Differential Observation and Integral Action in LTI State-Space Controllers and the PID Special Case

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Abstract. This paper makes the case that practical differentiation of measured state variables may be seen as an observation or estimation scheme for linear time invariant state space controllers. It is shown that although not having the separation property, the estimation error of this scheme – differential observation – converges to zero if the resulting closed loop system is strictly stable. On the basis of this concept, it is shown that PID controllers may be interpreted as a special case of state space controllers endowed with differential observation.

Keywords: state observation, differential observation, state space controllers, PID controllers.

1 Introduction

The goal of this paper is two fold. First, one wants to present the concept of differential observation as embodying an approach – beyond full state [1] and Luenberger observers [2] – to estimate states in state space controllers. Second, one wants to show that with the concept of differential observation and the well-known implementation of integral action in state space controllers, one is able to interpret PID controllers [3] as a particular case of state space controllers.

1.1 Differential observation

While not quite referred as such, practically differentiating a signal can be understood as estimating the derivative of the signal. This is so because differentiation is a non-causal operation. As it is believed that non-causal processes cannot be physically implemented, it follows that if one uses an electronic device, say a properly configured op-amp, to calculate the derivative of its input, what one gets is at most an approximation of the values that would be output by an ideal (non-causal) differentiator.

This is a well-known fact in the frequency domain, as the amplitude frequency response of an ideal differentiator should tend to infinity as ω tends to infinity. A practical differentiator must be band limited or must have an upper or corner frequency ω_c above which amplitude ceases to grow. The higher this frequency is, the smaller is the approximation error.

Assume that in a control system, plant variable x_1 is the derivative of variable x_2. One measures x_2 and practically differentiates the measure to obtain x_{s1}. The approximation error $e_{s1} = x_1 – x_{s1}$ cannot be calculated on-line (otherwise x_2 could be differentiated) but under appropriate conditions it can be bound and shown to converge to zero. This allows one to understand x_{s1} as an observation or estimate of x_1. Of course, there are other causes of uncertainty, as disturbances or measurement errors, but the differentiation error e_{s1} is intrinsic to the process.

Full-order or reduced order state observers are supposed to perform only causal operations, but, beyond being subject to disturbances or measurement errors as practical differentiation, they also have an intrinsic and irreducible cause of output error: the uncertainty about the plant model used in the observer. This is not the case of practical differentiators, as they are model free.

Summing up the above observations, we can interpret the operation of practical differentiators as differential observation of state variables at par with reduced-order observers. The output of practical differentiators, as the output of full or reduced order observers, can be seen as estimates of non-measured state variables. It follows that differential observation can be studied as a reduced order observation scheme and that one may compare it with the reduced-order or Luenberger observation scheme. The main, and more or less obvious at first sight, results of this study are:

– In principle, differential observation can be used together with full state feedback as full or reduced order observation can, with partial only state measurement.

– Differential observation does not have the separation property [3]. The poles resulting from upper or corner frequencies ω_{c1}, ω_{c2}, …, *do displace* the closed loop poles from the positions intended with full-state measurement.[1]This effect grows with the lowering of the corner frequencies and diminishes with their increase.

– The effect above can be easily assessed for strict stability of the regulated system modes and of estimation errors, therefore it turns out that differential observation can be incorporated into full state feedback designs as an appropriate observation scheme to consider.

1.2 Integral action

Inserting integral action in a state space controller is well understood. Here, one will be reviewing concepts.

A full state feedback does not warrant zero steady-state errors to references or disturbances that do not tend to zero. Among these, steps and ramps are of special interest to consider.

Steps are often good models not only for many references and disturbances but also, given linearity, to constant components of any input. Obtaining zero error in steady state to a step reference or disturbance for some state variable can be accomplished adding to the system, at most, one integral of the variable error.

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[1] Let one remark that the separation property *only* holds for full or Luenberger observers under the condition that the system model used by the observer perfectly matches the system.

Obtaining zero error in steady state to a ramp reference or disturbance for some state variable can be accomplished adding to the system, at most, two integrals of the variable error. These are well-known results: to get zero steady state error to a time power kt^n reference or disturbance the system transfer function must be of type n – which means that the loop transfer function must have n pure integrators.

Integral action is readily inserted into state-space controllers by associating to each variable for which a requirement of zero steady state error exists one or more states which will be an integral of the variable error, an integral of the integral of the variable error, etc. Interpreting these integrals as additional state variables of an augmented system state leads one to the following known result:

– Integral action can be included in a LTI state space controller by making the controller calculate the number of error integrals required to match steady-state error specifications. Understanding these integrals as additional variables in an augmented system allows one to establish the closed loop poles of this augmented system through a full-state feedback vector.

