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The notion of function held by basic education pre-service teachers

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Abstract: The current curricular guidelines for mathematics education in Portugal emphasize the relevance of working with different representations of functions to promote understanding. Given this relevance, we seek understanding about the notion of function held by 37 basic education pre-service teachers in their first year of a master's course. Data were collected through a task focusing on identifying functions in situations based on different representations. The content analysis technique was then adopted in the search for an understanding of the justifications given by the participants. The results achieved suggest it is easier for the pre-service teachers to identify examples that are not functions than examples that are functions. There is also a tendency for greater accuracy in the identification of examples expressed by tables than by algebraic expressions. The justifications presented show a notion of function as a relation between values of two non-empty sets, but without guaranteeing that this relation is single-valued.

Keywords: basic education; functions; preservice teachers

Introduction

The concept of function is one of the important concepts of Mathematics. According to the Portuguese syllabus, this concept is addressed in formal terms for the first time in the 7^{th} grade (age 12) and keeps being developed until the 12^{th} grade (age 17). However, the inherent characteristics of the concept of function are essential for the introduction of mathematical topics in the first two cycles of basic education. As examples we can consider the existence and uniqueness of the results of arithmetic operations studied at the 1^{st} cycle (grades 1-4), and the relationship between any geometric figure and its area, studied at the 2^{nd} cycle (grades 5-6).

As in most of the countries in the world, the Portuguese education system comprises twelve years before entering higher education. Of these years, the first nine correspond to basic education and the last three to secondary education. In basic education (consisting of three cycles: the first lasts four years - grades 1-4 -, has a unique responsible teacher and is also known as primary school; the second cycle lasts two years - grades 5-6 - and the third three years - grades 7-9), the mathematics' curriculum is the same for all students.

The current curricular guidelines suggest the use of different representations of a function (numerical, tabular, algebraic and graphical), assuming their relevance to the students' understanding. As the concept of function is complex, each representation offers an opportunity to understand what could not be understood in another representation. The connection between different representations creates a global vision, which is more than the union of the knowledge relative to each of the representations, and which allows the development of a deeper understanding. The relevance of the concept of function and the role of representations in their learning makes teachers' knowledge a very important issue. In this study, we seek understanding over the notion of function held by pre-service teachers in their first year of a master's degree in basic education.

Theoretical framework

The evolution of the notion of function and its integration in the school curriculum

The concept of function is seen as one of the most important in all mathematics (Ponte, 1992) and is one of most complex concepts not only of school mathematics but also at undergraduate level (Safuanov, 2015). The evolution of the concept of function goes back 4000 years (Kleiner, 2012) and there are many particular examples of functions that can be found throughout these years, such as counting, which implies a correspondence between a set of objects and a sequence of counting numbers; the four elementary arithmetical operations, which are functions of two variables; and the Babylonian tables of reciprocals, squares, square roots, cubic, and cubic roots (Ponte, 1992). However, the notion of function did not explicitly emerge until early in the eighteenth century. According to Kleiner (2012), this is due to two main reasons: lack of algebraic prerequisites and lack of motivation. For this author, a number of developments were fundamental to the rise of the function concept:

> "- extension of the concept of number to embrace real and (to some extent) even complex numbers (Bombelli, Stifel, et al.);
> - the creation of a symbolic algebra (Viète, Descartes, et al.);
> - the study of motion as a central problem of science (Kepler, Galileo, et al.);
> - the wedding of algebra and geometry (Fermat, Descartes, et al.)." (Kleiner, 2012, p. 104).

According to Ponte (1990), the "origin of the notion of function is confused with the beginnings of Infinitesimal Calculus" (p. 3). The term 'function' was used for the first time by Leibniz in his manuscripts of 1673 (Safuanov, 2015) to denote "the dependence on a curve of geometric quantities as subtangent and subnormal" (Ponte, 1990, p. 3). Also, the terms 'constant' and 'variable' were introduced by Leibniz (Safuanov, 2015).

In the correspondence between Leibniz and Bernoulli from 1694 to 1698, the term 'function' was adopted for the purpose of representing quantities dependent on some variable by means of an analytic expression (Ponte, 1990). The definition of function was first formulated by Bernoulli in 1718, when he considered a function of a certain variable as an amount that is a combination of that variable and constants (Kleiner, 2012). This definition was refined by Euler, a former student of Bernoulli, who replaced the term 'quantity' with 'analytic expression' in 1748 (Ponte, 1990). It was Euler who introduced the notation (x) for the concept of function in 1734 (Safuanov, 2015). The definition proposed by Euler led to several inconsistencies and limitations, since the same function can be represented by different analytical expressions, but it eventually came into force in the eighteenth and nineteenth centuries (Ponte, 1990).

