

$$|\nabla u| \leqslant g, \quad \mu \geqslant \alpha, \quad (\mu - \alpha)(|\nabla u| - g) = 0 \tag{1.4}$$

a.e. $(x,t) \in \Omega \times (0,T)$, with $u(0) = h \in \mathbb{K}_{g(0)}$ and $u = 0$ on $\partial\Omega \times (0,T)$, is uniquely solvable with $\mu \in L^\infty(\Omega \times (0,T))$ and $u \in L^\infty(0,T; \mathbb{K}_{g(t)} \cap H^2_{\mathrm{loc}}(\Omega)) \cap H^1(0,T; L^2(\Omega))$, and, in fact, it is equivalent to solve in (1.1), the variational inequality (1.2) with the same initial condition.

Considering the flux $\Phi = \mu \nabla u$, we can write (1.3) as a diffusion equation in $\Omega \times (0,T)$

$$\partial_t u - \nabla \cdot \Phi = f \tag{1.5}$$

and the constraints (1.4) can be written in the form

$$\mu \in \kappa_\alpha(|\nabla u|^2 - g^2)$$

where κ_α is the maximal monotone graph defined by $\kappa_\alpha(s) = \alpha$ if $s < 0$ and $\kappa_\alpha(0) = [\alpha, +\infty)$. In this form, this problem can be treated within the nonlinear semigroup theory, at least if g is time independent (see [2]), which however does not give much information on μ.

Recently, with $g \equiv 1$ and $\alpha = 0$, but with f being possibly a measure, equation (1.5) was interpreted as an evolution Monge–Kantorovich problem in [8]; its solution is the couple (Φ, u), where Φ is the transportation flux and u its potential. This was motivated by a generalization of the sandpile problem.

Formally, from (1.4) we can write $\mu = \alpha + \lambda$ with $\lambda = 0$ in the region $\{(x,y) : |\nabla u(x,t)| < g(x,t)\}$ and where, from (1.3), the solution u satisfies

$$\partial_t u - \alpha \Delta u = f \quad \text{in } \{|\nabla u| < g\}.$$

However, in general, we cannot expect that $\lambda \geqslant 0$ is a function and, in fact, it is only a measure. Following the approach of [1] for the stationary nonconstant gradient constraint, we consider here a more general class of linear differential operators L, including the examples of the constraint $|L\boldsymbol{u}| \leqslant g$ for possibly vector valued functions $\boldsymbol{u}$:

$$\begin{aligned}
&Lu = \nabla u \quad \text{(gradient)},\\
&Lu = \Delta u \quad \text{(Laplacian)},\\
&L\boldsymbol{u} = \nabla \times \boldsymbol{u} \quad \text{(curl)},\\
&L\boldsymbol{u} = D\boldsymbol{u} = \tfrac{1}{2}\left(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T\right) \quad \text{(symmetrized Jacobian)},\\
&Lu = (X_1 u, \dots, X_\ell u) = \boldsymbol{X} u \quad \text{(subelliptic gradient for vector fields } X_j).
\end{aligned}$$

In general, the formulation (1.3) must be extended in a duality sense of L^∞ and $(L^\infty)'$, the space of finitely additive, bounded and absolutely continuous measures λ, such that $\langle \lambda, \chi_\omega \rangle = \lambda(\omega)$, for all $\omega \subseteq \Omega \times (0,T)$, where χ_ω is the characteristic function of the measurable set ω; see [18, p. 118]. In §2 we give the precise formulation of the main results, which are exemplified with the above five examples, with applications in §3 and the proofs in §4.

§2. Assumptions and main results

Let Ω be a bounded open subset of $\mathbb{R}^d$ ($d \geqslant 2$) with Lipschitz boundary. For $T > 0$ and $t \in (0,T]$ we denote $Q_t = \Omega \times (0,t)$. We consider vector-valued functions $\boldsymbol{u} = (u_1, \dots, u_m)$ in the variables $(x,t) \in Q_T$ and, for a multi-index $\nu = (\nu_1, \dots, \nu_d)$, with $\nu_1, \dots, \nu_d \in \mathbb{N}_0$ and $|\nu| = \nu_1 + \cdots + \nu_d$, we denote by $\partial^\nu u_i = \frac{\partial^{|\nu|} u_i}{\partial x_1^{\nu_1} \cdots \partial x_d^{\nu_d}}$ the partial derivatives of u_i, for $i = 1, \dots, m$.

Let $L : \boldsymbol{W} \to L^2(\Omega)^\ell$ be the linear differential operator of order s given by

$$(L\boldsymbol{u})_j = \sum_{|\nu| \leqslant s} \sum_{k=1}^{m} \lambda^j_{\nu,k} \partial^\nu u_k, \quad 1 \leqslant j \leqslant \ell,$$

where $s, l, m \in \mathbb{N}$, $\nu \in \mathbb{N}_0^d$, $\lambda^j_{\nu,k} \in L^\infty(\Omega)$, and

$$\boldsymbol{W} = \left\{ \boldsymbol{v} \in L^2(\Omega)^m : L\boldsymbol{v} \in L^2(\Omega)^\ell \right\}$$

is a Hilbert space endowed with the graph norm.

As examples we have: the gradient; the Laplacian; the curl; the symmetrized Jacobian; and the subelliptic gradient.

Let $(\boldsymbol{V},\boldsymbol{H},\boldsymbol{V}')$ be a Gelfand triple with Hilbert spaces $\boldsymbol{H}\subseteq L^2(\Omega)^\ell$ and $\boldsymbol{V}$ a closed subspace of $\boldsymbol{W}$ such that $\|\boldsymbol{v}\|_{\boldsymbol{V}}:=\|L\boldsymbol{v}\|_{L^2(\Omega)^\ell}$ is a norm in $\boldsymbol{V}$ equivalent to the norm induced from $\boldsymbol{W}$.

We denote

$$\mathscr{V}=L^2\big(0,T;\boldsymbol{V}\big)\quad\text{and}\quad\mathscr{V}_\infty=L^\infty\big(0,T;\boldsymbol{V}_\infty\big),$$

endowed with the natural norms, where

$$\boldsymbol{V}_\infty=\Big\{\boldsymbol{v}\in\boldsymbol{V}:L\boldsymbol{v}\in L^\infty(\Omega)^\ell\Big\}.$$

For instance, if $L=\nabla$ we have $W=H^1(\Omega)$ and we can take $V=H^1_0(\Omega)$, but more examples can be considered as in §3 below.

Let

$$\boldsymbol{f}\in L^2(Q_T)^m,\quad g\in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big),\quad g\geqslant g_*>0,\tag{2.1}$$

and, for all $t\in[0,T]$, we define the convex set

$$\mathbb{K}_{g(t)}=\big\{\boldsymbol{v}\in\boldsymbol{V}:|L\boldsymbol{v}(x)|\leqslant g(x,t)\text{ a.e. in }\Omega\big\}\subseteq\mathscr{V}_\infty,\tag{2.2}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^\ell$. We also assume that

$$\boldsymbol{h}\in\mathbb{K}_{g(0)}.\tag{2.3}$$

Given $\boldsymbol{v}\in\mathscr{V}$, we say that

$$\boldsymbol{v}\in\mathbb{K}_g\quad\text{if }\boldsymbol{v}(t)\in\mathbb{K}_{g(t)}\quad\text{for a.e.}\quad t\in[0,T].$$

