Front. Math. China
XXXXXX

Weighted binary relations involving the core-EP inverse

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Abstract In this paper, we study a new binary relation defined on the set of rectangular complex matrices involving the weighted core-EP inverse and give its characterizations. This relation becomes a pre-order. Then, one-sided pre-orders associated to the weighted core-EP inverse are given from two perspectives. Finally, we make a comparison for these two sets of one-sided weighted pre-orders.

Keywords weighted core-EP inverse, core-EP inverse, pseudo core inverse, pre-order
MSC 15A09, 06A06

1 Introduction

A binary relation is a pre-order if it is reflexive and transitive; if it is also antisymmetric, then it is a partial order. The theory of partial orders (pre-orders) based on various generalized inverses has been increasingly investigated, such as, $*$-partial order [2], minus partial order [7, 16], sharp partial order [12], Drazin pre-order [11, 13], core partial order [1, 17, 19] and core-EP pre-order [5, 14, 18]. Meanwhile, weighted Drazin pre-order and one-sided weighted Drazin pre-order were studied by Hernández et al. [8, 9].

Motivated by the above papers, in this paper, our main goal is to study new binary relations defined by the weighted core-EP inverse.

Throughout this paper, $\mathbb{C}^{m\times n}$ is used to denote the set of all $m\times n$ complex matrices. For each complex matrix $A\in\mathbb{C}^{m\times n}$, A^* denotes the conjugate transpose of A, and $\mathscr{R}(A)$ denotes the range of A. The index of $A\in\mathbb{C}^{n\times n}$, denoted by $\mathrm{ind}(A)$, is the smallest non-negative integer k for which $\mathrm{rank}(A^k)=\mathrm{rank}(A^{k+1})$.

Recall that the core-EP inverse was proposed by Manjunatha Prasad and Mohana [10] for a square matrix of arbitrary index, as an extension of the core

Received July 1, 2019; accepted August 1, 2020
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inverse restricted to a square matrix of index at most 1 in [1]. Then, Gao and Chen [4] characterized the core-EP inverse (also known as the pseudo core inverse) in terms of three equations. Let $A \in \mathbb{C}^{n\times n}$ with $\mathrm{ind}(A) = k$, the core-EP inverse of A, denoted by $A^{\textcircled{\dagger}}$, is the unique solution of the system

$$XA^{k+1} = A^k,\ AX^2 = X,\ (AX)^* = AX.$$

The core-EP inverse is an outer inverse (resp. $\{2\}$-inverse), i.e., $A^{\textcircled{\dagger}}AA^{\textcircled{\dagger}} = A^{\textcircled{\dagger}}$, see [4].

Lemma 1. [4] *Let $A \in \mathbb{C}^{n\times n}$ with* $\mathrm{ind}(A) = k$. *Then* $A^{\textcircled{\dagger}} = A^D A^k (A^k)^\dagger$.

In [18], Wang introduced the core-EP pre-order as follows:

Definition 1. [18] *Let $A,\ B \in \mathbb{C}^{n\times n}$. Then $A \preceq^{\textcircled{\dagger}} B$ if $A^{\textcircled{\dagger}}A = A^{\textcircled{\dagger}}B$ and $AA^{\textcircled{\dagger}} = BA^{\textcircled{\dagger}}$.*

An extension of the core-EP inverse from a square matrix to a rectangular matrix was made by Ferreyra et al. [3] and was named the weighted core-EP inverse. Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$ $(W \neq 0)$ with $k = \max\{\mathrm{ind}(AW), \mathrm{ind}(WA)\}$, the W-weighted core-EP inverse $A^{\textcircled{\dagger},W}$ of A is the unique solution of the system

$$WAWX = P_{(WA)^k},\ \mathscr{R}(X) \subseteq \mathscr{R}((AW)^k).$$

Recently, Gao et al. [6] gave more representations of the weighted core-EP inverse of a rectangular complex matrix.

Lemma 2. [6] *Let $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. Then $A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$.*

Besides, Mosić [15] studied the weighted core-EP inverse of an operator between two Hilbert spaces as a generalization of the weighted core-EP inverse of a rectangular matrix. In addition, the author introduced some binary relations on the set of all Wg-Drazin invertible operators between two Hilbert spaces by means of the core-EP inverse of certain generalized Drazin invertible operators.

In this note, we focus on binary relations associated with the W-weighted core-EP inverse of rectangular complex matrices. The paper is organized as follows: in Section 2, a new binary relation $\preceq^{\textcircled{\dagger},W}$ on rectangular matrices is introduced and characterized. In Section 3, two sets of one-sided binary relations corresponding to $\preceq^{\textcircled{\dagger},W}$, namely, $\preceq^{\textcircled{\dagger},W,l}$, $\preceq^{\textcircled{\dagger},W,r}$ and $\preceq^{\textcircled{\dagger},W,L}$, $\preceq^{\textcircled{\dagger},W,R}$ are defined and compared, after which, a relationship diagram of $\preceq^{\textcircled{\dagger},W}$, $\preceq^{\textcircled{\dagger},W,l}$, $\preceq^{\textcircled{\dagger},W,r}$, $\preceq^{\textcircled{\dagger},W,L}$, $\preceq^{\textcircled{\dagger},W,R}$ is provided, and finally we conclude that $\preceq^{\textcircled{\dagger},W}$ is weaker than $\preceq^{\textcircled{d},W}$ defined in [15].