Let us note, *en passant*, that from a practical point of view, applying integral action is not a free lunch and the price to pay lies somewhere between slow modes of the error integrals and significant worsening of gain and phase margins. Furthermore, use of integral action requires some form of anti-reset windup to counter the effect of actuator saturation. With these convenient remarks in mind, integral action can be applied systematically in state space

1.3 Layers of variables and PID controllers

For the sake of generality, let one assume that a system has n state variables, of which $n/2$ are measured, let us call them the P variables. The other $n/2$ variables are derivatives of the P variables and we will call them the D variables – to be estimated through differential observation. Assume that it is meaningful to specify independent reference trajectories for the P variables as well as finite or null steady-state errors to be attained through integral action. Then, the controller must calculate $n/2$ error integrals, which one will call the I variables.

The D, P and I variables make up the augmented system state upon which a full feedback will be applied. Metaphorically, one can say that the control system has three layers of variables. This leads to the idea of a uniform LTI controller structure to be configured for systems with the properties above.

Now it can be recognized that a usual PID controller makes up a "vertical" slot of such a state space LTI controller. Actually one can see that a PD controller can be interpreted as a full state feedback with differential observation for a second order system with the property of one state variable being the derivative of the other. A PID controller will be the version with integral action of such a controller.

This is conceptually useful in design and teaching as well as in research. We do not need to distinguish between state space and PID controllers – the latter become a special case of the first ones, provided that differentiation is understood as a state variable observation scheme.

1.4 Plan of the paper

Section 2 develops the concept of differential observation in continuous time assuming a flat frequency response above the upper differentiation frequency. Properties of this observation scheme are given as well as the general expression of a state space controlled system with this observation scheme. From this, one obtains that the separation property does not hold for this scheme and practical considerations for its use are indicated.

Section 3 gives a review of integral action in state space controllers together with an example. In Section 4 one develops the interpretation of a PID controller as a state space controller with differential observation. Section 5 concludes with some remarks about further research.

2 Differential observation

If one could make an ideal differentiator, its transfer function would be

$$H_i(s) = \frac{Y(s)}{U(s)} = s \tag{1}$$

implying that the modulus of $H(\mathrm{j}\omega)$ should grow with ω without limit. For a practical differentiator, the modulus of $H_p(\mathrm{j}\omega)$ must cease to grow above some corner frequency ω_c. Here one will assume that the practical differentiator has as transfer function the series of (1) with a first-order low pass whose corner frequency is ω_c:

$$H_p(s) = \frac{Y(s)}{U(s)} = s\frac{\omega_c}{s+\omega_c} \tag{2}$$

This implies:

$$\lim_{\omega\to\infty}\left|H_p(\mathrm{j}\omega)\right| = \omega_c \tag{3}$$

Therefore the condition is fulfilled with $|H_p(\mathrm{j}\omega)|$ becoming constant for frequencies above ω_c. In the time domain, the output and input of such practical differentiator are related as:

$$\frac{dy}{dt} + \omega_c y = \omega_c \frac{du}{dt} \tag{4}$$

Now, let one assume that in a plant one wants to control, state variable x_1 is the derivative of state variable x_2:

$$\frac{dx_2}{dt} = x_1 \tag{5}$$

One can measure x_2 and differentiate it to obtain an estimate x_{s1} of x_1 with a practical differentiator as in (4). The relation between x_{s1} and x_2 becomes:

$$\frac{dx_{s1}}{dt} + \omega_c x_{s1} = \omega_c \frac{dx_2}{dt} \tag{6}$$

Therefore:

$$\frac{dx_{s1}}{dt} + \omega_c x_{s1} = \omega_c x_1 \tag{7}$$

So that x_{s1} equals x_1 passed through a first order low pass with corner frequency ω_c. The estimation error e_1 decreases with the value of ω_c:

$$e_1 = x_1 - x_{s1} = \frac{1}{\omega_c}\frac{dx_{s1}}{dt} \tag{8}$$

Let one assume that the plant is single input n^{th}-order:

$$\frac{dx}{dt} = Ax + Bu \tag{9}$$

Also, a full-state feedback $-K = -[k_1\ k_2\ ...\ k_n]$ has been calculated to set the eigenvalues of the regulated plant to the n desired values $P = [p_1\ p_2\ ...\ p_n]$ assuming full-state measurement. But, instead of full-state measurement, one measures x_2 to x_n and one substitutes x_1 by its estimate above to generate the feedback:

$$u = -[k_1\ k_2 \ldots k_n][x_{s1}\ x_2 \ldots x_n]^T \tag{10}$$

Because $x_{s1} = x_1 - e_1$, the state derivative becomes:

$$\begin{aligned}\frac{dx}{dt} &= Ax + B\left(-[k_1\ k_2 \ldots k_n][x_{s1}\ x_2 \ldots x_n]^T\right)\\ &= Ax + B\left(-[k_1\ k_2 \ldots k_n][x_1 - e_1\ x_2 \ldots x_n]^T\right)\\ &= (A - BK)x + Bk_1 e_1\end{aligned} \tag{11}$$