For $\alpha\geqslant 0$ we consider the variational inequality: to find

$$\boldsymbol{u}^\alpha\in\mathscr{V}_\infty\cap H^1(0,T;L^2(\Omega)^m)$$

satisfying

$$\begin{cases}\boldsymbol{u}^\alpha(t)\in\mathbb{K}_{g(t)},\ \boldsymbol{u}^\alpha(0)=\boldsymbol{h}\\ \displaystyle\int_\Omega\partial_t\boldsymbol{u}^\alpha(t)\cdot(\boldsymbol{v}-\boldsymbol{u}^\alpha(t))+\alpha\int_\Omega L\boldsymbol{u}^\alpha(t)\cdot L(\boldsymbol{v}-\boldsymbol{u}^\alpha(t))\geqslant\int_\Omega\boldsymbol{f}(t)\cdot(\boldsymbol{v}-\boldsymbol{u}^\alpha(t))\\ \hfill\forall\,\boldsymbol{v}\in\mathbb{K}_{g(t)},\quad\text{for a.e.}\quad t\in[0,T].\end{cases}\tag{2.4}$$

We define the Lagrange multiplier problem associated with this variational inequality: to find $(\lambda^\alpha,\boldsymbol{u}^\alpha)\in L^\infty(Q_T)'\times\big(\mathscr{V}_\infty\cap H^1\big(0,T;L^2(\Omega)^m\big)\big)$ such that

$$\int_{Q_T}\partial_t\boldsymbol{u}^\alpha\cdot\boldsymbol{v}+\langle\lambda^\alpha,L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v}\rangle+\alpha\int_{Q_T}L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v}=\int_{Q_T}\boldsymbol{f}\cdot\boldsymbol{v},\quad\boldsymbol{v}\in\mathscr{V}_\infty,\tag{2.5a}$$

$$\boldsymbol{u}^\alpha(0)=\boldsymbol{h}\quad\text{in }\Omega,\tag{2.5b}$$

$$\boldsymbol{u}^\alpha\in\mathbb{K}_g,\quad\lambda^\alpha\geqslant0\quad\text{in }L^\infty(Q_T)',\quad\lambda^\alpha(|L\boldsymbol{u}^\alpha|-g)=0\quad\text{in }L^\infty(Q_T)'.\tag{2.5c}$$

Here we denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the duality pairing between $L^\infty(Q_T)'$ and $L^\infty(Q_T)$ and we observe that $\lambda\varphi \in L^\infty(Q_T)'$ for $\lambda \in L^\infty(Q_T)'$ and $\varphi \in L^\infty(Q_T)$, by setting

$$\langle \lambda\varphi, \psi\rangle = \langle \lambda, \varphi\psi\rangle, \quad \forall \lambda \in L^\infty(Q_T)',\ \forall \varphi, \psi \in L^\infty(Q_T).$$

Theorem 2.1. *Assume that* (2.1)–(2.3) *are fulfilled and* $\alpha > 0$. *Then problem* (2.5) *has a solution*

$$(\lambda^\alpha, \boldsymbol{u}^\alpha) \in L^\infty(Q_T)' \times \big(\mathscr{V}_\infty \cap H^1\big(0,T;L^2(\Omega)^m\big)\big).$$

Theorem 2.2. *Under the assumptions* (2.1)–(2.3), *at least for a subsequence of* $\{(\lambda^\alpha, \boldsymbol{u}^\alpha)\}_{\alpha>0}$ *of solutions of problem* (2.5) *obtained in Theorem* 2.1, *we have*

$$\begin{aligned} &\lambda^\alpha \underset{\alpha\to 0}{\rightharpoonup} \lambda^0 \quad \text{in} \quad L^\infty(Q_T)', \\ &\boldsymbol{u}^\alpha \underset{\alpha\to 0}{\rightharpoonup} \boldsymbol{u}^0 \quad \text{in} \quad H^1\big(0,T;L^2(\Omega)^m\big) \quad \text{and in} \quad \mathscr{V}_\infty \text{ weak-}* \end{aligned}$$

and $(\lambda^0, \boldsymbol{u}^0)$ *solves problem* (2.5) *for* $\alpha = 0$.

Theorem 2.3. *Under the assumptions of the previous theorems, for* $\alpha \geqslant 0$ *the function* $\boldsymbol{u}^\alpha$ *is a unique solution of the variational inequality* (2.4).

§3. Lagrange multipliers for linear differential operators

Here we give various examples for the Lagrange multiplier problem (2.5), choosing appropriately the linear operator L and a variety of convex sets of type (2.2) in various functional settings.

3.1. A problem with the gradient constraint. Setting $Lv = \nabla v$, $H = L^2(\Omega)$, $W = H^1(\Omega)$, and $V = H^1_0(\Omega)$, as an immediate consequence of Theorems 2.1, 2.2, and 2.3 we can state the following result, which is applicable to the sandpile problem in the case of $\alpha = 0$ (see [11]).

Corollary 3.1. *Assume that* $f \in L^2(Q_T)$, $g \in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$ *with* $g \geqslant g_* > 0$ *and* $h \in \mathbb{K}_{g(0)}$. *Then the following Lagrange multiplier problem* (*with* $\alpha \geqslant 0$):

$$\int_{Q_T} \partial_t u^\alpha v + \langle \lambda^\alpha, \nabla u^\alpha \cdot \nabla v\rangle + \alpha \int_\Omega \nabla u^\alpha \cdot \nabla v = \int_{Q_T} fv, \quad \forall v \in L^\infty\big(0,T;W^{1,\infty}_0(\Omega)\big)$$

$$u^\alpha = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u^\alpha(0) = h \quad \text{in } \Omega$$

$$|\nabla u^\alpha| \leqslant g \quad \text{in } Q_T, \quad \lambda^\alpha \geqslant 0 \text{ and } \lambda^\alpha(|\nabla u^\alpha| - g) = 0 \quad \text{in } L^\infty(Q_T)'$$

has a solution $(\lambda^\alpha, u^\alpha) \in L^\infty(Q_T)' \times \big(L^\infty\big(0,T;W^{1,\infty}_0(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big)\big)$ *with* u^α *solving uniquely the corresponding variational inequality* (2.4).

3.2. A problem with the Laplacian constraint. Here we choose L to be the Laplace operator, $H = L^2(\Omega)$, and $W = \{v \in L^2(\Omega) : \Delta v \in L^2(\Omega)\}$. The usual norm in the subspace $V = H^2_0(\Omega)$ is equivalent to the norm

$$\|v\|_V = \|\Delta u\|_{L^2(\Omega)}$$

because Δ is an isomorphism between V and $L^2(\Omega)$. So, from Theorems 2.1, 2.2 and 2.3, we deduce the following statement.

Corollary 3.2. *Assume that $f \in L^2(Q_T)$, $g \in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$ with $g \geqslant g_* > 0$, and $h \in \mathbb{K}_{g(0)}$. Then the following Lagrange multiplier problem (with $\alpha \geqslant 0$):*

$$\int_{Q_T} \partial_t u^\alpha\, v + \langle \lambda^\alpha, \Delta u^\alpha\, \Delta v\rangle + \alpha \int_{Q_T} \Delta u^\alpha \Delta v = \int_{Q_T} f v, \quad \forall v \in L^\infty\big(0,T;H^2_0(\Omega)\big) \cap \mathscr{V}_\infty,$$

$$u^\alpha = \tfrac{\partial u^\alpha}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega \times (0,T), \quad u^\alpha(0) = h \text{ in } \Omega,$$

$$|\Delta u^\alpha| \leqslant g \text{ in } Q_T, \quad \lambda^\alpha \geqslant 0, \quad \text{and} \quad \lambda^\alpha(|\Delta u^\alpha| - g) = 0 \text{ in } L^\infty(Q_T)'$$

has a solution

$$(\lambda^\alpha, u^\alpha) \in L^\infty(Q_T)' \times \big(L^\infty\big(0,T;H^2_0(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big) \cap \mathscr{V}_\infty\big),$$

with u^α solving uniquely the corresponding variational inequality (2.4).