2 A pre-order defined by the weighted core-EP inverse

In this section, we define a new binary relation in terms of the weighted core-EP inverse and then give its characterizations.

Definition 2. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. It is said that $A \preceq^{\textcircled{\dagger},W} B$ if $(AW)A^{\textcircled{\dagger},W} = (BW)A^{\textcircled{\dagger},W}$ and $A^{\textcircled{\dagger},W}(WA) = A^{\textcircled{\dagger},W}(WB)$.*

Remark 1. *In general, $\preceq^{\textcircled{\dagger},W}$ does not preserve the rank function. Indeed, for fixed matrices A, W with WA being nilpotent, then $A \preceq^{\textcircled{\dagger},W} B$ always holds whatever B equals. Thus, the rank of B could be less, equal or greater than rank of A.*

In the following result, we give characterizations of $\preceq^{\textcircled{\dagger},W}$ involving the core-EP inverse.

Theorem 1. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = max\{\mathrm{ind}(WA), \mathrm{ind}(AW)\}$. Then the following conditions are equivalent:*

(a) $A \preceq^{\textcircled{\dagger},W} B$;

(b) $(AW)(A^{\textcircled{\dagger},W}W) = (BW)(A^{\textcircled{\dagger},W}W)$ *and* $(WA^{\textcircled{\dagger},W})(WA) = (WA^{\textcircled{\dagger},W})(WB)$;

(c) $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$ *and* $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$;

(d) $A(WA)^k = B(WA)^k$ *and* $(WA)^*(WA)^k = (WB)^*(WA)^K$;

(e) $A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}$ *and* $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$.

Proof. (a) $\Rightarrow$ (b) It is obvious from Definition 2.

(b) $\Rightarrow$ (c) By applying Lemma 2, $WA^{\textcircled{\dagger},W} = (WA)^{\textcircled{\dagger}}$. Thus, from $(WA^{\textcircled{\dagger},W})(WA) = (WA^{\textcircled{\dagger},W})(WB)$, it is clear that $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$. By Lemmas 1 and 2, as well as the fact that $W(AW)^D = (WA)^DW$, we have

$$\begin{aligned}(A^{\textcircled{\dagger},W}W)(AW)(AW)^D &= A[(WA)^{\textcircled{\dagger}}]^2W(AW)(AW)^D = A[(WA)^{\textcircled{\dagger}}]^2(WA)(WA)^DW\\ &= A[(WA)^{\textcircled{\dagger}}]^2(WA)^{k+2}[(WA)^D]^{k+2}W = A(WA)^k[(WA)^D]^{k+2}W\\ &= A[(WA)^D]^2W = (AW)^D.\end{aligned}$$

Post-multiply $(AW)(A^{\textcircled{\dagger},W}W) = (BW)(A^{\textcircled{\dagger},W}W)$ by $(AW)(AW)^D$, then we derive

$$(AW)(AW)^D = (BW)(AW)^D.$$

Hence, in view of Lemma 1,

$$\begin{aligned}(AW)(AW)^{\textcircled{\dagger}} &= (AW)(AW)^D(AW)^k[(AW)^k]^\dagger = (BW)(AW)^D(AW)^k[(AW)^k]^\dagger\\ &= (BW)(AW)^{\textcircled{\dagger}}.\end{aligned}$$

(c) $\Rightarrow$ (d) Post-multiplying $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$ by $(AW)^kA$, it comes immediately that $A(WA)^k = B(WA)^k$.

Pre-multiply $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$ by WA, then

$$WA(WA)^{\textcircled{\dagger}}(WA) = WA(WA)^{\textcircled{\dagger}}(WB).$$

Thus, $(WA)^*(WA)(WA)^{\textcircled{\dagger}} = (WB)^*(WA)(WA)^{\textcircled{\dagger}}$. Hence,

$$\begin{aligned}(WA)^*(WA)^k &= (WA)^*(WA)(WA)^{\textcircled{\dagger}}(WA)^k = (WB)^*(WA)(WA)^{\textcircled{\dagger}}(WA)^k\\ &= (WB)^*(WA)^k.\end{aligned}$$

(d) $\Rightarrow$ (e) From $A(WA)^k = B(WA)^k$, it follows that

$$A(WA)^{\textcircled{\dagger}} = A(WA)^k[(WA)^{\textcircled{\dagger}}]^{k+1} = B(WA)^k[(WA)^{\textcircled{\dagger}}]^{k+1} = B(WA)^{\textcircled{\dagger}}.$$

Then from $(WA)^*(WA)^k = (WB)^*(WA)^k$, it follows that $[(WA)^k]^*WA = [(WA)^k]^*WB$. Hence, by the definition of the core-EP inverse, we have

$$\begin{aligned}(WA)^{\textcircled{\dagger}}(WA) &= (WA)^{\textcircled{\dagger}}(WA)(WA)^{\textcircled{\dagger}}(WA)\\ &= (WA)^{\textcircled{\dagger}}[(WA)(WA)^{\textcircled{\dagger}}]^*(WA)\\ &= (WA)^{\textcircled{\dagger}}[(WA)^k((WA)^{\textcircled{\dagger}})^k]^*(WA)\\ &= (WA)^{\textcircled{\dagger}}[((WA)^{\textcircled{\dagger}})^k]^*[(WA)^k]^*(WA)\\ &= (WA)^{\textcircled{\dagger}}[((WA)^{\textcircled{\dagger}})^k]^*[(WA)^k]^*(WB)\\ &= (WA)^{\textcircled{\dagger}}(WB).\end{aligned}$$