Let one define A_1 and B_1 as row matrices equaling the first row of A and B, respectively:

$$A_1 = A(1,:) \qquad\qquad B_1 = B(1,:) \tag{12}$$

By (7) and (9) it follows that the derivative of the estimation error e_1 can be written:

$$
\begin{aligned}
\frac{de_1}{dt} &= \frac{dx_1}{dt} - \frac{dx_{s1}}{dt} \\
&= A_1 x + B_1 u - \left(\omega_c x_1 - \omega_c x_{s1}\right) \\
&= A_1 x + B_1\left(-[k_1\ k_2 \ldots k_n][x_{s1}\ x_2 \ldots x_n]^T\right) - \omega_c e_1 \\
&= A_1 x + B_1\left(-[k_1\ k_2 \ldots k_n][x_1 + e_1\ x_2 \ldots x_n]^T\right) - \omega_c e_1 \\
&= \left(A_1 - B_1 K\right)x + B_1 k_1 e_1 - \omega_c e_1 \\
&= \left(A_1 - B_1 K\right)x + \left(B_1 k_1 - \omega_c\right)e_1
\end{aligned}
\tag{13}
$$

One may consider that the regulated system has an augmented state $x_a = [x\ e_1]^T$. It follows that the state equation of the regulated system with full-state feedback and differential estimation of x_1 becomes:

$$
\begin{aligned}
&\frac{dx_a}{dt} = A_r x_a \\
&\begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{de_1}{dt} \end{bmatrix} = \begin{bmatrix} A - BK & Bk_1 \\ A_1 - B_1 K & B_1 k_1 - \omega_c \end{bmatrix} \begin{bmatrix} x \\ e_1 \end{bmatrix}
\end{aligned}
\tag{14}
$$

One may conclude that:

i) If the eigenvalues of A_r are strictly stable, the system is strictly stable and the estimation error converges to zero after an impulse disturbance in the state.

ii) The eigenvalues of A_r will be different from the intended P values. This amounts to say that differential estimation does not enjoy the separation property: the estimator does move the regulated system poles from the intended P positions. This effect diminishes with increasing ω_c, so, the amount of measurement noise in x_2 will basically determine its impact.

The above analysis can readily be generalized for m variables to estimate trough practical differentiation of m measured variables, equivalent results holding. An interesting example to consider is the classical inverted pendulum on a cart where one may want to estimate the pendulum angular velocity and the cart translation velocity from the angular and translation displacement measures.

3 Review of integral action in state space controllers

Let one assume again a n^{th}-order plant:

$$
\frac{dx}{dt} = Ax + Bu
\tag{15}
$$

with a full-state feedback and a servo signal for state variable x_i:

$$u = -Kx + k_i x_{ir} = -k_1 x_1 - \ldots + k_i (x_{ir} - x_i) - \ldots - k_n x_n \tag{16}$$

In (16) it is $K = [k_1,\ldots, k_i,\ldots,k_n]$, such that the closed loop eigenvalues are in intended positions given by a vector P. Also, x_{ir} is the reference variable for x_i and the feedback of variable x_i has been substituted by the error feedback $k_i(x_{ir}-x_i)$. The closed loop system is written:

$$\frac{dx}{dt} = (A - BK)x + Bk_i x_{ir} \tag{17}$$

To add integral action for variable x_i, one gets the controller to calculate a variable x_{n+1} equal to an integral of the error for some arbitrary $t = 0$:

$$x_{n+1}(t) = \int_0^t \left(x_{ir}(\tau) - x_i(\tau) \right) d\tau \tag{18}$$

One interprets (18) as making up together with (15) an augmented system with state $x_a = [x\ x_{n+1}]^T$:

$$\frac{dx_a}{dt} = \begin{bmatrix} \dfrac{dx}{dt} \\ \dfrac{dx_{n+1}}{dt} \end{bmatrix} = \begin{bmatrix} \vdots \\ \dfrac{dx_i}{dt} \\ \vdots \\ \dfrac{dx_{n+1}}{dt} \end{bmatrix} = \begin{bmatrix} & & & \vdots \\ & A & & 0 \\ & & & \vdots \\ \cdots & -1 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \vdots \\ x_i \\ \vdots \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} \vdots \\ 0 \\ \vdots \\ 1 \end{bmatrix} x_{ir}$$
$$= A_a x_a + B_a u + B_{ar} x_{ir}$$
(19)

For this augmented system, a gain vector K_a, with i-th entry k_{ai}, that places the closed loop eigenvalues at desired positions P_a can be calculated giving the closed loop system:

$$\frac{dx_a}{dt} = \left(A_a - B_a K_a \right) x_a + \left[B_a k_{ai} + B_{ar} \right] x_{ir} \tag{20}$$

It follows that integral action can be seamlessly integrated in a state space controller for any number of variables and with any number of integrators for each variable. The appearing practical limitation to this will be the degradation in gain and phase margins implied by increasing the number of pure integrators in the control loop.