Similarly, if we choose instead $V = H^2(\Omega) \cap H^1_0(\Omega)$ and assume that $\partial\Omega$ is of class $\mathscr{C}^{1,1}$, since then Δ is also an isomorphism between V and $L^2(\Omega)$, we can also solve the biharmonic Lagrange multiplier problem with Laplacian constraint and various boundary conditions.

3.3. Two problems with the curl constraint. Let $d = 3$ and set

$$\boldsymbol{V} = \big\{\boldsymbol{v} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{v} \in L^2(\Omega)^3,\ \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v} \cdot \boldsymbol{n}_{|\partial\Omega} = 0\big\}$$

or

$$\boldsymbol{V} = \big\{\boldsymbol{v} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{v} \in L^2(\Omega)^3,\ \nabla \cdot \boldsymbol{v} = 0,\ \boldsymbol{v} \times \boldsymbol{n}_{|\partial\Omega} = \boldsymbol{0}\big\}.$$

Here $L = \nabla\times$ is the curl operator,

$$\boldsymbol{H} = \big\{\boldsymbol{v} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{v} \in L^2(\Omega)^3,\ \nabla \cdot \boldsymbol{v} = 0\big\},$$
$$\boldsymbol{W} = \big\{\boldsymbol{v} \in L^2(\Omega)^3 : \nabla \times \boldsymbol{v} \in L^2(\Omega)^3\big\}$$

and the two possible choices of $\boldsymbol{V}$ are related to the boundary conditions. In both cases, $\boldsymbol{V}$ is closed in $H^1(\Omega)^3$. Next, the seminorm $\|\nabla \times \cdot\|_{L^2(\Omega)^3}$ is equivalent to the norm induced in $\boldsymbol{V}$ by the usual norm in $H^1(\Omega)^3$ (for details see [4]). Therefore, as a consequence of Theorems 2.1, 2.2, and 2.3 we have the following statement.

Corollary 3.3. *If* $\boldsymbol{f} \in L^2(Q_T)^3$, $g \in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$ *with* $g \geqslant g_* > 0$, *and* $\boldsymbol{h} \in \mathbb{K}_{g(0)}$, *then the following Lagrange multiplier problem* (*with* $\alpha \geqslant 0$):

$$\int\limits_{Q_T}\partial_t\boldsymbol{u}^\alpha\cdot\boldsymbol{v}+\langle\lambda^\alpha,\nabla\times\boldsymbol{u}^\alpha\cdot\nabla\times\boldsymbol{v}\rangle+\alpha\int\limits_{Q_T}\nabla\times\boldsymbol{u}^\alpha\cdot\nabla\times\boldsymbol{v}=\int\limits_{Q_T}f\cdot\boldsymbol{v},\quad \boldsymbol{v}\in\mathscr{V}_\infty,$$

$$\nabla\cdot\boldsymbol{u}^{\boldsymbol{\alpha}}=0 \text{ in } Q_T,\quad \boldsymbol{u}^\alpha\cdot\boldsymbol{n}=0 \text{ or } \boldsymbol{u}^\alpha\times\boldsymbol{n}=\boldsymbol{0} \text{ on } \partial\Omega\times(0,T),\quad \boldsymbol{u}^\alpha(0)=\boldsymbol{h} \text{ in } \Omega,$$

$$|\nabla\times\boldsymbol{u}^\alpha|\leqslant g \text{ in } Q_T,\quad \lambda^\alpha\geqslant 0 \text{ and } \lambda^\alpha(|\nabla\times\boldsymbol{u}^\alpha|-g)=0 \text{ in } L^\infty(Q_T)'$$

has a solution $(\lambda^\alpha,\boldsymbol{u}^\alpha)\in L^\infty(Q_T)'\times\big(\mathscr{V}_\infty\cap H^1\big(0,T;L^2(\Omega)^3\big)\big)$, *with* $\boldsymbol{u}^\alpha$ *solving uniquely the corresponding variational inequality* (2.4).

This curl constraint is related to type-II superconductivity models, where the region $\{|\nabla\times\boldsymbol{u}^\alpha|=g\}$ corresponds to the critical state of the superconductor (see [12] and [9]).

3.4. Stokes flow for thick fluids. Let $d=2,3$ and $L=D$, where

$$D\boldsymbol{u}=\tfrac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$$

and so $\boldsymbol{W}=\big\{\boldsymbol{v}\in L^2(\Omega)^d:\, D\boldsymbol{v}\in L^2(\Omega)^{d^2}\big\}$.

Put $\boldsymbol{V}=\overline{\mathbb{J}}^{H^1(\Omega)^d}$, where $\mathbb{J}=\big\{\boldsymbol{v}\in\mathscr{C}_0^\infty(\Omega)^d:\nabla\cdot\boldsymbol{v}=0\big\}$. By using Korn's inequality, it is well known that $\|D\boldsymbol{u}\|_{L^2(\Omega)^d}$ is a norm in $\boldsymbol{V}$ equivalent to the norm of $H^1(\Omega)^d$.

Hence, as a consequence of Theorems 2.1, 2.2, and 2.3, we obtain the existence of a generalized Lagrange multiplier for the Stokes flow of a thick fluid with viscosity $\alpha\geqslant 0$ considered in [13].

Corollary 3.4. *If* $\boldsymbol{f}\in L^2(Q_T)^d$, $g\in W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$ *with* $g\geqslant g_*>0$, *and* $\boldsymbol{h}\in\mathbb{K}_{g(0)}$, *then the following Lagrange multiplier problem* (*with* $\alpha\geqslant 0$):

$$\int\limits_{Q_T}\partial_t\boldsymbol{u}^\alpha\cdot\boldsymbol{v}+\langle\lambda^\alpha,D\boldsymbol{u}^\alpha\cdot D\boldsymbol{v}\rangle+\alpha\int\limits_{Q_T}D\boldsymbol{u}^\alpha\cdot D\boldsymbol{v}=\int\limits_{Q_T}\boldsymbol{f}\cdot\boldsymbol{v},\quad \boldsymbol{v}\in\mathscr{V}_\infty,$$

$$\nabla\cdot\boldsymbol{u}^{\boldsymbol{\alpha}}=0 \text{ in } Q_T,\quad \boldsymbol{u}^\alpha=\boldsymbol{0} \text{ on } \partial\Omega\times(0,T),\quad \boldsymbol{u}^\alpha(0)=\boldsymbol{h} \text{ in } \Omega,$$

$$|D\boldsymbol{u}^\alpha|\leqslant g \text{ in } Q_T,\quad \lambda^\alpha\geqslant 0, \text{ and } \lambda^\alpha(|D\boldsymbol{u}^\alpha|-g)=0 \text{ in } L^\infty(Q_T)'$$

has a solution $(\lambda^\alpha,\boldsymbol{u}^\alpha)\in L^\infty(Q_T)'\times\big(\mathscr{V}_\infty\cap H^1\big(0,T;L^2(\Omega)^{d^2}\big)\big)$, *with* $\boldsymbol{u}^\alpha$ *solving uniquely the corresponding variational inequality* (2.4).

3.5. Constraint on first order vector fields. Suppose that Ω is connected with a $\mathscr{C}^\infty$ boundary and $L=(X_1,\dots,X_\ell)=\boldsymbol{X}$ is a family of Lipschitz vector fields on $\mathbb{R}^d$ that connect the space. We shall assume that the structure of L

supports the Sobolev–Poincaré embedding $V \hookrightarrow L^2(\Omega)$, when V is the closure of $\mathscr{D}(\Omega)$ in

$$W = \{v \in L^2(\Omega) : X_j v \in L^2(\Omega),\ j = 1, \dots, \ell\},$$

with the graph norm.