The notion of function has evolved due to its association with the notions of continuity and serial development. One of these developments resulted from Fourier's work, which addressed problems of heat conduction in objects in which he considered body temperature to be a function of two variables (time and space). Fourier conjectured that for any function it would be possible to achieve trigonometric series development at an appropriate interval. This statement was not proved by Fourier, but by Dirichlet, who formulated sufficient conditions for the representability of a function by a Fourier series (Ponte, 1990). In 1837 Dirichlet "then separated the concept of function from its analytical representation, formulating it in terms of arbitrary correspondence between numerical sets" (Ponte, 1990, p. 4). Thus, a function would consist only of a correspondence between two variables, such that for all the value of the independent variable one and only one value of the dependent variable is associated (Ponte, 1990). It was with the development of Cantor's theory of sets that the notion of function come to include anything that was an arbitrary correspondence between any sets, numeric or not. From the notion of correspondence to the notion of relation (Ponte, 1990).

Functions and their representations

Different representations in the teaching and learning of mathematics

Representations are assumed as central for students' learning (NCTM, 2000) and are often used to emphasize important mathematical concepts (Mitchell, Charalambous, & Hill, 2014). Conceptualized as entities that symbolize or stand for other entities (Duval, 2006; Goldin & Kaput, 1996), different representations can elucidate different aspects of the concept. They can help the students who are trying to make sense of the concept, offering some support to organize their ideas and develop mental models of the concept (Mitchell, Charalambous, & Hill, 2014). Simultaneously, the use of different representations can also create the opportunity to consider student diversity, creating space for different ways of reasoning and different preferences (Dreher, Kuntze, & Lerman, 2016). Consequently, representations can make abstract concepts more accessible to the students (Flores, 2002) and foster the connection between procedures and concepts (NCTM, 2000). This is the main reason why working with different representations and the connections among them plays a key role for learners in building up conceptual knowledge in the mathematics classroom (Dreher, Kuntze, & Lerman, 2016). However, as emphasized by Rocha (2016), the mathematical learning does not take place automatically just because the students use different representations. The representations are not inherently transparent (Meira, 1998). Thus, the students need opportunities to reflect on their actions and the teacher's guidance to make connections between representations and underlying mathematical ideas (Stein & Bovalino, 2001). In addition, many times, the teachers only use one representation or do not articulate the different representations used (Nachlieli & Tabach, 2012). This is why Mitchell, Charalambous and Hill (2014) address the ability to teach with representations as a critical component of teaching mathematics well. Dreher, Kuntze and Lerman (2016) go further, highlighting the relevance of specific knowledge and views about using multiple representations and the need to pay attention to it in the professional development of pre-service teachers. After all, only the combination of different representations affords the development of a rich concept image (Tall, 1988) and this requires the teachers' ability to recognize that and to design rich mathematical activities (Dreher, Kuntze, & Lerman, 2016).

Different representations in the teaching and learning of functions

The different representations of mathematical concepts are of great importance in student learning (Viseu, Fernandes, & Martins, 2017), because each of these representations adds or highlights something that is hidden or not prominent in other representations. Thus, the exploration of the different representations in learning is a requirement for a deeper understanding.

In the case of the concept of function, the main representations are the numerical, tabular, graphical and algebraic representations, each of which reveals specific aspects and properties of functions.

Numeric and tabular representations. These representations, which some authors (such as Cuoco, 2001, and Rocha, 2016) consider as distinct and others (such as Goos and Benninson, 2008, and Lesser, 2001) as the same representation, are based on one or more pairs of values of the variables involved in functional relationship. When several pairs of values are presented, the representation facilitates generalization, that is, facilitates discovery of a law of formation, which is characteristic of algebraic representation. In this representation, verifying that we have a function requires the student to analyse the numerical values that are given in the table, according to the relation in question. For Brown and Mehilos (2010), tabular representation facilitates the passage from concrete to abstract, giving meaning to algebraic variables and expressions.