(e) $\Rightarrow$ (a) Note that $(AW)A^{\textcircled{\dagger},W} = A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}} = (BW)A^{\textcircled{\dagger},W}$ as well as $A^{\textcircled{\dagger},W}(WA) = A[(WA)^{\textcircled{\dagger}}]^2(WA) = A[(WA)^{\textcircled{\dagger}}]^2(WB) = A^{\textcircled{\dagger},W}(WB)$. □

From Theorem 1, it is clear that if $A \preceq^{\textcircled{\dagger},W} B$, then $WA \preceq^{\textcircled{\dagger}} WB$. However the converse is not true in general, see Example 1.

Example 1. *Let* $A = \begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix}1&0\\0&0\\1&0\end{bmatrix} \in \mathbb{C}^{3\times 2}$ *and* $W = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix} \in \mathbb{C}^{2\times 3}$. *Then*

$$WA = \begin{bmatrix}1&0\\0&0\end{bmatrix},\ (WA)^{\textcircled{\dagger}} = \begin{bmatrix}1&0\\0&0\end{bmatrix},\ WB = \begin{bmatrix}1&0\\0&0\end{bmatrix}.$$

Hence $WA \preceq^{\textcircled{\dagger}} WB$. *However* $A(WA)^{\textcircled{\dagger}} \neq B(WA)^{\textcircled{\dagger}}$. *That is to say,* $A \not\preceq^{\textcircled{\dagger},W} B$.

Theorem 2. *The binary relation* $\preceq^{\textcircled{\dagger},W}$ *defined on* $\mathbb{C}^{m\times n}$ *is a pre-order.*

Proof. From Definition 2, it follows immediately that $\preceq^{\textcircled{\dagger},W}$ is reflexive. Now, suppose that $A,\ B,\ C \in \mathbb{C}^{m\times n}$ satisfy $A \preceq^{\textcircled{\dagger},W} B$ and $B \preceq^{\textcircled{\dagger},W} C$. Note that $A \preceq^{\textcircled{\dagger},W} B$ is equivalent to

$$A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}} \text{ and } (WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB) \tag{1}$$

by applying Theorem 1. Likewise, $B \preceq^{\textcircled{\dagger},W} C$ is equivalent to

$$B(WB)^{\textcircled{\dagger}} = C(WB)^{\textcircled{\dagger}} \text{ and } (WB)^{\textcircled{\dagger}}(WB) = (WB)^{\textcircled{\dagger}}(WC). \tag{2}$$

Thus,

$$\begin{aligned}A(WA)^{\textcircled{\dagger}} &\overset{(1)}{=\!=} B(WA)^{\textcircled{\dagger}} = BWA[(WA)^{\textcircled{\dagger}}]^2 = BW[A(WA)^{\textcircled{\dagger}}](WA)^{\textcircled{\dagger}}\\ &\overset{(1)}{=\!=} BW[B(WA)^{\textcircled{\dagger}}](WA)^{\textcircled{\dagger}} = BWB(WA)^{\textcircled{\dagger}}(WA)^{\textcircled{\dagger}}.\end{aligned}$$

Note that $WB(WA)^{\textcircled{\dagger}} = WBWA[(WA)^{\textcircled{\dagger}}]^2 = (WB)^2[(WA)^{\textcircled{\dagger}}]^2 = \cdots = (WB)^k[(WA)^{\textcircled{\dagger}}]^k$, where $k = \max\{\mathrm{ind}(WA), \mathrm{ind}(WB)\}$.
Thus,

$$A(WA)^{\textcircled{\dagger}} = B(WB)^k[(WA)^{\textcircled{\dagger}}]^{k+1} = B(WB)^{\textcircled{\dagger}}(WB)^{k+1}[(WA)^{\textcircled{\dagger}}]^{k+1}.$$

Hence, by (2),

$$\begin{aligned} A(WA)^{\textcircled{\dagger}} &= C(WB)^{\textcircled{\dagger}}(WB)^{k+1}[(WA)^{\textcircled{\dagger}}]^{k+1} = C(WB)^k[(WA)^{\textcircled{\dagger}}]^{k+1} \\ &= C(WA)^{\textcircled{\dagger}}, \text{ because of} \\ (WA)^{\textcircled{\dagger}} &= WA[(WA)^{\textcircled{\dagger}}]^2 \overset{(1)}{=\!=} WB[(WA)^{\textcircled{\dagger}}]^2 = \cdots == (WB)^k[(WA)^{\textcircled{\dagger}}]^{k+1}. \end{aligned}$$