As an example, we have an Hörmander operator

$$X_j = \sum_{i=1}^{d} \gamma_{ij} \partial_{x_i}, \qquad j = 1, \dots, \ell,$$

with $\gamma_{ij} \in \mathscr{C}^\infty(\overline{\Omega})$ such that the Lie algebra generated by these ℓ vector fields has dimension d (see [5, 6]). For other classes of vector fields, namely, those associated with degenerate subelliptic operators, see for instance [3, 6].

As a consequence of Theorems 2.1, 2.2, and 2.3, we have the following.

Corollary 3.5. *Under the above assumptions, if*

$$f \in L^2(Q_T), \quad g \in W^{1,\infty}\big(0,T; L^\infty(\Omega)\big)$$

with $g \geqslant g_* > 0$, *and* $h \in \mathbb{K}_{g(0)}$, *then the Lagrange multiplier problem* (*with* $\alpha \geqslant 0$)

$$\int_{Q_T} \partial_t u^\alpha v + \langle \lambda^\alpha, \boldsymbol{X} u^\alpha \cdot \boldsymbol{X} v\rangle + \alpha \int_{Q_T} \boldsymbol{X} u^\alpha \cdot \boldsymbol{X} v = \int_{Q_T} f v,$$
$$v \in \mathscr{V}_\infty, \quad u^\alpha(0) = h \ in\ \Omega,$$
$$|\boldsymbol{X} u^\alpha| \leqslant g \ in\ Q_T, \quad \lambda^\alpha \geqslant 0 \quad and \quad \lambda^\alpha(|\boldsymbol{X} u^\alpha| - g) = 0 \ in\ L^\infty(Q_T)'$$

has a solution $(\lambda^\alpha, u^\alpha) \in L^\infty(Q_T)' \times \big(\mathscr{V}_\infty \cap H^1\big(0,T; L^2(\Omega)\big)\big)$, *with* u^α *solving uniquely the corresponding variational inequality* (2.4). □

§4. Existence of Lagrange multipliers

For $0 < \varepsilon < 1$, we consider the continuous function

$$k_\varepsilon(s) = \begin{cases} 0 & \text{if } s \leqslant 0; \\ e^{\frac{s}{\varepsilon}} - 1 & \text{if } 0 < s \leqslant \frac{1}{\varepsilon}; \\ e^{\frac{1}{\varepsilon^2}} - 1 & \text{if } s > \frac{1}{\varepsilon}. \end{cases}$$

By applying a general result for evolution quasilinear operators of monotone type (see, for instance, [15, Theorem 8.9, p. 224 or Theorem 8.30, p. 243]), we have the following result.

Proposition 4.1. *Under the assumptions* (2.1)–(2.3), *the problem*

$$\begin{cases} \langle \partial_t \boldsymbol{u}^{\varepsilon\alpha}(t), \boldsymbol{v}\rangle_{\boldsymbol{V}'\times\boldsymbol{V}} + \displaystyle\int_\Omega \big(k_\varepsilon(|\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}(t)|^2 - g^2(t)) + \alpha\big)\, \boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}(t)\cdot \boldsymbol{L}\boldsymbol{v} \\ \qquad\qquad\qquad\qquad = \displaystyle\int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v} \quad \forall\, \boldsymbol{v}\in\boldsymbol{V}, \quad \forall t\in(0,T], \\ \boldsymbol{u}^{\varepsilon\alpha}(0) = \boldsymbol{h} \end{cases} \tag{4.1}$$

has a unique solution $\boldsymbol{u}^{\varepsilon\alpha}\in\mathscr{V}$ *such that* $\partial_t\boldsymbol{u}^{\varepsilon\alpha}\in\mathscr{V}' = L^2(0,T;\boldsymbol{V}')$ *and so* $\boldsymbol{u}^{\varepsilon\alpha}\in\mathscr{C}([0,T];L^2(\Omega)^m)$.

From now on, we denote $k_\varepsilon(|\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}|^2 - g^2)$ by $\widehat{k}_{\varepsilon\alpha}$.

Lemma 4.1. *Under the assumptions* (2.1)–(2.3), *there exists* $C>0$, *independent of* ε *and* α, *such that*:

$$\|\boldsymbol{u}^{\varepsilon\alpha}\|_{L^\infty(0,T,L^2(\Omega)^m)} \leqslant C, \tag{4.2}$$

$$\|\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}\|_{L^2(0,T,L^2(\Omega)^l)} \leqslant \frac{C}{\alpha}, \tag{4.3}$$

$$\|\widehat{k}_{\varepsilon\alpha}|\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}|^2\|_{L^1(Q_T)} \leqslant C, \tag{4.4}$$

$$\|\widehat{k}_{\varepsilon\alpha}\|_{L^1(Q_T)} \leqslant C, \tag{4.5}$$

$$\|\widehat{k}_{\varepsilon\alpha}\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}\|_{\left(L^\infty(Q_T)^\ell\right)'} \leqslant C, \tag{4.6}$$

$$\|\partial_t\boldsymbol{u}^{\varepsilon\alpha}\|_{L^2(Q_T)^m} \leqslant C. \tag{4.7}$$

Proof. Using $\boldsymbol{u}^{\varepsilon\alpha}$ as a test function in (4.1), we have

$$\tfrac12\int_\Omega |\boldsymbol{u}^{\varepsilon\alpha}(t)|^2 + \int_{Q_t}\Big(\widehat{k}_{\varepsilon\alpha}+\alpha\Big)|\boldsymbol{L}\boldsymbol{u}^{\varepsilon\alpha}|^2 \leqslant \tfrac12\int_{Q_t}|f|^2 + \tfrac12\int_{Q_t}|\boldsymbol{u}^{\varepsilon\alpha}|^2 + \tfrac12\int_\Omega|\boldsymbol{h}|^2 \tag{4.8}$$

from where we obtain

$$\int_\Omega |\boldsymbol{u}^{\varepsilon\alpha}(t)|^2 \leqslant \|\boldsymbol{f}\|^2_{L^2(Q_T)^m} + \|\boldsymbol{h}\|^2_{L^2(\Omega)^m} + \int_{Q_t}|\boldsymbol{u}^{\varepsilon\alpha}|^2$$

which shows, by Grönwall's inequality, that

$$\int_\Omega |\boldsymbol{u}^{\varepsilon\alpha}(t)|^2 \leqslant \Big(\|\boldsymbol{f}\|_{L^2(Q_T)^m} + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)}\Big)e^T,$$

proving (4.2).

To prove (4.3) and (4.4), we go back to (4.8) getting

$$\alpha\int_{Q_t}|L\boldsymbol{u}^{\varepsilon\alpha}|^2\leqslant\frac{1}{2}\left(\|\boldsymbol{f}\|^2_{L^2(Q_T)^m}+\||\boldsymbol{h}\|^2_{L^2(\Omega)}\right)(Te^T+1),$$

$$\int_{Q_t}\widehat{k}_{\varepsilon\alpha}|L\boldsymbol{u}^{\varepsilon\alpha}|^2\leqslant\frac{1}{2}\left(\|\boldsymbol{f}\|^2_{L^2(Q_T)^m}+\||\boldsymbol{h}\|^2_{L^2(\Omega)}\right)(Te^T+1)$$

and, observing that $\widehat{k}_{\varepsilon\alpha}|L\boldsymbol{u}^{\varepsilon\alpha}|^2\geqslant\widehat{k}_{\varepsilon\alpha}g^2\geqslant\widehat{k}_{\varepsilon\alpha}g_*^2$, we obtain (4.5).