Graphical representation. This representation consists of representing all the points the coordinates of which satisfy the functional relation in question. Compared with other representations, it reveals certain properties of functions, such as zeros, its sign, monotony, etc. In terms of finding whether a relation is a function, the criterion that any vertical line intercepts the graph in at most one point is adopted. According to Friedlander and Tabach (2001), the graphical representation is intuitive and appealing owing to its visual character.

Algebraic representation. This representation, in addition to the sets involved in the function, involves an algebraic relationship between the variables considered in the function. It is a highly compact and abstract representation in which the algebraic law plays a fundamental role in the study of the function, requiring the student to manipulate the relation algebraically to study function properties, including verifying that it is a function. For Friedlander and Tabach (2001), algebraic representation is precise, general and effective in the presentation of mathematical patterns and models and is often the only way to justify general statements.

Research methodology

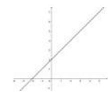
With the main goal of ascertaining the notion of function held by future basic education teachers, we proposed to 37 pre-service teachers, a task with relations between two variables defined through graphs, tables and algebraic expressions that should be classified as functions or not functions. The data were collected by one of the authors of this work, in the academic year 2018/2019, during one of the courses taken by the pre-service teachers. Some of the participants in this study were attending the first year of the master's course in pre-school and basic education (primary school) (M1, $n = 23$) and the others attending the master's course in 1st cycle of basic teaching and Natural Sciences and Mathematics teaching in the 2nd cycle of basic teaching (M2, $n = 14$).

Usually, students who apply to these two master's courses, offered by the same university, have a Basic Education Undergraduate degree. In terms of their learning process, every pre-service teacher studied mathematics at least up to the 9th grade, where they learned the topic of functions using their different representations. More specifically, in the 7th grade, the notion of function is introduced for the first time, the linear function is studied in the 8th grade, and the inverse proportionality and quadratic function of type $f(x) = ax^2,\ a \neq 0$ are studied in the 9th grade. In the task, eight items were proposed to the pre-service teachers, in which they were asked to identify whether each example represents a function or not (Figure 1).

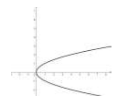
In the task proposed, items a), b), d) and g) involve the representation of relations through graphs; items c) and e) involve the representation of relations through tables; and items f) and h) involve the representation of relations through algebraic expressions. For each of these items, the 37 pre-service teachers were asked to identify if the example represented or not a function, justifying their answer.

Determine whether each relation is a function. Justify your answer.

***a*)**



***b*)**



***c*)**

x	y
1	2
2	3
3	2
4	5

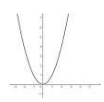
***d*)**

***e*)**

x	y
1	3
1	4
2	6
3	5

***f*)** $x = -2$

***g*)**



***h*)** $y = x^2 - 2x$

Figure 1. Task proposed to the preservice teachers

The answers given by the pre-service teachers were classified as correct (C), partially correct (PC) or incorrect (I). An answer was considered C if, besides the correct classification as function or not function, a correct justification was given. In the cases in which the identification was correct, but no justification or inadequate justification was given, the answer was considered to be PC. Moreover, the frequencies of different types of answers to the different items, according to the master's degree and type of representation used, were considered.

Results

In Table 1 we summarize the answers of the pre-service teachers to the different items of the proposed task, according to the three types of answers considered (C, PC, I) and the situation of having no answer (NA). Globally, considering the eight items all together, 16% of the answers were C, 51% were PC, 20% were I and 13% were NA.

These results highlighted some difficulties for pre-service teachers in identifying functions. This difficulty is reflected by the fact that the frequency of answers classified as partially correct - due to the absence of justification (21.3%) or inadequate justification (26%) - is higher than for the other types of response (Table 2).

Table 1: Frequency of the different types of answers of the pre-service teachers to the 8 items ($n = 37$)

Type of answer	Graphs								Tables				Algebraic expressions			
	a)		b)		d)		g)		c)		e)		f)		h)	
	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2
C	4	1	3	4	5	4	2	–	3	–	9	8	2	2	1	–
PC	16	13	10	3	7	7	12	13	7	9	4	6	11	8	13	12
I	2	–	3	7	4	3	4	1	9	5	6	–	6	4	3	2
NA	1	–	7	–	7	–	5	–	4	–	4	–	4	–	6	–

C-Correct; PC-Partially correct; I-Incorrect; NA-Without any answer.