Since $(WA)^{\textcircled{\dagger}} = (WA)^{\textcircled{\dagger}}WA(WA)^{\textcircled{\dagger}} = (WA)^{\textcircled{\dagger}}[WA(WA)^{\textcircled{\dagger}}]^* = (WA)^{\textcircled{\dagger}}[(WB)^k((WA)^{\textcircled{\dagger}})$ $= (WA)^{\textcircled{\dagger}}[(WB)^k((WA)^{\textcircled{\dagger}})^k]^*WB(WB)^{\textcircled{\dagger}} = (WA)^{\textcircled{\dagger}}WB(WB)^{\textcircled{\dagger}}$, then

$$\begin{aligned} (WA)^{\textcircled{\dagger}}(WA) &= (WA)^{\textcircled{\dagger}}WB = (WA)^{\textcircled{\dagger}}WB(WB)^{\textcircled{\dagger}}WB = (WA)^{\textcircled{\dagger}}WB(WB)^{\textcircled{\dagger}}WC \\ &= (WA)^{\textcircled{\dagger}}(WC). \end{aligned}$$

Hence $A \preceq^{\textcircled{\dagger},W} C$ and the transitivity holds. This completes the proof. □

In general, the binary relation $\preceq^{\textcircled{\dagger},W}$ is not antisymmetric as Example 2 shows.

Example 2. *Let* $A = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix} \in \mathbb{C}^{4\times 3}$, $B = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&2 \\ 0&0&0 \end{bmatrix} \in \mathbb{C}^{4\times 3}$ *and* $W = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} \in \mathbb{C}^{3\times 4}$. *Then it is easy to verify that* $A \preceq^{\textcircled{\dagger},W} B$ *and* $B \preceq^{\textcircled{\dagger},W} A$. *However* $A \neq B$.

Ferreyra et al. [3] established a simultaneous unitarily upper-triangularization of $A \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$.

Lemma 3. [3] *Let* $A \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$. *Then there exist unitary matrices* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{n\times n}$ *such that*

$$A = U \begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix} V^* \text{ and } W = V \begin{bmatrix} W_1 & W_{12} \\ O & W_2 \end{bmatrix} U^*, \tag{3}$$

where A_1, $W_1 \in \mathbb{C}^{t\times t}$ *are non-singular matrices and* A_2W_2, W_2A_2 *are nilpotent of indices* $\mathrm{ind}(AW)$ *and* $\mathrm{ind}(WA)$, *respectively.*
In this case,

$$A^{\textcircled{\dagger},W} = U \begin{bmatrix} (W_1A_1W_1)^{-1} & O \\ O & O \end{bmatrix} V^*. \tag{4}$$

By applying Lemma 3, we have the following result.

Theorem 3. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. If A and W are written as in* (3)*, then the following conditions are equivalent:*
(a) $A \preceq^{\textcircled{\dagger},W} B$;
(b) *there exist* $B_{12} \in \mathbb{C}^{t\times(n-t)},\ B_2 \in \mathbb{C}^{(m-t)\times(n-t)}$ *such that* $B = U\begin{bmatrix} A_1 & B_{12} \\ O & B_2 \end{bmatrix}V^*$ *and*

$$W_1A_{12} + W_{12}A_2 = W_1B_{12} + W_{12}B_2.$$

Proof. Consider the following partition of B:

$$B = U\begin{bmatrix} B_1 & B_{12} \\ B_3 & B_2 \end{bmatrix}V^*. \tag{5}$$

In view of (3)-(5), $AWA^{\textcircled{\dagger},W} = U\begin{bmatrix} W_1^{-1} & O \\ O & O \end{bmatrix}V^*$ and

$$BWA^{\textcircled{\dagger},W} = U\begin{bmatrix} B_1(W_1A_1)^{-1} & O \\ B_3(W_1A_1)^{-1} & O \end{bmatrix}V^*.$$

From $AWA^{\textcircled{\dagger},W} = BWA^{\textcircled{\dagger},W}$, it follows clearly that $B_1 = A_1$ and $B_3 = O$. The equality $A^{\textcircled{\dagger},W}WA = A^{\textcircled{\dagger},W}WB$ leads to $W_1A_{12} + W_{12}A_2 = W_1B_{12} + W_{12}B_2$. Hence $(a) \Rightarrow (b)$ holds. The converse is straightforward. □

3 One-sided pre-orders

In this section, we consider two sets of one-sided weighted binary relations involving the core-EP inverse and then make a comparison of them. According to Definition 2, it is natural to define the left and right-sided relations associated to $\preceq^{\textcircled{\dagger},W}$ as follows:

Definition 3. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$. It is said that*
(a) $A \preceq^{\textcircled{\dagger},W,l} B$ *if* $A^{\textcircled{\dagger},W}(WA) = A^{\textcircled{\dagger},W}(WB)$;
(b) $A \preceq^{\textcircled{\dagger},W,r} B$ *if* $(AW)A^{\textcircled{\dagger},W} = (BW)A^{\textcircled{\dagger},W}$.

It is clear that $A \preceq^{\textcircled{\dagger},W} B$ if and only if $A \preceq^{\textcircled{\dagger},W,l} B$ and $A \preceq^{\textcircled{\dagger},W,r} B$ according to Definitions 2 and 3.

Analogously to Theorem 1, we get characterizations of one-sided weighted core-EP pre-orders.

Theorem 4. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k = \mathrm{ind}(WA)$. Then the following conditions are equivalent:*
(a) $A \preceq^{\textcircled{\dagger},W,l} B$;
(b) $(WA^{\textcircled{\dagger},W})(WA) = (WA^{\textcircled{\dagger},W})(WB)$;
(c) $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$;
(d) $(WA)^*(WA)^k = (WB)^*(WA)^k$.