Note that if $\boldsymbol{\zeta}\in\left(L^\infty(Q_T)\right)^\ell$, then

$$\begin{aligned}\left|\int_{Q_t}\widehat{k}_{\varepsilon\alpha}L\boldsymbol{u}^{\varepsilon\alpha}\cdot\boldsymbol{\zeta}\right|&\leqslant\left(\int_{Q_t}\widehat{k}_{\varepsilon\alpha}^{\frac12}\,|L\boldsymbol{u}^{\varepsilon\alpha}|\,\widehat{k}_{\varepsilon\alpha}^{\frac12}\right)\|\boldsymbol{\zeta}\|_{(L^\infty(Q_T))^\ell}\\&\leqslant\|\widehat{k}_{\varepsilon\alpha}|L\boldsymbol{u}^{\varepsilon\alpha}|^2\|^{\frac12}_{L^1(Q_T)}\|\widehat{k}_{\varepsilon\alpha}\|^{\frac12}_{L^1(Q_T)}\|\boldsymbol{\zeta}\|_{(L^\infty(Q_T))^\ell}\end{aligned}$$

and (4.6) follows from (4.4) and (4.5).

Using the Galerkin approximation, we can take $\partial_t\boldsymbol{u}^{\varepsilon\alpha}$ formally as a test function in (4.1) to obtain

$$\int_{Q_t}|\partial_t\boldsymbol{u}^{\varepsilon\alpha}|^2+\tfrac12\int_{Q_t}(\widehat{k}_{\varepsilon\alpha}+\alpha)\partial_t|L\boldsymbol{u}^{\varepsilon\alpha}|^2\leqslant\tfrac12\int_{Q_t}|f|^2+\tfrac12\int_{Q_t}|\partial_t\boldsymbol{u}^{\varepsilon\alpha}|^2,$$

and hence

$$\int_{Q_t}|\partial_t\boldsymbol{u}^{\varepsilon\alpha}|^2+\int_{Q_t}\widehat{k}_{\varepsilon\alpha}\,\partial_t|L\boldsymbol{u}^{\varepsilon\alpha}|^2+\alpha\int_\Omega|L\boldsymbol{u}^{\varepsilon\alpha}(t)|^2\leqslant\int_{Q_t}|f|^2+\alpha\int_\Omega|L\boldsymbol{h}|^2.$$

Consequently

$$\begin{aligned}\int_{Q_t}|\partial_t\boldsymbol{u}^{\varepsilon\alpha}|^2+\int_{Q_t}\widehat{k}_{\varepsilon\alpha}\,\partial_t\left(|L\boldsymbol{u}^{\varepsilon\alpha}|^2-g^2\right)+\alpha\int_\Omega|L\boldsymbol{u}^{\varepsilon\alpha}(t)|^2&\\\leqslant\int_{Q_t}|f|^2+\alpha\int_\Omega|L\boldsymbol{h}|^2-2\int_{Q_t}\widehat{k}_{\varepsilon\alpha}\,g\,\partial_t g,&\end{aligned}$$

$$\begin{aligned}\int_{Q_t}\widehat{k}_{\varepsilon\alpha}\,\partial_t\left(|L\boldsymbol{u}^{\varepsilon\alpha}|^2-g^2\right)&=\int_{Q_t}\partial_t\left[\phi_\varepsilon\left(|L\boldsymbol{u}^{\varepsilon\alpha}|^2-g^2\right)\right]\\&=\int_\Omega\phi_\varepsilon\left(|L\boldsymbol{u}^{\varepsilon\alpha}(t)|^2-g^2(t)\right)\geqslant0,\end{aligned}$$

where we set $\phi_\varepsilon(s) = \int_0^s k_\varepsilon(\tau)\,d\tau$ and, since $\phi_\varepsilon(|L\boldsymbol{h}|^2 - g^2) = 0$, we have (4.7) from

$$
\begin{aligned}
&\|\partial_t \boldsymbol{u}^{\varepsilon\alpha}\|^2_{L^2(Q_T)^m} \\
&\leqslant \|f\|^2_{L^2(Q_T)^m} + \alpha\|L\boldsymbol{h}\|^2_{L^2(\Omega)^l} + 2\,\|\widehat{k}_{\varepsilon\alpha}\|_{L^1(Q_T)}\|g\|_{L^\infty(Q_T)}\|\partial_t g\|_{L^\infty(Q_T)}. \qquad \square
\end{aligned}
$$

As a consequence of this lemma, we see that there exists a subsequence of $\{\boldsymbol{u}^{\varepsilon\alpha}\}_\varepsilon$ converging to a function $\boldsymbol{u}^\alpha \in \mathscr{V} \cap H^1\big(0,T;L^2(\Omega)^m\big)$ such that

$$
\begin{aligned}
&\boldsymbol{u}^{\varepsilon\alpha} \underset{\varepsilon\to 0}{\rightharpoonup} \boldsymbol{u}^\alpha \quad \text{in} \quad L^\infty\big(0,T,L^2(\Omega)^m\big) \text{ weak-}*, \\
&L\boldsymbol{u}^{\varepsilon\alpha} \underset{\varepsilon\to 0}{\rightharpoonup} L\boldsymbol{u}^\alpha \quad \text{in} \quad L^2(Q_T)^\ell, \\
&\widehat{k}_{\varepsilon\alpha} \underset{\varepsilon\to 0}{\rightharpoonup} \lambda^\alpha \quad \text{in} \quad (L^\infty(Q_T))', \\
&\widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha} \underset{\varepsilon\to 0}{\rightharpoonup} \Lambda^\alpha \quad \text{in} \quad \Big(L^\infty(Q_T)^\ell\Big)', \\
&\partial_t\boldsymbol{u}^{\varepsilon\alpha} \underset{\varepsilon\to 0}{\rightharpoonup} \partial_t\boldsymbol{u}^\alpha \quad \text{in} \quad L^2(Q_T)^m.
\end{aligned} \tag{4.9}
$$

Lemma 4.2. *Under the assumptions* (2.1)–(2.3), *we have* $\boldsymbol{u}^\alpha \in \mathbb{K}_g$.

Proof. Consider

$$
\begin{aligned}
A_\varepsilon &= \big\{(x,t) \in Q_T : 0 \leqslant |L\boldsymbol{u}^{\varepsilon\alpha}(x,t)|^2 - g^2(x,t) < \sqrt{\varepsilon}\big\}, \\
B_\varepsilon &= \big\{(x,t) \in Q_T : \sqrt{\varepsilon} \leqslant |L\boldsymbol{u}^{\varepsilon\alpha}(x,t)|^2 - g^2(x,t)\big\}.
\end{aligned}
$$

Then

$$
|B_\varepsilon| = \int_{B_\varepsilon} 1 \leqslant \int_{B_\varepsilon} \frac{\widehat{k}_{\varepsilon\alpha}}{k_\varepsilon(\sqrt{\varepsilon})} \leqslant \frac{1}{e^{\frac{1}{\sqrt{\varepsilon}}} - 1} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} \leqslant \frac{C}{e^{\frac{1}{\sqrt{\varepsilon}}} - 1}
$$

and

$$
\begin{aligned}
\int_{Q_T} \big(|L\boldsymbol{u}^{\varepsilon\alpha}| - g\big)^+ &= \int_{A_\varepsilon} \big(|L\boldsymbol{u}^{\varepsilon\alpha}| - g\big) + \int_{B_\varepsilon} \big(|L\boldsymbol{u}^{\varepsilon\alpha}| - g\big) \\
&\leqslant \int_{A_\varepsilon} \frac{1}{g_*}\big(|L\boldsymbol{u}^{\varepsilon\alpha}|^2 - g^2\big) + \int_{B_\varepsilon} \big(|L\boldsymbol{u}^{\varepsilon\alpha}| - g\big) \\
&\leqslant \frac{|Q_T|}{g_*}\sqrt{\varepsilon} + |B_\varepsilon|^{\frac{1}{2}}\ \big\||L\boldsymbol{u}^{\varepsilon\alpha}| - g\big\|^{\frac{1}{2}}_{L^2(Q_T)} \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}
$$

Since $\boldsymbol{\xi} \mapsto \big(|\boldsymbol{\xi}| - g\big)^+$ is a convex function, by the lower semicontinuity we have

$$\int_{Q_T} \big(|L\boldsymbol{u}^\alpha| - g\big)^+ \leqslant \varliminf_{\varepsilon\to 0} \int_{Q_T} \big(|L\boldsymbol{u}^{\varepsilon\alpha}| - g\big)^+ = 0$$

and we may conclude that $|L\boldsymbol{u}^\alpha| \leqslant g$ a.e. in Q_T. $\square$

Proof of Theorem 2.1. We will denote by $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ the duality pairing between $\big(L^\infty(Q_T)^\ell\big)'$ and $L^\infty(Q_T)^\ell$ and by $\langle \cdot\,,\,\cdot \rangle$ the duality pairing between $L^\infty(Q_T)'$ and $L^\infty(Q_T)$.