Table 2: Frequency of answers without justification or with an inadequate justification ($n = 37$)

	Graphs								Tables				Algebraic expressions			
	a)		b)		d)		g)		c)		e)		f)		h)	
	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2	M1	M2
No justification	7	7	3	3	1	3	1	3	3	3	3	5	4	3	10	4
Inadequate justification	9	6	7	–	6	4	6	4	4	6	1	1	7	5	3	8

Now, in Table 3, the analysis is deepened by attending to the master's course being taken by the students and to the type of representation. In terms of the master's courses involved in the study, it can be seen that, in all the eight items, the M2 students present a greater number of C, PC and I answers, while M1 students present a greater number of NA.

Table 3. Percentage of the different types of answers in each of the courses and in each type of representation (*n*=37)

Type of answer	Master course		Type of representation		
	M1	M2	Graphical	Tabular	Algebraic
C	15,8	17,0	15,54	27,03	6,76
PC	43,5	59,8	52,03	35,14	59,46
I	20,0	23,2	18,92	27,03	20,27
NA	20,7	00,0	13,51	10,80	13,51

C-Correct; PC-Partially correct; I-Incorrect; NA-Without any answer.

Considering the type of representation, it can be seen that the number of correct answers in the tabular representation is higher than the ones in graphical and algebraic representations. This arose from the absence of justifications or, in the case of graphical and algebraic representations, inadequate justifications that result in a classification of PC for these answers.

In the next section, some of the students' answers are presented and analysed according to the representation used.

Identifying functions in graphical representations

Among the items where the information is presented through a graph, item a) represents a function and the pre-service teachers justified it saying that: "each object corresponds to one and only one image" (A8, M1; A6, M2). In item g), where the relation represented is also a function, the same justification was given by two pre-service teachers from M1, but by none from M2.

The relations presented in items b) and d) are not functions. For item b), some pre-service teachers stated that it was not a function since "there cannot be two images for each object" (A13, M1) and "to each object corresponds more than one image" (A12, M2). In item d), they said the relation presented is not a function because "to each object corresponds more than one image" (A8, M1); "a unique object has all the images" (A13, M1); "to the object 3 correspond several images" (A1, M2).

Considering the justifications given, we can notice that some pre-service teachers, from both masters' courses enrolled in the study, have a precise notion of function. It is not possible to infer a similar conclusion from PC answers. For items a) and g), some students answered that the relation "is a function", without any justification (A1, M1; A4, M2), while others provided an inadequate justification for their response, as the following statements illustrate: "it is a function because there are two variables involved" (A2, M1; A14, M2); "to each object corresponds an image" (A21, M1; A6, M2).

The same type of answers was obtained for items b) and d). For item b), some students just answered that the example presented "is not a function" (A10, M1; A5, M2), while others considered that it is not a function "since we do know the values of x and y" (A16, M1), or "because to a domain correspond two domains" (A19, M1).

Regarding item d), some students just answered that the example presented "is not a function" (A11, M1; A9, M2), while others added as justification to that answer the statement "because it is a line" (A1, M1; A14, M2), "because there is no correspondence between points of both sets" (A7, M1), or "because it is a line that corresponds to $x = 3$ and in order to be a function there must be two variables" (A15, M2).

Analysing the incorrect answers, in M1 there are students that considered that the example presented in item a) does not represent a function "because there is no correspondence between the points" (A7). For item b), some of the incorrect answers in M1 were due to the idea that the relation presented is a function "because to one coordinate corresponds one and only one object" (A12). Among the incorrect answers by students of M2 to this item, one can find that the relation presented is a function "since there are two variables and it corresponds to a parabola" (A15) or simply "because it is a parabola" (A8).

As far item d) is concerned, the incorrect answers state that the situation represents a function because, for M1 students, "it is a line" (A23) or "to a coordinate corresponds only one object" (A12). The M2 students that incorrectly answered this item also considered that it was a function, because "it is a constant function" (A8).

Last but not least, to item g), the unique incorrect answer that has a justification states that the relation is not a function "since it is a curve" (A11, M1).

Identifying functions in tabular representations

Two of the eight items, items c) and e), present relations using tables. In the case of item c), the relation is a function, since, as A8's answer states, "to each object corresponds one and only one image" (A8, M1), but it was identified as such by only three pre-service teachers. The situation presented in item e) does not represent a function, since "there is more than one y to the same x" (A23, M1); "to the object 1 correspond two images" (A13, M2).