Theorem 5. *Let* $A,\ B \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *with* $k = max\{\mathrm{ind}(WA), \mathrm{ind}(AW)\}$. *Then the following conditions are equivalent:*

(a) $A \preceq^{\textcircled{\dagger},W,r} B$;
(b) $(AW)(A^{\textcircled{\dagger},W}W) = (BW)(A^{\textcircled{\dagger},W}W)$;
(c) $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$;
(d) $A(WA)^k = B(WA)^k$;
(e) $A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}$.

In light of Theorem 4, Theorem 5 and the proof of Theorem 2, we have the following result.

Theorem 6. *The binary relations* $\preceq^{\textcircled{\dagger},W,l}$ *and* $\preceq^{\textcircled{\dagger},W,r}$ *defined on* $\mathbb{C}^{m\times n}$ *are both pre-orders.*

Proof. Suppose that $A, B \in \mathbb{C}^{m\times n}$. According to Theorem 4, $A \preceq^{\textcircled{\dagger},W,l} B$ is equivalent to $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$. Similarly, according to Theorem 5, $A \preceq^{\textcircled{\dagger},W,r} B$ is equivalent to $A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}$. By the proof of Theorem 2, it is known that both $\preceq^{\textcircled{\dagger},W,l}$ and $\preceq^{\textcircled{\dagger},W,r}$ are pre-orders. □

However, it is possible that we also define the the right and left-sided relations associated to $\preceq^{\textcircled{\dagger},W}$ as follows:

Definition 4. *Let* $A,\ B \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$. *It is said that*

(a) $A \preceq^{\textcircled{\dagger},W,L} B$ *if* $(WA^{\textcircled{\dagger},W})(WA) = (WA^{\textcircled{\dagger},W})(WB)$ *and* $(WA)(WA^{\textcircled{\dagger},W}) = (WB)(WA^{\textcircled{\dagger},W})$;
(b) $A \preceq^{\textcircled{\dagger},W,R} B$ *if* $(AW)(A^{\textcircled{\dagger},W}W) = (BW)(A^{\textcircled{\dagger},W}W)$ *and* $(A^{\textcircled{\dagger},W}W)(AW) = (A^{\textcircled{\dagger},W}W)(BW)$.

In what follows, $\preceq^{\textcircled{\dagger},W,L}$ and $\preceq^{\textcircled{\dagger},W,R}$ are characterized respectively.

Theorem 7. *Let* $A,\ B \in \mathbb{C}^{m\times n}$ *and* $W \in \mathbb{C}^{n\times m}$ *with* $k = \mathrm{ind}(WA)$. *Then the following conditions are equivalent:*

(a) $A \preceq^{\textcircled{\dagger},W,L} B$;
(b) $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$ *and* $(WA)(WA)^{\textcircled{\dagger}} = (WB)(WA)^{\textcircled{\dagger}}$;
(c) $WA \preceq^{\textcircled{\dagger}} WB$;
(d) $(WA)^*(WA)^k = (WB)^*(WA)^k$ *and* $(WA)^{k+1} = (WB)(WA)^k$.

Proof. (a) ⇔ (b) According to Definition 4, $A \preceq^{\textcircled{\dagger},W,L} B$ if and only if $(WA^{\textcircled{\dagger},W})(WA) = (WA^{\textcircled{\dagger},W})(WB)$ and $(WA)(WA^{\textcircled{\dagger},W}) = (WB)(WA^{\textcircled{\dagger},W})$. Observe that $(WA^{\textcircled{\dagger},W}) = (WA)^{\textcircled{\dagger}}$ by Lemma 2. Then (a) ⇔ (b) follows.

(b) ⇔ (c) It is clear by applying Definition 1.

(b) ⇒ (d) Pre-multiply $(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB)$ by $[(WA)^k]^*(WA)$, then

$$\begin{aligned}[(WA)^k]^*(WA) &= [(WA)^k]^*(WA)(WA)^{\textcircled{\dagger}}(WA) = [(WA)^k]^*(WA)(WA)^{\textcircled{\dagger}}(WB) \\ &= [(WA)^k]^*(WB).\end{aligned}$$

Note that $(WA)^*(WA)^k = (WB)^*(WA)^k$ if and only if $[(WA)^k]^*(WA) = [(WA)^k]^*(WB)$ by making an involution.

Post-multiply $(WA)(WA)^{\textcircled{\dagger}} = (WB)(WA)^{\textcircled{\dagger}}$ by $(WA)^{k+1}$, then

$$(WA)^{k+1} = (WA)(WA)^{\textcircled{\dagger}}(WA)^{k+1} = (WB)(WA)^{\textcircled{\dagger}}(WA)^{k+1} = (WB)(WA)^k.$$

(d) $\Rightarrow$ (b) From $[(WA)^k]^*(WA) = [(WA)^k]^*(WB)$, it follows that

$$\begin{aligned}(WA)^{\textcircled{\dagger}}(WA) &= (WA)^{\textcircled{\dagger}}(WA)(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}[(WA)^k((WA)^{\textcircled{\dagger}})^k]^*(WA) \\ &= (WA)^{\textcircled{\dagger}}[((WA)^{\textcircled{\dagger}})^k]^*[(WA)^k]^*(WA) = (WA)^{\textcircled{\dagger}}[((WA)^{\textcircled{\dagger}})^k]^*[(WA)^k]^*(W \\ &= (WA)^{\textcircled{\dagger}}(WB).\end{aligned}$$