First we prove that

$$\langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^\alpha \rangle\!\rangle = \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle. \tag{4.10}$$

From

$$\int_{Q_T} \partial_t \boldsymbol{u}^{\varepsilon\alpha} \cdot \boldsymbol{v} + \int_{Q_T} \widehat{k}_{\varepsilon\alpha}\, L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{v} + \alpha \int_{Q_T} L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{v} = \int_{Q_T} \boldsymbol{f}\boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathscr{V} \supset \mathscr{V}_\infty, \tag{4.11}$$

using $\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha$ as a test function, we obtain

$$\begin{aligned} \int_{Q_T} \partial_t \boldsymbol{u}^{\varepsilon\alpha} \cdot \boldsymbol{u}^{\varepsilon\alpha} - \int_{Q_T} \partial_t \boldsymbol{u}^{\varepsilon\alpha} \cdot \boldsymbol{u}^\alpha + \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 - \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{u}^\alpha \\ + \alpha \int_{Q_T} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 - \alpha \int_{Q_T} L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{u}^\alpha = \int_{Q_T} \boldsymbol{f} \cdot (\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha). \end{aligned} \tag{4.12}$$

But, since (4.9) implies $\boldsymbol{u}^\alpha(0) = \boldsymbol{h}$,we have

$$\begin{aligned} \int_{Q_T} \partial_t \boldsymbol{u}^\alpha \cdot \boldsymbol{u}^\alpha &= \tfrac12 \int_\Omega |\boldsymbol{u}^\alpha(T)|^2 - \tfrac12 \int_\Omega |\boldsymbol{h}|^2 \\ &\leqslant \varliminf_{\varepsilon\to 0} \tfrac12 \int_\Omega |\boldsymbol{u}^{\varepsilon\alpha}(T)|^2 - \tfrac12 \int_\Omega |\boldsymbol{h}|^2 = \varliminf_{\varepsilon\to 0} \int_{Q_T} \partial_t \boldsymbol{u}^{\varepsilon\alpha} \cdot \boldsymbol{u}^{\varepsilon\alpha}, \end{aligned}$$

$$\int_{Q_T} |L\boldsymbol{u}^\alpha|^2 \leqslant \varliminf_{\varepsilon\to 0} \int_{Q_T} |L\boldsymbol{u}^{\varepsilon\alpha}|^2$$

and therefore

$$\varlimsup_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 \leqslant \langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^\alpha \rangle\!\rangle.$$

Then we also have

$$
\begin{aligned}
0 &\leqslant \varlimsup_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^{\alpha})|^2 \\
&= \varlimsup_{\varepsilon\to 0} \Big(\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 - 2\int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{u}^{\alpha} + \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\alpha}|^2 \Big) \\
&\leqslant -\langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^\alpha \rangle\!\rangle + \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle.
\end{aligned}
\tag{4.13}
$$

Since, by the definition of $\widehat{k}_{\varepsilon\alpha}$,

$$
\widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 \geqslant \widehat{k}_{\varepsilon\alpha} g^2, \tag{4.14}
$$

also

$$
\langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle \leqslant \langle \lambda^\alpha, g^2 \rangle = \lim_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} g^2 \leqslant \varlimsup_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 \leqslant \langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^\alpha \rangle\!\rangle,
$$

proving (4.10).

We can rewrite (4.12) as follows:

$$
\begin{aligned}
&\int_{Q_T} \partial_t(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha) \cdot (\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha) + \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2 + \alpha \int_{Q_T} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2 \\
&\qquad = \int_{Q_T} \boldsymbol{f} \cdot (\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha) - \int_{Q_T} \partial_t \boldsymbol{u}^\alpha \cdot (\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha) \\
&\qquad\qquad - \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha) - \alpha \int_{Q_T} L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha),
\end{aligned}
$$

obtaining

$$
0 \leqslant \alpha \varlimsup_{\varepsilon\to 0} \int_{Q_T} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2 \leqslant -\langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^\alpha \rangle\!\rangle + \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle = 0,
$$

which yields the strong convergence of $L\boldsymbol{u}^{\varepsilon\alpha}$ to $L\boldsymbol{u}^\alpha$ in $L^2(Q_T)^\ell$. Consequently, we also have

$$
\lim_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2 = 0. \tag{4.15}
$$

So, for any $\boldsymbol{v} \in \mathscr{V}_\infty$ we also have

$$\Big|\int_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)\cdot L\boldsymbol{v}\Big| \leqslant \Big(\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2\Big)^{\frac12} \|\widehat{k}_{\varepsilon\alpha}\|_{L^1(Q_T)} \|L\boldsymbol{v}\|_{L^\infty(Q_T)^\ell} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus, since

$$\int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha}\cdot L\boldsymbol{v} = \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)\cdot L\boldsymbol{v} + \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\alpha}\cdot L\boldsymbol{v} \underset{\varepsilon\to 0}{\longrightarrow} \langle \lambda^\alpha, L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v}\rangle,$$

letting ε tend to zero in (4.11) with $\boldsymbol{v} \in \mathscr{V}_\infty$, we obtain (2.5a).

Given $\zeta \in L^\infty(Q_T)$, let $\zeta^+ = \max\{\zeta, 0\}$ and $\zeta^- = \max\{-\zeta, 0\}$. Observing that in

$$\begin{aligned}\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2\zeta^\pm &= \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^\alpha)|^2\zeta^\pm \\ &\quad + 2\int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha}\cdot L\boldsymbol{u}^\alpha\zeta^\pm - \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\alpha}|^2\zeta^\pm,\end{aligned} \tag{4.16}$$

the first term of the second member vanishes as $\varepsilon \to 0$, by (4.15), and using (4.10) we have

$$\lim_{\varepsilon\to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2\zeta^\pm = \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2\zeta^\pm\rangle. \tag{4.17}$$

Using this observation, inequality (4.14) and Lemma 4.2, we find

$$\begin{aligned}\langle \lambda^\alpha g^2, \zeta^\pm\rangle = \overline{\lim_{\varepsilon\to 0}} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} g^2\zeta^\pm &\leqslant \lim_{\varepsilon\to 0}\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2\zeta^\pm \\ &\leqslant \langle \lambda^\alpha |L\boldsymbol{u}^\alpha|^2, \zeta^\pm\rangle \leqslant \langle \lambda^\alpha g^2, \zeta^\pm\rangle\end{aligned}$$

which proves that $\lambda^\alpha(|L\boldsymbol{u}^\alpha|^2 - g^2) = 0$ in $L^\infty(Q_T)'$ and, consequently, also $\lambda^\alpha(|L\boldsymbol{u}^\alpha| - g) = 0$ in $L^\infty(Q_T)'$, because $|L\boldsymbol{u}^\alpha| + g \geqslant g_* > 0$.