Considering PC answers, in item c), while some students stated only that the relation "is a function" (A7, M1; A15, M2), others justified their answer, writing that "for each x corresponds a y" (A20, M1), or "each object has one image" (A6, M2). Among incorrect answers to this item, there were statements that the relation is not a function because "it just presents a table with some information" (A16, M1), or "every object has more than an image" (A8, M2).

For item e), some of the PC answers just state that "it is not a function" (A19, M1; A7, M2), while in others one can find justifications, such as the answer stating that it is not a function because "you have twice the same value of x" (A11, M1), or "there are two equal objects" (A8, M2).

In the incorrect answers to this item, some students stated that it is a function because "it has two variables x and y" (A2, M1), or because "to a coordinate corresponds only one object" (A12, M1).

Identifying functions in algebraic representations

Of the eight items, two, items, f) and h), present relations using an algebraic representation. For item f), the relation is not a function, and it was identified correctly by only two pre-service teachers from each Master's degree, giving reasons such as "if we imagine its representation on the Cartesian graphic, to the object -2 correspond several images" (A17, M1) or "every image corresponds to a unique object" (A3, M2). In the situation presented in item h), only one pre-service teacher of M1 gave a correct answer, considering that it represents a function because "every object has one and only one image" (A8, M1).

Analysing the PC answers given to item f), we observe that some students state only that the relation "is not a function" (A21, M1; A10, M2), while others justified that conclusion by saying that it is so because "it is a line" (A1, M1), or "it has only one variable" (A14, M2).

Among the incorrect answers to this item, there are justifications for the fact that the relation is a function, including "this value exists in the abscissa x" (A6, M1), or "to the object corresponds an image" (A6, M2).

For h), some PC answers state only that the relation "is a function" (A17, M1; A10, M2), while others present justifications such as "it has two variables" (A5, M1), or "to each value of x corresponds one value of y" (A8, M2).

In the incorrect answers to this item, some students wrote only that "it is not a function" (A19, M1; A7, M2).

Conclusions and implications for mathematics education

The results of this study suggest that most of the pre-service teachers do not have a precise notion of the concept of function. They face difficulties in identifying the essential attributes of a function, strongly evidenced in their written records or in the oral use of the expression 'one and only one', and also in the exploration of different representations, such as tables, graphs and analytical expressions. These results are partially consistent with the ones achieved in other studies and pointing to an incomplete understanding of the concept of function (Steele et al, 2013). However, the difficulties identified in the present study are related not only to the formal definition of the concept, but also to its use.

Usually, in the school context the two essential attributes of the concept of function are not emphasized: 1) it is a binary relation; 2) it is single-valued. Teaching strategies and textbooks tend to state that, in a function, each x element corresponds to one and only one y element. This compressed form of defining a function, in which those two attributes of a function are not explicit, is an obstacle to the understanding of this definition, and contributes to the difficulties faced by the pre-service teachers in justifying the cases where a function is represented.

In general terms, pre-service teachers are more comfortable distinguishing whether a given relationship is or not a function when a tabular representation is presented, than when a graphical representation or even an algebraic representation is involved. This may be due to a greater use of tables, rather than graphs or analytical expressions, in the teaching strategies (especially at early levels). However, it can also be due to the fact that graphical and algebraic representations require more than correspondence between elements. These representations require knowledge about the relationship between ordered pairs and the distinction between variables. It can also happen that the initial learning prevails more over time, but it may simply be that tables are easier for the students to understand.

The algebraic representation seems to be the most difficult one for the pre-service teachers. This is also the representation with the highest number of partially correct answers. This might be due to the difficulty in justifying the answer. In fact, partially correct answers are the most frequent type of answer in all representations.

This analysis about the relation between the representation used and the concept of function is the main contribution brought about by this study, once the approach from most of the studies simply points to some preference for the use of numeric and algebraic representations based on the fact they are the ones used more often (Steele et al, 2013), and fail to actually analyse each representation.

The difficulty that pre-service teachers reveal in connecting the essential attributes of a function across different representations points to a greater use of a single representation when studying the topic of functions, as suggested by Carraher and Schliemann (2007).

In graphical terms, Markovits, Eylon, and Bruckeimer (1998) consider that the students tend to manifest a misconception of linearity that is the idea that the graph of a function is a straight line. This may be related to the predominance in teaching of this type of graph. In this study, such a conjecture is not verified, as evidenced by the similarity between the number of correct answers in the case where the graph is a straight line and in the case where it is a parabola with a horizontal axis of symmetry.