Post-multiplying $(WA)^{k+1} = (WB)(WA)^k$ by $[(WA)^{\textcircled{\dagger}}]^{k+1}$, we thus have $(WA)(WA)^{\textcircled{\dagger}} = (WB)(WA)^{\textcircled{\dagger}}$. □

Theorem 8. *Let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$ with $k =max\{\mathrm{ind}(WA), \mathrm{ind}(AW)\}$. Then the following conditions are equivalent:*
(a) $A \preceq^{\textcircled{\dagger},W,R} B$;
(b) $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$ *and* $(WA)^{\textcircled{\dagger}}(WA)W = (WA)^{\textcircled{\dagger}}(WB)W$;
(c) $A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}$ *and* $(WA)^{\textcircled{\dagger}}(WA)W = (WA)^{\textcircled{\dagger}}(WB)W$;
(d) $A(WA)^k = B(WA)^k$ *and* $W^*(WA)^*(WA)^k = W^*(WB)^*(WA)^k$.

Proof. (a) $\Rightarrow$ (b) In view of Definition 4, $A \preceq^{\textcircled{\dagger},W,R} B$ if and only if $(AW)(A^{\textcircled{\dagger},W}W) = (BW)(A^{\textcircled{\dagger},W}W)$ and $(A^{\textcircled{\dagger},W}W)(AW) = (A^{\textcircled{\dagger},W}W)(BW)$. By Lemmas 1 and 2, we have

$$\begin{aligned}(AW)(AW)^{\textcircled{\dagger}} &= (AW)(AW)^D(AW)^k[(AW)^k]^\dagger \\ &= A[(WA)^D(WA)^k]W[(AW)^k]^\dagger \\ &= A[(WA)^{\textcircled{\dagger}}(WA)^k]W[(AW)^k]^\dagger \\ &= [A(WA)^{\textcircled{\dagger}}W](AW)^k[(AW)^k]^\dagger \\ &= (AW)(A^{\textcircled{\dagger},W}W)(AW)^k[(AW)^k]^\dagger \\ &= (BW)(A^{\textcircled{\dagger},W}W)(AW)^k[(AW)^k]^\dagger \\ &= (BW)A[(WA)^{\textcircled{\dagger}}]^2W(AW)^k[(AW)^k]^\dagger \\ &= (BW)A[(WA)^{\textcircled{\dagger}}]^2(WA)^{k+2}[(WA)^D]^2W[(AW)^k]^\dagger \\ &= (BW)A(WA)^k[(WA)^D]^2W[(AW)^k]^\dagger \\ &= (BW)(AW)^D(AW)^k[(AW)^k]^\dagger \\ &= (BW)(AW)^{\textcircled{\dagger}} \text{ and}\end{aligned}$$

$$\begin{aligned}(WA)^{\textcircled{\dagger}}(WA)W &= (WA)^{\textcircled{\dagger}}(WA)^2[(WA)^{\textcircled{\dagger}}]^2WAW = (WA)^{\textcircled{\dagger}}WAW(A^{\textcircled{\dagger},W}WAW) \\ &= (WA)^{\textcircled{\dagger}}WAW(A^{\textcircled{\dagger},W}WBW) = (WA)^{\textcircled{\dagger}}(WB)W.\end{aligned}$$

(b) $\Rightarrow$ (c) Post-multiply $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$ by $A(WA)^{\textcircled{\dagger}}$, then

$$\begin{aligned}(AW)(AW)^{\textcircled{\dagger}}A(WA)^{\textcircled{\dagger}} &= (AW)(AW)^{\textcircled{\dagger}}A(WA)^{k+1}[(WA)^{\textcircled{\dagger}}]^{k+2} \\ &= (AW)(AW)^{\textcircled{\dagger}}(AW)^{k+1}A[(WA)^{\textcircled{\dagger}}]^{k+2} \\ &= (AW)^{k+1}A[(WA)^{\textcircled{\dagger}}]^{k+2} \\ &= A(WA)^{k+1}[(WA)^{\textcircled{\dagger}}]^{k+2} \\ &= A(WA)^{\textcircled{\dagger}}\end{aligned}$$

and analogously

$$(BW)(AW)^{\textcircled{\dagger}}A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}.$$

(c) $\Rightarrow$ (d) From $A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}}$, it follows that

$$A(WA)^k = A(WA)^{\textcircled{\dagger}}(WA)^{k+1} = B(WA)^{\textcircled{\dagger}}(WA)^{k+1} = B(WA)^k.$$

Since $(WA)^{\textcircled{\dagger}}(WA)W = (WA)^{\textcircled{\dagger}}(WB)W$, then $W^*(WA)^*[(WA)^{\textcircled{\dagger}}]^* = W^*(WB)^*[(WA)^{\textcircled{\dagger}}$
Hence,

$$\begin{aligned}W^*(WA)^*(WA)^k &= W^*(WA)^*[(WA)^{\textcircled{\dagger}}]^*WA^*(WA)^k = W^*(WB)^*[(WA)^{\textcircled{\dagger}}]^*WA^*(WA)^k \\ &= W^*(WB)^*(WA)^k.\end{aligned}$$