Since $\lambda^\alpha \geqslant 0$ is immediate and since $\boldsymbol{u}^\alpha(t) \in \mathbb{K}_{g(t)}$ for almost all $t \in [0,T]$, we have $\boldsymbol{u}^\alpha \in \mathscr{V}_\infty$. Recalling (4.11), it remains to show that, for $\alpha > 0$,

$$\langle\!\langle \Lambda^\alpha, L\boldsymbol{v}\rangle\!\rangle = \langle \lambda^\alpha, L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v}\rangle, \quad \boldsymbol{v} \in \mathscr{V}_\infty. \tag{4.18}$$

But

$$
\begin{aligned}
\langle\!\langle \Lambda^\alpha, L\boldsymbol{v}\rangle\!\rangle &= \lim_{\varepsilon\to 0}\int\limits_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha}\cdot Lv \\
&= \lim_{\varepsilon\to 0}\int\limits_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha}-\boldsymbol{u}^\alpha)\cdot Lv + \lim_{\varepsilon\to 0}\int\limits_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v} = \langle \lambda^\alpha, L\boldsymbol{u}^\alpha\cdot L\boldsymbol{v}\rangle,
\end{aligned}
$$

because

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\Big|\int\limits_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha}-\boldsymbol{u}^\alpha)\cdot L\boldsymbol{v}\Big| \qquad (4.19)\\
&\leqslant \lim_{\varepsilon\to 0}\Big(\int\limits_{Q_T} \widehat{k}_{\varepsilon\alpha}|L(\boldsymbol{u}^{\varepsilon\alpha}-\boldsymbol{u}^\alpha)|^2\Big)^{\frac12}\|\widehat{k}_{\varepsilon\alpha}\|^{\frac12}_{L^1(Q_T)}\|L\boldsymbol{v}\|_{L^\infty(Q_T)^\ell} = 0. \qquad \square
\end{aligned}
$$

Remark 4.1. Note that (4.18), in general, does not imply that the flux Λ^α is given by

$$\Lambda^\alpha = \lambda^\alpha L\boldsymbol{u}^\alpha.$$

It is an interesting open question to characterize when this property occurs beyond the only known example $L=\nabla$ for special cases of g and f as in [17] and already referred to in the Introduction.

Proof of Theorem 2.2. From Lemma 4.2 and estimates (4.5), (4.6), and (4.7), we also have independently of $\alpha>0$:

$$
\begin{aligned}
\|L\boldsymbol{u}^\alpha\|_{L^\infty(Q_T)^\ell} &\leqslant \|g\|_{L^\infty(Q_T)},\\
\|\lambda^\alpha\|_{L^\infty(Q_T)'} &\leqslant \varliminf_{\varepsilon\to 0}\|\widehat{k}_{\varepsilon\alpha}\|_{L^\infty(Q_T)'} \leqslant C,\\
\|\boldsymbol{\Lambda}^\alpha\|_{(L^\infty(Q_T)^\ell)'} &\leqslant \varliminf_{\varepsilon\to 0}\|\widehat{k}_{\varepsilon\alpha}L\boldsymbol{u}^{\varepsilon\alpha}\|_{\boldsymbol{L}^\infty(Q_T)'} \leqslant C,\\
\|\partial_t\boldsymbol{u}^\alpha\|_{L^2(Q_T)^m} &\leqslant \varliminf_{\varepsilon\to 0}\|\partial_t\boldsymbol{u}^{\varepsilon\alpha}\|_{L^2(Q_T)^m} \leqslant C,
\end{aligned}
$$

and therefore $\boldsymbol{u}^\alpha$ is also uniformly bounded in $\mathscr{W}=\big\{\boldsymbol{v}\in\mathscr{V}:\partial_t\boldsymbol{v}\in\mathscr{V}'\big\}$ and so in $\mathscr{C}\big([0,T];L^2(\Omega)\big)^m$. Consequently, there exist $\boldsymbol{u}^0\in\mathscr{V}_\infty$, $\lambda^0\in L^\infty(Q_T)'$ and $\Lambda^0\in\big(L^\infty(Q_T)^\ell\big)'$ such that, for a subsequence, we have

$$
\begin{aligned}
L\boldsymbol{u}^\alpha &\underset{\alpha\to 0}{\rightharpoonup} L\boldsymbol{u}^0 \quad \text{in} \quad L^\infty(Q_T)^\ell \ \text{weak-}*,\\
\lambda^\alpha &\underset{\alpha\to 0}{\rightharpoonup} \lambda^0 \quad \text{in} \quad (L^\infty(Q_T))',\\
\Lambda^\alpha &\underset{\alpha\to 0}{\rightharpoonup} \Lambda^0 \quad \text{in} \quad \Big(L^\infty(Q_T)^\ell\Big)',
\end{aligned}
$$

$$\partial_t \boldsymbol{u}^\alpha \underset{\alpha\to 0}{\rightharpoonup} \partial_t \boldsymbol{u}^0 \quad \text{in} \quad L^2(Q_T)^m,$$
$$\boldsymbol{u}^\alpha(t) \underset{\alpha\to 0}{\rightharpoonup} \boldsymbol{u}^0(t) \quad \text{in} \quad L^2(Q_T)^m \quad \text{for all } t \in [0,T].$$

Letting $\alpha \to 0$ in (2.5a), we obtain

$$\int_{Q_T} \partial_t \boldsymbol{u}^0 \cdot \boldsymbol{v} + \langle\!\langle \Lambda^0, L\boldsymbol{v} \rangle\!\rangle = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{v}, \quad \boldsymbol{v} \in \mathscr{V}_\infty. \tag{4.20}$$

We easily conclude that $\boldsymbol{u}^0(0) = \boldsymbol{h}$ and $\lambda^0 \geqslant 0$. Furthemore, if ω is any measurable subset of Q_T, then

$$\int_\omega |L\boldsymbol{u}^0| \leqslant \varliminf_{\alpha\to 0} \int_\omega |L\boldsymbol{u}^\alpha| \leqslant \int_\omega g,$$

and we conclude that $|L\boldsymbol{u}^0| \leqslant g$ a.e. in Q_T.

Observe that by (2.5) and (4.20) we obtain

$$\begin{aligned} &\varlimsup_{\alpha\to 0} \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle \\ &\quad = \varlimsup_{\alpha\to 0} \Big(\int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^\alpha - \tfrac{1}{2} \int_\Omega \big(|\boldsymbol{u}^\alpha(T)|^2 - |\boldsymbol{h}|^2 \big) - \alpha \int_{Q_T} |L\boldsymbol{u}^\alpha|^2 \Big) \\ &\quad \leqslant \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^0 - \tfrac{1}{2} \int_\Omega \big(|\boldsymbol{u}^0(T)|^2 - |\boldsymbol{h}|^2 \big) \\ &\quad = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{u}^0 - \int_{Q_T} \partial_t \boldsymbol{u}^0 \cdot \boldsymbol{u}^0 = \langle\!\langle \Lambda^0, L\boldsymbol{u}^0 \rangle\!\rangle. \end{aligned} \tag{4.21}$$

Therefore

$$0 = \lim_{\alpha\to 0} \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 - g^2 \rangle \leqslant \langle\!\langle \Lambda^0, L\boldsymbol{u}^0 \rangle\!\rangle - \langle \lambda^0, g^2 \rangle \leqslant \langle\!\langle \Lambda^0, L\boldsymbol{u}^0 \rangle\!\rangle - \langle \lambda^0, |L\boldsymbol{u}^0|^2 \rangle.$$