According to Duval (2006), linking the different representations of functions is not easy. Representations are mobilized and developed only if they are transformed into other representations. Thus, highlighting the importance of connections between different representations is central for students' understanding of mathematical concepts. The use of different representations promotes an understanding about what is mathematically relevant in a representation and helps to convert it to another representation and to identify the specific function from the information on that representation.

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References

Brown, A., & Mehilos, M. (2010). Using tables to bridge arithmetic and algebra. *Mathematics Teaching in the Middle School, 15*(9), 532-538.

Carraher, D., & Schliemann, A. (2007). Early algebra and algebraic reasoning. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 669-705). Charlotte, N.C: Information Age Publishing, Inc. and NCTM.

Cuoco, A. (2001). Preface. In A. Cuoco, & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. ix-xiii). Reston: NCTM.

Dreher, A., Kuntze, S., & Lerman, S. (2016). Why use multiple representations in the mathematics classroom? Views of English and German preservice teachers. *International Journal of Science and Mathematics Education, 14* (Suppl 2), S363-S382.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103-131.

Flores, A. (2002). Geometric representations in the transition from arithmetic to algebra. In F. Hitt (Ed.), *Representations and mathematics visualization* (pp. 9-29). Mexico: North American Chapter of IGPME, Cinvestav-IPN.

Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In A. Cuoco, & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 173-185). Reston: NCTM.

Goldin, G., & Kaput, J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. Steffe, P. Nesher, P. Cobb, G. Goldin, & J. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). Hillside, NJ: Lawrence Erlbaum Associates Inc.

Goos, M., & Bennison, A. (2008). Surveying the technology landscape: Teachers' use of technology in secondary mathematics classrooms. *Mathematics Education Research Journal*, *20*(3), 102-130.

Kleiner, I. (2012). A brief history of the function concept. In I. Kleiner (Ed.), *Excursions in the history of mathematics* (pp. 103-124). New York, NY: Springer, 103-124.

Lesser, L. (2001). Representations of reversal: An exploration of Simpson's paradox. In A. Cuoco, & F. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 129-145). Reston: NCTM.

Markovits, Z., Eylon, B., & Bruckeimer, M. (1998). Difficulties students have with the function concept. In A. Coxford, & A. Shulte (Eds.), *The ideas of algebra: K-12* (pp. 43-60). Reston, VA: National Council of Teachers of Mathematics.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education, 29*(2), 121-142.

Mitchell, R., Charalambous, C., & Hill, H. (2014). Examining the task and knowledge demands needed to teach with representations. *Journal of Mathematics Teacher Education, 17*, 37-60.

Nachlieli, T., & Tabach, M. (2012). Growing mathematical objects in the classroom - the case of function. *International Journal of Educational Research, 51/52*, 10-27.

National Council of Teachers of Mathematics (NCTM) (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ponte, J. (1990). O conceito de função no currículo de matemática [The concept of function in mathematics curriculum]. *Educação e Matemática*, *15*, 3-9.

Ponte, J. (1992). The history of concept of function and some educational implications. *The Mathematics Educator*, *3*(2), 3-8.

Rocha, H. (2016). Teacher's representational fluency in a context of technology use.

Teaching Mathematics and its Applications, *35*(2), 53-64.

Safuanov, I. (2015). The history of the concept of a function and its teaching in Russia. In K. Krainer, & N. Vondrová (Eds.), *Proceedings of the ninth congress of the European society for research in mathematics education* (pp.1866-1872). Prague: Charles University in Prague, Faculty of Education and ERME.

Steele, M., Hillen, A., & Smith, M. (2013). Developing mathematical knowledge for teaching in a methods course: The case of function. *Journal of Mathematics Teacher Education, 16*(6), 451-482.

Stein, M., & Bovalino, J. (2001). Manipulatives: One piece of the puzzle. *Mathematics Teaching in the Middle School, 6*, 356-359.

Tall, D. (1988). Concept image and concept definition. In J. de Lange, & M. Doorman (Eds.), *Senior secondary mathematics education* (pp. 37-41). Utrecht: OW & OC.

Viseu, F., Fernandes, J., & Martins, P. (2017). Exploration of the concept of function in tables and graphs by students of the 3rd cycle. *REIPE, E*(1), 11-15.