(d) $\Rightarrow$ (a) Since $A(WA)^k = B(WA)^k$, then

$$\begin{aligned}(AW)(A^{\textcircled{\dagger},W}W) &= (AW)(A[(WA)^{\textcircled{\dagger}}]^2W) = A(WA)^{\textcircled{\dagger}}W = A(WA)^k[(WA)^{\textcircled{\dagger}}]^{k+1}W \\ &= B(WA)^k[(WA)^{\textcircled{\dagger}}]^{k+1}W = B(WA)^{\textcircled{\dagger}}W = (BW)(A^{\textcircled{\dagger},W}W).\end{aligned}$$

By making an involution on $W^*(WA)^*(WA)^k = W^*(WB)^*(WA)^k$, we obtain

$$[(WA)^k]^*WAW = [(WA)^k]^*WBW.$$

Therefore,

$$\begin{aligned}(A^{\textcircled{\dagger},W}W)(AW) &= A[(WA)^{\textcircled{\dagger}}]^2WAW = A[(WA)^{\textcircled{\dagger}}]^2([(WA)^{\textcircled{\dagger}}]^k)^*[(WA)^k]^*WAW \\ &= A[(WA)^{\textcircled{\dagger}}]^2([(WA)^{\textcircled{\dagger}}]^k)^*[(WA)^k]^*WBW = A[(WA)^{\textcircled{\dagger}}]^2WBW \\ &= (A^{\textcircled{\dagger},W}W)(BW).\end{aligned}$$

This completes the proof. □

From Definitions 2 and 4 as well as Theorem 1, it is easy to verify that $A \preceq^{\textcircled{\dagger},W} B$ if and only if $A \preceq^{\textcircled{\dagger},W,L} B$ and $A \preceq^{\textcircled{\dagger},W,R} B$. It is worth mentioning that only one of $A \preceq^{\textcircled{\dagger},W,L} B$ and $A \preceq^{\textcircled{\dagger},W,R} B$ is not sufficient to prove $A \preceq^{\textcircled{\dagger},W} B$, see Examples 1 and 3 in conjunction with Theorems 7 and 8.

Example 3. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$ *and* $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 3}$. *Then*

$$WA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ (WA)^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ WB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ AW = (AW)^{\textcircled{\dagger}} = BW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, $(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}$ *and* $(WA)^{\textcircled{\dagger}}(WA)W = (WA)^{\textcircled{\dagger}}(WB)W$, *i.e.,* $A \preceq^{\textcircled{\dagger},W,R} B$. *However,* $(WA)^{\textcircled{\dagger}}(WA) \neq (WA)^{\textcircled{\dagger}}(WB)$. *That is to say,* $A \npreceq^{\textcircled{\dagger},W} B$.

Using the fact that $\preceq^{\textcircled{\dagger}}$ is a pre-order, we derive the following result.

Theorem 9. *The binary relations* $\preceq^{\textcircled{\dagger},W,L}$ *and* $\preceq^{\textcircled{\dagger},W,R}$ *defined on* $\mathbb{C}^{m\times n}$ *are both pre-orders.*

Proof. Let $A, B \in \mathbb{C}^{m\times n}$. Observe that $A \preceq^{\textcircled{\dagger},W,L} B$ if and only if

$$(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB) \text{ and } (WA)(WA)^{\textcircled{\dagger}} = (WB)(WA)^{\textcircled{\dagger}}$$

by Theorem 7, and $A \preceq^{\textcircled{\dagger},W,R} B$ if and only if

$$A(WA)^{\textcircled{\dagger}} = B(WA)^{\textcircled{\dagger}} \text{ and } (WA)^{\textcircled{\dagger}}(WA)W = (WA)^{\textcircled{\dagger}}(WB)W$$

by Theorem 8. According to the proof of Theorem 2, it is easy to check that $\preceq^{\textcircled{\dagger},W,L}$ and $\preceq^{\textcircled{\dagger},W,R}$ are both pre-orders. □

Remark 2. *Observe that* $A \preceq^{\textcircled{\dagger},W,L} B$ *if and only if* $WA \preceq^{\textcircled{\dagger}} WB$ *in Theorem 7. However, in general,* $A \preceq^{\textcircled{\dagger},W,R} B$ *is not equivalent to* $AW \preceq^{\textcircled{\dagger}} BW$ *in Theorem 8 as the following example shows.*

Example 4. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $W = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 3}$. *Then* $WA = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $(WA)^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $AW = (AW)^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $BW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Thus,*

$$\begin{aligned} &(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}}, \\ &(AW)^{\textcircled{\dagger}}(AW) = (AW)^{\textcircled{\dagger}}(BW), \\ &(WA)^{\textcircled{\dagger}}(WA)W \neq (WA)^{\textcircled{\dagger}}(WB)W. \end{aligned}$$

Hence $AW \preceq^{\textcircled{\dagger}} BW$. *However,* $A \npreceq^{\textcircled{\dagger},W,R} B$.