On the other hand, using (4.18), we have

$$\begin{aligned} 0 &\leqslant \varlimsup_{\alpha\to 0} \langle \lambda^\alpha, |L(\boldsymbol{u}^\alpha - \boldsymbol{u}^0)|^2 \rangle \\ &= \varlimsup_{\alpha\to 0} \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle - \lim_{\alpha\to 0} 2\langle \lambda^\alpha, L\boldsymbol{u}^\alpha \cdot L\boldsymbol{u}^0 \rangle + \lim_{\alpha\to 0} \langle \lambda^\alpha, |L\boldsymbol{u}^0|^2 \rangle \\ &\leqslant -\langle\!\langle \Lambda^0, L\boldsymbol{u}^0 \rangle\!\rangle + \langle \lambda^0, |L\boldsymbol{u}^0|^2 \rangle. \end{aligned} \tag{4.22}$$

and then

$$\langle \lambda^0, |L\boldsymbol{u}^0|^2 \rangle = \langle \lambda^0, g^2 \rangle = \langle\!\langle \Lambda^0, L\boldsymbol{u}^0 \rangle\!\rangle = \lim_{\alpha\to 0} \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle. \tag{4.23}$$

Letting $\alpha \to 0$ in (2.5a), for $\alpha > 0$, written in the form

$$\int_{Q_T} \partial_t \boldsymbol{u}^\alpha \cdot \boldsymbol{v} + \langle \lambda^\alpha, L(\boldsymbol{u}^\alpha - \boldsymbol{u}^0) \cdot L\boldsymbol{v} \rangle + \langle \lambda^\alpha, L\boldsymbol{u}^0 \cdot L\boldsymbol{v} \rangle + \alpha \int_{Q_T} L\boldsymbol{u}^\alpha \cdot L\boldsymbol{v} = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{v},$$

we obtain

$$\int_{Q_T} \partial_t \boldsymbol{u}^0 \cdot \boldsymbol{v} + \langle \lambda^0, L\boldsymbol{u}^0 \cdot L\boldsymbol{v} \rangle = \int_{Q_T} \boldsymbol{f} \cdot \boldsymbol{v}, \quad \boldsymbol{v} \in \mathscr{V}_\infty,$$

and therefore $(\lambda^0, \boldsymbol{u}^0)$ will solve (2.5a) for $\alpha = 0$, provided we can show that

$$\lim_{\alpha \to 0} \langle \lambda^\alpha, L(\boldsymbol{u}^\alpha - \boldsymbol{u}^0) \cdot L\boldsymbol{v} \rangle = 0, \quad \boldsymbol{v} \in \mathscr{V}_\infty. \tag{4.24}$$

We start by observing that, for fixed $\alpha > 0$,

$$\lim_{\varepsilon \to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^0) \cdot L\boldsymbol{v} = \langle\!\langle \Lambda^\alpha, L\boldsymbol{v} \rangle\!\rangle - \langle \lambda^\alpha, L\boldsymbol{u}^0 \cdot L\boldsymbol{v} \rangle = \langle \lambda^\alpha, L(\boldsymbol{u}^\alpha - \boldsymbol{u}^0) \cdot L\boldsymbol{v} \rangle$$

and

$$\lim_{\varepsilon \to 0} \left| \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^0) \cdot L\boldsymbol{v} \right| \leqslant \lim_{\varepsilon \to 0} \Big(\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^0)|^2 \Big)^{\frac12} \|\widehat{k}_{\varepsilon\alpha}\|^{\frac12}_{L^1(Q_T)} \|L\boldsymbol{v}\|_{L^\infty(Q_T)^\ell}.$$

Then, using (4.17) and (4.22), we have

$$\begin{aligned}
&\overline{\lim_{\alpha \to 0}} \lim_{\varepsilon \to 0} \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L(\boldsymbol{u}^{\varepsilon\alpha} - \boldsymbol{u}^0)|^2 \\
&\quad = \overline{\lim_{\alpha \to 0}} \lim_{\varepsilon \to 0} \Big(\int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^{\varepsilon\alpha}|^2 - 2 \int_{Q_T} \widehat{k}_{\varepsilon\alpha} L\boldsymbol{u}^{\varepsilon\alpha} \cdot L\boldsymbol{u}^0 + \int_{Q_T} \widehat{k}_{\varepsilon\alpha} |L\boldsymbol{u}^0|^2 \Big) \\
&\quad = \lim_{\alpha \to 0} \big(\langle \lambda^\alpha, |L\boldsymbol{u}^\alpha|^2 \rangle - 2 \langle\!\langle \Lambda^\alpha, L\boldsymbol{u}^0 \rangle\!\rangle + \langle \lambda^\alpha, |L\boldsymbol{u}^0|^2 \rangle \big) = 0.
\end{aligned}$$

Finally, since $\lambda^0 \geqslant 0$ and $\boldsymbol{u}^0 \in \mathbb{K}_g$, we have $\langle \lambda^0, \big(|L\boldsymbol{u}^0| - g\big)\zeta^\pm \rangle \leqslant 0$, for any $\zeta \in L^\infty(Q_T)$. From the inequality $\widehat{k}_{\varepsilon\alpha}\big(|L\boldsymbol{u}^{\varepsilon\alpha}|^2 - g^2\big)\zeta^\pm \geqslant 0$ arguing as in (4.17), we obtain $\langle \lambda^0, \big(|L\boldsymbol{u}^0|^2 - g^2\big)\zeta^\pm \rangle \geqslant 0$ and, afterwards, $\langle \lambda^0, \big(|L\boldsymbol{u}^0| - g\big)\zeta^\pm \rangle \geqslant 0$. Therefore $(\lambda^0, \boldsymbol{u}^0)$ also solves (2.5b) and (2.5c) for $\alpha = 0$. ☐

Proof of Theorem 2.3. First we observe that, given $\boldsymbol{v} \in \mathscr{V}_\infty$ such that $\boldsymbol{v} \in \mathbb{K}_g$, using $\boldsymbol{v} - \boldsymbol{u}^\alpha$ as test function in (2.5), for $\alpha \geqslant 0$, we get

$$\int_{Q_T} \partial_t \boldsymbol{u}^\alpha \cdot (\boldsymbol{v} - \boldsymbol{u}^\alpha) + \langle \lambda^\alpha, L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{v} - \boldsymbol{u}^\alpha) \rangle + \alpha \int_{Q_T} L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{v} - \boldsymbol{u}^\alpha) = \int_{Q_T} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}^\alpha).$$

But using (2.5c), we obtain

$$\begin{aligned}\langle \lambda^\alpha, L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{v}-\boldsymbol{u}^\alpha)\rangle &\leqslant \langle \lambda^\alpha, |L\boldsymbol{u}^\alpha||L\boldsymbol{v}| - |L\boldsymbol{u}^\alpha|^2\rangle \\ &\leqslant \langle \lambda^\alpha(g-|L\boldsymbol{u}^{\boldsymbol{\alpha}}|), |L\boldsymbol{u}^\alpha|\rangle = 0\end{aligned} \tag{4.25}$$

and we obtain the variational inequality

$$\int_{Q_T} \partial_t \boldsymbol{u}^\alpha \cdot (\boldsymbol{v}-\boldsymbol{u}^\alpha) + \alpha \int_{Q_T} L\boldsymbol{u}^\alpha \cdot L(\boldsymbol{v}-\boldsymbol{u}^\alpha) \geqslant \int_{Q_T} \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}^\alpha), \quad \boldsymbol{v}\in\mathbb{K}_g. \tag{4.26}$$

But it is well known (see, for instance, Remark 2.12 of [10]), by using appropriate test functions, that (4.26) is equivalent to (2.4).

The proof of the uniqueness of solution of the variational inequality (2.4) is standard. □

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Поступило 25 апреля 2019 г.

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