4 PID controllers as state space controllers with differential observation

The above analysis allows one to interpret a PID controller as a state space controller with differential observation and integral action. Let one assume a second order plant where state variable x_1 is the derivative of state variable x_2 and only x_2 is measured:

$$\begin{bmatrix} \dfrac{dx_1}{dt} \\ \dfrac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{21}$$

Let one assume that one wants:

i) A full state regulatory feedback for the above system with a servo signal for y to follow a reference y_r.

ii) Differential observation of x_1.

iii) Integral action on the error $e = y_r - y$.

The equations defining the state space controller will be:

$$\begin{aligned} &\frac{dx_{s1}(t)}{dt} + \omega_c x_{s1}(t) = \omega_c \frac{dx_2(t)}{dt} \\ &x_3(t) = \int_0^t \big(y_r(\tau) - y(\tau)\big)d\tau = \int_0^t \big(y_r(\tau) - x_2(\tau)\big)d\tau \\ &u(t) = -k_1 x_{s1}(t) - k_2 x_2(t) - k_3 x_3(t) + k_2 y_r(t) \\ &\quad\;\; = k_2\big(y_r(t) - y(t)\big) - k_3 \int_0^t \big(y_r(\tau) - y(\tau)\big)d\tau - k_1 x_{s1}(t) \end{aligned} \tag{22}$$

It may be recognized that one can summarily describe the above controller by the PI on error D on output rule

$$u(t) = K_p e(t) + K_i \int_0^t e(\tau)\,d\tau + K_d \frac{dy(t)}{dt} \tag{23}$$

where $K_p = k_2$, $K_i = -k_3$, $K_d = -k_1$ and practical differentiation is abbreviated as ideal differentiation.

Some remarks are in order. First, it is clear that PID control rules can only be interpreted as full state feedbacks for second or first order systems – in the latter, derivative action will not be used. Application of PID control rules to third or higher order systems is somehow a kind of "state-deficient" state feedback. Second, any of the possible variations of PID control rules can be accommodated in this interpretation. For example, an also derivative on error PID can be obtained by adding to the command signal in (22) the result of practically differentiating the reference signal. This configuration may be useful for references that change as ramps, but for references that change as steps suffers from the setpoint derivative quick.

Interpretation of state space controllers with differential observation as “stacks” of PID control rules is also possible. For an example, one may refer the many instances in the literature where an inverted pendulum on a cart is stabilized with “PID control”. Stabilization is obtained by differentiating the displacement signals, therefore obtaining estimates of velocities, and computing the command variable as a value that represents a full state feedback. Integral action may as well be added.

5 Conclusion and further research

In this paper, one has presented practical differentiation as a scheme to observe or estimate state variables that are derivatives of measured variables in linear time invariant state space controllers. It was shown that, although this scheme does not enjoy the separation property – intended regulatory closed loop eigenvalues are displaced by the observation eigenvalues –, the estimation error converges to zero if the resulting closed loop is strictly stable.

One has also shown that understanding practical differentiation as a scheme to observe or estimate state variables allows one to interpret PID control rules as full state feedback controllers – with reference following and integral action – for first and second order systems.

There are several aspects which one would like to investigate further. First, it will be in order to extend the approach to discrete time controllers. The fact that these are band limited to the Nyquist frequency makes one expect differences with respect to the continuous time version presented here. An interesting one is that, differently from the continuous time case, high-frequency behavior of discrete differentiation is unique. The above analysis is predicated on a constant high frequency response of practical differentiators and it will be more complicated if more than one pole is considered in the model of a practical differentiator.

Second, the displacement of intended eigenvalues by the observation eigenvalues may be subject to scrutiny in order to quantify its effects in behavior as a function of the corner frequency value of practical differentiators.

Of course, the practical application of differential observation must be assessed. As PID rules are ubiquitously applied, it follows that according to the interpretation developed in this paper, people *do* apply – maybe unwittingly – differential observation.

With relation to full state and Luenberger observers, a differential observer presents the advantage of being model free – one does not to have a system model to get the needed estimates of variables. On the other hand, to design a full state feedback for intended closed loop regulatory eigenvalues one needs a system model – the observation scheme not withstanding. So, if a system model is necessary after all, why not to use a Luenberger observer, with better performance? It may be the case that

PID control rules allow for a simple and effective way to tune experimentally the majority of control loops without much theory and numeric calculations.

As a last remark, one may presume, based on state variables basis change, that for second order systems the condition of one state variable being the derivative of the other may be lifted without loss of the possibility to get a stabilizing full state feedback.

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