Likewise, we can illustrate that $A \preceq^{\textcircled{\dagger},W,R} B$ *but* $AW \npreceq^{\textcircled{\dagger}} BW$ *by letting*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2},\ B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \in \mathbb{C}^{3\times 2},\ W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \in \mathbb{C}^{2\times 3}.$$

Here we omit the details.

Let us make a comparison of these two sets of one-sided weighted core-EP pre-orders.

Firstly, it is clear that if $A \preceq^{\textcircled{\dagger},W,L} B$, then $A \preceq^{\textcircled{\dagger},W,l} B$, but the converse may not be true, see the example below.

Example 5. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 3}$. *Then* $WA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $WB = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ *and* $(WA)^{\textcircled{\dagger}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *It is clear that*

$$(WA)^{\textcircled{\dagger}}(WA) = (WA)^{\textcircled{\dagger}}(WB),\ i.e.,\ A \preceq^{\textcircled{\dagger},W,l} B.$$

However, $A \npreceq^{\textcircled{\dagger},W,L} B$ *in light of Theorem 7, as* $(WA)(WA)^{\textcircled{\dagger}} \neq (WB)(WA)^{\textcircled{\dagger}}$.

Secondly, if $A \preceq^{\textcircled{\dagger},W,R} B$, then $A \preceq^{\textcircled{\dagger},W,r} B$. But the converse may not be true, as the following example shows.

Example 6. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 3}$. *It is easy to very that*

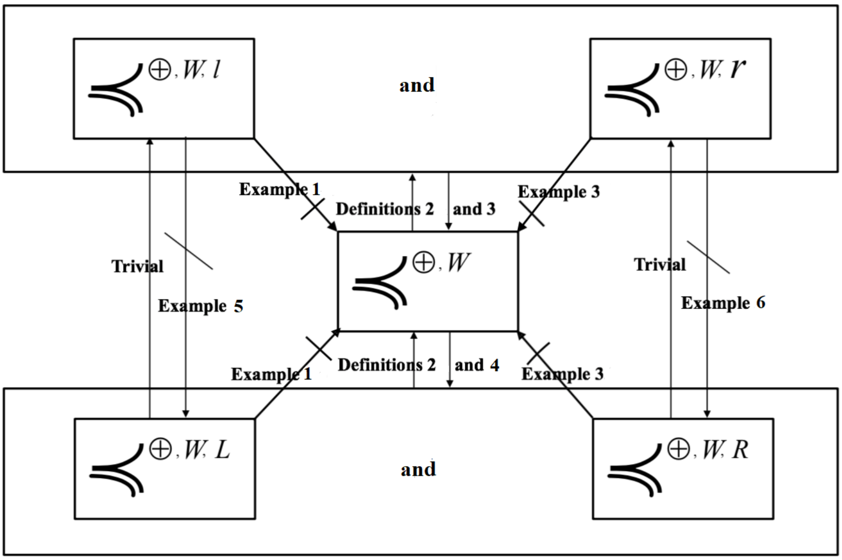
$$(AW)(AW)^{\textcircled{\dagger}} = (BW)(AW)^{\textcircled{\dagger}},\ i.e.,\ A \preceq^{\textcircled{\dagger},W,r} B.$$

However, $(WA)^{\textcircled{\dagger}}(WA)W \neq (WA)^{\textcircled{\dagger}}(WB)W$. *Hence* $A \npreceq^{\textcircled{\dagger},W,R} B$ *in view of Theorem 8.*

A relationship diagram of

$$\preceq^{\textcircled{\dagger},W},\ \preceq^{\textcircled{\dagger},W,L},\ \preceq^{\textcircled{\dagger},W,R},\ \preceq^{\textcircled{\dagger},W,l},\ \preceq^{\textcircled{\dagger},W,r}$$

is provided as follows.



Finally, applying [15, Definition 3.1] to complex matrices, let $A,\ B \in \mathbb{C}^{m\times n}$ and $W \in \mathbb{C}^{n\times m}$, then (one-sided) pre-orders with respect to the W-weighted core-EP inverse are given by

$$A \preceq^{\textcircled{d},W,l} B \text{ if } AW \preceq^{\textcircled{\dagger}} BW;$$
$$A \preceq^{\textcircled{d},W,r} B \text{ if } WA \preceq^{\textcircled{\dagger}} WB;$$
$$A \preceq^{\textcircled{d},W} B \text{ if } A \preceq^{\textcircled{d},W,l} \text{ and } A \preceq^{\textcircled{d},W,r}.$$

We conclude that $A \preceq^{\textcircled{d},W} B$ is stronger than $A \preceq^{\textcircled{\dagger},W} B$.

In fact, it is clear that $A \preceq^{\textcircled{d},W} B$ yields $A \preceq^{\textcircled{\dagger},W} B$, however, the converse is not true in general, for example, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $B = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \end{bmatrix} \in \mathbb{C}^{3\times 2}$, $W = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \in \mathbb{C}^{2\times 3}$ as in Example 4, we thus have $A \preceq^{\textcircled{\dagger},W} B$, whereas $AW \npreceq^{\textcircled{\dagger}} BW$ and hence $A \npreceq^{\textcircled{d},W} B$.

Acknowledgements The authors are highly grateful to the anonymous referees who provided helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (Grant No.11771076) and Sponsored by Shanghai Sailing Program (Grant No. 20YF1433100).

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